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SURFACES IN $\mathbb{S}^3$ AND $\mathbb{H}^3$ VIA SPINORS

Bertrand MOREL

Abstract

We generalize the spinorial characterization of isometric immersions of surfaces in $\mathbb{R}^3$ given by T. Friedrich to surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$. The main argument is the interpretation of the energy-momentum tensor associated with a special spinor field as a second fundamental form. It turns out that such a characterization of isometric immersions in terms of a special section of the spinor bundle also holds in the case of hypersurfaces in the Euclidean 4-space.

1. Introduction

It is well known that a description of a conformal immersion of an arbitrary surface $M^2 \hookrightarrow \mathbb{R}^3$ by a spinor field φ on M^2 satisfying the inhomogenous Dirac equation

$$D\varphi = H\varphi, \tag{1}$$

(where D stands for the Dirac operator and H for the mean curvature of the surface), is possible. Recently, many authors investigated such a description (see for example [9],[10],[14]).

In fact, it is clear that any oriented immersed surface $M^2 \hookrightarrow \mathbb{R}^3$ inherits from $\mathbb{R}^3$ a solution of Equation (1), the surface M being endowed with the induced metric and the induced spin structure. Moreover, the solution has constant length. This solution is obtained by the restriction to the surface of a parallel spinor field on $\mathbb{R}^3$. In [6], T. Friedrich clarifies the above-mentioned representation of surfaces in $\mathbb{R}^3$ in a geometrically invariant way by proving the following:

THEOREM 1.1 (FRIEDRICH [6]) *Let (M^2, g) be an oriented, 2-dimensional manifold and $H : M \to \mathbb{R}$ a smooth function. Then the following data are equivalent:*

1. *An isometric immersion $(\widetilde{M}^2, g) \to \mathbb{R}^3$ of the universal covering $\widetilde{M}^2$ into the Euclidean space $\mathbb{R}^3$ with mean curvature H.*

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2. *A solution φ of the Dirac equation*

$$D\varphi = H\varphi\,,$$

with constant length $|\varphi| \equiv 1$.

3. *A pair (φ, T) consisting of a symmetric endomorphism T of the tangent bundle TM such that $\mathrm{tr}(T) = H$ and a spinor field φ satisfying, for any $X \in \Gamma(TM)$, the equation*

$$\nabla_X\varphi + T(X)\cdot\varphi = 0\,.$$

In this paper, we prove the analoguous characterizations for surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$ (Theorems 4.1 and 4.2). They are obtained by studying the equation of restrictions to a surface of real and imaginary Killing spinor fields (compare with [6]).

We note that the involved symmetric endomorphism T is the energy-momentum tensor associated with the restricted Killing spinor which describes the immersion.

Finally, the case of the hypersurfaces of $\mathbb{R}^4$ is treated (Theorem 5.3).

2. Restricting Killing spinor fields to a surface

Let N^3 be a 3-dimensional oriented Riemannian manifold, with a fixed spin structure. Denote by ΣN the spinor bundle associated with this spin structure. If M^2 is an oriented surface isometrically immersed into N^3, denote by ν its unit normal vector field. Then M^2 is endowed with a spin structure, canonically induced by that of N^3. Denote by ΣM the corresponding spinor bundle. The following proposition is essential for what follows (see for example [2],[5],[12],[15]):

Proposition 2.1 *There exists an identification of $\Sigma N_{|M}$ with ΣM, which after restriction to M, sends every spinor field $\psi \in \Gamma(\Sigma N)$ to the spinor field denoted by $\psi^* \in \Gamma(\Sigma M)$. Moreover, if $\underset{N}{\cdot}$ (resp. $\cdot$) stands for Clifford multiplication on ΣN (resp. ΣM), then one has*

$$(X \underset{N}{\cdot} \nu \underset{N}{\cdot} \psi)^* = X\cdot\psi^*\,, \tag{2}$$

for any vector field X tangent to M.

Another important formula is the well-known spinorial Gauss formula: if ∇^N and ∇ stand for the covariant derivatives on $\Gamma(\Sigma N)$ and $\Gamma(\Sigma M)$ respectively, then, for all $X \in TM$ and $\psi \in \Gamma(\Sigma N)$

$$(\nabla^N_X\psi)^* = \nabla_X\psi^* + \frac{1}{2}h(X)\cdot\psi^*, \tag{3}$$

where h is the second fundamental form of the immersion $M \hookrightarrow N$ viewed as a symmetric endomorphism of the tangent bundle of M.

Assume now that N^3 admits a non-trivial Killing spinor field of Killing constant $\eta \in \mathbb{C}$, i.e., a spinor field $\Phi \in \Gamma(\Sigma N)$ satisfying

$$\nabla^N_Y \Phi = \eta Y \underset{N}{\cdot} \Phi \tag{4}$$

for all vector field Y on N. Recall that η has to be real or pure imaginary and that Φ never vanishes on N, as a non-trivial parallel section for a modified connection (see [4],[5]). In what follows, we will consider the model spaces, with their standard metrics, $\mathbb{R}^3$ with $\eta = 0$, $\mathbb{S}^3$ with $\eta = 1/2$, and $\mathbb{H}^3$ with $\eta = i/2$ which are characterized by the fact that they admit a maximal number of linearly independant Killing spinor fields with constant η.

Let (e_1, e_2) be a positively oriented local orthonormal basis of $\Gamma(TM)$ such that (e_1, e_2, ν) is a positively oriented local orthonormal basis of $\Gamma(TN)_{|M}$. Denote by

$$\omega_3 = -e_1 \underset{N}{\cdot} e_2 \underset{N}{\cdot} \nu$$

the complex volume form on the complex Clifford bundle $\mathbb{C}lN$ and $\omega = e_1 \cdot e_2$ the real volume form on $\mathbb{C}lM$. Recall that ω_3 acts by Clifford multiplication as the identity on ΣN. Therefore, denoting $\varphi := \Phi^*$, formula (2) yields

$$(e_1 \underset{N}{\cdot} \Phi)^* = (-e_1 \underset{N}{\cdot} e_1 \underset{N}{\cdot} e_2 \underset{N}{\cdot} \nu \underset{N}{\cdot} \Phi)^* = e_2 \cdot \varphi = -e_1 \cdot \omega \cdot \varphi$$

$$(e_2 \underset{N}{\cdot} \Phi)^* = (-e_2 \underset{N}{\cdot} e_1 \underset{N}{\cdot} e_2 \underset{N}{\cdot} \nu \underset{N}{\cdot} \Phi)^* = -e_1 \cdot \varphi = -e_2 \cdot \omega \cdot \varphi$$

and

$$(\nu \underset{N}{\cdot} \Phi)^* = (-\nu \underset{N}{\cdot} e_1 \underset{N}{\cdot} e_2 \underset{N}{\cdot} \nu \underset{N}{\cdot} \Phi)^* = \omega \cdot \varphi \, .$$

Then, these last relations with Equations (3) and (4) show that

$$\forall X \in TM \, , \qquad \nabla_X \varphi + \frac{1}{2} h(X) \cdot \varphi + \eta X \cdot \omega \cdot \varphi = 0 \tag{5}$$

Recall that the spinor bundle ΣM splits into

$$\Sigma M = \Sigma^+ M \oplus \Sigma^- M$$

where $\Sigma^\pm M$ is the ± 1-eigenspace for the action of the complex volume forme $\omega_2 = i\,\omega$. Under this decomposition, we will denote $\varphi = \varphi^+ + \varphi^-$, and define $\overline{\varphi} := \varphi^+ - \varphi^-$. Therefore Equation (5) is equivalent to

$$\nabla_X \varphi + \frac{1}{2} h(X) \cdot \varphi - i\eta X \cdot \overline{\varphi} = 0 \, .$$

The ambient spinor bundle ΣN can be endowed with a Hermitian inner product $(\ ,)_N$ for which Clifford multiplication by any vector tangent to N is skew-symmetric. This product induces another Hermitian inner product on ΣM, denoted by $(\ ,)$ making the identification of Proposition 2.1 an isometry. Now, relation (2) shows that Clifford multiplication by any vector tangent to M is skew-symmetric with respect to $(\ ,)$.

PROPOSITION 2.2 *If $\eta \in \mathbb{R}$, then φ has constant length. If $\eta \in i\mathbb{R}^*$, then for all vector X tangent to M,*

$$X|\varphi|^2 = 2\Re(i\eta X \cdot \overline{\varphi}, \varphi) \ .$$

Proof . — Since Clifford multiplication by any vector tangent to M is skew-symmetric with respect to $(\ ,)$, we have $\Re(Y \cdot \varphi, \varphi) = 0$ for all $Y \in TM$. Taking this fact into account and computing

$$X|\varphi|^2 = 2\Re(\nabla_X \varphi, \varphi)$$

with the help of formula (5), completes the proof. □

Recalling that the Dirac operator D is defined on $\Gamma(\Sigma M)$ by

$$D = e_1 \cdot \nabla_{e_1} + e_2 \cdot \nabla_{e_2} \ ,$$

we compute directly that

$$D\varphi = H\varphi + 2\eta\, \omega \cdot \varphi = H\varphi - 2i\eta\overline{\varphi}$$

where H is the mean curvature of the immersion $M \hookrightarrow N$. It is well known that the action of the Dirac operator satisfies $(D\varphi)^{\pm} = D\varphi^{\mp}$ (see [11],[5]). Therefore, we note that

$$D(\varphi^{\pm}) = (H \pm 2i\eta)\varphi^{\mp} \ . \tag{6}$$

We have as in [6] the following

PROPOSITION 2.3 *Let M^2 be a minimal surface in N^3. Then the restriction of any Killing spinor Φ with constant η on N^3 restricts to an eigenspinor $\varphi^\star$ on the surface M:*

$$D\varphi^\star = 2\eta\varphi^\star$$

Moreover, if η is real, then $\varphi^\star$ has constant length.

Proof . — Since $H = 0$, we have

$$D(\varphi^{\pm}) = \pm 2i\eta\varphi^{\mp} \ .$$

Therefore, it suffices to define $\varphi^\star = \varphi^+ + i\varphi^-$. □

3. Solutions of the restricted Killing spinor equation

Let (M^2, g) be an oriented, 2-dimensional Riemannian manifold with a spin structure. We endow the spinor bundle ΣM with a Hermitian inner product $(\ ,)$ for which Clifford multiplication by any vector tangent to M is skew-symmetric.

We study now some properties of a given solution $\varphi \in \Gamma(\Sigma M)$ of the following equation

$$\nabla_X \varphi + T(X) \cdot \varphi - i\eta X \cdot \overline{\varphi} = 0 \,, \tag{7}$$

or equivalently

$$\nabla_X \varphi + T(X) \cdot \varphi + \eta X \cdot \omega \cdot \varphi = 0 \,, \tag{8}$$

where T stand for a symmetric endomorphism of the tangent bundle of M, and $\eta \in \mathbb{R} \cup i\mathbb{R}$.

In view of the preceding section and for reasons which will become clearer later, we will call this equation the *restricted Killing spinor equation*. The following proposition shows the role of solutions of the restricted Killing spinor equation in the theory of surfaces in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$. In fact, we see that the integrability conditions for such sections of the spinor bundle are exactly the Gauß and Codazzi-Mainardi equations.

In the following, (e_1, e_2) denotes a positively oriented local orthonormal basis of $\Gamma(TM)$.

Proposition 3.1 *Assume that (M^2, g) admits a non trivial solution of Equation (7) and let $S = 2T$, then*

$$(\nabla_X S)(Y) = (\nabla_Y S)(X) \qquad \textit{(Codazzi-Mainardi Equation)},$$

and

$$R_{1212} - \det(S) = 4\eta^2 \qquad \textit{(Gauß Equation)},$$

where $R_{1212} = g(R(e_1, e_2)\, e_2, e_1)$, and R is the Riemann tensor of M.

Proof . — Let φ a non-trivial solution of (7). We compute the action of the spinorial curvature tensor $\mathcal{R}$ on φ defined for all $X, Y \in TM$ by

$$\mathcal{R}(X,Y)\varphi = \nabla_X \nabla_Y \varphi - \nabla_Y \nabla_X \varphi - \nabla_{[X,Y]} \varphi \,.$$

Since it is skew-symmetric and $\dim M = 2$, with the help of formula (8), we only compute

$$\begin{aligned}
\nabla_{e_1} \nabla_{e_2} \varphi &= \nabla_{e_1}(-T(e_2) \cdot \varphi - \eta e_2 \cdot \omega \cdot \varphi) \\
&= \nabla_{e_1}(-T(e_2) \cdot \varphi - \eta e_1 \cdot \varphi) \\
&= -\nabla_{e_1} T(e_2) \cdot \varphi - T(e_2) \cdot \nabla_{e_1} \varphi - \eta \nabla_{e_1} e_1 \cdot \varphi - \eta e_1 \cdot \nabla_{e_1} \varphi \\
&= -\nabla_{e_1} T(e_2) \cdot \varphi + T(e_2) \cdot T(e_1) \cdot \varphi - \eta T(e_2) \cdot e_2 \cdot \varphi \\
&\quad -\eta \nabla_{e_1} e_1 \cdot \varphi + \eta e_1 \cdot T(e_1) \cdot \varphi - \eta^2 e_1 \cdot e_2 \cdot \varphi
\end{aligned}$$

as well as

$$\begin{aligned}
\nabla_{e_2} \nabla_{e_1} \varphi &= -\nabla_{e_2} T(e_1) \cdot \varphi + T(e_1) \cdot T(e_2) \cdot \varphi + \eta T(e_1) \cdot e_1 \cdot \varphi \\
&\quad +\eta \nabla_{e_2} e_2 \cdot \varphi - \eta e_2 \cdot T(e_2) \cdot \varphi + \eta^2 e_1 \cdot e_2 \cdot \varphi \,.
\end{aligned}$$

So, taking into account that $[e_1, e_2] = \nabla_{e_1} e_2 - \nabla_{e_2} e_1$, a straightforward computation gives

$$\begin{aligned} \mathcal{R}(e_1, e_2)\varphi &= \Big((\nabla_{e_2} T)(e_1) - (\nabla_{e_1} T)(e_2)\Big) \cdot \varphi \\ &\quad -\Big(T(e_1) \cdot T(e_2) - T(e_2) \cdot T(e_1)\Big) \cdot \varphi \\ &\quad -2\eta^2 e_1 \cdot e_2 \cdot \varphi \end{aligned} \tag{9}$$

On the other hand, it is well known that this spinorial curvature tensor corresponds to the Riemann tensor R of M via the relation

$$\mathcal{R}(e_1, e_2)\varphi = -\frac{1}{2} R_{1212} e_1 \cdot e_2 \cdot \varphi \ . \tag{10}$$

Now, it is easy to see that

$$T(e_1) \cdot T(e_2) - T(e_2) \cdot T(e_1) = 2 \det(T) e_1 \cdot e_2$$

and therefore, if we put $S = 2T$ and define the *function*

$$G := R_{1212} - \det(S) - 4\eta^2$$

and the *vector field*

$$C := (\nabla_{e_1} S)(e_2) - (\nabla_{e_2} S)(e_1),$$

Equations (9) and (10) yield

$$C \cdot \varphi = G e_1 \cdot e_2 \cdot \varphi \ .$$

Note that $e_1 \cdot e_2 \cdot \varphi = -i\,\overline{\varphi}$, hence

$$C \cdot \varphi^{\pm} = \pm i G \varphi^{\mp} \ .$$

Applying two times this relation, it suffices to note that

$$||C||^2 \varphi^{\pm} = -G^2 \varphi^{\pm} \ ,$$

and so $C = 0$ and $G = 0$. □

Note that up to rescaling, we can take $\eta = 0$, $1/2$, or $i/2$. The case $\eta = 0$ is treated in [6] and is the starting point of the proof of Theorem 1.1. We will discuss the cases $\eta = 1/2$ and $\eta = i/2$ separately. We begin by

LEMMA 3.2 *Let φ be a non trivial solution of the restricted Killing spinor equation (7). Then*

- *if $\eta = 1/2$, φ has constant norm and the symmetric endomorphism T, viewed as a covariant symmetric 2-tensor, is given by*

$$T(X, Y) = \frac{1}{2} \Re(X \cdot \nabla_Y \varphi + Y \cdot \nabla_X \varphi, \varphi / |\varphi|^2)$$

- *if* $\eta = i/2$, φ *satisfies* $X|\varphi|^2 = -\Re(X\cdot\overline{\varphi},\varphi)$ *and one has*

$$T(X,Y)|\varphi|^2 = \frac{1}{2}\Re(X\cdot\nabla_Y\varphi + Y\cdot\nabla_X\varphi,\varphi) + \frac{1}{2}\Big(|\varphi^-|^2 - |\varphi^+|^2\Big)g(X,Y)$$

Proof . — The first claim of each case is proved in Proposition 2.2. Let $T_{jk} = g(T(e_j),e_k)$, then, for $j = 1,2$,

$$\nabla_{e_j}\varphi = -\sum_{k=1}^{2} T_{jk}e_k\cdot\varphi + i\eta e_j\cdot\overline{\varphi}\ .$$

Taking Clifford multiplication by e_l and the scalar product with φ, we get

$$\Re(e_l\cdot\nabla_{e_j}\varphi,\varphi) = -\sum_{k=1}^{2} T_{jk}\Re(e_l\cdot e_k\cdot\varphi,\varphi) + \Re(i\eta e_l\cdot e_j\cdot\overline{\varphi},\varphi)\ .$$

Since $\Re(e_l\cdot e_k\cdot\varphi,\varphi) = -\delta_{lk}|\varphi|^2$, it follows, by symmetry of T

$$\Re(e_l\cdot\nabla_{e_j}\varphi + e_j\cdot\nabla_{e_l}\varphi,\varphi) = 2T_{lj}|\varphi|^2 - 2\Re(i\eta\overline{\varphi},\varphi)\delta_{lj}\ .$$

This completes the proof by taking $\eta = 1/2$ or $\eta = i/2$. □

Now, we prove that the necessary conditions on a spinor field $\psi \in \Gamma(\Sigma M)$ obtained in the previous section (i.e. Proposition 2.2 and Equation (6)) are enough to prove that ψ is a solution of the restricted Killing spinor equation.

The case $\boldsymbol{\eta = 1/2}$: Consider a non-trivial spinor field ψ of constant length, which satisfies $D\psi^\pm = (H \pm i)\psi^\mp$. Define the following 2-tensors on (M^2,g)

$$T^\pm(X,Y) = \Re(\nabla_X\psi^\pm, Y\cdot\psi^\mp)\ .$$

First note that

$$\mathrm{tr}T^\pm = -\Re(D\psi^\pm,\psi^\mp) = -\Re((H\pm i)\psi^\mp,\psi^\mp) = -H|\psi^\mp|^2\ . \tag{11}$$

We have the following relations

$$\begin{aligned}
T^\pm(e_1,e_2) &= \Re(\nabla_{e_1}\psi^\pm, e_2\cdot\psi^\mp) = \Re(e_1\cdot\nabla_{e_1}\psi^\pm, e_1\cdot e_2\cdot\psi^\mp)\\
&= \Re(D\psi^\pm, e_1\cdot e_2\cdot\psi^\mp) - \Re(e_2\cdot\nabla_{e_2}\psi^\pm, e_1\cdot e_2\cdot\psi^\mp)\\
&= \Re((H\pm i)\psi^\mp, e_1\cdot e_2\cdot\psi^\mp) + \Re(\nabla_{e_2}\psi^\pm, e_1\cdot\psi^\mp)\\
&= |\psi^\mp|^2 + T^\pm(e_2,e_1)\ .
\end{aligned} \tag{12}$$

Lemma 3.3 *The* 2*-tensors* $T^\pm$ *are related by the equation*

$$|\psi^+|^2T^+ = |\psi^-|^2T^-$$

Proof . — This relation is trivial at any point $p \in M$ where $|\psi^+|^2$ or $|\psi^-|^2$ vanishes. Therefore we can assume in the following that both spinors ψ^+ and ψ^- are not zero in the neighbourhood of a point in M.

With respect to the scalar product $\Re(\, , \,)$, the spinors

$$e_1 \cdot \frac{\psi^-}{|\psi^-|} \quad \text{and} \quad e_2 \cdot \frac{\psi^-}{|\psi^-|}$$

form a local orthonormal basis of $\Gamma(\Sigma^+ M)$. Hence, in this basis, we can write

$$\begin{aligned} \nabla_X \psi^+ &= \Re(\nabla_X \psi^+, e_1 \cdot \frac{\psi^-}{|\psi^-|})\, e_1 \cdot \frac{\psi^-}{|\psi^-|} + \Re(\nabla_X \psi^+, e_2 \cdot \frac{\psi^-}{|\psi^-|})\, e_2 \cdot \frac{\psi^-}{|\psi^-|} \\ &= \frac{T^+(X)}{|\psi^-|^2} \cdot \psi^- \end{aligned}$$

where the vector field $T^+(X)$ is defined by

$$g(T^+(X), Y) = T^+(X, Y)\,, \qquad \forall Y \in TM\,.$$

In the same manner, we can show that

$$\nabla_X \psi^- = \frac{T^-(X)}{|\psi^+|^2} \cdot \psi^+\,.$$

Since ψ has constant length, for all vector X tangent to M, we have

$$\begin{aligned} 0 &= X|\psi|^2 = X(|\psi^+|^2 + |\psi^-|^2) \\ &= 2\Re(\nabla_X \psi^+, \psi^+) + 2\Re(\nabla_X \psi^-, \psi^-) \\ &= 2\Re(W(X) \cdot \psi^-, \psi^+) \end{aligned} \tag{13}$$

with

$$W(X) = \frac{T^+(X)}{|\psi^-|^2} - \frac{T^-(X)}{|\psi^+|^2}\,.$$

To conclude, it suffices to note that Equations (11) and (12) imply W is traceless and symmetric, and that Equation (13) implies that W has rank less or equal to 1. This obviously implies $W = 0$. □

Proposition 3.4 *Assume that there exists on (M^2, g) a non-trivial solution ψ of the equation $D\psi = H\psi - i\overline{\psi}$ with constant length. Then such a solution satisfies the restricted Killing spinor equation with $\eta = 1/2$.*

Proof . — Let $F := T^+ + T^-$. Lemma 3.3 and the begining of its proof imply

$$\frac{F}{|\psi|^2} = \frac{T^+}{|\psi^-|^2} = \frac{T^-}{|\psi^+|^2}\,.$$

Hence $F/|\psi|^2$ is well defined on the whole surface M, and

$$\nabla_X \psi = \nabla_X \psi^+ + \nabla_X \psi^- = \frac{F(X)}{|\psi|^2} \cdot \psi \tag{14}$$

where the vector field $F(X)$ is defined by $g(F(X), Y) = F(X, Y)$, $\forall Y \in TM$. Note that by Equation (12), the 2-tensor F is not symmetric. Define now the symmetric 2-tensor

$$T(X, Y) = -\frac{1}{2|\psi|^2} \left(F(X, Y) + F(Y, X)\right) \ .$$

Observe that T is defined as in Lemma 3.2. It is straigthforward to show that

$$T(e_1, e_1) = -F(e_1, e_1)/|\psi|^2 \quad , \quad T(e_2, e_2) = -F(e_2, e_2)/|\psi|^2 \ ,$$

$$T(e_1, e_2) = -F(e_1, e_2)/|\psi|^2 + \frac{1}{2} \quad \text{and} \quad T(e_2, e_1) = -F(e_2, e_1)/|\psi|^2 - \frac{1}{2}$$

once more by Equation (12). Taking into account these last relations in Equation (14), we conclude

$$\nabla_X \psi = -T(X) \cdot \psi - \frac{1}{2} X \cdot \omega \cdot \psi \ .$$

□

The case $\eta = i/2$:

PROPOSITION 3.5 *Assume that there exists on (M^2, g) a nowhere vanishing solution ψ of the equation $D\psi = H\psi + \overline{\psi}$. Then, if this solution satisfies*

$$X|\psi|^2 = -\Re(X \cdot \overline{\psi}, \psi) \ , \qquad \forall X \in \Gamma(TM),$$

then it is solution of the restricted Killing spinor equation with $\eta = i/2$.

Proof . — Defining the 2-tensors $T^{\pm}$ as in the previous case, we get

$$\mathrm{tr} T^{\pm} = -(H \mp 1)|\psi^{\mp}|^2 \ , \tag{15}$$

and

$$T^{\pm}(e_1, e_2) = T^{\pm}(e_2, e_1) \ . \tag{16}$$

First note that

$$-\Re(X \cdot \overline{\psi}, \psi) = -\Re(X \cdot \psi^+, \psi^-) + \Re(X \cdot \psi^-, \psi^+) = 2\Re(X \cdot \psi^-, \psi^+) \ .$$

Therefore, following the proof of Lemma 3.3, we get

$$\Re(X \cdot \psi^-, \psi^+) = \Re(W(X) \cdot \psi^-, \psi^+) \tag{17}$$

with

$$W(X) = \frac{T^+(X)}{|\psi^-|^2} - \frac{T^-(X)}{|\psi^+|^2} \ .$$

As in the previous case, Equations (15), (16) and (17) imply that $W - \mathrm{Id}_{TM}$ is a symmetric, traceless endomorphism of rank not greater than 1, hence $W = \mathrm{Id}_{TM}$ and we have the relation

$$|\psi^+|^2 T^+ - |\psi^-|^2 T^- = |\psi^+|^2 |\psi^-|^2 g \, .$$

Therefore, if we define the symmetric 2-tensor $F = T^+ + T^- + \frac{1}{2}(|\psi^+|^2 - |\psi^-|^2)g$, we have on the whole surface M

$$\frac{F}{|\psi|^2} = \frac{T^+ + T^- + (|\psi^+|^2 - |\psi^-|^2)g}{|\psi^+|^2 + |\psi^-|^2} = \frac{T^-}{|\psi^+|^2} + \frac{1}{2}g = \frac{T^+}{|\psi^-|^2} - \frac{1}{2}g \, .$$

On the other hand, we get

$$\nabla_X \psi = \nabla_X \psi^+ + \nabla_X \psi^- = \frac{T^+(X)}{|\psi^-|^2} \cdot \psi^- + \frac{T^-(X)}{|\psi^+|^2} \cdot \psi^+ \, .$$

These two last equations imply

$$\nabla_X \psi = \frac{F(X)}{|\psi|^2} \cdot (\psi^+ + \psi^-) + \frac{1}{2} X \cdot \psi^- - \frac{1}{2} X \cdot \psi^+ \, ,$$

which is equivalent to

$$\nabla_X \psi = -T(X) \cdot \psi - \frac{1}{2} X \cdot \overline{\psi} \, .$$

Naturally, we put $T = -\frac{F}{|\psi|^2}$ and note that T is defined as in Lemma 3.2. □

4. Surfaces in $\mathbb{S}^3$ or $\mathbb{H}^3$

We are now able to generalize Theorem 1.1 to surfaces in $\mathbb{S}^3$ or $\mathbb{H}^3$. In section 2, we saw that an oriented, immersed surface $M^2 \hookrightarrow \mathbb{S}^3$ (resp. $\mathbb{H}^3$) inherits an induced metric g, a spin structure, and a solution φ of

$$D\varphi = H\varphi - i\overline{\varphi} \qquad (\text{resp. } D\varphi = H\varphi + \overline{\varphi}\,) \tag{18}$$

with constant length (resp. with $X|\varphi|^2 = -\Re(X \cdot \overline{\varphi}, \varphi)$ for all vector X tangent to M). This spinor field φ on M^2 is the restriction of a real (resp. imaginary) Killing spinor field in $\mathbb{S}^3$ (resp. $\mathbb{H}^3$). Section 3 shows that at least locally the converse is true. Assume that there exists a solution of Equation (18) on an oriented, 2-dimensional Riemannian manifold (M^2, g) endowed with a spin structure, for a given function $H : M \to \mathbb{R}$. Then this solution satisfies the restricted Killing spinor equation with a well defined endomorphism $T : TM \to TM$ with $\mathrm{tr}T = H$. Moreover, there exists an isometric immersion $(M^2, g) \hookrightarrow \mathbb{S}^3$ (resp. $\mathbb{H}^3$) with second fundamental form $S = 2T$.

THEOREM 4.1 *Let (M^2, g) be an oriented, 2-dimensional manifold and $H : M \to \mathbb{R}$ a smooth function. Then the following data are equivalent:*

1. *An isometric immersion $(\tilde{M}^2, g) \to \mathbb{S}^3$ of the universal covering $\tilde{M}^2$ into the 3-dimensional round sphere $\mathbb{S}^3$ with mean curvature H.*

2. *A solution φ of the Dirac equation*

$$D\varphi = H\varphi - i\overline{\varphi}$$

with constant length.

3. *A pair (φ, T) consisting of a symmetric endomorphism T such that $\mathrm{tr}(T) = H$ and a spinor field φ satisfying the equation*

$$\nabla_X\varphi + T(X)\cdot\varphi - \frac{i}{2}X\cdot\overline{\varphi} = 0\,.$$

THEOREM 4.2 *Let (M^2, g) be an oriented, 2-dimensional manifold and $H : M \to \mathbb{R}$ a smooth function. Then the following data are equivalent:*

1. *An isometric immersion $(\tilde{M}^2, g) \to \mathbb{H}^3$ of the universal covering $\tilde{M}^2$ into the 3-dimensional hyperbolic space $\mathbb{H}^3$ with mean curvature H.*

2. *A nowhere vanishing solution φ of the Dirac equation*

$$D\varphi = H\varphi + \overline{\varphi}$$

satisfying

$$X|\varphi|^2 = -\Re(X\cdot\overline{\varphi}, \varphi) \quad \forall X \in \Gamma(TM).$$

3. *A pair (φ, T) consisting of a symmetric endomorphism T such that $\mathrm{tr}(T) = H$ and a spinor field φ satisfying the equation*

$$\nabla_X\varphi + T(X)\cdot\varphi + \frac{1}{2}X\cdot\overline{\varphi} = 0 \quad \forall X \in \Gamma(TM).$$

REMARK 4.3 It has been pointed out to us that the case of surfaces in $\mathbb{S}^3$ has already been treated by Leonard Voss (*Diplomarbeit, Humboldt-Universität zu Berlin*, unpublished).

5. Hypersurfaces in $\mathbb{R}^4$

We conclude by giving a characterization of hypersurfaces in the Euclidean 4-space in terms of a special section of the intrinsic spinor bundle of the hypersurface, in a very similar way to that of Theorem 1.1.

Let M^3 be an oriented hypersurface isometrically immersed into $\mathbb{R}^4$, denote by ν its unit normal vector field. Then M^3 is endowed with a spin structure, canonically induced by that of $\mathbb{R}^4$. Denote by ΣM the corresponding spinor bundle and $\Sigma^+\mathbb{R}^4$ the bundle of positive spinors in $\mathbb{R}^4$. We then have the anologous result of Proposition 2.1:

PROPOSITION 5.1 *There exists an identification of* $\Sigma^+\mathbb{R}^4$ *with* ΣM*, which after restriction to* M*, sends every spinor field* $\psi \in \Gamma(\Sigma^+\mathbb{R}^4)$ *to the spinor field denoted by* $\psi^* \in \Gamma(\Sigma M)$*. Moreover, if* $\underset{\mathbb{R}^4}{\cdot}$ *(resp.* $\cdot$*) stands for Clifford multiplication on* $\Sigma^+\mathbb{R}^4$ *(resp.* ΣM*), then one has*

$$(X \underset{\mathbb{R}^4}{\cdot} \nu \underset{\mathbb{R}^4}{\cdot} \psi)^* = X \cdot \psi^* \ , \tag{19}$$

for any vector field X *tangent to* M*.*

Recall the following definition

DEFINITION 5.2 *A symmetric 2-tensor* $T \in S^2(M)$ *is called a* Codazzi tensor *if it satisfies the Codazzi-Mainardi equation, i.e.*

$$(\nabla_X T)(Y) = (\nabla_Y T)(X) \qquad \forall X, Y \in \Gamma(TM) \ ,$$

(T *being viewed in this formula via the metric* g *as a symmetric endomorphism of the tangent bundle).*

We now prove the following

THEOREM 5.3 *Let* (M^3, g) *be an oriented, 3-dimensional Riemannian manifold. Then the following data are equivalent:*

1. *An isometric immersion* $(\widetilde{M}^3, g) \to \mathbb{R}^4$ *of the universal covering* $\widetilde{M}^3$ *into the Euclidean space* $\mathbb{R}^4$ *with second fundamental form* h*.*
2. *A pair* (φ, T) *consisting of a Codazzi tensor* T *such that* $2T = h$ *and a non trivial spinor field* φ *satisfying, for any* $X \in \Gamma(TM)$*, the equation*

$$\nabla_X \varphi + T(X) \cdot \varphi = 0 \, .$$

Proof . — Let (M^3, g) be an oriented hypersurface isometrically immersed into $\mathbb{R}^4$ with second fundamental form h. Let ψ be any parallel positive spinor field on $\mathbb{R}^4$. Denote by $\varphi := \psi^* \in \Gamma(\Sigma M)$ the restriction of ψ given by Proposition 5.1. Then Gauß formula (3) yields

$$\nabla_X \varphi + \frac{1}{2} h(X) \cdot \varphi = 0 \ .$$

Since h is a second fundamental form, it is clear that $T = \frac{1}{2}h$ is a Codazzi tensor and that (φ, T) give the desired pair.

Conversely, if (M^3, g) is an oriented, 3-dimensional Riemannian manifold admitting such a pair (φ, T), then obviously Codazzi-Mainardi equation holds for $h = 2T$.

Therefore, the action of the spinorial curvature tensor on the spinor φ is given by

$$\mathcal{R}(X,Y)\varphi = \Big(T(Y)\cdot T(X) - T(X)\cdot T(Y)\Big)\cdot\varphi \tag{20}$$

Let (e_1, e_2, e_3) be a positively oriented local orthonormal basis of $\Gamma(TM)$. Then Equation (20) yields

$$\sum_{k\neq l}\Big(\mathcal{R}_{ijkl} + 4T_{il}T_{jk} - 4T_{ik}T_{jl}\Big)e_k\cdot e_l\cdot\varphi = 0$$

which imply in dimension 3 that each componant

$$\mathcal{R}_{ijkl} + 4T_{il}T_{jk} - 4T_{ik}T_{jl}$$

is zero, since for $1\leq k<l\leq 3$ and $1\leq k'<l'\leq 3$,

$$\Re(e_k\cdot e_l\cdot\varphi, e_{k'}\cdot e_{l'}\cdot\varphi) = \pm\delta_{kk'}\delta_{ll'}|\varphi|^2\ .$$

Therefore $h=2T$ satisfies the Gauß equation. □

REMARK 5.4 *Let (φ, T) be a pair as in Theorem 5.3 (2). Then necessarily the Codazzi tensor T has to be defined as the energy-momentum tensor associated with the spinor field φ (see for example [7], [8] or [13]). Such a special spinor field is then called a* Codazzi Energy-Momentum spinor*, and generalizes the notion of Killing spinors (see [1], [3], [13]).*

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