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CONSTANT CURVATURE $(2+1)$-SPACETIMES AND PROJECTIVE STRUCTURES

Francesco BONSANTE

Abstract

Nous illustrons une classification des espaces-temps $(2+1)$ globalement hyperboliques à courbure constant, en termes de certaines structures projectives complexes portées par les surfaces de niveau de leur temps *cosmologique canonique*. Ceci dérive d'une théorie des *rotations de Wick canoniques*, développée en collaboration avec Riccardo Benedetti [6], qui sera egalement brièvement illustrée.

1. Introduction

Given an orientable surface S we are interested in the space $\mathcal{MS}_k(S)$ of Lorentzian structures on $S\times\mathbb{R}$ of constant curvature k such that $S\times\{0\}$ is a complete Cauchy surface (see Sec. 2). The results we will explain in this work have been achieved by R. Benedetti and the author within a more general research [6]. We will present a general overview on that theory, stressing on the description of $\mathcal{MS}_k(S)$ arising in that framework. Proofs will be just outlined, omitting technical details.

In a seminal work [17], G. Mess gave a full classification of the space $\mathcal{MS}_k(S)$ assuming that the surface was compact and $k\in\{0,-1\}$. The de Sitter case (corresponding to $k=1$) was carried over some years later by K. Scannell [18], who developed an original Mess remark. The key result of their works is that the space $\mathcal{MS}_k(S)$ is homeomorphic to the cotangent bundle of the Teichmuller space of S, provided that the genus of S is greater than 2.

In [7], Benedetti and Guadagnini pointed out the cosmological time as fundamental tool to better understand flat globally hyperbolic spacetimes classified by Mess. In fact, the cosmological time turns to be an important object also in [18]. A remarkable fact is that, in both the contexts, level surfaces of the cosmological time are obtained by *grafting* a hyperbolic surface F (homeomorphic to S) along a measured geodesic lamination λ. Moreover Mess parameters are explicitely related to the pair (F,λ) (actually they furnish good parameters for the space $\mathcal{MS}_k(S)$).

A very similar behaviour holds in the Anti de Sitter framework, even if in this case cosmological time is a $C^{1,1}$-function until it reaches the value $\pi/2$. Anyway, also in this case level surfaces for values $< \pi/2$ are obtained by grafting a hyperbolic surface along a measured geodesic lamination, and these data determine the spacetime.

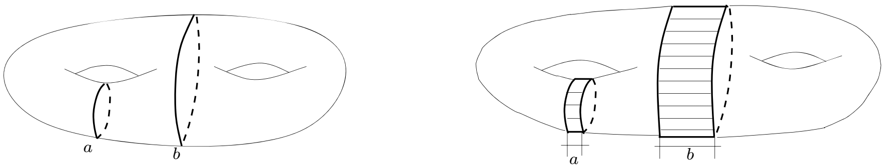


Figure 1: The grafting along a weighted multicurve. The annuli carry a Euclidean metric.

Let us recall that grafting is a procedure to obtain from a hyperbolic surface $F \cong S$ another Riemannian structure on S equipped with a complex projective structure. An important result, due to Thurston, is that the map

$$(F, \lambda) \mapsto Gr_\lambda(F)$$

gives rise to a bijection between the space $\mathcal{T}_g \times \mathcal{ML}_g$ of measured geodesic laminations on hyperbolic structures, and the space $\mathcal{P}(S)$ of complex projective structures on S.

When S is not compact, one could try to generalize Mess and Thurston constructions. In fact, the notion of measured geodesic lamination can be implemented for every hyperbolic surface, and it is not difficult to see that Mess and Thurston constructions work as well. But in this case they do not give rise to a complete classification of $\mathcal{MS}_k(S)$ or $\mathcal{P}(S)$ (*i.e.*, there are globally hyperbolic spacetimes of constant curvature such that the cosmological level sets are not obtained by grafting a hyperbolic surface along a measured geodesic lamination).

There are two natural problems arising from this remark:

1) To find out a more general notion of measured geodesic lamination coinciding with the usual one in the compact case, that allows to generalize Mess and Thurston constructions to obtain complete classifications of $\mathcal{MS}_k(S)$ as well as $\mathcal{P}(S)$.

2) To make explicit the identifications between $\mathcal{P}(S)$ and $\mathcal{MS}_k(S)$ for compact S (arising from Thurston and Mess parameterizations) in order to see whether they can be generalized in the non-compact case.

In [16], Kulkarni and Pinkall introduced the notion of measured geodesic lamination on a straight convex set that allows to carry out a complete classification of projective structures on a surface S with non-abelian fundamental group. Actually, they showed that the Thurston construction could be applied to these more general laminations and every projective structure could be constructed in such a way.

In [6], we showed that also Mess constructions could be applied to these laminations and they leads to a complete classification of $\mathcal{MS}_k(S)$.

In the flat case, the proof is based on [4], that points out a clear picture of the universal covering spaces and the linear holonomies of globally hyperbolic flat spacetimes. On the other hand, in [11] the universal covering spaces are classified in terms of measured geodesic laminations on straight convex sets.

The proofs in de Sitter and the Anti de Sitter case are carried over both by developing Mess-Scannell ideas in this more general case, and by using an explicit map

$$\mathcal{MS}_0(S) \to \mathcal{MS}_k(S)$$

that solves question *2)* . In fact, such a map is constructed by developing a *canonical Wick rotation and rescaling theory.*

Before going further, let us briefly introduce those notions. In general, given a manifold M, a no-where vanishing vector field X, and a pair of positive functions α, β, the Wick rotation is an operation transforming Riemannian metrics on M into Lorentzian metrics that make X a timelike vector field. Namely, given a Riemannian metric g the metric $h = W_{(X,\alpha,\beta)}(g)$ obtained by the Wick rotation of g along X with rescaling function α and β is determined by the following properties

1. $X^{\perp_g} = X^{\perp_h} = X^{\perp}$.
2. $h|_{X^\perp} = \alpha g|_{X^\perp}$.
3. $h(X,X) = -\beta g(X,X)$.

Clearly, the Wick rotation can be also regarded as an operation transforming Lorentzian metrics making X a timelike vector field into Riemannian metrics.

On the other hand the rescaling is a similar operation depending on a vector field X and two positive functions α, β and acting on the space of Lorentzian metrics that make X a timelike vector field. The main difference with the Wick rotation is that it preserves the signature of the metrics. Namely, the rescaled metric $h = R_{(X,\alpha,\beta)}(g)$ is determined by properties *1., 2.* (the same used to define the Wick rotation) and

Let us outline the scheme we follow to develop the announced Wick rotation rescaling theory.

– We prove that every maximal globally hyperbolic *flat* spacetime $\cong S \times \mathbb{R}$ is equipped with a $C^{1,1}$ cosmological time (provided that $\pi_1(S)$ is not abelian).

– We point out a canonical Wick Rotation on $M(>1) := T^{-1}((1, +\infty))$ directed along the gradient of T that yields a hyperbolic metric. Moreover this hyperbolic structure extends to a (complex) projective structure (in a sense that we will make precise) on the level surface $M(1) = T^{-1}(1)$.

– We point out a canonical rescaling on $M(<1) := T^{-1}((0,1))$ directed along the gradient of T, that yields a de Sitter metric. Throughout this work, $M^{(1)}$ denotes $M(<1)$ equipped with such a metric. $M^{(1)}$ turns to be a maximal globally hyperbolic, level surfaces $M(a)$ of T are Cauchy surfaces of $M^{(1)}$, and the cosmological time of $M^{(1)}$ is an explicit function of T.

– We point out a canonical rescaling on M directed along the gradient of T, that yields an Anti de Sitter structure. We denote such a structure by $M^{(-1)}$. Level surfaces of T are Cauchy surfaces of $M^{(-1)}$ and the cosmological time of $M^{(-1)}$ is an explicit function of T. $M^{(-1)}$ is not maximal so, $\mathcal{N}(M^{(-1)})$ denotes its maximal extension.

Remark 1.1. — In this context the word *canonical* means that a function $f : M \to N$ between two flat globally hyperbolic spacetimes is an isometry, if and only if it is an isometry for the respective Wick rotated (or rescaled) structures.

Now we can state the main classification theorem.

Theorem 1.2. — *Let S be a surface with non-abelian fundamental group. Then the maps*

$$\mathcal{MS}_0(S) \ni M \mapsto M(1) \in \mathcal{P}(S)$$
$$\mathcal{MS}_0(S) \ni M \mapsto M^{(1)} \in \mathcal{MS}_1(S)$$
$$\mathcal{MS}_0(S) \ni M \mapsto \mathcal{N}(M^{(-1)}) \in \mathcal{MS}_{-1}(S)$$

are bijective.

Finally, let us illustrate the contents of each section.

In Sec. 2 we state basic notations and recall fundamental facts about Lorentzian geometry. In particular we introduce the Klein models of constant curvature Lorentzian geometries, and we describe isometries, geodesics, and the duality between points and planes (that will play a fundamental rôle in our constructions).

In Sec. 3 we study flat globally hyperbolic spacetimes. We recall the basic results of [4] we need, and then we introduce the measured geodesic laminations on straight convex sets. Following [11], we give a complete classification of flat globally hyperbolic spacetimes (provided that the fundamental group of the Cauchy surface is not abelian).

In Sec. 4 we describe the canonical Wick rotation on $M(> 1)$. Even if the proof is omitted, we describe the general scheme and the main ideas to achieve that result. Then, we recall the basic facts of [16], and we try to relate the projective structure on $M(1)$ (that we get through the Wick rotation procedure), with the Kulkarni-Pinkall theory. All the objects associated to $M(1)$ in Kulkarni-Pinkall framework (Thurston metric, canonical stratification, H-hull, measured geodesic lamination on a straight convex set) are recovered in a very explicit way.

In Sec. 5 we describe the canonical de Sitter rescaling. In [18] Scannell associated to every projective structure on S a de Sitter structure on $S \times \mathbb{R}$. We will show that $M^{(1)}$ is the de Sitter structure associated to $M(1)$.

In Sec. 6 we describe the canonical Anti de Sitter rescaling. In order to deduce that every Anti de Sitter spacetime is obtained by rescaling a flat one, we construct the inverse rescaling. Finally we treat two problems:

1. In [17] the classification of $\mathcal{MS}_{-1}(S)$ is related to the earthquake theory on S. We try to generalize its remark to this more general situation. In particular we determine those spacetimes whose closure in the anti de Sitter boundary (that is canonically identified to $\mathbb{RP}^1 \times \mathbb{RP}^1$) is the graph of a homeomorphism.

2. The class of maximal globally hyperbolic Anti de Sitter spacetimes is invariant under time-orientation reversing. We will see that the sub-class of those corresponding to measured geodesic laminations on the whole $\mathbb{H}^2$ is not invariant for that operation.

2. Basic notations

Spacetimes and cosmological time

In this section we quickly state basic facts about Lorentzian geometry and give the definition of cosmological time. For a more general introduction to the Lorentzian geometry we refer to [5, 15].

Throughout this paper manifolds are supposed to be *connected* and *orientable.* A Lorentzian metric on a manifold M is a symmetric 2-form η with signature equal to $(n, 1)$. This means that locally we can define a frame, say $e_0, \ldots, e_n$, such that the matrix of η with respect to such a frame is $\mathrm{diag}(-1, 1, \ldots, 1)$.

Since the metric is not degenerated, there exists a unique Levi-Civita connection associated to η (we mean a symmetric connection ∇ such that $\nabla\eta = 0$). So, we can consider geodesics and curvature tensors just as in the Riemannian case.

A tangent vector $v \in \mathrm{T}_pM$ is said *spacelike* (resp.*timelike, null*) if $\eta_p(v, v) > 0$ (resp. $\eta_p(v, v) < 0$, $\eta_p(v, v) = 0$). A path $c : I \to M$ is *spacelike* (resp.*timelike, causal*) if its speed vector is spacelike (resp. timelike, non-spacelike) at every point. Clearly there are paths that are neither spacelike nor causal. On the other hand, if

c is a geodesic in M then $\eta(\dot{c}, \dot{c})$ is constant. By consequence, geodesics are either spacelike or causal.

Either the set of timelike vectors $\mathcal{C} \subset TM$ is connected or it is formed by two connected components. In the latter case we say that (M, η) is a time-orientable Lorentzian manifold. The choice of a connected component of $\mathcal{C}$ is a time-orientation. A *spacetime* is a Lorentzian manifold equipped with a time-orientation. Let M be a spacetime and $\mathcal{C}_+$ be the chosen component. A non-spacelike vector $v \in \mathrm{T}_p M$ is said *future directed* (resp. *past directed*) if it is not zero and it lies (resp. it does not lie) in the closure of $\mathcal{C}_+$. A causal path $c : I \to M$ is *future directed* (resp. *past directed*) if its speed vector is future directed (resp. past directed).

Given a point p in a spacetime M the future of p, denoted by $\mathrm{I}^+(p)$, is the set of the final points of future-directed timelike curves starting from p. An analogous definition holds for the past of p, that is denoted by $\mathrm{I}^-(p)$. Finally, if we replace timelike by causal curves we obtain the causal future of p (denoted by $\mathrm{J}^+(p)$) and the causal past of p (denoted by $\mathrm{J}^-(p)$). In general, $\mathrm{I}^+(p)$ and $\mathrm{I}^-(p)$ are open sets in M, whereas $\mathrm{J}^+(p)$ and $\mathrm{J}^-(p)$ are neither open nor closed.

If $c : I \to M$ is a causal path the Lorentzian length of c is

$$\ell(c) = \int_I \sqrt{-\eta(\dot{c}(t), \dot{c}(t))}\mathrm{d}t$$

Given $p \in M$ and $q \in \mathrm{J}^-(p)$ the Lorentzian distance $d(p, q)$ between them is the sup of the Lorentzian lengths of the causal curves joining them.

The *cosmological time* on M is the function

$$\tau_M : M \ni p \mapsto \sup\{d(p, q) | q \in \mathrm{J}^-(p)\} \in (0, +\infty]$$

In general τ_M is a very degenerated function (for instance if M is geodetically complete then $\tau_M = +\infty$).

We say that τ_M is *regular* if satisfies the following properties

1) it takes finite values.

2) it decreases to 0 along every inextensible past-directed causal curve.

The notion of regular cosmological time was pointed out in [1]. Being regular implies stronger regularity conditions.

THEOREM 2.1. — [1] *Suppose τ_M is regular. Then, it is continuous and twice-differentiable almost every-where. Level surfaces of τ_M are future Cauchy surfaces.*

■

A *Cauchy surface* S in M is an embedded surface such that every inextensible causal curve meets S exactly in one point. Spacetimes that admit a Cauchy surface are said *globally hyperbolic*. Being globally hyperbolic is the strongest causality condition and implies topological constraints on M.

THEOREM 2.2. — [14] *Let M be globally hyperbolic spacetime. Two Cauchy surfaces S and S' in M are homeomorphic. Moreover M is homeomorphic to $S \times \mathbb{R}$.*

In this work we will be mainly concerned with constant curvature (2+1)-spacetimes containing a *complete* Cauchy surface (that means a spacelike Cauchy surface S that is complete for the induced Riemannian structure). In order to carry out a reasonable classification of such spacetimes we need to restrict this class.

We say that a constant curvature spacetime M containing a Cauchy surface S is *maximal* if every isometrical embedding of M into a constant curvature spacetime M' *sending S onto a Cauchy surface of M'* is an isometry. The following theorem assures that every constant curvature globally hyperbolic spacetime is obtained as a regular neighbourhood of a Cauchy surface into a maximal one. So, we restrict ourselves to study the class of maximal spacetimes.

THEOREM 2.3. — [12] *For every constant curvature globally hyperbolic spacetime M there exists a unique maximal one M' and an isometric embedding*

$$\phi : M \to M'$$

sending a Cauchy surface of M onto a Cauchy surface of M'.

Now, let us fix an orientable surface S and $k \in \{-1, 0, 1\}$: the object we are going to study is the set, denoted by $\mathcal{MS}_k(S)$, of Lorentzian metrics on $S \times \mathbb{R}$ that make it a maximal globally hyperbolic spacetime of constant curvature equal to k, considered up to the action of the homotopy trivial diffeomorphisms of $S \times \mathbb{R}$.

Before investigating these structure spaces, we state some elementary facts about the constant curvature geometries.

For any choice of $k \in \{-1, 0, 1\}$ we will present an isotropic model, that is a Lorentzian manifold $\mathbb{X}_k$ of constant curvature equal to k such that the isometry group acts transitively on it and the stabilizer of a point is the group $O(2,1)$. An interesting property of an isotropic manifold $\mathbb{X}$ is that every isometry between two open sets of $\mathbb{X}$ extends to an isometry of the whole $\mathbb{X}$.

Every spacetime M of constant curvature equal to k is equipped with an atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ such that φ_i is an isometry of U_i with an open set of $\mathbb{X}_k$. Since $\mathbb{X}_k$ is isotropic, the changes of charts are restrictions of isometries of $\mathbb{X}_k$. So the atlas determines a $(\mathbb{X}_k, ISO(\mathbb{X}_k))$-structure on M (see [8] for an introduction to this topic).

So, every spacetime of constant curvature k is equipped with a developing map

$$dev : \tilde{M} \to \mathbb{X}_k$$

(that is a local isometry) and a holonomy representation

$$h : \pi_1(M) \to ISO(\mathbb{X}_k)$$

such that dev is h-equivariant. The developing map is determined up to post-composition with isometries of $\mathbb{X}_{-1}$ and h is determined up to conjugation in $ISO(\mathbb{X}_k)$. Moreover, the data (dev, h) determine the isometry class of M.

Minkowski space

The $(2+1)$-Minkowski space, $\mathbb{X}_0$, is $\mathbb{R}^{2+1}$ equipped with the standard flat form

$$\eta = -\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2\,.$$

Every tangent space of $\mathbb{X}_0$ is canonically identified to $\mathbb{R}^{2+1}$ provided with the standard Minkowski product

$$\langle v, w\rangle = -v_0w_0 + v_1w_1 + v_2w_2$$

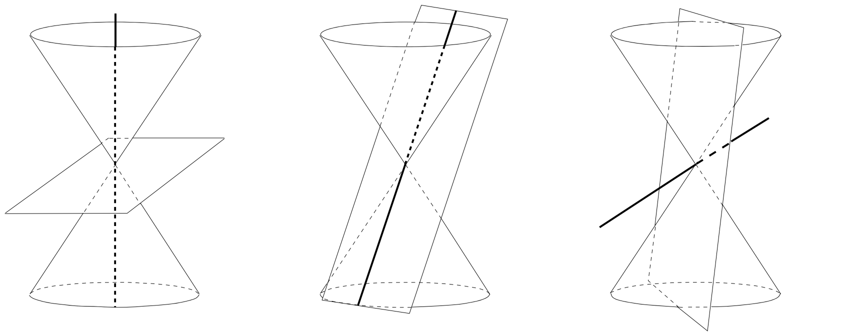
Throughout this work we will use this identification without mentioning it.

We consider on $\mathbb{X}_0$ the standard orientation and the time-orientation such that a timelike vector v is future-directed if and only if $v_0 > 0$.

An isometry of $\mathbb{X}_0$ is an affine map whose linear part preserves $\langle\cdot,\cdot\rangle$. In particular, if $\mathbb{R}^3$ denotes the group of translations of $\mathbb{X}_0$ and $O(2,1)$ is the group of linear maps preserving the Minkowski form, we have $ISO(\mathbb{X}_0) = \mathbb{R}^3 \rtimes O(2,1)$ (the action of $O(2,1)$ on $\mathbb{R}^3$ being the natural one). The connected component of $O(2,1)$, denoted by $SO^+(2,1)$, is the group of orientation preserving and time-orientation preserving linear transformations. Therefore, it is not difficult to see that it has index 4 in $O(2,1)$ and is contained in two index 2 subgroups: the group of the orientation preserving isometries $SO(2,1)$, and the group of the time-orientation preserving isometries $O^+(2,1)$.

Geodesics in $\mathbb{X}_0$ are straight lines. Up to isometries, they are of three types: spacelike, timelike or null. Totally geodesic planes are affine planes: they are classified by the restriction of the form η_0 on them. So, they can be spacelike (if the restriction of η_0 gives rise to a flat Riemannian metric), timelike (if the restriction of the metric gives rise to a flat $1+1$ Lorentzian metric), or null (if the restriction of the metric is a degenerated metric).

Since the form $\langle\cdot,\cdot\rangle$ is not degenerated, the orthogonality yields a duality between planes and lines through 0. Fig. 2 shows that the plane dual to a line is spacelike (resp. timelike, null) if the line is timelike (resp. spacelike, null).



a) A spacelike plane and the dual timelike line.

b) A null plane and the dual null line.

c) A timelike plane and the dual spacelike line.

Figure 2: The duality between lines and planes in $\mathbb{X}_0$.

The set of future directed unit timelike vectors is an embedded spacelike surface isometric to the hyperbolic plane $\mathbb{H}^2$. We call it the *hyperboloid model* of $\mathbb{H}^2$. Clearly the group $O^+(2,1)$ acts by isometries on it and every isometry of $\mathbb{H}^2$ can be realized in such a way. It follows that $SO^+(2,1)$ is naturally identified to $PSL(2,\mathbb{R})$.

In this model, geodesics on $\mathbb{H}^2$ are the intersection of $\mathbb{H}^2$ with timelike planes. Sometimes we will consider the *Klein model* that is obtained by projecting the hyperbolid model on $\mathbb{RP}^2$. In this model geodesics are the intersection of $\mathbb{H}^2$ with projective lines. The boundary of $\mathbb{H}^2$ in $\mathbb{RP}^2$ is the set of null directions and isometries of $\mathbb{H}^2$ extend to homeomorphisms of $\overline{\mathbb{H}}^2$.

De Sitter space

We present an isotropic, non-simply connected model of the de Sitter geometry. Anyway, in order to obtain the simply connected one, it will be sufficient to consider its universal covering space.

Consider in the $(3+1)$-Minkowski space the set $\hat{\mathbb{X}}_1$ of unit spacelike directions. It is a connected Lorentzian hypersurface homeomorphic to $S^2 \times \mathbb{R}$. The group $O(3,1)$ acts by isometries on it, the action is transitive, and the stabilizer of a point is isomorphic to $O(2,1)$. It follows that $\hat{\mathbb{X}}_1$ is an isotropic Lorentzian manifold. Since the centrum of $O(3,1)$ is the group $\{\pm Id\}$, also the Lorentzian manifold $\mathbb{X}_1 = \hat{\mathbb{X}}_1/\{\pm Id\}$ is isotropic. Notice that $\mathbb{X}_1$ embeds in $\mathbb{RP}^3$ and its image is the complementary of $\overline{\mathbb{H}}^3$.

$\mathbb{X}_1$ is homeomorphic to the oriented fibre bundle on $\mathbb{RP}^2$. In particular it is not time-orientable and $\hat{\mathbb{X}}_1$ is its time-orientation covering space (in fact, its universal covering space).

By the above discussion it follows that the isometry group of $\mathbb{X}_1$ is $O^+(3,1)$.

The main advantage to use this model is that geodesics are projective lines. A complete spacelike geodesic is a projecive line that is entirely contained in $\mathbb{X}_1$ (and in particular complete spacelike geodesics are closed curves with length equal to 2π). A complete timelike geodesic cuts $\partial\mathbb{H}^3$ in two points (and in this case its Lorentzian length is $+\infty$). Finally a complete null geodesic line is a projective line tangent to $\mathbb{H}^3$.

The duality between lines and 3-planes in $\mathbb{R}^{3+1}$ induces a bijection between spacelike directions (that are points in $\mathbb{X}_1$) and Lorentzian 3 linear spaces. On the other hand the intersection of $\mathbb{H}^3$ with a Lorentzian 3 linear space is a totally geodesic plane. So there is a bijective correspondence between points in $\mathbb{X}_1$ and totally geodesic planes of $\mathbb{H}^3$.

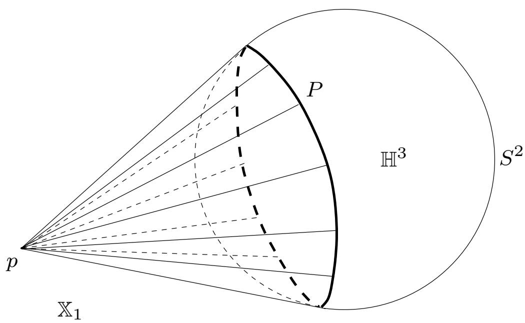


Figure 3: The plane in $\mathbb{H}^3$ dual to a point $p \in \mathbb{X}_1$.

Another way to describe such a correspondence is to fix a point $p \in \mathbb{X}_1$ and to consider the set of null lines starting at p. They are tangent to $\partial\mathbb{H}^3$ in some point. The locus of such points is exactly the boundary of the dual plane of p (see Fig. 3).

Finally, let us remark that the duality between points in $\mathbb{X}_1$ and planes in $\mathbb{H}^3$ lifts to a duality between points in $\hat{\mathbb{X}}_1$ and *oriented* planes of $\mathbb{H}^3$.

Anti de Sitter space

If η_0 denote the bilinear symmetric form on the space of 2×2 real matrices such that

$$\eta_0(x, x) = -\det x$$

then we have $SL(2,\mathbb{R}) = \{x | \eta_0(x, x) = -1\}$. Since the signature of η_0 is $(2, 2)$ the restriction of η_0 to $SL(2,\mathbb{R})$ is a Lorentzian product. Notice that both right and left multiplications induce an isometry action of $SL(2,\mathbb{R})$ on $M(2 \times 2,\mathbb{R})$. Thus, the left action of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ on $SL(2,\mathbb{R})$ given by

$$(a, b) \cdot x = axb^{-1}$$

induces a homomorphism

$$SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \to ISO(SL(2,\mathbb{R}),\eta_0)$$

whose kernel is the subgroup $\{(Id, Id), (-Id, -Id)\}$. By looking at that action we get that $SL(2,\mathbb{R})$ is an isotropic Lorentzian manifold and every isometry of $SL(2,\mathbb{R})$ is represented by a pair $(a,b) \in SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. The center of the isometry group is a $\mathbb{Z}_2$-group generated by $[-Id, Id] = [Id, -Id]$. In particular the form η_0 induces on $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/(Id, -Id)$ an isotropic Lorentzian metric η of constant curvature -1 (as an explicit computation shows). Notice that the isometry group of $PSL(2,\mathbb{R})$ is $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R}) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\langle(Id, -Id), (-Id, -Id)\rangle$. Throughout this work we will denote by $\mathbb{X}_{-1}$ the Lorentzian manifold $(PSL(2,\mathbb{R}), \eta)$.

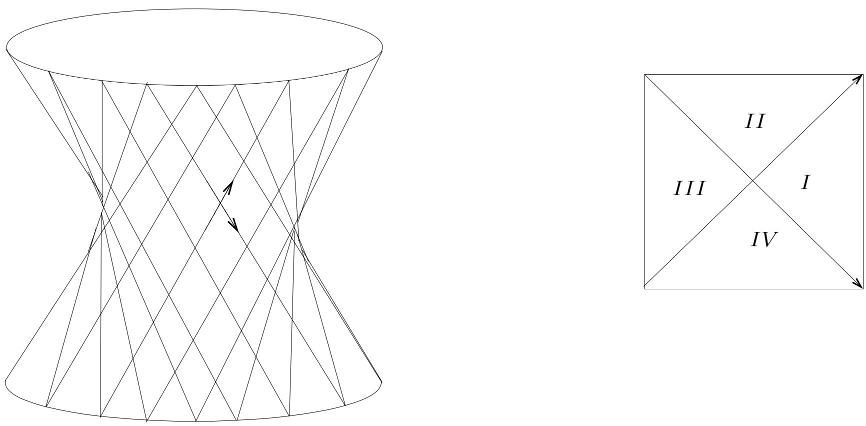


Figure 4: The left and right foliations on $\partial\mathbb{X}_{-1}$.

The boundary of $\mathbb{X}_{-1}$ in $\mathbb{RP}^3$ consists in the projective classes of rank 1 matrices. Notice that we have a canonical map

$$\partial\mathbb{X}_{-1} \ni x \mapsto (\mathrm{Im} x, \ker x) \in \mathbb{RP}^1 \times \mathbb{RP}^1$$

that induces a product structure on $\partial\mathbb{X}_{-1}$. In particular, $\partial\mathbb{X}_{-1}$ is homeomorphic to a torus and $\mathbb{X}_{-1}$ is a solid torus. Moreover the leaves of both left and right foliations are complete projective lines in $\mathbb{RP}^3$. The tangent vectors of these foliations separate $T_p\partial\mathbb{X}_{-1}$ in 4-quadrants. We say that a vector $v \in T_p\mathbb{X}_{-1}$ is spacelike if it lies either in the first or in the third quadrant. It is null if it lies on a line tangent to either the left or the right foliation. Finally it is timelike otherwise. By making this choice we obtain that limit of spacelike (resp. timelike) vectors in $\mathbb{X}_{-1}$ are non-timelike (resp. non-spacelike).

Clearly, the action of $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ extends on $\overline{\mathbb{X}}_{-1}$. Moreover if we consider the product action of $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ on $\mathbb{RP}^1 \times \mathbb{RP}^1$, then the canonical identification is a $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$-equivariant map.

In this model geodesics of $\mathbb{X}_{-1}$ are projective lines. In particular, timelike geodesics are projective lines entirely contained in $\mathbb{X}_{-1}$ with length equal to π. Spacelike geodesics are projective lines cutting $\partial\mathbb{X}_{-1}$ in 2 points and have infinite length. Finally null lines are projective lines tangent to $\partial\mathbb{X}_{-1}$.

On the other hand totally geodesic planes are projective planes. In particular, spacelike planes are compression disks whose boundary is a spacelike curve in $\partial\mathbb{X}_{-1}$.

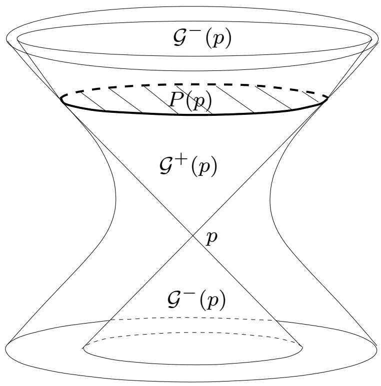


Figure 5: The duality between spacelike planes and points in $\mathbb{X}_{-1}$.

The form η_0 on $M(2\times 2,\mathbb{R})$ induces a duality between projective planes and points in $\mathbb{RP}^3$. If we take a point in $\mathbb{X}_{-1}$ its dual plane $P(p)$ is a spacelike plane: its boundary is given by the final points of null rays starting from p. Every timelike geodesic starting from p orthogonally cuts $P(p)$ at time $\pi/2$. In particular, points of $P(p)$ parametrizes timelike geodesic through p. Notice that the future of a point in $\mathbb{X}_{-1}$ is the whole $\mathbb{X}_{-1}$, so it is not a nice notion. For this reason we define the geodesic timelike locus $\mathcal{G}(p)$ of p as the set of points that can be reached from p by means of a timelike geodesic. Sometimes it will be useful to consider the decomposition

$$\mathcal{G}(p) = \mathcal{G}^+(p) \cup P(p) \cup \mathcal{G}^-(p)$$

where $\mathcal{G}^+(p)$ is the set of points that can be reached from p by means of a future directed geodesic of length $<\pi/2$ (and $\mathcal{G}^-(p)$ is defined in the same way by replacing the future with the past).

Geodesics through Id are 1-parameter subgroup: hyperbolic subgroups are spacelike geodesics, parabolic subgroups are null geodesics and elliptic subgroups are timelike geodesics. Hence, the dual plane of Id is the set of elliptic transformations that are rotations of π around some point in $\mathbb{H}^2$. If we associate to $x\in P(Id)$ its fixed point in $\mathbb{H}^2$ we get a homeomorphism

$$P(Id)\to\mathbb{H}^2$$

that turns to be an isometry (by consequence, every spacelike plane is isometric to $\mathbb{H}^2$). The standard embedding $\varphi : \mathbb{H}^2 \to \mathbb{X}_{-1}$ is, by definition, the inverse of this map. It is $PSL(2,\mathbb{R})$-equivariant in the following sense

$$\varphi(Ax) = (A, A)\varphi(x).$$

Moreover, the standard embedding extends to a map

$$\varphi : \overline{\mathbb{H}}^2 \to \overline{\mathbb{X}}_{-1}$$

sending a point $p \in \mathbb{RP}^1 = \partial\mathbb{H}^2$ to the point $(p,p) \in \partial\mathbb{X}_{-1}$.

3. Maximal flat spacetimes

In this section we give a full description of maximal flat spacetimes containing a complete Cauchy surface. In the first part we describe their universal coverings and their holonomies, following [4]. In the second part we use measured geodesic laminations on straight convex sets of $\mathbb{H}^2$, to give a full classification of $\mathcal{MS}_0(S)$ at least when $\pi_1(S)$ is not abelian.

The first step to describe maximal flat spacetimes containing a complete Cauchy surface is to prove that the developing map is an embedding. We will see that the completeness of the Cauchy surface is an essential hypothesis at this step.

Lemma 3.1. — [17] *Let M be a simply connected flat spacetime containing a complete Cauchy surface S. Then the restriction of the developing map on S is an embedding and the image is the graph of a function on the horizontal plane $\{x_0 = 0\}$.*

Proof : The composition of D with the orthogonal projection on the horizontal plane $\pi : S \to \{x_0 = 0\}$ turns to lengthen the lengths. Since S is complete then it is a covering map so, it is a homeomorphism.

∎

Consider a spacetime M containing a complete Cauchy surface S. The developing map sends $\tilde{S}$ onto a complete spacelike hypersurface (still denoted by $\tilde{S}$) of $\mathbb{X}_0$ and the holonomy group acts freely and properly on it. We can consider the domain of dependence $\mathcal{U}$ of that hypersurface, namely, the set of points p in $\mathbb{X}_0$ such that every causal curve starting from p meets $\tilde{S}$. Hence, $\tilde{S}$ is a Cauchy surface of $\mathcal{U}$ and the holonomy group acts freely and properly on $\mathcal{U}$. By construction, $\mathcal{U}/\pi_1(M)$ turns to be a maximal flat spacetime containing S. By the uniqueness of the maximal extension, we get that M is isometric to $\mathcal{U}/\pi_1(M)$ so that $\tilde{M}$ is isometric to $\mathcal{U}$.

The definition of $\mathcal{U}$ implies that a point p does not lie in $\mathcal{U}$ if and only if a null geodesic ray l starting from p does not meet $\tilde{S}$. An argument close to that used in the proof of Lemma 3.1 shows that the null plane P containing l does not meet

$\tilde{S}$. So, $\mathcal{U}$ is contained either in the future or in the past of P. In particular, $\mathcal{U}$ is a convex set.

Let us point out a notion that will play a fundamental rôle in the following part of this section. A *(future complete) regular domain* $\mathcal{U}$ is a convex subset of $\mathbb{X}_0$ such that

1. $\mathcal{U} = \bigcap_{P \text{ null support plane}} \mathrm{I}^+(P)$.

2. At least two non-parallel null support planes exist.

THEOREM 3.2. — [4] *Let M be a maximal globally hyperbolic spacetime containing a Cauchy surface S. Then, up to changing the time orientation, one of the following sentences holds*

1. *$\tilde{M} = \mathbb{X}_0$ and the holonomy group acts by spacelike translations on $\mathbb{X}_0$ (and in particular $\pi_1(M)$ is isomorphic to either $\{0\}$, or $\mathbb{Z}$, or $\mathbb{Z}^2$);*

2. *$\tilde{M} = \mathrm{I}^+(P)$ and the holonomy acts by spacelike translations (in particular, $\pi_1(M)$ is either $\{0\}$ or $\mathbb{Z}$);*

3. *$\tilde{M} = \mathrm{I}^+(P) \cap \mathrm{I}^-(Q)$ where P and Q are parallel null planes: in this case the holonomy group is isomorphic to either $\{0\}$ or $\mathbb{Z}$;*

4. *$\tilde{M}$ is the future of a spacelike line and the holonomy group is isomorphic to either $\{0\}$, or $\mathbb{Z}$, or $\mathbb{Z}^2$;*

5. *$\tilde{M}$ is a regular domain different from the future of a spacelike line and the linear holonomy $\pi_1(M) = \pi_1(S) \to SO^+(2,1)$ is a faithful and discrete representation.*

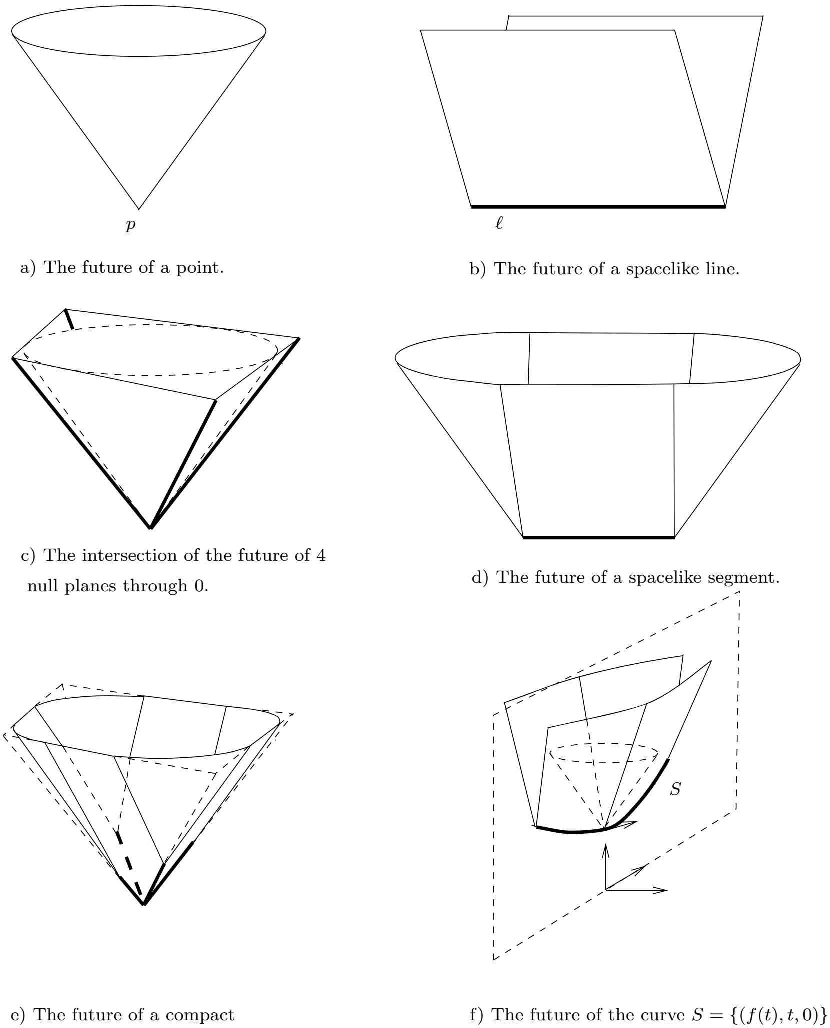
COROLLARY 3.3. — *Let S be a surface with non-abelian fundamental group. Then the universal covering $\tilde{M}$ of $M \in \mathcal{MS}_0(S)$ is a regular domain different from the future of a spacelike line and the linear holonomy is a faithful and discrete representation*

$$h_L : \pi_1(S) \to SO(2,1)\,.$$

In [6] we showed that, in fact, $\mathbb{H}^2/h_L(\pi_1(S))$ is homeomorphic to S.

We have focused on regular domains because they are equipped with regular cosmological time. More precisely, the following statement holds.

PROPOSITION 3.4. — [10, 4] *Let T be the cosmological time on a regular domain $\mathcal{U}$. For every $p \in \mathcal{U}$ there exists a unique $r(p) \in \partial\mathcal{U}$ such that $T(p)$ is the length of the timelike geodesic segment $[p, r(p)]$. Moreover the following facts hold.*



a) The future of a point.

b) The future of a spacelike line.

c) The intersection of the future of 4 null planes through 0.

d) The future of a spacelike segment.

e) The future of a compact spacelike tree.

f) The future of the curve $S = \{(f(t), t, 0)\}$ where f is a convex 1-Lipshitz function.

Figure 6: Examples of regular domains.

1. *The plane passing at $r(p)$ orthogonal to the segment $p - r(p)$ is a support plane for $\mathcal{U}$ and this property characterizes the point $r(p)$.*
2. *The function $r : \mathcal{U} \to \mathbb{X}_0$ is locally Lipschitz. The function T is $\mathrm{C}^{1,1}$ and concave. Its Lorentzian gradient is the unitary past directed timelike vector*

$$\nabla_L T(p) = \frac{-1}{T(p)}(p - r(p))\,.$$

3. *Level surfaces of T are complete convex Cauchy surfaces.*

■

Remarks 3.5. —

1. If $f : M \to N$ is a map between smooth manifolds, then being locally Lipchitz is a property depending only on the differentiable structure (actually we can choose arbitrary Riemannian metrics and the result does not depend on this choice). In particular the point 2. of the statement makes sense.
2. The point 1. implies that $r(tp + (1-t)r(p)) = r(p)$ for $t > 0$. So the integral line of the gradient of the cosmological time is a future complete geodesic starting from $r(p)$.
3. The behaviour of the function T is very close to that carried by the distance from a convex body in Euclidean (and hyperbolic) geometry. However the function T is defined inside the regular domain whereas the distance function is defined outside the convex body.

Summarizing, given a regular domain, we consider the following maps

- The cosmological time T;
- The *retraction* $r : \mathcal{U} \to \partial\mathcal{U}$
- The *Gauss map* $N : \mathcal{U} \to \mathbb{H}^2$ sending p to $-\nabla_L T(p)$

The map N is called the Gauss map since it is the Gauss map of every level surface $\mathcal{U}(a) = T^{-1}(a)$.

The image of the retraction Σ is called the *initial singularity*: because of point 1. of Proposition 3.4, Σ coincides with the set of points in $\partial\mathcal{U}$ admitting a spacelike support plane.

For the same reason the image of the Gauss map $\mathcal{H}_{\mathcal{U}}$ is the set of points in $\mathbb{H}^2$ orthogonal to some spacelike support plane of $\mathcal{U}$.

Since $\mathcal{U}$ is the intersection of the future of its *null support planes* (whose orthogonal directions are identified to points in $\partial\mathbb{H}^2$), it is not hard to see that $\mathcal{H}_{\mathcal{U}}$ is the convex hull of such points. In particular, it is a *straight convex set* (*i.e.*, a convex set in $\mathbb{H}^2$ that is the convex hull of a set of points in $\partial\mathbb{H}^2$).

Given a point $p \in \Sigma$, Proposition 3.4 implies that $\mathcal{F}_p := N(r^{-1}(p))$ coincides with the set of points in $\mathbb{H}^2$ orthogonal to some support plane of $\mathcal{U}$ *through* p. In particular, it turns out to be a straight convex set. The geodesic segment joining two points $p, p' \in \Sigma$ is spacelike. Moreover its orthogonal plane through 0 determines a geodesic in $\mathbb{H}^2$ that separates $\mathcal{F}_p$ from $\mathcal{F}_{p'}$. So, $\mathcal{H}_{\mathcal{U}}$ is the union of straight convex sets that can intersect each other only along a boundary component. In particular, the set

$$\mathcal{L}_{\mathcal{U}} = \bigcup_{p\in\Sigma\,:\,\dim\mathcal{F}_p=1} \mathcal{F}_p \cup \bigcup_{p\in\Sigma\,:\,\dim\mathcal{F}_p=2} \partial\mathcal{F}_p \cup \partial\mathcal{H}_{\mathcal{U}}$$

determines a geodesic lamination on $\mathcal{H}_{\mathcal{U}}$ according to Kulkarni Pinkall definition.

Before going further, let us recall the definition of *measured geodesic lamination on a straight convex set* given in [16]. If we fix a straight convex set $\mathcal{H}$ of $\mathbb{H}^2$, a geodesic lamination on it is just a closed subset provided with a geodesic foliation, such that every boundary component of $\mathcal{H}$ is a leaf of this foliation. In particular, $\mathcal{L}_{\mathcal{U}}$ furnishes an example of measured geodesic lamination on $\mathcal{H}_{\mathcal{U}}$.

Given a geodesic lamination $\mathcal{L}$ on a straight convex set $\mathcal{H}$ a *transverse measure* is the assignment of a positive measure μ_k to each transverse arc k such that

1. $\operatorname{supp}\mu_k = k \cap \mathcal{L}$;
2. If $k' \subset k$ then $\mu_{k'} = \mu_k|_{k'}$;
3. If k and k' are homotopic through a path of transverse arcs then the homotopy sends μ_k to $\mu_{k'}$;
4. $\mu_k(k) = +\infty$ if and only if an end-point of k lies on $\partial\mathcal{H}$.

Notice that 1., 2., 3. are the usual requirements for measured geodesic laminations. Instead the point 4. expresses the maximality of $\mathcal{H}$.

The simplest example of measured geodesic lamination on $\mathbb{H}^2$ is a finite union of disjoint geodesics equipped with a positive number (weight). The corresponding measure on a transverse arc is concentrated on the intersection (that is a finite set), and the measure of each intersection point is equal to the weight of the leaf through it. Clearly, also finite geodesic laminations on straight convex sets can be equipped with a measure in the same way, with the only difference that boundary leaves take the weight $+\infty$.

We include in the picture also the *degenerated* lamination that is given, by definition, by a single geodesic equipped with the weight $+\infty$. Even if, strictly speaking, it is not a right lamination, we will see that constructions we will implement with measured geodesic laminations could be applied to this case in a very natural way.

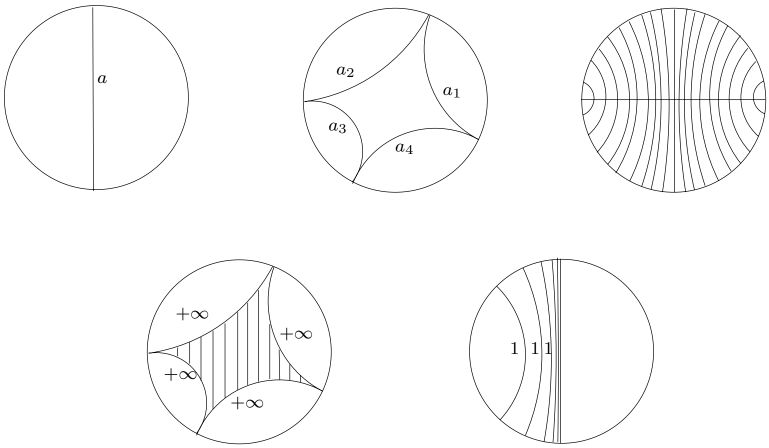


Figure 7: Some examples of measured geodesic laminations on straight convex sets.

Let us go back to the original topic. We want to classify regular domains in terms of measured geodesic laminations on straight convex sets. Actually, given a regular domain we have already constructed a geodesic lamination $\mathcal{L}_{\mathcal{U}}$ on $\mathcal{H}_{\mathcal{U}}$. Now, we construct a transverse measure $\mu_{\mathcal{U}}$ on it.

Given a transverse arc $k \subset \mathcal{H}_{\mathcal{U}}$ let $\hat{k}$ denote $N^{-1}(k) \cap \mathcal{U}(1)$. In [11] we proved that $\hat{k}$ is a locally rectifiable arc. Denote by t the arc-length parameter of $\hat{k}$. Since r is locally Lipschitz then the path $r(t) = r(\hat{k}(t))$ is differentiable almost everywhere. Its derivative $\dot{r}$ is spacelike (and orthogonal to $\mathcal{F}_{r(t)}$) almost everywhere. So we put

$$\mu_k = N_*(|\dot{r}|\mathrm{d}t)\,.$$

It is not difficult to see that in this way a transverse measure $\mu_{\mathcal{U}}$ on $\mathcal{L}_{\mathcal{U}}$ is defined.

Theorem 3.6. — [11] *The association*

$$\mathcal{U} \mapsto (\mathcal{H}_{\mathcal{U}}, \mathcal{L}_{\mathcal{U}}, \mu_{\mathcal{U}})$$

establishes a bijection between regular domains up to translation, and measured geodesic laminations on straight convex sets.

Let us sketch how the proof of Theorem 3.6 can be carried out. Given a regular domain $\mathcal{U}$ we have to recognize it by using only the measured geodesic lamination $(\mathcal{H}_{\mathcal{U}}, \mathcal{L}_{\mathcal{U}}, \mu_{\mathcal{U}})$.

Let us fix a point $p_0 \in \Sigma$: up to translations we can suppose $p_0 = 0$. Now, let us fix a point $x_0 \in \mathcal{F}_{p_0}$, and, for every point $x \in \mathcal{H}_{\mathcal{U}}$, consider a transverse arc $c : [0, a] \to \mathbb{H}^2$ joining x_0 to x. If $c(t) \in \mathcal{L}_{\mathcal{U}}$ let $v(t)$ be the unit spacelike vector

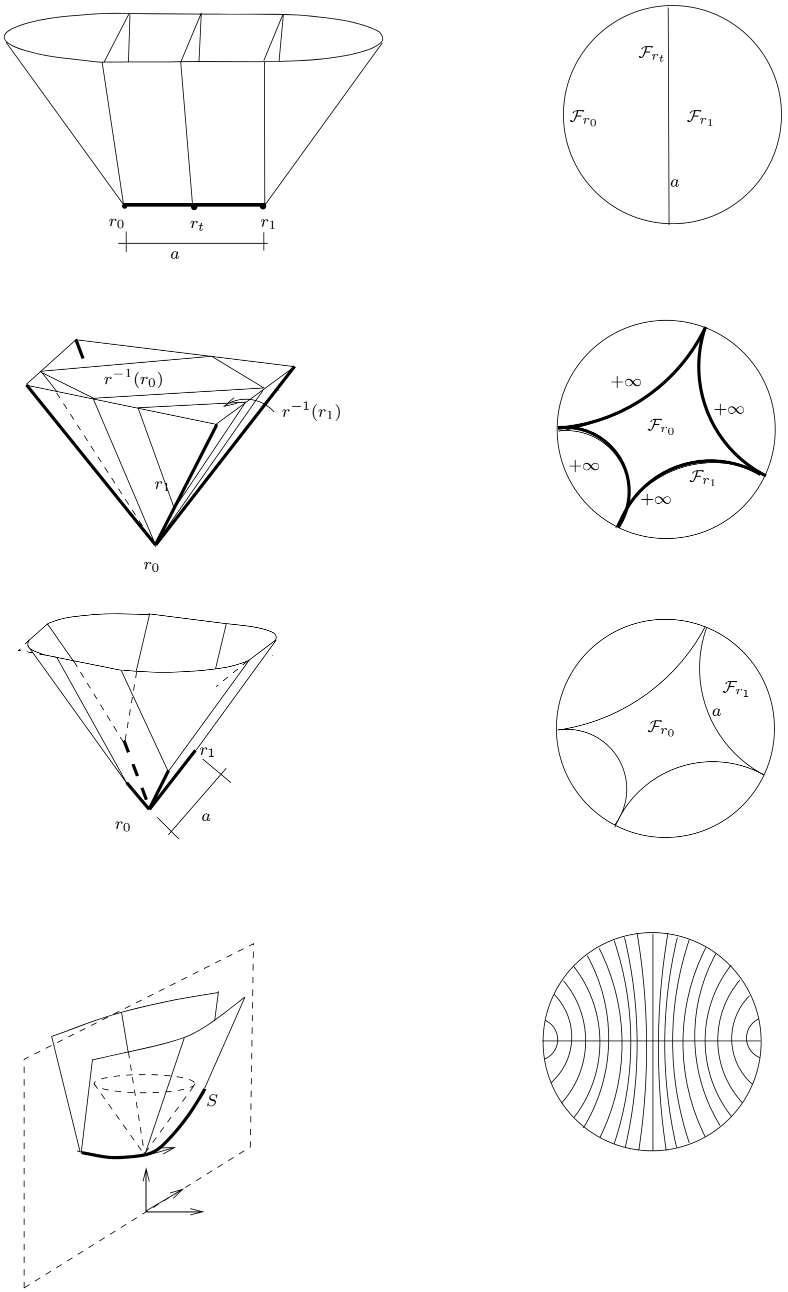


Figure 8: Measured geodesic laminations associated to some regular domains.

in $\mathbb{R}^2$ orthogonal to the leaf through $c(t)$ and pointing as c, otherwise let us put $v(t) = 0$. Then, we can define

$$\rho(x) = \int_{(0,a)} v(t)\mathrm{d}\mu_c(t).$$

It is not hard to see that $\rho(x)$ is independent of the choice of the arc c. Moreover, by definition, we can easily check that $\rho(x) \in \Sigma$ and $x \in \mathcal{F}_{\rho(x)}$.

If $P(x, \rho(x))$ denotes the spacelike plane passing through $\rho(x)$ and orthogonal to x, we obtain

$$\mathcal{U} = \bigcap_{x \in \mathcal{H}_{\mathcal{U}}} \mathrm{I}^+(P(x, \rho(x))) .$$

■

Now let us take $M \in \mathcal{MS}_0(S)$ and assume that $\pi_1(S)$ is non-abelian. We have seen that its universal covering $\tilde{M}$ is a regular domain different from the future of a spacelike line and its linear holonomy $h : \pi_(S) \rightarrow SO(2,1)$ is a discrete and faithful representation such that $\mathbb{H}^2/h(\pi_1(S))$ is homeomorphic to S. Let $(\mathcal{H}, \mathcal{L}, \mu)$ be the measured geodesic lamination associated to $\tilde{M}$. We have that it is $\pi_1(S)$-invariant in the following sense:

1. $h(\gamma)(\mathcal{H}) = \mathcal{H}$ for every $\gamma \in \pi_1(S)$;
2. $h(\gamma)$ preserves $\mathcal{L}$ and sends leaves to leaves;
3. $h(\gamma)$ sends μ_k to $\mu_{h(\gamma)(k)}$

Conversely, let us take a discrete and faithful representation $h : \pi_1(S) \rightarrow SO(2,1)$ and a measured geodesic lamination on a straight convex set $(\mathcal{H}, \mathcal{L}, \mu)$ that is $h(\pi_1(S))$-invariant. Let $\mathcal{U}$ be the regular domain associated to $(\mathcal{H}, \mathcal{L}, \mu)$ and let us take $p_0 \in \Sigma$ such that $\mathcal{F}_{p_0}$ is not a weighted leaf of $\mathcal{L}$. Up to translating $\mathcal{U}$, we can suppose $p_0 = 0$. Then, let us define

$$\tau : \pi_1(S) \rightarrow \mathbb{R}^{2+1}$$

by setting $\tau(\gamma)$ be the point on Σ corresponding to the leaf $\gamma(\mathcal{F}_{p_0})$. We have that τ is a $\mathbb{R}^{2+1}$-valued cocycle, so that the map

$$\rho : \pi_1(S) \ni \gamma \mapsto h(\gamma) + \tau(\gamma) \in Iso(\mathbb{X}_0)$$

is a homomorphism. In particular, $\mathcal{U}$ turns to be ρ-invariant and the quotient $\mathcal{U}/\rho(\pi_1(S))$ is a maximal globally hyperbolic flat spacetime.

Now we can give the classification statement we are looking for. Let us consider the set of pairs (h, λ) where h is a faithful and discrete representation $\pi_1(S) \rightarrow SO(2,1)$ such that $\mathbb{H}^2/h(\pi_1(S)) = S$ and λ is a measured geodesic lamination on a straight convex set that is invariant by $h(\pi_1(S))$. On this set let us consider the action of $SO(2,1)$ given by the rule

$$\gamma \cdot (h, \lambda) = (\gamma h \gamma^{-1}, \gamma_* \lambda)$$

and denote by $\mathcal{T}(S) \times \mathcal{ML}$ the set of equivalence classes of that action.

Corollary 3.7. — *Let S be a surface with non-abelian fundamental group. For $M \in \mathcal{MS}_0(S)$ denote by h_M its linear holonomy and by $\lambda_M = (\mathcal{H}_M, \mathcal{L}_M, \mu_M)$ the measured geodesic lamination associated to the universal covering space. Then the map*

$$M \mapsto [h_M, \lambda_M]$$

induces a bijection between $\mathcal{MS}_0$ and the set $\mathcal{T}(S) \times \mathcal{ML}$.

4. Hyperbolic Wick Rotation and projective structures

Theorem 4.1. — [6] *Given a regular domain $\mathcal{U}$, the Wick Rotation on $\mathcal{U}(> 1)$ along the gradient of the cosmological time T with rescaling functions*

$$\alpha = \frac{1}{T^2 - 1} \qquad \beta = \frac{1}{(T^2 - 1)^2} \tag{1}$$

yields a hyperbolic metric.

The developing map extends to a map

$$D : \mathcal{U}(\geq 1) \to \overline{\mathbb{H}}^3$$

such that $D : \mathcal{U}(1) \to S^2$ is a $\mathrm{C}^{1,1}$-local homeomorphism.

The isometry group of $\mathcal{U}$ (as Lorentzian manifold) coincides with the isometry group of $\mathcal{U}(> 1)$ (as hyperbolic manifold).

Corollary 4.2. — *If M is a spacetime, whose universal covering is a regular domain, then the Wick Rotation on $M(> 1)$ with rescaling functions as in (1) yields a hyperbolic metric.*

If $h : \pi_1(M) \to PSL(2, \mathbb{C})$ denotes the holonomy of such a hyperbolic structure, the local homeomorphism

$$D : \tilde{M}(1) \to S^2$$

obtained by extending the developing map of $M(> 1)$, is h-equivariant, providing a developing map for a complex projective structure on $M(1)$.

We sketch the basic steps to prove Theorem 4.1 .

First, we consider the case when the regular domain $\mathcal{U}_0$ is the future of a spacelike segment I of length A less than π (the corresponding lamination is $(\mathbb{H}^2, l_0, A)$

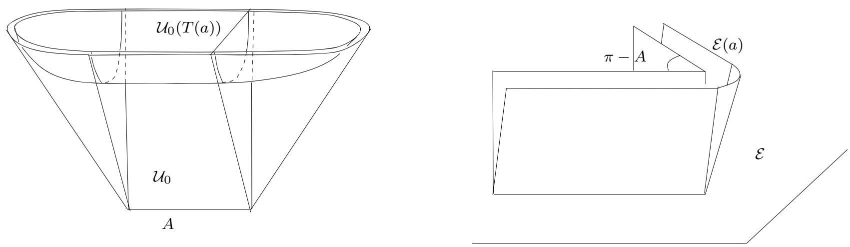


Figure 9: The shape of a T- level surface in $\mathcal{U}_0$ and the shape of a δ-level surface in $\mathcal{E}$

where l_0 is the geodesic dual to the direction of I). Now we bend $\mathbb{H}^2$ in $\mathbb{H}^3$ along l_0 in such a way that the bending angle is A. On the non-convex region $\mathcal{E}$ bounded by this bent surface the distance δ fom the boundary is a $\mathrm{C}^{1,1}$-submersion. The key remark is that the functions δ and T have the same qualitative behaviour. More precisely, the gradient of both of them is a unitary vector (timelike in Lorentzian case) and the integral lines of them are geodesics. Moreover the shape of the level surfaces of δ is quite similar to the shape of the level surfaces of T in $\mathcal{U}$. Actually they are formed by two negatively constant curved half planes joint each other by an Euclidean band. By a more careful analysis, we get that for every $a > 0$ there exists a unique $T(a) > 1$ such that the level surfaces $\mathcal{E}(a)$ and $\mathcal{U}(T(a))$ are related by a homothety. More precisely $T(a) = tgh(1/a)$ and the factor of the homothety

$$\mathcal{U}(T(a)) \to \mathcal{E}(a)$$

is $(T(a)^2 - 1)^{-1/2}$. So, we can construct a homeomorphism

$$D : \mathcal{U}(> 1) \to \mathcal{E}$$

such that

1. $D^*(\delta) = arctgh(1/T)$.

2. D sends the integral lines of the gradient of T to the integral lines of the gradient of δ.

3. $D| : \mathcal{U}(t) \to \mathcal{E}(arctgh(1/t))$ is a homothety.

The map D turns to be $\mathrm{C}^{1,1}$ and the pull-back of the hyperbolic metric on $\mathcal{E}$ is obtained by the Wick Rotation on $\mathcal{U}(> 1)$ along the gradient of T with rescaling functions given in (1). This concludes the proof of the first step.

The following step is to prove the Theorem under the assumption that the measured geodesic lamination associated to $\mathcal{U}$ is finite. In this case for every point

$p \in \mathcal{U}$ we can easily construct an isometry embedding of a neighbourhood U of p into $\mathcal{U}_0$ *sending the cosmological time of* $\mathcal{U}$ *onto the cosmological time of* $\mathcal{U}_0$. Since the result of the Wick rotation on U does depend only on the metric on U and on the cosmological time on U this step follows from the previous one.

When the lamination λ associated to $\mathcal{U}$ is general, we fix a point $p \in \mathcal{U}$ and construct a sequence of finite laminations λ_n that approximate λ *in a neighbourhood of* $N(p)$. A small neighbourhood of p in $\mathcal{U}$ is contained in every regular domain $\mathcal{U}_n$ corresponding to λ_n in such a way that the cosmological time T_n of $\mathcal{U}_n$ converges to the cosmological time of $\mathcal{U}$ in C^1-topology. By means of this fact we get that the developing map D_n of the hyperbolic structure of $\mathcal{U}_n(> 1)$ converges to a local C^1-homeomorphism on U and the pull-back of the hyperbolic metric is obtained by the Wick Rotation on the gradient of T with rescaling functions given by (1).

In order to deepen the result of Theorem 4.1, we are going to describe the complex projective structure on $M(1)$ from the Kulkarni-Pinkall point of view. For the sake of the completeness, let us recall the basic points of their classification of the complex projective structures. Let us take a projective structure on a surface S and consider the developing map

$$D : \tilde{S} \to S^2 .$$

Pulling back the standard metric of S^2 on $\tilde{S}$ is not a well-defined operation (*i.e.*, it depends on the choice of the developing map). Nevertheless, by the compactness of S^2, the completion $\hat{S}$ of $\tilde{S}$ with respect to such a metric is well-defined. By looking at $\hat{S}$ we can focus on three cases that yield very different descriptions:

1) $\tilde{S}$ is complete: in this case D is a homeomorphism so that S is S^2 (equipped with the standard structure). In this case we say that S is of *elliptic* type.

2) $\hat{S} \setminus \tilde{S}$ consists only of one point: in this case $\tilde{S}$ is projectively equivalent to $\mathbb{R}^2$ and the holonomy action preserves the standard Euclidean metric (so, S is equipped with a Euclidean structure). In this case we say that S is of *parabolic* type.

3) $\hat{S} \setminus \tilde{S}$ contains at least 2 points: in this case we say that S is of *hyperbolic* type.

Clearly, the most interesting case is the third one (that, for instance, includes the case when $\pi_1(S)$ is not abelian). In this case, by developing a Thurston idea, Kulkarni and Pinkall constructed a canonical stratification of $\tilde{S}$. Let us quickly explain their procedure.

A *disk* in $\tilde{S}$ is a set Δ such that $D|_\Delta$ is injective and the image of Δ is a round disk in S^2 (this notion is well defined because maps in $PSL(2, \mathbb{C})$ send round disks onto round disks). Given a *maximal* disk Δ (maximal with respect to the inclusion), we can consider its closure $\overline{\Delta}$ in $\hat{S}$.

The developing map D sends $\overline{\Delta}$ onto the closed disk $\overline{D(\Delta)}$. Hence, if g_Δ denotes the pull-back on Δ of the standard hyperbolic metric on $D(\Delta)$, we can regard the boundary of Δ in $\hat{S}$ as its ideal boundary.

Since Δ is supposed to be maximal it is not hard to see that $\overline{\Delta}$ is not contained in $\tilde{S}$. So, denote by Λ_Δ the set of point in $\overline{\Delta}\setminus\tilde{S}$ and denote by $\hat{\Delta}$ the convex hull in (Δ, g_Δ) of Λ_Δ (by maximality Λ_Δ contains at least two points).

Proposition 4.3. — [16] *For every point $p\in\tilde{S}$ there exists a unique maximal disk Δ containing p such that $p\in\hat{\Delta}$.*

So, $\{\hat{\Delta}|\Delta \text{ is a maximal disk}\}$ is a partition of $\tilde{S}$. Following [16], we call it the *canonical stratification* of $\tilde{S}$. Clearly the stratification is invariant under the action of $\pi_1(S)$.

Let g be the Riemannian metric on $\tilde{S}$ that, at a point $p\in\tilde{S}$, coincides with the metric g_Δ, where Δ is the maximal disk such that $p\in\hat{\Delta}$. It is a conformal metric (*i.e.* it makes D a conformal map). Moreover, it is $\mathrm{C}^{1,1}$ and invariant under the action of $\pi_1(S)$. So, it induces a metric on S, called the *Thurston metric* on S.

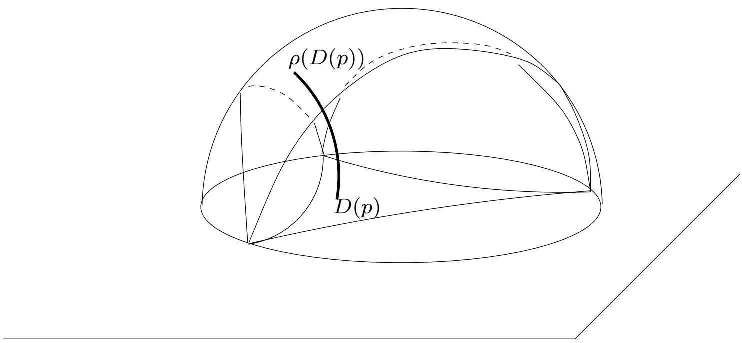


Figure 10: The construction of the H-hull.

By means of the canonical stratification, we can construct a hyperbolic structure on $S\times(0,+\infty)$. In fact, we construct an h-equivariant local homeomorphism

$$dev:\tilde{S}\times(0,+\infty)\to\mathbb{H}^3$$

(where h is the holonomy of $\tilde{S}$).

For $p\in\tilde{S}$ let $\Delta(p)$ denote the maximal disk such that $p\in\hat{\Delta}(p)$. The boundary of $D(\Delta(p))$ can be regarded as the boundary of a plane $P(p)$ of $\mathbb{H}^3$. Denote by $\rho:\overline{\mathbb{H}}^3\to P$ the nearest point retraction. Then $dev(p,\cdot)$ parameterizes in arc-length the geodesic ray of $\mathbb{H}^3$ with end-points $\rho(D(p))\in P$ and $D(p)$ (see Fig. 10).

PROPOSITION 4.4. — [16] *The map dev is a* $C^{1,1}$ *developing map for a hyperbolic structure on* $S \times (0,1)$*. Moreover, it extends to a map*

$$\overline{dev} : \tilde{S} \times (0, +\infty] \to \overline{\mathbb{H}}^3$$

such that $dev|_{\tilde{S}\times\{0\}}$ *is a developing map for the complex projective structure on* S*.*

We call such a hyperbolic structure the *H-hull* of S and denote it by $H(\tilde{S})$. Notice that $H(\tilde{S})$ is never complete as hyperbolic manifold. Let $P(\tilde{S})$ denote the boundary of $H(\tilde{S})$ in its completion.

The map *dev* extends on $P(\tilde{S})$. Thus, given a path c on $P(\tilde{S})$ we can define its length as the length of $dev(c)$. In such a way $P(\tilde{S})$ can be equipped with a path-metric distance. In [16] it is shown that $P(\tilde{S})$ is isometric to a straight convex set of $\mathbb{H}^2$. Moreover, the developing map

$$dev : P(\tilde{S}) \to \mathbb{H}^3$$

is the bending of $P(\tilde{S})$ along a measured $\lambda(\tilde{S})$.

THEOREM 4.5. — [16] *The correspondence*

$$\tilde{S} \mapsto (P(\tilde{S}), \lambda(\tilde{S}))$$

induces a bijection among the Moebius-equivalence classes of simply connected projective manifolds of hyperbolic type and the set of measured geodesic laminations on straight convex sets (up to the natural $PSL(2,\mathbb{R})$*-action).*

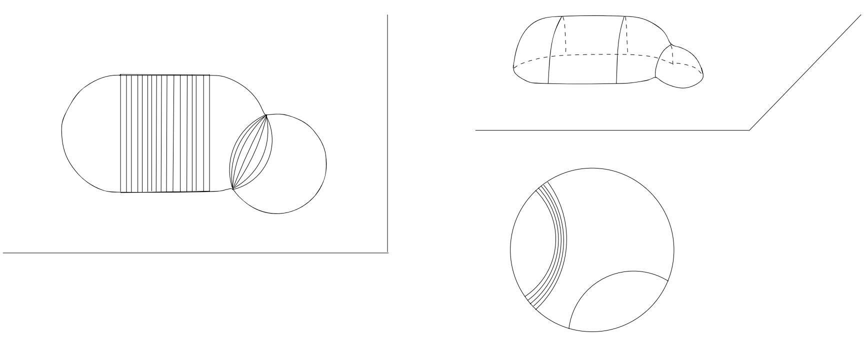


Figure 11: The H-hull of the natural projective structure on a simply connected open set in $\mathbb{C}$.

Remark 4.6. — When the developing map is an embedding $\tilde{S} \to S^2$ and the boundary of $\tilde{S}$ in S^2 is a Jordan curve, we can give a simpler description of Kulkarni-Pinkall constructions. In this case, we can consider the convex hull K of $\partial\tilde{S}$ in $\mathbb{H}^3$. Then,

1. $H(\tilde{S})$ is the component of $\mathbb{H}^3 - K$ close to $\tilde{S}$.
2. $P(\tilde{S})$ is the component of ∂K on which $\tilde{S}$ retracts.
3. The canonical stratification of $\tilde{S}$ is obtained by taking the inverse images of the faces of $P(\tilde{S})$ through the nearest point retraction.
4. The lamination associated to $\tilde{S}$ is obtained by depleating $P(\tilde{S})$.

If we take a surface S with non-abelian fundamental group we have seen that projective structures of S are of hyperbolic type. Moreover the pleated set $P(\tilde{S})$ is not a geodesic (*i.e.*, the interior of $P(\tilde{S})$ is non-empty). By looking at the construction of the H-hull and of $P(\tilde{S})$ we can see that there exists a natural retraction

$$\rho : \overline{H(\tilde{S})} \to P(\tilde{S})$$

such that for every point p in the interior of $P(\tilde{S})$ the inverse image $\rho^{-1}(p)$ is a geodesic segment joining the point p to a point on $\tilde{S}$. In particular we get that $\pi_1(S)$ acts free and properly discontinuously on the interior of $P(\tilde{S})$, and the quotient $P(S) = P(\tilde{S})/\pi_1(S)$ is homeomorphic to S.

Corollary 4.7. — *Let S be a surface with non-abelian fundamental group. For a projective structure F on S denote by h_F the hyperbolic holonomy of $P(F)$ and by $\lambda_F = (P(\tilde{F}), \mathcal{L}_F, \mu_F)$ the measured geodesic lamination associated to the universal covering. Then the map*

$$F \mapsto [h_F, \lambda_F]$$

induces a bijection between $\mathcal{P}(S)$ and the set $\mathcal{T}(S) \times \mathcal{ML}$.

Now, let us go back to our original problem. We have taken a flat spacetimes M containing a Cauchy surface homeomorphic to S (we have assumed $\pi_1(S)$ is non-Abelian) and we have constructed a hyperbolic structure on $M(>1)$ and a projective structure on $M(1)$.

Theorem 4.8. — [6] *The projective structure on $M(1)$ is of hyperbolic type.*

- *The canonical decomposition of $\tilde{M}(1)$ is given by the fibers of the retraction on the initial singularity.*
- *The Thurston metric on $M(1)$ coincides with the metric carried by $M(1)$ as spacelike slice in M.*

– *The* H*-hull of* $M(1)$ *is* $M(>1)$ *equipped with the hyperbolic metric.*

– *The measured geodesic lamination associated to* $\tilde{M}(1)$ *(as projective structure) is the same geodesic lamination associated to* $\tilde{M}$ *(as regular domain).*

– *The hyperbolic holonomy of* $P(M(1))$ *coincides with the linear holonomy of* M*.*

Corollary 4.9. — *The map*

$$\mathcal{MS}_0(S) \ni M \mapsto M(1) \in \mathcal{P}(S)$$

is bijective.

5. De Sitter rescaling

In [18], Scannell associated to every projective structure on S a de Sitter spacetime $U(S) \cong S \times \mathbb{R}$. In some sense, $U(S)$ is dual to $H(S)$. Let us sketch the main steps of the construction of $U(S)$.

The basic idea is to define a local homeomorphism

$$\tilde{S} \times \mathbb{R} \to \mathbb{X}_1$$

that is equivariant under the holonomy of S (this makes sense because the isometry group of $\mathbb{X}_1$ is $PSL(2,\mathbb{C})$). Let us recall that $\mathbb{X}_1$ is the complementary region in $\mathbb{RP}^3$ of $\overline{\mathbb{H}}^3$ and is identified with the set of planes of $\mathbb{H}^3$. Given $p \in \tilde{S}$ let $\Delta(p)$ be the maximal disk around p such that $p \in \hat{\Delta}(p)$. We have seen that there exists a plane P in $\mathbb{H}^3$ such that

$$\partial P = \partial D(\Delta(p)) .$$

Denote by $r(p)$ the corresponding dual point in $\mathbb{X}_1$. Now, we define

$$dev' : \tilde{S} \times (0, +\infty) \to \mathbb{X}_1$$

in such a way that $dev'(p, \cdot)$ is the arc length parameterization of the unique geodesic segment from $r(p)$ towards $D(p)$ (see Fig. 12).

Proposition 5.1. — [18] *The map* dev' *is a* $\mathrm{C}^{1,1}$*-developing map for a de Sitter structure on* $S \times (0, +\infty)$*. Moreover the coordinate* $t \in (0, +\infty)$ *coincides with the cosmological time of such a spacetime and level surfaces are Cauchy surfaces.*

Spacetimes arising in this way are called *standard.* spacetime

Theorem 5.2. — [18] *Every maximal de Sitter spacetime containing a complete Cauchy surface homeomorphic to* S *is standard.*

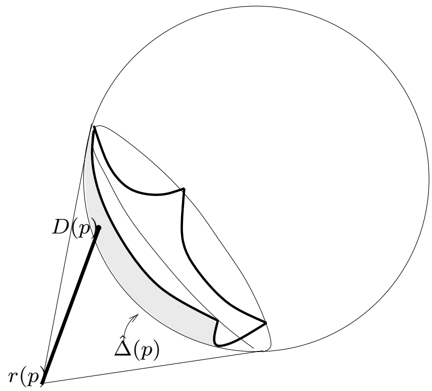


Figure 12: Construction of a standard de Sitter spacetime.

REMARKS 5.3. —

1. Scannell associated de Sitter spacetimes also to non-hyperbolic projective structures. Anyway, since we are mainly interested to the hyperbolic case, we will omit these details.

2. Scannell was concerned with compact surfaces. Anyway its construction works with any surface and the proof of Theorem 5.2 works only by assuming the completeness of the Cauchy surface.

Given a regular domain $\mathcal{U}$ in $\mathbb{X}_0$, we have seen that a suitable Wick Rotation yields a hyperbolic structure on $\mathcal{U}(>1)$ that is the H-hull of a projective structure on $\mathcal{U}(1)$. Now, we are going to see that a suitable rescaling on $\mathcal{U}(<1)$ yields the standard de Sitter spacetime associated to S.

THEOREM 5.4. — *Let $\mathcal{U}$ be a regular domain. Then the rescaling on $\mathcal{U}(<1)$ along the gradient of the cosmological time T with rescaling functions*

$$\alpha = \frac{1}{1-T^2} \qquad\qquad \beta = \frac{1}{(1-T^2)^2} \tag{2}$$

yields a de Sitter structure on $\mathcal{U}(<1)$, that we will denote by $\mathcal{U}^{(1)}$.

The cosmological time on $\mathcal{U}^{(1)}$ is given by the following formula

$$\theta = arctghT\,.$$

$\mathcal{U}^{(1)}$ is the standard spacetimes corresponding to the projective structure on $\mathcal{U}(1)$. Moreover, there exists a local homeomorphism

$$dev : \mathcal{U} \to \mathbb{RP}^3$$

such that its restriction on $\mathcal{U}(<1)$ (resp. $\mathcal{U}(1)$, $\mathcal{U}(>1)$) is a developing map for $\mathcal{U}^{(1)}$ (resp. the projective structure on $\mathcal{U}(1)$, the hyperbolic structure on $\mathcal{U}(>1)$).

The isometries of $\mathcal{U}^{(1)}$ coincide with the isometries of $\mathcal{U}$.

The steps of the proof of this theorem are the same as in the proof of Theorem 4.1. We first consider the case that $\mathcal{U}$ is the future of a spacelike segment of length less than π (and in this case the picture is similar to the previous one, see Fig. 13). Then we get the result for regular domains associated to finite laminations. Finally, by using an approximation argument, we get the full statement.

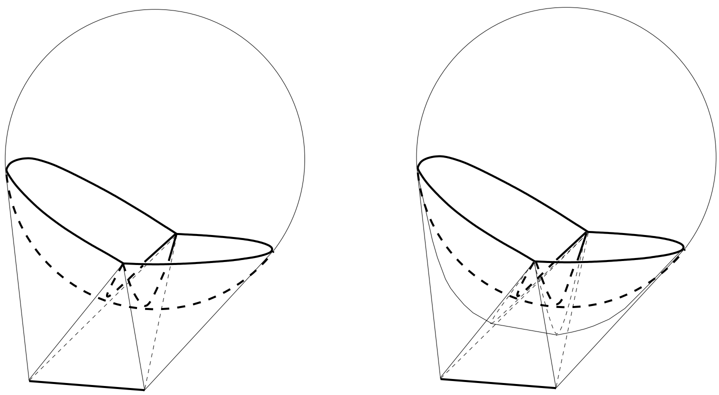


Figure 13: The de Sitter spacetime obtained by rescaling $\mathcal{U}_0$.

By Theorem 3.6, Theorem 4.5, and Theorem 5.2 we have that the correspondence

$$\mathcal{U} \mapsto \mathcal{U}^{(1)}$$

induces a bijection between regular domains and simply connected standard spacetimes associated to hyperbolic projective structures on a disk. In particular, given a simply connected standard spacetime (associated to a projective structure of hyperbolic type) U the level surface $U(a)$ of its cosmological time is isometric (up to a rescaling factor) to some level surface of a regular domain, that is complete (see [4]). It follows that $U(a)$ is complete.

Corollary 5.5. — *M is a maximal de Sitter spacetime containing a complete Cauchy surface $\cong S$ if and only if $\tilde{M}$ is a standard spacetime.*

Corollary 5.6. — *Given a maximal flat spacetime containing a complete Cauchy surface homeomorphic to S, the rescaling on $M(<1)$ along the gradient of its cosmological time with rescaling functions given in (2) yields a maximal de Sitter spacetime $M^{(1)}$ containing a complete Cauchy surface $\cong S$. Moreover the correspondence*

$$\mathcal{MS}_0(S) \ni M \mapsto M^{(1)} \in \mathcal{MS}_1(S)$$

is a bijection.

6. Anti de Sitter rescaling

First of all let us introduce the notion of *standard Anti de Sitter spacetime.* Given a no-where timelike embedded closed curve C in $\partial\mathbb{X}_{-1}$, we have

1. C is a meridian with respect to $\mathbb{X}_{-1}$.
2. There exists a spacelike plane P whose boundary is disjoint from C (see [17]).

In particular the set

$$\mathcal{Y}(C) = \{p \in \mathbb{X}_{-1} | \partial P(p) \cap C = \varnothing\}$$

is non-empty ($P(p)$ denotes the dual plane of p). We call it the *Cauchy development of* C. It is easy to see that $\mathcal{Y}(C)$ is an open convex subset and given two points $p, q \in \mathcal{Y}(C)$ there exists a unique geodesic segment joining them. An Anti de Sitter spacetime is called *standard* if its universal covering is isometric to the Cauchy development of some curve C.

Proposition 6.1. — [17] *If M is a maximal Anti de Sitter spacetime containing a complete Cauchy surface S then its universal covering is a* standard *spacetime.*

The holonomy $\rho = (\rho_L, \rho_R) : \pi_1(S) \to PSL(2, \mathbb{R})^2$ is a pair of faithful and discrete representations.

So, the class of standard spacetimes is quite interesting for our purposes. Let us collect some simple facts about them

1. If $\overline{\mathcal{Y}}(C)$ is the closure of $\mathcal{Y}(C)$ in $\overline{\mathbb{X}}_{-1}$ then $\overline{\mathcal{Y}}(C) \cap \partial\mathbb{X}_{-1} = C$. So, the Cauchy development determines the curve (*i.e.*, if $\mathcal{Y}(C) = \mathcal{Y}(C')$ then $C = C'$).
2. The boundary of $\mathcal{Y}(C)$ in $\partial\mathbb{X}_{-1}$ is formed by two components: the future boundary $\partial_+\mathcal{Y}(C)$ and the past boundary $\partial_-\mathcal{Y}(C)$. Both of them are achronal surfaces and for every point $p \in \partial_\pm\mathcal{Y}(C)$ there exists a null geodesic ray contained in $\partial_\pm\mathcal{Y}(C)$ joining p to some point in C.
3. Let P be a plane disjoint from C and consider the convex hull $\mathcal{K}(C)$ of C in $\mathbb{R}^3 = \mathbb{RP}^3 \setminus P$. Since C is no-where timelike, for every point $p \in C$ the null plane through p is a support plane for $\mathcal{K}(C)$. Thus, $\mathcal{K}(C)$ is contained in $\overline{\mathbb{X}}_{-1}$. Moreover we have $\mathcal{K}(C) \cap \partial\mathbb{X}_{-1} = C$.

If C is different from the boundary of a spacelike plane, $\mathcal{K}(C)$ has non-empty interior. $\partial\mathcal{K}(C) \setminus C$ is formed by two components: the future and the past boundaries (resp. $\partial_+\mathcal{K}(C)$ and $\partial_-\mathcal{K}(C)$) that are achronal surfaces. When C is the boundary of a plane P, by definition we put $\partial_+\mathcal{K}(C) = \partial_-\mathcal{K}(C) = P$.

The interior of $\mathcal{K}(C)$ is contained in $\mathcal{Y}(C)$. A point in $\partial_\pm\mathcal{K}(C)$ lies in $\mathcal{Y}(C)$ if and only if no null support plane passes through it. In this case then a neighbourhood U of p in $\partial_\pm\mathcal{K}(C)$ carries a $\mathrm{C}^{0,1}$-distance such that U is isometric to an open set of $\mathbb{H}^2$.

Remark 6.2. — In some sense $\mathcal{K}(C)$ has the same features of the convex hull in $\mathbb{H}^3$ of some Jordan curve in S^2. Let us stress that some important differences occur. First of all it is not difficult to see that C is a Lipschitzian curve. Moreover, in hyperbolic case the boundary of the convex hull is formed by two components that are isometric to $\mathbb{H}^2$. In Anti de Sitter case, the distance is defined only on $\partial_\pm\mathcal{K}(C)\cap\mathcal{Y}(C)$ and in general *is not complete.*

A remarkable subset of $\mathcal{Y}(C)$ is the *past part* that is is the past of $\mathcal{K}(C)$ in $\mathcal{Y}(C)$. We will denote such a set by $\mathcal{P}(C)$. It is the convex region bounded by $\partial_-\mathcal{Y}(C)$ and $\partial_+\mathcal{K}(C)$.

Proposition 6.3. — [6] *The cosmological time τ of $\mathcal{P}(C)$ is a $\mathrm{C}^{1,1}$-function taking values on $(0,\pi/2)$.*

For every $p\in\mathcal{P}(C)$ there exists a unique $\rho_-(p)\in\partial_-\mathcal{Y}(C)$ such that $\tau(p)$ is the length of the geodesic segment between p and $\rho_-(p)$.

The gradient of τ is a unit timelike vector. The integral line of τ through p is a geodesic of length $\pi/2$ joining $\rho_-(p)$ to $\rho_+(p)\in\partial_+\mathcal{K}(C)$. The plane $P(\rho_-(p))$ is a support plane for $\mathcal{P}(C)$ passing through $\rho_+(p)$ and $P(\rho_+(p))$ is a support plane passing through $\rho_-(p)$.

Now we can state the main theorem that allows to relate $\mathcal{M}_0(S)$ to $\mathcal{M}_{-1}(S)$.

Theorem 6.4. — [6] *Let $\mathcal{U}$ be a regular domain, then the rescaling on $\mathcal{U}$ along the gradient of the cosmological time T with rescaling functions*

$$\alpha=\frac{1}{T^2+1}\qquad\qquad \beta=\frac{1}{(T^2+1)^2}\tag{3}$$

yields a Anti de Sitter structure denoted by $\mathcal{U}^{(-1)}$. The cosmological time on $\mathcal{U}^{(-1)}$ is given by

$$\tau=arctg(T)$$

(and in particular takes values in $(0,\pi/2)$).

Level surfaces of τ are complete Cauchy surfaces of $\mathcal{U}^{(-1)}$. Finally, $\mathcal{U}^{(-1)}$ is the past part of its maximal extension.

The proof of this theorem follows the same steps as the proofs of Theorems 4.1, 5.4. Let us just describe what is $\mathcal{U}^{(-1)}$ when $\mathcal{U}$ is the future of a geodesic segment of length A.

In P, Q are spacelike planes of $\mathbb{X}_{-1}$ cutting each other along a geodesic we can define the notion of dihedral angle between them. Namely, since they intersect each other the dual points $p(P)$ and $p(Q)$ are related by a spacelike geodesic segment (that is unique). Then the angle between P and Q is the length of that segment.

Remark 6.5. — By using the duality between planes in $\mathbb{H}^3$ and point in $\mathbb{X}_1$ we can interpret the classical notion of dihedral angle in $\mathbb{H}^3$ in the same way. An important difference is that in $\mathbb{X}_1$ spacelike geodesics are closed curves of length 2π so, in hyperbolic geometry the length of a angle is well-defined $\mod 2\pi$. On the other hand, in $\mathbb{X}_{-1}$ spacelike geodesics are open curves of infinite length so, the angles are well-defined numbers in $(0, +\infty)$.

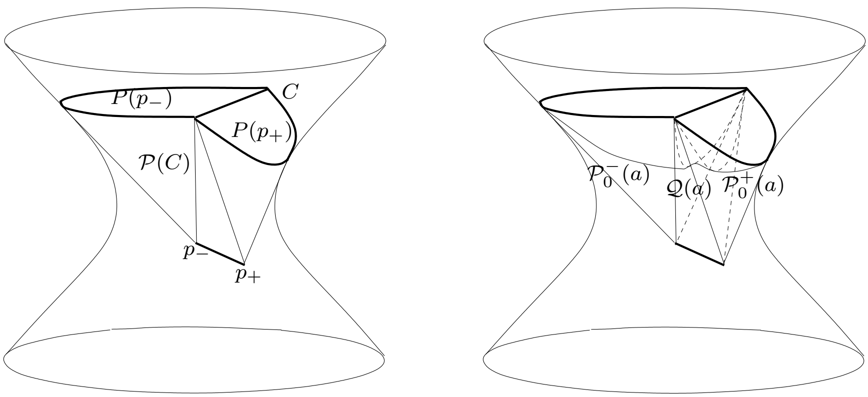


Figure 14: The Anti de Sitter spacteime obtained by recsaling $\mathcal{U}_0$.

Since the notion of angles between planes has been defined, bending a plane along a finite measured lamination makes sense. Now, let us take a geodesic l on $P_- = P(p_-)$ and bend P_- along (l, A) to obtain a surface S convex in the past (see Fig. 14). Then consider the dual point p_+ of the plane P_+ forming an angle A with P_- along l. It is not difficult to see that points on the segment $[p_-, p_+]$ correspond to the support planes of the bent surface along l. The boundary C_0 of S is a no-where timelike curve (in fact, it is spacelike) so we can consider the past part $\mathcal{P}_0$ of the Cauchy development $\mathcal{Y}(C)$ of C. The future boundary of $\mathcal{P}_0$ is S whereas the past boundary contains the segment $[p_-, p_+]$. For every point $p \in \mathcal{P}_0$ there exists a unique timelike geodesic ray trough it with past end-point $\rho_-(p)$ in $[p_-, p_+]$ and future end-point $\rho_+(p)$ in S such that $P(\rho_-(p))$ is a support plane for S. It follows that $\tau(p)$ is the Lorentzian length of $[\rho_-(p), p]$. Now consider the decomposition of the level surface $\mathcal{P}_0(a) = \tau^{-1}(a)$ in three pieces $\mathcal{P}_0^{\pm}(a) = \rho^{-1}(p_\pm) \cap \mathcal{P}_0(a)$ and $\mathcal{Q}_0(a) = \rho^{-1}(a)((p_-, p_+)) \cap \mathcal{P}_0(a)$. The map

$$\rho_+ : \mathcal{P}_0^{\pm}(a) \to P_\pm$$

is a homothety so $\mathcal{P}_0^{\pm}(a)$ are negatively constant curved half planes. By analyzing the map

$$\mathcal{Q}_0(a) \ni x \mapsto (\rho_-(x), \rho_+(x)) \in l \times (p_-, p_+)$$

we get that $\mathcal{Q}_0(a)$ is a Euclidean band with width depending on a. So the shape of $\mathcal{P}_0(a)$ is similar to the shape of $\mathcal{U}_0(a)$ and we can argue as in the proof of Theorem 4.1.

Remark 6.6. — In general Fig. 14 shows that τ is neither convex nor concave.

Corollary 6.7. — *Let S be a surface with non-abelian fundamental group. If $M \in \mathcal{MS}_0(S)$ then rescaling M along the gradient of the cosmological time T with rescaling functions given in (3) yields a Anti de Sitter globally hyperbolic spacetime $M^{(-1)}$ that is the past part of its maximal extension $\mathcal{N}(M^{-1})$.*

Given a regular domain $\mathcal{U}$, the rescaling on $\mathcal{U}^{(-1)}$ along the gradient of τ with rescaling functions

$$\alpha = \frac{1}{\cos^2 \tau} \qquad\qquad \beta = \frac{1}{\cos^4 \tau} \tag{4}$$

is the inverse of that defined on $\mathcal{U}$ so it yields $\mathcal{U}$ again. By consequence the correspondence

$$\mathcal{MS}_0(S) \ni M \mapsto \mathcal{N}(M^{-1}) \in \mathcal{MS}_{-1}(S)$$

is injective. In order to prove that it is surjective, we should prove that given the Cauchy development $\mathcal{Y}(C)$ of a no-where timelike curve C the rescaling on its past part $\mathcal{P}(C)$ along the gradient of the cosmological time τ with rescaling functions given in (4) yields a regular domain.

First suppose that $\partial_+\mathcal{K}(C)$ is a complete spacelike surface. Then it is obtained by bending $\mathbb{H}^2$ along a measured lamination λ.

Now, let us take the flat regular domain $\mathcal{U}_\lambda$ associated to λ. By looking at the proof of Theorem 6.4, we get that the future boundary of $\mathcal{U}_\lambda^{(-1)}$ is obtained by bending $\mathbb{H}^2$ along λ. It follows that $\mathcal{U}_\lambda^{(-1)} = \mathcal{P}(C)$.

The proof of the general case is more complicated.

The first step is to prove that the rescaling on $\mathcal{P}(C)$ along the gradient of τ with rescaling functions given by (4) yields a flat metric. The proof of this step follows from the following lemma.

Lemma 6.8. — *Let C be a no-where timelike curve in $\partial\mathbb{X}_{-1}$ and p be a point in $\mathcal{P}(C)$. There exists a no-where timelike curve C' such that $\partial_+\mathcal{K}(C')$ is complete and an isometric embedding of a neighbourhood U of p in $\mathcal{P}(C)$*

$$\phi : U \to \mathcal{P}(C')$$

such that $\phi^*(\tau') = \tau$ (where τ and τ' respectively denote the cosmological time on $\mathcal{P}(C)$ and on $\mathcal{P}(C')$).

Let us denote by $\mathcal{P}^{(0)}$ the flat spacetime obtained by the above rescaling. By an explicit computation we get that the cosmological time T of $\mathcal{P}^{(0)}$ is given by

$$T = tg\tau$$

and level surfaces of T are Cauchy surfaces.

The second step is to prove that the T-level surfaces $\mathcal{P}^{(0)}(a)$ are complete. This follows from the following proposition.

Proposition 6.9. — [6] *The level surfaces $\mathcal{P}_C(a)$ of τ in $\mathcal{P}(C)$ are complete.*

The proof of this proposition is based on two remarks:

1. Take a point $p_0 \in \mathcal{P}_C(a)$ then every point $p \in \mathcal{P}_C(a)$ is related to p_0 by a spacelike geodesic segment. Denote by $h(p)$ the length of such a segment. Then the function $p \mapsto h(p)$ is *proper.*

2. If II denote the secund fundamental form on $\mathcal{P}_C(a)$, there exists a constant $k = k(a)$ such that
$$|II(v, v)| \leq k \langle v, v\rangle$$
for every $v \in T\mathcal{P}_C(a)$.

By using point 2. an explicit computation shows that the distance $d(p, p_0)$ is greater than $Mh(p)$ where $M = M(a)$ is a constant depending only on a. From point 1. it follows that balls centered in p_0 are compact and Proposition 6.9 follows.

Since $\mathcal{P}^{(0)}(a)$ is related to $\mathcal{P}_C(tga)$ by a homothety, it is complete. So, the developing map
$$D : \mathcal{P}^{(0)} \to \mathbb{X}_0$$
is an embedding. Moreover, if we fix a point $p \in \partial_-\mathcal{Y}(C)$ we have that $D(\mathcal{P}_C(a) \cap \rho_-^{-1}(r))$ is, up to translations, a straight convex set on the surface $tg(a) \cdot \mathbb{H}^2 \subset \mathrm{I}^+(0)$ (see Fig. 15). Then, we get that at least two null (non-parallel) lines in $\mathbb{X}_0$ does not intersect $\mathcal{P}^{(0)}(tga) = D(\mathcal{P}_C(a))$ and its maximal extension is a regular domain $\mathcal{U}$. Moreover, the same remark shows that the restriction on the cosmological time $T_{\mathcal{U}}$ on $\mathcal{U}$ coincides with the cosmological time T on $\mathcal{P}^{(0)}$. In particular the developing map gives rise to an isometrical embedding

$$D_a : \mathcal{P}^{(0)}(a) \to \mathcal{U}(a)\,.$$

By the completeness of $\mathcal{P}^{(0)}(a)$ we argue that D_a is an isometry. Since the range of T is $(0, +\infty)$ we obtain that $\mathcal{P}^{(0)}$ is isometric to $\mathcal{U}$.

Theorem 6.10. — *Let M be the past part of a standard Anti de Sitter spacetime, and τ denote its cosmological time. Then the rescaling on M along the gradient of τ with rescaling functions given in (4) yields a maximal flat globally hyperbolic spacetime.*

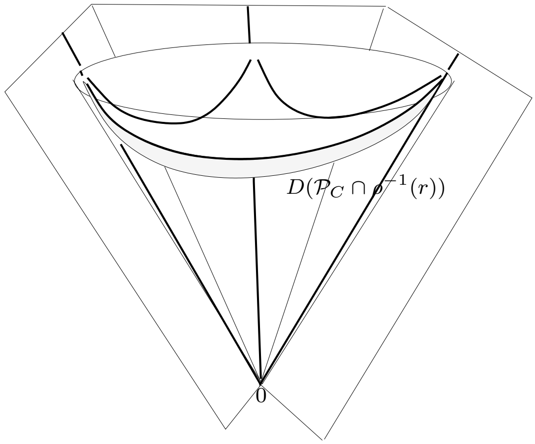


Figure 15: The image of $D(\mathcal{P}_C(a) \cap \rho^{-1}(r)$ is a straight convex set on $tga \cdot \mathbb{H}^2$.

Corollary 6.11. — *If C is a no-where timelike curve in $\partial\mathbb{X}_{-1}$ then $\partial_+\mathcal{K}(C) \cap \mathcal{Y}(C)$ is isometric to a straight convex set $\mathcal{H}_C$ in $\mathbb{H}^2$. The bending locus provided a measured geodesic lamination λ_C on it. The pair $(\mathcal{H}_C, \lambda_C)$ determines uniquely $\mathcal{Y}(C)$.*

Finally, by Theorems 6.4 and 6.10 we obtain the following classification result.

Corollary 6.12. — *Let S be a surface with non-abelian fundamental group. Then the map*

$$\mathcal{MS}_0(S) \ni M \mapsto \mathcal{N}(M^{(-1)}) \in \mathcal{MS}_{(-1)}(S)$$

is bijective.

Standard simply connected Anti de Sitter spacetimes can be described either in terms of a no-where timelike curve (by looking at the adherence set in $\partial\mathbb{X}_{-1}$) or in terms of a measured geodesic lamination on a straight convex set (by looking $\partial_+\mathcal{K}(C)$). In the last part of this section we sketch some applications concerning with this remark.

Relation with the earthquake theory.

Mess pointed out an interesting relation between the classification of anti de Sitter structures on $S \times \mathbb{R}$ and the earthquake theory on S (assuming S to be a compact surface)

Theorem 6.13. — [17] *Take a compact surface S of genus $g \geq 2$. If $M \in \mathcal{MS}_0(S)$, the holonomy $\rho = (\rho_L, \rho_R) : \pi_1(S) \to PSL(2, \mathbb{R})^2$ consists of two discrete and faithful representations. The curve C is the graph of the homeomorphism* *conjugating ρ_L to ρ_R. Moreover if λ is the measured lamination on $S_L = \mathbb{H}^2/\rho_L$ such that the left earthquake along λ yields $S_R = \mathbb{H}^2/\rho_R$ then the future boundary S_+ of the past part of M is obtained by the left earthquake $\mathcal{E}$ on S_L along $\lambda/2$. Moreover the bending locus of S_+ is the lamination $\mathcal{E}_*(\lambda/2)$.*

This theorem points out a nice relationship between the boundary curve C and the future boundary of the past part. We are going to generalize this result in the general case. For this purpose it is convenient to generalize the notion of left (right) earthquake. If λ is a measured geodesic lamination on a straight convex set $\mathcal{H}$, the *(generalized) earthquake* along λ is defined as a map

$$\mathcal{E}_\lambda : \mathrm{int}\mathcal{H} \to \mathbb{H}^2 \,.$$

In [13] Epstein and Marden pointed out a general procedure to associate to every geodesic lamination on $\mathbb{H}^2$ a cocycle

$$\beta_\lambda : \mathbb{H}^2 \times \mathbb{H}^2 \to PSL(2, \mathbb{R}) \,,$$

i.e., a map satisfying the cocycle relation

$$\beta_\lambda(x, y) \circ \beta_\lambda(y, z) = \beta_\lambda(x, z) \,.$$

(Actually a more general procedure was pointed out for complex-valued measured geodesic lamination.) It turns out that the left earthquake $\mathcal{E}$ along λ can be expressed in the very convenient way in terms of λ, namely

$$\mathcal{E}(x) = \beta(x_0, x)x$$

where x_0 is a fixed point. Now the construction of the cocycle applies to measured geodesic lamination on straight convex set. I particular if λ is defined on $\mathcal{H}$ the the corresponding cocycle is defined on $\mathrm{int}\mathcal{H} \times \mathrm{int}\mathcal{H}$. The generalized earthquake is the map $\mathcal{E}_\lambda : \mathrm{int}\mathcal{H} \to \mathbb{H}^2$ defined by $\mathcal{E}_\lambda(x) = \beta_\lambda(x_0, x)x$.

In general $\mathcal{E}_\lambda$ is injective but not surjective. Its image $\mathcal{E}_\lambda(\mathcal{H})$ is the interior of a straight convex set and we can push forward the lamination λ to obtain a lamination $\mathcal{E}_\lambda(\lambda)$ on $\mathcal{E}_\lambda(\mathcal{H})$.

Clearly, a similar construction works to define the generalized right earthquake. In general if $\mathcal{E}_\lambda$ is the left earthquake on $\mathcal{H}$ along λ then the right earthquake on $\mathcal{E}_\lambda(\mathcal{H})$ along $\mathcal{E}_\lambda(\lambda)$ is the inverse of $\mathcal{E}_\lambda$.

These notions are suitable in our framework, because every measured geodesic lamination λ on a straight convex set $\mathcal{H}$ gives rise to an earthquake, that transform λ in another measured geodesic lamination on a straight convex set.

When a geodesic lamination is invariant under the action of a group $\Gamma < PSL(2, \mathbb{R})$ then the following relation holds

$$\beta_\lambda(\gamma x, \gamma y) = \gamma \beta_\lambda(x, y)\gamma^{-1} \,.$$

So the map

$$h_\lambda : \Gamma \ni \gamma \mapsto B(x_0, \gamma x_0)\gamma$$

is a representation and $\mathcal{E}_\lambda$ is h_λ-equivariant. We say that h_λ is a deformation of Γ by the left earthquake along λ (in the same way the deformation by the right earthquake is defined).

PROPOSITION 6.14. — [6] *Given M be a maximal flat spacetime containing a complete Cauchy surface S. Denote by h_0 the linear holonomy and by λ the measured geodesic lamination on a straight convex set corresponding to $\tilde{M}$. Then the holonomy of $M^{(-1)}$ is the pair of representations (h_-, h_+) such that h_- (resp. h_+) is the deformation of h_0 by a right (resp. left) earthquake along $\lambda/2$.*

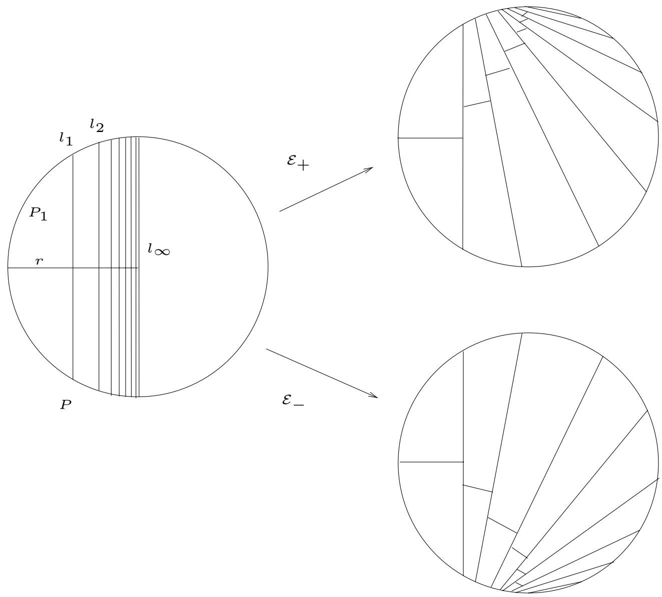


Figure 16: A measured lamination on a straight convex set yielding surjective right and left earthquakes.

By means of generalized earthquake, we can characterize the standard spacetimes, whose boundary curve C is the graph of a homeomorphism, in terms of $\partial_+\mathcal{P}(C)$.

PROPOSITION 6.15. — [6] *Let λ_C be the measured geodesic lamination on a convex set $\mathcal{H}_C$ associated to a standard spacetime $\mathcal{Y}(C)$ (*i.e., *$\partial_+\mathcal{P}(C)$ is obtained by bending $\mathcal{H}_C$ along λ_C). Then C is the graph of a homeomorphism if and only if both the left and the right earthquakes along λ are surjective maps*

$$\mathcal{E}_\pm : \mathrm{int}(\mathcal{H}_C) \to \mathbb{H}^2 .$$

Moreover in this case the left earthquake $\hat{\mathcal{E}}$ on $\mathbb{H}^2$ along $2\mathcal{E}_-(\lambda)$ is a true earthquake (extending to a homeomorphism of $\mathbb{P}^1 = \partial\mathbb{H}^2$) and we have

$$C = \{(x, \hat{\mathcal{E}}(x)) | x \in \mathbb{P}^1\} .$$

In Figure 16 it is shown an example of measured geodesic lamination on a straight convex set different from $\mathbb{H}^2$ that produces surjective earthquakes. It follows that there exist curves C that are graphs of homeomorphisms, such that $\partial_+\mathcal{P}(C)$ is not complete.

Completeness of $\partial_+\mathcal{P}(C)$ and time-orientation reversing. — Another interesting question is to characterize the curves C such that $\partial_+\mathcal{P}(C)$ is a complete surface. This problem seems to be quite hard, and we want just to remark that this property is not invariant by time-orientation reversing. We are going to provide a nice example that shows this fact in a very explicit way. Take the finite area hyperbolic structure on $S = S^2 \setminus \{0, 1, \infty\}$ with holonomy $\rho : \pi_1(S) \to PSL(2, \mathbb{R})$. Take the invariant geodesic lamination on $\mathbb{H}^2$ obtained by tessellating $\mathbb{H}^2$ by ideal triangles. Now put the weight 1 on each leaf and denote by λ such a geodesic lamination. We know that the left and the right earthquakes $\mathcal{E}_\pm$ along λ are not surjective and produce representations

$$\rho_\pm : \pi_1(S) \to PSL(2, \mathbb{R})$$

that are convex co-compact. The image of $\mathcal{E}_\pm$ is exactly the convex core of $\rho_\pm$. Moreover, by the symmetry of the picture, we have that ρ_- and ρ_+ are conjugated. It follows that the pair (ρ_-, ρ_+) preserves a plane P in $\mathbb{X}_{-1}$ and we have

$$P/(\rho_-, \rho_+) = \mathbb{H}^2/\rho_- = \mathbb{H}^2/\rho_+ .$$

Denote by Λ the limit set in ∂P of the action of (ρ_-, ρ_+) on P. It is a Cantor set. Any set in $\partial\mathbb{X}_{-1}$ invariant under (ρ_-, ρ_+) conatins Λ.

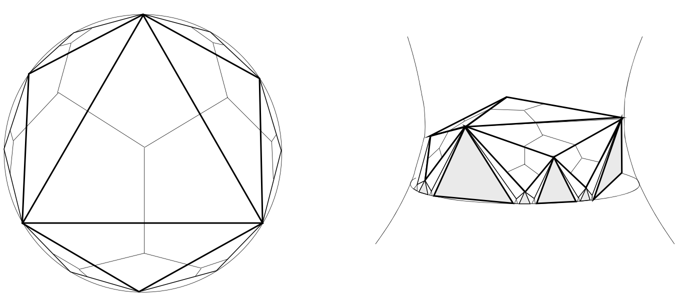


Figure 17: The bending of $\mathbb{H}^2$ into $\mathbb{X}_{-1}$ along λ.

Now S_+ denotes the surface obtained by bending $\mathbb{H}^2$ in $\mathbb{X}_{-1}$ along λ and denote by C the boundary in $\partial\mathbb{X}_{-1}$ of S_+. Clearly S_+ is the future boundary of $\mathcal{K}(C)$. By Proposition 6.14 we have that C is invariant under (ρ_-, ρ_+) and so C contains λ. On the other hand, let $\alpha \in \pi_1(S)$ be the loop around a puncture. We have that $\rho_-(\alpha)$ and $\rho_+(\alpha)$ are hyperbolic transformations and with fixed points (x_-, x_+) and (y_-, y_+).

Clearly (x_-, y_-) and (x_+, y_+) are contained in Λ. On the other hand by a careful analysis of the bending procedure we get [6]

$$(x_+, y_-) \in C$$

So we obtain that the curve C contains the segment on a left leaf with endpoints (x_-, y_-) and (x_+, y_-) and the segment of the right leaf with end points (x_+, y_+) and (x_+, y_-). It follows that the past boundary of $\mathcal{K}(C)$ admits null support planes (see Fig.17).

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