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Abstract. We study the discretization of an elliptic partial differential equation, posed on a two- or three-dimensional domain with smooth boundary, endowed with a generalized Robin boundary condition which involves the Laplace–Beltrami operator on the boundary surface. The boundary is approximated with piecewise polynomial faces and we use isoparametric finite elements of arbitrary order for the discretization. We derive optimal-order error bounds for this non-conforming finite element method in both L^2- and H^1-norm. Numerical examples illustrate the theoretical results.

Keywords. generalized Robin boundary conditions, Laplace–Beltrami operator, isoparametric finite elements, finite element method, error analysis.

1. Introduction

1.1. The generalized Robin boundary value problem

In this paper, we study the following second-order partial differential equation endowed with a boundary condition including the Laplace–Beltrami operator

$$\begin{cases} -\Delta u + \kappa u = f & \text{in } \Omega \,, \\ \dfrac{\partial u}{\partial \nu} + \alpha u - \beta \Delta_\Gamma u = g & \text{on } \Gamma = \partial\Omega \,, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) is a domain with curved boundary $\Gamma = \partial\Omega$, $\alpha > 0, \beta > 0$ and $\kappa \geq 0$ are given constants and f, g are given functions on Ω and $\partial\Omega$, respectively.

The generalized Robin problem (1.1) is studied in [15] (with $\kappa = 0$). The authors prove existence and uniqueness of the weak solution and analyze the regularity of the solution given the regularity of f and g. It turns out that the solution to the generalized problem possesses better regularity properties than the solution to the standard Robin problem, that is (1.1) with $\beta = 0$. Moreover, they analyze the *conforming* finite element discretization of (1.1) and prove optimal-order error bounds in both L^2- and H^1-norm. However, in [15] the authors have to assume that Ω can be represented exactly by the finite element mesh such that the numerical domain coincides with the exact domain or, equivalently, that the finite element space V_h is contained in the solution space V. Two different cases are considered: either Γ is polyhedral, or of class $C^{1,1}$. In the first case, they have to introduce mixed boundary conditions, because the generalized boundary condition cannot be imposed on the entire boundary (see [15, Remark 3.1]). In the second case, it is restrictive to assume that the computational mesh is capable of representing the boundary exactly.

The purpose of this paper is to generalize the results of [15] to *non-conforming* finite elements, where the additional error that stems from the approximation of the geometry is taken into account. Based on a polyhedral approximation of Ω, on which linear finite elements can be used, we construct a piecewise polynomial approximation domain and isoparametric finite elements of arbitrary order. Since the finite element space is no longer contained in the solution space, we cannot compare the finite element solution and the exact solution directly. To overcome this, we lift the finite element solution to the solution space to be able to analyze the error of the method.

The above setting allows us to treat different types of boundary conditions in a unified setting. Here we focus on the generalized Robin problem, and the convergence results for the isoparametric finite element discretization of (1.1) with the standard Robin boundary condition ($\beta = 0$) or Neumann boundary condition ($\alpha = \beta = 0$) are obtained as a consequence. We derive error bounds between the exact solution and the lifted finite element solution that are optimal with respect to the regularity of the right-hand side functions f and g. Under suitable regularity assumptions, the error satisfies optimal-order error bounds.

1.2. **Applications**

The problem (1.1) has applications for example in heat conduction processes, see [13], or in the context of Schrödinger operators [12]. Generalized Robin boundary conditions appear also in the context of domain decomposition methods [11, 18] and in the Schwarz waveform relaxation algorithm [10, 14]. A more comprehensive list of applications can be found in [15].

1.3. **Outline of the paper**

In Section 2, we introduce basic notations and derive a variational form of the generalized Robin problem. In Section 3, the approximation of the geometry is described, followed by the isoparametric finite element method in Section 4. In Section 5, we derive error estimates in both L^2- and H^1-norm. We begin by stating the main results in Section 5.1, followed by a convergence proof for the H^1-estimate that is clearly separated into stability and consistency, and finally the proof of the L^2-estimate. We finish with some numerical experiments in two and three space dimensions in Section 6.

2. **Continuous problem**

2.1. **Preliminaries**

Let $\Omega \subset \mathbb{R}^n$, ($n = 2, 3$) be an open, bounded and connected domain with sufficiently smooth boundary $\Gamma = \partial\Omega$. In the following, we require Γ at least of class C^2. For a more thorough introduction to the following concepts and definitions, we refer to [7, Section 2], where more details about the following concepts can be found, cf. [6, 8].

The outer unit normal on Γ is denoted by ν. The tangential gradient of a function w defined on some open neighborhood of Γ is given by

$$\nabla_\Gamma w = \nabla w - (\nabla w \cdot \nu)\,\nu$$

and depends on values of w on Γ only. The Laplace–Beltrami operator is given by

$$\Delta_\Gamma w = \nabla_\Gamma \cdot \nabla_\Gamma w = \sum_{j=1}^n (\nabla_\Gamma)_j (\nabla_\Gamma)_j\, w\,.$$

We denote by $d : \mathbb{R}^n \to \mathbb{R}$ the signed distance function

$$d(x) = \begin{cases} -\mathrm{dist}(x, \Gamma) & \text{if } x \in \Omega\,, \\ 0 & \text{if } x \in \Gamma\,, \\ \mathrm{dist}(x, \Gamma) & \text{otherwise}, \end{cases}$$

where $\mathrm{dist}(x, \Gamma) = \inf\{|x - y| : y \in \Gamma\}$ denotes the distance of x to Γ. Since Γ is a C^2-manifold, there exists a $\delta > 0$ and a strip

$$\mathcal{U}_\delta = \{x \in \mathbb{R}^n : |d(x)| < \delta\} \tag{2.1}$$

such that for each $x \in \mathcal{U}_\delta$ there exists a unique $p(x) \in \Gamma$ such that

$$x = p\,(x) + d(x)\nu(p(x))\,, \tag{2.2}$$

see [7, Section 2]. $p(x)$ is the closest point to x on Γ.

We let $c > 0$ denote a generic constant that assumes different values on different occurrences. We use the standard notation for Sobolev spaces, i.e. $H^0(\Omega) := L^2(\Omega) = \{u : \Omega \to \mathbb{R} : \int_\Omega u^2 \mathrm{d}x < \infty\}$, $H^{k+1}(\Omega) := \{u \in L^2(\Omega) : \nabla u \in H^k(\Omega)^n\}$. It is well known that the trace γu of a function $u \in H^k(\Omega)$ is in $H^{k-1/2}(\Gamma)$ if $\Gamma \in C^{k-1,1}$. Due to the Laplace–Beltrami operator in the boundary condition of (1.1), it turns out that we need $\gamma u \in H^1(\Gamma)$ to derive a weak formulation. Therefore $H^1(\Omega)$ is not the suitable weak solution space. Instead, we work with the space

$$H^k(\Omega; \Gamma) := \left\{ u \in H^k(\Omega) : \gamma u \in H^k(\Gamma) \right\}$$

endowed with the norm

$$\|u\|_{H^k(\Omega;\Gamma)} = \left(\|u\|_{H^k(\Omega)}^2 + \|\gamma u\|_{H^k(\Gamma)}^2 \right)^{1/2}. \tag{2.3}$$

Recall that for a function $w \in H^k(\Gamma)$, the $H^k(\Gamma)$-norm is defined using tangential derivatives, i.e.

$$\|w\|_{H^k(\Gamma)} = \left(\|w\|_{L^2(\Gamma)}^2 + \|\nabla_\Gamma w\|_{H^{k-1}(\Gamma)^n}^2 \right)^{1/2}.$$

It is shown in [15, Lemma 2.5] that the space $H^k(\Omega; \Gamma)$ with the inner product that induces (2.3) is a Hilbert space.

2.2. **Variational form**

To derive the weak formulation, we make use of the integration by parts formula on Γ: for $w \in H^1(\Gamma)$, we have (see [7])

$$\int_\Gamma -\Delta_\Gamma u w \mathrm{d}\boldsymbol{\sigma} = \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma w \mathrm{d}\boldsymbol{\sigma}\,. \tag{2.4}$$

We multiply (1.1) with a test function φ, integrate over Ω and obtain

$$\int_\Omega \nabla u \cdot \nabla \varphi + \kappa u \varphi \mathrm{d}x - \int_\Gamma \frac{\partial u}{\partial \nu} \gamma \varphi \mathrm{d}\boldsymbol{\sigma} = \int_\Omega f \varphi \mathrm{d}x\,.$$

Substituting the boundary condition and using (2.4) with $w = \gamma \varphi$, we arrive at

$$\int_\Omega \nabla u \cdot \nabla \varphi + \kappa u \varphi \mathrm{d}x + \alpha \int_\Gamma (\gamma u)(\gamma \varphi) \mathrm{d}\boldsymbol{\sigma} + \beta \int_\Gamma \nabla_\Gamma (\gamma u) \cdot \nabla_\Gamma (\gamma \varphi) \mathrm{d}\boldsymbol{\sigma} = \int_\Omega f \varphi \mathrm{d}x + \int_\Gamma g(\gamma \varphi) \mathrm{d}\boldsymbol{\sigma}\,.$$

We use the following notation for bilinear forms defined on $H^1(\Omega; \Gamma) \times H^1(\Omega; \Gamma)$:

$$m^\Omega(u, v) = \int_\Omega uv \mathrm{d}x\,, \tag{2.5}$$

$$a^\Omega(u, v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x\,,$$

$$m^\Gamma(u, v) = \int_\Gamma (\gamma u)(\gamma v) \mathrm{d}\boldsymbol{\sigma}\,,$$

$$a^\Gamma(u, v) = \int_\Gamma \nabla_\Gamma (\gamma u) \cdot \nabla_\Gamma (\gamma v) \mathrm{d}\boldsymbol{\sigma}\,,$$

$$a(u, v) = a^\Omega(u, v) + \kappa m^\Omega(u, v) + \alpha m^\Gamma(u, v) + \beta a^\Gamma(u, v)\,.$$

The right hand side is denoted by

$$\ell(\varphi) = \int_\Omega f \varphi \mathrm{d}x + \int_\Gamma g(\gamma \varphi) \mathrm{d}\boldsymbol{\sigma}\,.$$

The variational form thus reads: find $u \in V = H^1(\Omega; \Gamma)$ such that

$$a(u, \varphi) = \ell(\varphi) \tag{2.6}$$

for all $\varphi \in H^1(\Omega; \Gamma)$. The following regularity result is proved in [15].

Proposition 2.1. *Let $\alpha, \beta > 0$, $\kappa \geq 0$ and $j \geq 1$. If $\Gamma \in C^{j,1}$, $f \in H^{j-1}(\Omega)$, $g \in H^{j-1}(\Gamma)$, then there exists a unique solution $u \in H^{j+1}(\Omega; \Gamma)$ that satisfies the a priori bound*

$$\|u\|_{H^{j+1}(\Omega;\Gamma)} \leq c \left(\|f\|_{H^{j-1}(\Omega)} + \|g\|_{H^{j-1}(\Gamma)} \right).$$

Let us remark that for the standard Robin boundary value problem, i.e. (1.1) with $\beta = 0$, we need $g \in H^{j-1/2}(\Gamma)$ to have $u \in H^{j+1}(\Omega)$, and the trace theorem then yields $\gamma u \in H^{j+1/2}(\Gamma)$, so the generalized problem requires less regularity in the data to produce a more regular solution, cf. [15, Remark 3.5].

3. Domain approximation

Before we describe the finite element method, we need to construct an approximation of Ω and Γ. We follow the construction of [8], which is based on [16], [2] and [3].

3.1. Linear approximation

Let $\Omega_h^{(1)}$ be a polyhedral approximation of Ω with boundary $\Gamma_h^{(1)} = \partial \Omega_h^{(1)}$. We construct $\Omega_h^{(1)}$ such that the faces of $\Gamma_h^{(1)}$ are simplices whose vertices lie on Γ (triangles in $\mathbb{R}^3$ and straight lines in $\mathbb{R}^2$). We construct a quasi-uniform triangulation $\mathcal{T}_h^{(1)}$ of $\Omega_h^{(1)}$ consisting of simplices (tetrahedrons on $\mathbb{R}^3$ and triangles in $\mathbb{R}^2$). We set

$$h = \max\{\operatorname{diam}(T) : T \in \mathcal{T}_h^{(1)}\}$$

and assume that $h \leq h_0$, where h_0 is sufficiently small such that $\Gamma_h^{(1)} \subset \mathcal{U}_\delta$, where $\mathcal{U}_\delta$ is defined in (2.1).

3.2. Exact triangulation

Before we define the computational domain, we define an *exact triangulation* of Ω. We denote by $\widehat{T}$ the unit n-simplex. For each $T \in \mathcal{T}_h^{(1)}$, there exists an affine transformation $\Phi_T : \mathbb{R}^n \to \mathbb{R}^n$ that maps $\widehat{T}$ onto T, which we write as

$$\Phi_T(\widehat{x}) = B_T \widehat{x} + b_T,$$

where $B_T \in \mathbb{R}^{n \times n}$, $b_T \in \mathbb{R}^n$. Φ_T is exactly the map used for linear finite elements. We now call T^c a curved simplex if there exists a C^1-mapping Φ_T^c that maps $\widehat{T}$ onto T^c which is of the form

$$\Phi_T^c = \Phi_T + \varrho_T,$$

where Φ_T is an affine map as defined above and $\varrho_T : \widehat{T} \to \mathbb{R}^N$ is a C^1-mapping satisfying

$$C_T := \sup_{\widehat{x} \in \widehat{T}} |D\varrho_T(\widehat{x}) B_T^{-1}| \leq C < 1. \tag{3.1}$$

There are several ways to define ϱ_T. We follow the construction of [8], based on [4]. Note that each $T \in \mathcal{T}_h^{(1)}$ is either an internal simplex with at most one node on the boundary, or T has more than one node on the boundary. In the first case, we set $\varrho_T = 0$. For the latter case, we denote by l the

number of nodes of T that lie on the boundary $\Gamma_h^{(1)}$. The vertices $x_1^T, \ldots, x_{n+1}^T$ of T are ordered such that $x_1^T, \ldots, x_l^T$ lie on $\Gamma_h^{(1)}$. For each $x^T \in T$, there is a unique representation

$$x^T = \sum_{j=1}^{n+1} \lambda_j x_j^T$$

in barycentric coordinates. Note that

$$\lambda_{n+1} = 1 - \sum_{j=1}^{n} \lambda_j \,.$$

We write $\widehat{x}^T = (\lambda_1, \ldots, \lambda_N)$ for the coordinates of x in $\widehat{T}$. We introduce

$$\lambda^*(\widehat{x}) = \sum_{j=1}^{l} \lambda_j \,, \quad \widehat{\sigma} = \{\widehat{x} \in \widehat{T} : \lambda^*(\widehat{x}) = 0\} \,.$$

We have $\lambda^*(\widehat{x}) = 0$ if $\widehat{x}$ is a node which is not belonging to the boundary (or if $\widehat{x}$ is on the edge between such nodes in the three-dimensional case, when $l = 2$), and $\lambda^*(\widehat{x}) = 1$ if $\widehat{x} \in T \cap \Gamma_h^{(1)}$.

We denote by τ_T the face of $\Gamma_h^{(1)}$ that corresponds to the boundary face of T, i. e. $\tau_T = T \cap \Gamma_h^{(1)}$. For $\widehat{x} \notin \widehat{\sigma}$, we denote the projection of $x = \Phi_T(\widehat{x})$ onto τ_T by

$$y(\widehat{x}) = \sum_{j=1}^{l} \frac{\lambda_j}{\lambda^*} x_j^T \,.$$

Then, using the normal projection p defined in (2.2), we define ϱ_T by

$$\varrho_T(\widehat{x}) = \begin{cases} (\lambda^*(\widehat{x}))^{k+2}(p(y(\widehat{x})) - y(\widehat{x})) \,, & \text{if } \widehat{x} \notin \widehat{\sigma} \,, \\ 0 \,, & \text{if } \widehat{x} \in \widehat{\sigma} \,. \end{cases}$$

Basic regularity properties of the above maps are stated and proved in [8]. In particular, it is shown that ρ_T satisfies (3.1) for $h \leq h_0$ sufficiently small.

3.3. Computational domain and lifts

We can now define the higher-order computational domain $\Omega_h^{(k)}$ for $k \geq 1$. Let $T \in \mathcal{T}_h^{(1)}$ and $\varphi_1^k, \ldots, \varphi_{n_k}^k$ be the Lagrangian basis functions of degree k on $\widehat{T}$ corresponding to the nodal points $\widehat{x}^1, \ldots, \widehat{x}^{n_k}$ on $\widehat{T}$. Here, n_k denotes the number of nodal points on each element, for example $n_k = 4$ or $n_k = 10$ for linear or quadratic finite elements in three dimensions. Then, we define a parametrization of a polynomial simplex $T^{(k)}$ by

$$\Phi_T^{(k)}(\widehat{x}) = \sum_{j=1}^{n_k} \Phi_T^c(\widehat{x}^j) \varphi_j^k(\widehat{x}) \,.$$

Note that, by the Lagrangian property, we have

$$\Phi_T^{(k)}(\widehat{x}^l) = \Phi_T^c(\widehat{x}^l) \,.$$

We can apply this to each $T \in \mathcal{T}_h^{(1)}$ and then define $\Omega_h^{(k)}$ as the union of elements in $\mathcal{T}_h^{(k)}$, defined by

$$\mathcal{T}_h^{(k)} := \{T^{(k)} : T \in \mathcal{T}_h^{(1)}\} \,, \quad T^{(k)} := \{\Phi_T^{(k)}(\widehat{x}) : \widehat{x} \in \widehat{T}\} \,.$$

For $k = 1$, this notation is consistent with the notation of $\Omega_h^{(1)}$ in the previous subsection.

Definition 3.1. For a function $w_h : \Omega_h^{(k)} \to \mathbb{R}$, its lift $w_h^l : \Omega \to \mathbb{R}$ is defined by $w_h^l = w_h \circ (\Phi_T^{(k)})^{-1}$, i.e.

$$w_h^l \left(\Phi_T^{(k)}(x) \right) = w_h(x), \quad x \in \Omega_h^{(k)}.$$

For a continuous function $w : \Omega \to \mathbb{R}$, its inverse lift is defined by $w^{-l} = w \circ \Phi_T^{(k)}$.

The following lemma states that both the L^2-norm and the H^1-seminorm of functions on $\Omega_h^{(k)}$ and their lifts are equivalent.

Proposition 3.2. *There exists a constant $c > 0$ independent of h (but depending on k, n and the geometry of Ω), such that for all $w_h : \Omega_h^{(k)} \to \mathbb{R}$*

$$\frac{1}{c} \|w_h\|_{L^2(\Omega_h^{(k)}; \Gamma_h^{(k)})} \leq \|w_h^l\|_{L^2(\Omega; \Gamma)} \leq c \|w_h\|_{L^2(\Omega_h^{(k)}; \Gamma_h^{(k)})},$$

$$\frac{1}{c} \|\nabla w_h\|_{L^2(\Omega_h^{(k)})} \leq \|\nabla w_h^l\|_{L^2(\Omega)} \leq c \|\nabla w_h\|_{L^2(\Omega_h^{(k)})},$$

$$\frac{1}{c} \|\nabla_{\Gamma_h}(\gamma_h w_h)\|_{L^2(\Gamma_h^{(k)})} \leq \|\nabla_\Gamma(\gamma w_h^l)\|_{L^2(\Gamma)} \leq c \|\nabla_{\Gamma_h}(\gamma_h w_h)\|_{L^2(\Gamma_h^{(k)})}.$$

Proof. See [8, Proposition 4.9] for the bulk estimate and [3] for the estimate on the boundary. ∎

4. **The isoparametric finite element method**

In this section we introduce the finite element method. We use piecewise polynomial finite element functions of degree k, which lead to isoparametric finite elements. Isoparametric finite elements are also used in [8] in the context of bulk–surface equations; the traces of isoparametric bulk finite element functions on the boundary can be considered as surface finite elements, see e.g. [6, 7].

From now on, we write Ω_h and Γ_h instead of $\Omega_h^{(k)}$ and $\Gamma_h^{(k)}$. We collect the nodes $x_1, \ldots, x_N \in \mathbb{R}^n$ of the triangulation in a vector $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^{nN}$ such that exactly the first N_Γ nodes $x_1, \ldots, x_{N_\Gamma}$ lie on Γ. We use Lagrangian basis functions $\varphi_1, \ldots, \varphi_N$, which are defined elementwise such that their pullback to the reference element is polynomial of degree k. The basis functions satisfy the property $\varphi_j(x_k) = \delta_{jk}$ for $1 \leq j, k \leq N$. The finite element space is then defined as

$$V_h = \text{span} \{\varphi_1, \ldots, \varphi_N\}.$$

Recall that, as opposed to [15], the finite element space V_h is *not* contained in $V = H^1(\Omega; \Gamma)$. The right-hand side functions are approximated with appropriate functions $f_h : \Omega_h \to \mathbb{R}$ and $g_h : \Gamma_h \to \mathbb{R}$. If f and g are continuous, one could use the inverse lifts or the finite element interpolations, for example.

We use the following discrete analogues of the bilinear forms defined in (2.5):

$$m_h^\Omega(u_h, v_h) = \int_{\Omega_h} u_h v_h \mathrm{d}x,$$

$$a_h^\Omega(u_h, v_h) = \int_{\Omega_h} \nabla u_h \cdot \nabla v_h \mathrm{d}x,$$

$$m_h^\Gamma(u_h, v_h) = \int_{\Gamma_h} (\gamma_h u_h)(\gamma_h v_h) \mathrm{d}\boldsymbol{\sigma}_h,$$

$$a_h^\Gamma(u_h, v_h) = \int_{\Gamma_h} \nabla_{\Gamma_h}(\gamma_h u_h) \cdot \nabla_{\Gamma_h}(\gamma_h v_h) \mathrm{d}\boldsymbol{\sigma}_h,$$

$$a_h(u_h, v_h) = a_h^\Omega(u_h, v_h) + \kappa m_h^\Omega(u_h, v_h) + \alpha m_h^\Gamma(u_h, v_h) + \beta a_h^\Gamma(u_h, v_h).$$

Here, γ_h denotes the discrete trace operator on Γ_h, $\mathrm{d}\boldsymbol{\sigma}_h$ denotes the discrete surface measure on Γ_h (see [8, 3, 7] for further details). Moreover, we denote

$$\ell_h(\varphi_h) = \int_{\Omega_h} f_h \varphi_h \mathrm{d}x + \int_{\Gamma_h} g_h (\gamma_h \varphi_h) \mathrm{d}\boldsymbol{\sigma}_h \,.$$

The bilinear forms are defined on $V_h \times V_h$ and ℓ_h is defined on V_h.

The discretized formulation of (2.6) now reads: given $f_h, g_h \in V_h$, find $u_h \in V_h$ such that

$$a_h(u_h, \varphi_h) = \ell_h(\varphi_h) \tag{4.1}$$

for all $\varphi_h \in V_h$. Since a_h is coercive and bounded and V_h is a (finite-dimensional) Hilbert space, we get existence and uniqueness of the discrete solution by the Lax-Milgram lemma.

4.1. **Matrix–vector formulation**

We derive a matrix–vector formulation of the discretized problem. First, we note that (4.1) is equivalent to: find $u_h \in V_h$ such that

$$a_h(u_h, \varphi_j) = \ell_h(\varphi_j)$$

for all basis functions φ_j, $j = 1, \ldots, N$. The functions f_h and g_h, which are assumed to be finite element functions, can be written as $f_h(\cdot) = \sum_{j=1}^{N} f_h(x_j)\varphi_j(\cdot)$, $g_h(\cdot) = \sum_{j=1}^{N_\Gamma} g_h(x_j)\varphi_j(\cdot)$. We collect the nodal values in vectors

$$\mathbf{f} = (f_h(x_j))_{j=1}^{N} \,, \quad \mathbf{g} = (g_h(x_j))_{j=1}^{N_\Gamma} \,.$$

We define the bulk and surface mass and stiffness matrices:

$$(\mathbf{M}_\Omega)_{jk} = \int_{\Omega_h} \varphi_j \varphi_k \mathrm{d}x \,,$$

$$(\mathbf{A}_\Omega)_{jk} = \int_{\Omega_h} \nabla\varphi_j \cdot \nabla\varphi_k \mathrm{d}x \,, \quad 1 \le j, k \le N \,,$$

$$(\mathbf{M}_\Gamma)_{jk} = \int_{\Gamma_h} (\gamma_h \varphi_j)(\gamma_h \varphi_k) \mathrm{d}\boldsymbol{\sigma}_h \,,$$

$$(\mathbf{A}_\Gamma)_{jk} = \int_{\Gamma_h} \nabla_{\Gamma_h}(\gamma_h \varphi_j) \cdot \nabla_{\Gamma_h}(\gamma_h \varphi_k) \mathrm{d}\boldsymbol{\sigma}_h \,, \quad 1 \le j, k \le N_\Gamma \,.$$

We introduce the matrix $\boldsymbol{\gamma} = (I_{N_\Gamma}, 0) \in \mathbb{R}^{N_\Gamma \times N}$, where I_{N_Γ} denotes the identity matrix of size $N_\Gamma \times N_\Gamma$. For a finite element function w_h with nodal values collected in a vector $\mathbf{w}$, $\boldsymbol{\gamma}\mathbf{w} \in \mathbb{R}^{N_\Gamma}$ is the vector of the nodal values on the boundary nodes.

Proposition 4.1. *Let $u_h(\cdot) = \sum_{j=1}^{N} u_j \varphi_j(\cdot) \in V_h$ denote the finite element solution to (4.1) and $\mathbf{u} = (u_j)_{j=1}^{N}$ the vector of nodal values. Then the spatially discretized problem (4.1) is equivalent to the linear system*

$$\mathbf{Ku} = \mathbf{b} \,, \tag{4.2}$$

where $\mathbf{K} = \boldsymbol{\gamma}^{\mathrm{T}}(\alpha\mathbf{M}_\Gamma + \beta\mathbf{A}_\Gamma)\boldsymbol{\gamma} + \kappa\mathbf{M}_\Omega + \mathbf{A}_\Omega$ and $\mathbf{b} = \mathbf{M}_\Omega\mathbf{f} + \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{M}_\Gamma\mathbf{g}$.

Proof. Follows from linearity and a direct computation. ∎

The following properties of $\mathbf{K}$ are needed in the error analysis.

Lemma 4.2. *For a finite element function $w_h = \sum_{j=1}^{N} w_j \varphi_j$ with corresponding nodal vector $\mathbf{w} \in \mathbb{R}^n$, the a_h-norm of w_h, defined by $\|w_h\|_{a_h} = (a_h(w_h, w_h))^{1/2} = (\mathbf{w}^{\mathrm{T}}\mathbf{Kw})^{1/2}$ and the $H^1(\Omega_h, \Gamma_h)$-norm are equivalent.*

Proof. For $\alpha = \beta = \kappa = 1$, we have $\|w_h\|_{a_h} = \|w_h\|_{H^1(\Omega_h;\Gamma_h)}$. In the general case, denote $c_1 = \min(\alpha,\beta,\kappa,1)$ and $c_2 = \max(\alpha,\beta,\kappa,1)$ and we have

$$c_1\|w_h\|_{a_h} \leq \|w_h\|_{H^1(\Omega_h;\Gamma_h)} \leq c_2\|w_h\|_{a_h}\,.$$

$\blacksquare$

Remark 4.3. If the right-hand side functions f and g are not approximated with finite element functions, the vector $\mathbf{b}$ in (4.2) is defined by integrals over Ω and Γ, which then have to be approximated with quadrature rules. In this paper, we do not intend analyzing these numerical integration errors and therefore assume that f and g are approximated with finite element functions f_h and g_h. This is not fully practical for $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$, cf. [5, 8]. We will carefully carry out the error analysis such that this approximation error is taken into account. If f and g are continuous, f_h and g_h can be chosen as finite element interpolations of f and g, and provided that f and g are sufficiently regular this interpolation error is of the same order as the order of the finite element method.

Definition 4.4. For a function $w \in H^2(\Omega)$, its finite element interpolation $\widetilde{I}_h w \in V_h$ is given by

$$\widetilde{I}_h w(\cdot) = \sum_{j=1}^{N} w(x_j)\varphi_j(\cdot)\,.$$

The lifted finite element interpolation $I_h w : \Omega \to \mathbb{R}$ is then defined as

$$I_h w = \left(\widetilde{I}_h w\right)^l\,.$$

Note that since $n \in \{2,3\}$, we have $H^2(\Omega) \subset C^0(\Omega)$, so the pointwise evaluation is well-defined. The following two approximation properties are crucial in order to prove optimal-order error bounds with respect to the regularity of the exact solution.

Proposition 4.5. *Let $k \geq 1$. There exists a constant c independent of h and j, such that for all $2 \leq j \leq k+1$*

$$\|w - I_h w\|_{L^2(\Omega;\Gamma)} \leq ch^j\|w\|_{H^j(\Omega;\Gamma)}\,,$$
$$\|w - I_h w\|_{H^1(\Omega;\Gamma)} \leq ch^{j-1}\|w\|_{H^j(\Omega;\Gamma)}$$

for all $w \in H^j(\Omega;\Gamma)$.

Proof. See [2, Corollary 4.1] and [3]. $\blacksquare$

Proposition 4.6. *For any $u_h, w_h \in V_h$ with lifts $u_h^l, w_h^l \in V_h^l \subset H^1(\Omega)$, we have the following estimates:*

$$\left|m_h^\Omega(u_h, w_h) - m^\Omega(u_h^l, w_h^l)\right| \leq ch^k\|u_h^l\|_{L^2(\Omega)}\|w_h^l\|_{L^2(\Omega)}\,,$$
$$\left|m_h^\Omega(u_h, w_h) - m^\Omega(u_h^l, w_h^l)\right| \leq ch^{k+1}\|u_h^l\|_{H^1(\Omega)}\|w_h^l\|_{H^1(\Omega)}\,,$$
$$\left|a_h^\Omega(u_h, w_h) - a^\Omega(u_h^l, w_h^l)\right| \leq ch^k\|u_h^l\|_{H^1(\Omega)}\|w_h^l\|_{H^1(\Omega)}\,.$$

The traces of u_h, w_h on Γ_h and their lifts on Γ satisfy

$$\left|m_h^\Gamma(u_h, w_h) - m^\Gamma(u_h^l, w_h^l)\right| \leq ch^{k+1}\|u_h^l\|_{L^2(\Gamma)}\|w_h^l\|_{L^2(\Gamma)}\,,$$
$$\left|a_h^\Gamma(u_h, w_h) - a^\Gamma(u_h^l, w_h^l)\right| \leq ch^{k+1}\|\nabla_\Gamma u_h^l\|_{L^2(\Gamma)}\|\nabla_\Gamma w_h^l\|_{L^2(\Gamma)}\,.$$

For $u, w \in H^2(\Omega)$ with inverse lifts u^{-l}, w^{-l}, we have

$$\left| a_h^\Omega(u^{-l}, w^{-l}) - a^\Omega(u, w) \right| \le ch^{k+1} \|u\|_{H^2(\Omega)} \|w\|_{H^2(\Omega)} \,.$$

Proof. See [9, Lemma 7.15] or in the proof of [8, Lemma 6.2]. ∎

5. Error analysis

In this section, we analyze the error of the isoparametric finite element method. Since the exact solution and the numerical solution are defined on different domains, we cannot compare them directly. Instead, we compare the exact solution to the lift of the numerical solution. We derive optimal-order error estimates for finite elements of arbitrary order $k \ge 1$, with respect to both the regularity of the solution and the approximation of the data.

We begin by stating the main results of this paper. The proof of the following theorems follows down below and is clearly separated into stability and consistency.

5.1. Statement of the main result

Theorem 5.1. *Let $j \ge 1$ be a natural number, $f \in H^{j-1}(\Omega)$, $g \in H^{j-1}(\Gamma)$, let $u \in H^{j+1}(\Omega; \Gamma)$ be the solution of (2.6). Denote by $u_h : \Omega_h^{(k)} \to \mathbb{R}$ the numerical solution to (4.1) computed with isoparametric finite elements of order $k \ge 1$, f_h and g_h approximations to f and g. Then, the error between the exact solution and the lifted finite element solution is bounded by*

$$\|u - u_h^l\|_{H^1(\Omega; \Gamma)} \le C h^{\min(k,j)} + c\|f - f_h^l\|_{L^2(\Omega)} + c\|g - g_h^l\|_{L^2(\Gamma)} \,,$$

where C and c are independent of h, C depends on $\|f\|_{L^2(\Omega)}$, $\|g\|_{L^2(\Gamma)}$ and $\|u\|_{H^{\min(k,j)+1}(\Omega;\Gamma)}$ and c is independent of these quantities.

In particular: If $j \ge k$ and f_h and g_h are chosen such that $\|f - f_h^l\|_{L^2(\Omega)} \le ch^k$ and $\|g - g_h^l\|_{L^2(\Gamma)} \le ch^k$, then the error is bounded by

$$\|u - u_h^l\|_{H^1(\Omega; \Gamma)} \le C h^k \,, \tag{5.1}$$

where C depends on the regularity of f and g and on $\|u\|_{H^{k+1}(\Omega;\Gamma)}$.

Remark 5.2. The assumptions in the second part of Theorem 5.1 are satisfied if $f \in H^k(\Omega)$, $g \in H^k(\Gamma)$ for $k \ge 2$. In this case f_h and g_h can be chosen as finite element interpolations of f and g. The interpolation errors are then bounded using Proposition 4.5, and we arrive at (5.1).

For the L^2-estimate, we need slightly more assumptions, see Remark 5.6.

Theorem 5.3. *Let $j \ge 1$ be a natural number, $f \in H^{j-1}(\Omega) \cap H^1(\Omega)$, $g \in H^{j-1}(\Gamma)$, let $u \in H^{j+1}(\Omega; \Gamma)$ be the solution of (2.6). Denote by $u_h : \Omega_h^{(k)} \to \mathbb{R}$ the numerical solution of (4.1) computed with isoparametric finite elements of order $k \ge 1$. Then, the error between the exact solution and the lifted finite element solution is bounded by*

$$\|u - u_h^l\|_{L^2(\Omega; \Gamma)} \le C h^{\min(k,j)+1} + c\|f - f_h^l\|_{L^2(\Omega)} + c\|g - g_h^l\|_{L^2(\Gamma)} + ch^{k+1}\|f - f_h^l\|_{H^1(\Omega)} \,,$$

where C and c are independent of h, C depends on $\|f\|_{H^1(\Omega)}$, $\|g\|_{L^2(\Gamma)}$ and $\|u\|_{H^{\min(k,j)+1}(\Omega;\Gamma)}$ and c is independent of these quantities.

In particular: If $j \ge k$ and f_h and g_h are chosen such that $\|f - f_h^l\|_{L^2(\Omega)} \le ch^{k+1}$, $\|f - f_h^l\|_{H^1(\Omega)} \le c$ and $\|g - g_h^l\|_{L^2(\Gamma)} \le ch^{k+1}$, then the error is bounded by

$$\|u - u_h^l\|_{L^2(\Omega; \Gamma)} \le C h^{k+1} \,.$$

Remark 5.4. The assumptions in the second part of Theorem 5.3 are satisfied if $f \in H^{k+1}(\Omega)$, $g \in H^{k+1}(\Gamma)$ for $k \geq 1$ with $f_h = \widetilde{I}_h f$, $g_h = \widetilde{I}_h g$, see Proposition 4.5.

The proof of Theorems 5.1 and 5.3 follows down below and is clearly separated into stability and consistency.

5.2. Stability

The finite element interpolation $u_h^* : \Omega_h^{(k)} \to \mathbb{R}$ of the exact solution, which corresponds to the nodal vector $\mathbf{u}^* = (u(x_j))_{j=1}^N$, satisfies the numerical scheme up to a defect $\mathbf{d}$, which corresponds to a finite element function $d_h \in V_h$:

$$\mathbf{K}\mathbf{u}^* = \mathbf{b} + (\mathbf{M}_\Omega + \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{M}_\Gamma\boldsymbol{\gamma})\mathbf{d}. \tag{5.2}$$

Note that $\mathbf{K}$ is symmetric and positive definite and thus both $\mathbf{K}^{-1}$ and $\mathbf{K}^{-1/2}$ exist. Subtracting (5.2) from (4.2), we find that the error $\mathbf{e} = \mathbf{u} - \mathbf{u}^*$ satisfies

$$\mathbf{K}\mathbf{e} = -(\mathbf{M}_\Omega + \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{M}_\Gamma\boldsymbol{\gamma})\mathbf{d}.$$

We test this equation with $\mathbf{e}$ and obtain

$$\|\mathbf{e}\|_{\mathbf{K}}^2 := \mathbf{e}^{\mathrm{T}}\mathbf{K}\mathbf{e} = -\mathbf{e}^{\mathrm{T}}(\mathbf{M}_\Omega + \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{M}_\Gamma\boldsymbol{\gamma})\mathbf{d}.$$

The defect will be estimated in the dual norm induced by the bilinear form a_h:

$$\|\mathbf{d}\|_\star := \|\mathbf{K}^{-1/2}(\mathbf{M}_\Omega + \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{M}_\Gamma\boldsymbol{\gamma})\mathbf{d}\|_2 = \sup_{0 \neq \mathbf{w} \in \mathbb{R}^N} \frac{\mathbf{d}^{\mathrm{T}}(\mathbf{M}_\Omega + \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{M}_\Gamma\boldsymbol{\gamma})\mathbf{K}^{-1/2}\mathbf{w}}{(\mathbf{w}^{\mathrm{T}}\mathbf{w})^{1/2}}$$

$$= \sup_{0 \neq \mathbf{z} \in \mathbb{R}^N} \frac{\mathbf{d}^{\mathrm{T}}(\mathbf{M}_\Omega + \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{M}_\Gamma\boldsymbol{\gamma})\mathbf{z}}{(\mathbf{z}^{\mathrm{T}}\mathbf{K}\mathbf{z})^{1/2}} = \sup_{0 \neq \varphi_h \in V_h} \frac{\int_{\Omega_h} d_h \varphi_h \mathrm{d}x + \int_{\Gamma_h} (\gamma_h d_h)(\gamma_h \varphi_h)\mathrm{d}\boldsymbol{\sigma}_h}{\|\varphi_h\|_{a_h}}.$$

With the Cauchy–Schwarz and Young inequality, we obtain

$$\|\mathbf{e}\|_{\mathbf{K}}^2 = -\mathbf{e}^{\mathrm{T}}(\mathbf{M}_\Omega + \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{M}_\Gamma\boldsymbol{\gamma})\mathbf{d} = -\mathbf{e}^{\mathrm{T}}\mathbf{K}^{1/2}\mathbf{K}^{-1/2}(\mathbf{M}_\Omega + \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{M}_\Gamma\boldsymbol{\gamma})\mathbf{d} \tag{5.3}$$

$$\leq \|\mathbf{K}^{1/2}\mathbf{e}\|_2\|\mathbf{K}^{-1/2}(\mathbf{M}_\Omega + \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{M}_\Gamma\boldsymbol{\gamma})\mathbf{d}\|_2$$

$$= \|\mathbf{e}\|_{\mathbf{K}}\|\mathbf{d}\|_\star.$$

We thus have shown that

$$\|\mathbf{e}\|_{\mathbf{K}} \leq \|\mathbf{d}\|_\star.$$

5.3. Consistency

In this section, we bound the dual norm of the defect in order to obtain an optimal order H^1-estimate. In order to prove error bounds of order j, we assume that the solution $u \in H^{j+1}(\Omega; \Gamma)$, which is provided if Γ is a C^{j+1}-manifold, $g \in H^{j-1}(\Gamma)$, $f \in H^{j-1}(\Omega)$ (see Proposition 2.1). Note that since $j \geq 1$ and the dimension $n \in \{2, 3\}$, we have $H^{j+1}(\Omega) \subseteq C^0(\Omega)$ and the finite element interpolation $\widetilde{I}_h u$ of u is well-defined.

Proposition 5.5. *Under the assumptions of Theorem 5.1, the defect is bounded by*

$$\|\mathbf{d}\|_\star \leq Ch^{\min(k,j)} + c\|f_h^l - f\|_{L^2(\Omega)} + c\|g_h^l - g\|_{L^2(\Gamma)}, \tag{5.4}$$

where $C = C(\|f\|_{L^2(\Omega)}, \|g\|_{L^2(\Gamma)}, \|u\|_{H^{j+1}(\Omega;\Gamma)})$.

Proof. The defect equation (5.2) is equivalent to
$$m_h^\Omega(d_h, \varphi_h) + m_h^\Gamma(d_h, \varphi_h) = \alpha m_h^\Gamma(\widetilde{I}_h u, \varphi_h) + \beta a_h^\Gamma(\widetilde{I}_h u, \varphi_h) + \kappa m_h^\Omega(\widetilde{I}_h u, \varphi_h) + a_h^\Omega(\widetilde{I}_h u, \varphi_h) - \ell_h(\varphi_h)$$
for all finite element functions $\varphi_h \in V_h$. Since $\varphi_h^l \in H^1(\Omega; \Gamma)$, the exact solution u satisfies
$$0 = \alpha m^\Gamma(u, \varphi_h^l) + \beta a^\Gamma(u, \varphi_h^l) + \kappa m^\Omega(u, \varphi_h^l) + a^\Omega(u, \varphi_h^l) - \ell(\varphi_h^l) \, .$$
Subtracting both equations yields
$$
\begin{aligned}
m_h^\Omega(d_h, \varphi_h) + m_h^\Gamma(d_h, \varphi_h) = {} & \alpha \left(m_h^\Gamma(\widetilde{I}_h u, \varphi_h) - m^\Gamma(u, \varphi_h^l) \right) \\
& + \beta \left(a_h^\Gamma(\widetilde{I}_h u, \varphi_h) - a^\Gamma(u, \varphi_h^l) \right) \\
& + \kappa \left(m_h^\Omega(\widetilde{I}_h u, \varphi_h) - m^\Omega(u, \varphi_h^l) \right) \\
& + \left(a_h^\Omega(\widetilde{I}_h u, \varphi_h) - a^\Omega(u, \varphi_h^l) \right) \\
& + \left(\ell_h(\varphi_h) - \ell(\varphi_h^l) \right) \, .
\end{aligned}
$$
We estimate the five terms separately.
(i) We write
$$
\begin{aligned}
& m_h^\Gamma(\widetilde{I}_h u, \varphi_h) - m^\Gamma(u, \varphi_h^l) \\
& = m_h^\Gamma(\widetilde{I}_h u, \varphi_h) - m^\Gamma(I_h u, \varphi_h^l) + m^\Gamma(I_h u - u, \varphi_h^l) \, .
\end{aligned}
$$
With the Cauchy–Schwarz inequality and Proposition 4.5 we obtain for the second term:
$$
\begin{aligned}
m^\Gamma(I_h u - u, \varphi_h^l) & \leq \|\gamma(I_h u - u)\|_{L^2(\Gamma)} \|\gamma \varphi_h^l\|_{L^2(\Gamma)} \\
& \leq \|I_h u - u\|_{L^2(\Omega; \Gamma)} \|\varphi_h^l\|_{H^1(\Omega; \Gamma)} \\
& \leq c h^{j+1} \|u\|_{H^{j+1}(\Omega; \Gamma)} \|\varphi_h^l\|_{H^1(\Omega; \Gamma)} \, .
\end{aligned}
$$
For the first term, we use Proposition 4.6 and then Proposition 4.5:
$$
\begin{aligned}
& \left| m_h^\Gamma(\widetilde{I}_h u, \varphi_h) - m^\Gamma(I_h u, \varphi_h^l) \right| \\
& \leq c h^{k+1} \|\gamma I_h u\|_{L^2(\Gamma)} \|\gamma \varphi_h^l\|_{L^2(\Gamma)} \\
& \leq c h^{k+1} \|I_h u\|_{L^2(\Omega; \Gamma)} \|\varphi_h^l\|_{H^1(\Omega; \Gamma)} \\
& \leq c h^{k+1} \left(\|I_h u - u\|_{L^2(\Omega; \Gamma)} + \|u\|_{L^2(\Omega; \Gamma)} \right) \|\varphi_h^l\|_{H^1(\Omega; \Gamma)} \\
& \leq c h^{k+1} \left(c h^{j+1} \|u\|_{H^{j+1}(\Omega; \Gamma)} + \|u\|_{H^1(\Omega; \Gamma)} \right) \|\varphi_h^l\|_{H^1(\Omega; \Gamma)} \\
& \leq c h^{k+1} \|u\|_{H^{j+1}(\Omega; \Gamma)} \|\varphi_h^l\|_{H^1(\Omega; \Gamma)} \, .
\end{aligned}
$$
(ii) Similarly, we write
$$
\begin{aligned}
& a_h^\Gamma(\widetilde{I}_h u, \varphi_h) - a^\Gamma(u, \varphi_h^l) \\
& = a_h^\Gamma(\widetilde{I}_h u, \varphi_h) - a^\Gamma(I_h u, \varphi_h^l) + a^\Gamma(I_h u - u, \varphi_h^l) \, .
\end{aligned}
$$
We then proceed as in the first step and obtain
$$\left| a_h^\Gamma(\widetilde{I}_h u, \varphi_h) - a^\Gamma(u, \varphi_h^l) \right| \leq c h^{\min(k+1, j)} \|u\|_{H^{j+1}(\Omega; \Gamma)} \|\varphi_h^l\|_{H^1(\Omega; \Gamma)} \, .$$
(iii,iv) Using Propositions 4.5 and 4.6, we obtain analogously
$$\left| m_h^\Omega(\widetilde{I}_h u, \varphi_h) - m^\Omega(u, \varphi_h^l) \right| \leq c h^{\min(k, j)+1} \|u\|_{H^{j+1}(\Omega; \Gamma)} \|\varphi_h^l\|_{H^1(\Omega; \Gamma)}$$
and
$$\left| a_h^\Omega(\widetilde{I}_h u, \varphi_h) - a^\Omega(u, \varphi_h^l) \right| \leq c h^{\min(k, j)} \|u\|_{H^{j+1}(\Omega; \Gamma)} \|\varphi_h^l\|_{H^1(\Omega; \Gamma)} \, .$$

(v) For the last term we note that

$$\ell_h(\varphi_h) - \ell(\varphi_h^l) = m_h^\Omega(f_h, \varphi) - m^\Omega(f, \varphi_h^l)$$
$$+ m_h^\Gamma(g_h, \gamma_h \varphi_h) - m^\Gamma(g, \gamma \varphi_h^l).$$

We write

$$m_h^\Omega(f_h, \varphi_h) - m^\Omega(f, \varphi_h^l) = m_h^\Omega(f_h, \varphi_h) - m^\Omega(f_h^l, \varphi_h^l) + m^\Omega(f_h^l - f, \varphi_h^l).$$

By the Cauchy–Schwarz inequality the last term is bounded by

$$m^\Omega(f_h^l - f, \varphi_h^l) \leq \|f_h^l - f\|_{L^2(\Omega)} \|\varphi_h^l\|_{H^1(\Omega;\Gamma)}.$$

For the first term we use Proposition 4.6:

$$\left| m_h^\Omega(f_h, \varphi_h) - m^\Omega(f_h^l, \varphi_h^l) \right| \leq ch^k \|f_h^l\|_{L^2(\Omega)} \|\varphi_h^l\|_{L^2(\Omega)}$$
$$\leq ch^k \left(\|f_h^l - f\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right) \|\varphi_h^l\|_{H^1(\Omega;\Gamma)},$$

so that we obtain the bound

$$\left| m_h^\Omega(f_h, \varphi_h) - m^\Omega(f, \varphi_h^l) \right| \leq ch^k \|f\|_{L^2(\Omega)} \|\varphi_h^l\|_{H^1(\Omega;\Gamma)} + c\|f_h^l - f\|_{L^2(\Omega)} \|\varphi_h^l\|_{H^1(\Omega;\Gamma)}.$$

In a similar fashion we estimate

$$\left| m_h^\Gamma(g_h, \varphi_h) - m^\Gamma(g, \varphi_h^l) \right| \leq \left(ch^{k+1} \|g\|_{L^2(\Gamma)} + \|g_h^l - g\|_{L^2(\Gamma)} \right) \|\varphi_h^l\|_{H^1(\Omega;\Gamma)}.$$

Adding the five estimates, using definition (5.3) of the dual norm together with the coercivity of a_h, we obtain (5.4). ∎

Now we can prove Theorem 5.1.

Proof of Theorem 5.1. The error is decomposed in the following way:

$$u - u_h^l = (u - I_h u) + \left(I_h u - u_h^l \right).$$

With Proposition 4.5, we obtain $\|u - I_h u\|_{H^1(\Omega;\Gamma)} \leq ch^j \|u\|_{H^{j+1}(\Omega;\Gamma)}$. For the second term, we note that $\widetilde{I}_h u - u_h$ is the finite element function corresponding to the nodal vector $\mathbf{u}^* - \mathbf{u} = -\mathbf{e}$, so using Proposition 3.2 and Lemma 4.2, we obtain

$$\|I_h u - u_h^l\|_{H^1(\Omega;\Gamma)} \leq c\|\widetilde{I}_h u - u_h\|_{H^1(\Omega_h;\Gamma_h)} \leq c\|\mathbf{e}\|_{\mathbf{K}} \leq c\|\mathbf{d}\|_\star,$$

so the result follows from Proposition 5.5. ∎

5.4. L^2-estimate

In order to derive an optimal-order L^2-estimate, we apply the Aubin–Nitsche trick.

Proof of Theorem 5.3. Consider the dual problem: for $\eta \in L^2(\Omega;\Gamma)$, find $z_\eta \in H^1(\Omega;\Gamma)$ such that

$$a(z_\eta, \psi) = m^\Omega(\eta, \psi) + m^\Gamma(\gamma\eta, \gamma\psi) \quad \forall \psi \in H^1(\Omega;\Gamma).$$

This is the weak formulation of (1.1) with $f = \eta$, $g = \gamma\eta$. Since $\eta \in L^2(\Omega;\Gamma)$, we have $z_\eta \in H^2(\Omega;\Gamma)$ and z_η satisfies the a priori estimate (see Proposition 2.1)

$$\|z_\eta\|_{H^2(\Omega;\Gamma)} \leq c\|\eta\|_{L^2(\Omega;\Gamma)}. \tag{5.5}$$

With $\eta = e = u - u_h^l$ and writing $z = z_e$ for brevity, we have

$$\|e\|_{L^2(\Omega;\Gamma)}^2 = m^\Omega(e, e) + m^\Gamma(\gamma e, \gamma e) = a(e, z) \tag{5.6}$$
$$= a(u - u_h^l, z - I_h z) + a(u, I_h z) - a(u_h^l, I_h z)$$
$$= a(u - u_h^l, z - I_h z) + \ell(I_h z) - a(u_h^l, I_h z - z) - a(u_h^l, z)$$

$$\begin{aligned}
&= a(u - u_h^l, z - I_h z) + \ell(I_h z) - \ell_h(\widetilde{I}_h z) + a_h(u_h, \widetilde{I}_h z) \\
&\quad - a(u_h^l, I_h z - z) - a(u_h^l - u, z) - a(u, z) \\
&= a(u - u_h^l, z - I_h z) \\
&\quad + \left(\ell(I_h z) - \ell_h(\widetilde{I}_h z) \right) \\
&\quad + \left(a_h(u_h, \widetilde{I}_h z - z^{-l}) - a(u_h^l, I_h z - z) \right) \\
&\quad + \left(a_h(u_h - u^{-l}, z^{-l}) - a(u_h^l - u, z) \right) \\
&\quad + \left(a_h(u^{-l}, z^{-l}) - a(u, z) \right) .
\end{aligned}$$

We estimate the five terms separately.

(i) Using the boundedness of a, Theorem 5.1, Proposition 4.5 and the a priori bound (5.5), we obtain

$$\begin{aligned}
a(u - u_h^l, z - I_h z) &\leq c\|u - u_h^l\|_{H^1(\Omega;\Gamma)} \|z - I_h z\|_{H^1(\Omega;\Gamma)} \\
&\leq ch\|z\|_{H^2(\Omega;\Gamma)} \|u - u_h^l\|_{H^1(\Omega;\Gamma)} \\
&\leq ch\|e\|_{L^2(\Omega;\Gamma)} \left(Ch^{\min(k,j)} + c\|f - f_h^l\|_{L^2(\Omega)} + c\|g - g_h^l\|_{L^2(\Gamma)} \right) \\
&\leq \left(Ch^{\min(k,j)+1} + ch\|f - f_h^l\|_{L^2(\Omega)} + ch\|g - g_h^l\|_{L^2(\Gamma)} \right) \|e\|_{L^2(\Omega;\Gamma)} .
\end{aligned}$$

(ii) We write

$$\begin{aligned}
\ell(I_h z) - \ell_h(\widetilde{I}_h z) &= m^\Omega(f, I_h z) - m_h^\Omega(f_h, \widetilde{I}_h z) \\
&\quad + m^\Gamma(g, I_h z) - m_h^\Gamma(g_h, \widetilde{I}_h z) \\
&= m^\Omega(f - f_h^l, I_h z) + \left(m^\Omega(f_h^l, I_h z) - m_h^\Omega(f_h, \widetilde{I}_h z) \right) \\
&\quad + m^\Gamma(g - g_h^l, I_h z) + \left(m^\Gamma(g_h^l, I_h z) - m_h^\Gamma(g_h, \widetilde{I}_h z) \right) .
\end{aligned}$$

Using Cauchy–Schwarz, Proposition 4.5 and (5.5), we see that the first term is bounded by

$$\begin{aligned}
m^\Omega(f - f_h^l, I_h z) &\leq \|f - f_h^l\|_{L^2(\Omega)} \left(\|I_h z - z\|_{L^2(\Omega;\Gamma)} + \|z\|_{L^2(\Omega;\Gamma)} \right) \\
&\leq \|f - f_h^l\|_{L^2(\Omega;\Gamma)} \left(ch^2 + 1 \right) \|z\|_{H^2(\Omega;\Gamma)} \\
&\leq c\|f - f_h^l\|_{L^2(\Omega)} \|e\|_{L^2(\Omega;\Gamma)} .
\end{aligned}$$

For the third term we proceed similarly and obtain

$$\begin{aligned}
m^\Gamma(g - g_h^l, I_h z) &\leq \|g - g_h^l\|_{L^2(\Gamma)} \|I_h z\|_{L^2(\Gamma)} \\
&\leq \|g - g_h^l\|_{L^2(\Gamma)} \left(\|I_h z - z\|_{L^2(\Omega;\Gamma)} + \|z\|_{L^2(\Omega;\Gamma)} \right) \\
&\leq c\|g - g_h^l\|_{L^2(\Gamma)} \|e\|_{L^2(\Omega;\Gamma)} .
\end{aligned}$$

For the second term we use Propositions 4.6 and 4.5 to obtain

$$\begin{aligned}
m^\Omega(f_h^l, I_h z) - m_h^\Omega(f_h, \widetilde{I}_h z) &\leq ch^{k+1} \left(\|f_h^l - f\|_{H^1(\Omega)} + \|f\|_{H^1(\Omega)} \right) \|I_h z\|_{H^1(\Omega)} \qquad (5.7) \\
&\leq ch^{k+1} \left(\|f_h^l - f\|_{H^1(\Omega)} + \|f\|_{H^1(\Omega)} \right) \|e\|_{L^2(\Omega;\Gamma)} .
\end{aligned}$$

With Proposition 4.6, we similarly obtain

$$m^\Gamma(g_h^l, I_h z) - m_h^\Gamma(g_h, \widetilde{I}_h z) \leq ch^{k+1} \left(\|g\|_{L^2(\Gamma)} + \|g_h^l - g\|_{L^2(\Gamma)} \right) \|e\|_{L^2(\Omega;\Gamma)} .$$

(iii) With Proposition 4.6 and Theorem 5.1, we obtain

$$a_h(u_h, \widetilde{I_h z - z^{-l}}) - a(u_h^l, I_h z - z)$$

$$\leq ch^k \|u_h^l\|_{H^1(\Omega;\Gamma)} \|I_h z - z\|_{H^1(\Omega;\Gamma)}$$

$$\leq ch^k \|u_h^l - u + u\|_{H^1(\Omega;\Gamma)} ch\|z\|_{H^2(\Omega;\Gamma)}$$

$$\leq ch^{k+1} \left(\|u_h^l - u\|_{H^1(\Omega;\Gamma)} + \|u\|_{H^1(\Omega;\Gamma)} \right) \|z\|_{H^2(\Omega;\Gamma)}$$

$$\leq ch^{k+1} \left(Ch^{\min(k,j)} + c\|f - f_h^l\|_{L^2(\Omega)} + c\|g - g_h^l\|_{L^2(\Gamma)} \right) \|e\|_{L^2(\Omega;\Gamma)}.$$

(iv) Using the same arguments, we obtain for the fourth term

$$a(u_h - u^{-l}, z^{-l}) - a(u_h^l - u, z)$$

$$\leq ch^k \|u_h^l - u\|_{H^1(\Omega;\Gamma)} \|z\|_{H^1(\Omega;\Gamma)}$$

$$\leq ch^k \left(Ch^{\min(k,j)} + c\|f - f_h^l\|_{L^2(\Omega)} + c\|g - g_h^l\|_{L^2(\Gamma)} \right) \|e\|_{L^2(\Omega;\Gamma)}.$$

(v) For the fifth term, using $u, z \in H^2(\Omega)$, we have with Proposition 4.6 and Theorem 5.1

$$a_h(u^{-l}, z^{-l}) - a(u, z) \leq ch^{k+1} \|u\|_{H^2(\Omega;\Gamma)} \|z\|_{H^2(\Omega;\Gamma)}$$

$$\leq ch^{k+1} \|u\|_{H^{k+1}(\Omega;\Gamma)} \|e\|_{L^2(\Omega;\Gamma)}.$$

Inserting all the bounds into (5.6) gives the bound:

$$\|e\|_{L^2(\Omega;\Gamma)} \leq Ch^{\min(k,j)+1} + c\|f - f_h^l\|_{L^2(\Omega)} + c\|g - g_h^l\|_{L^2(\Gamma)} + ch^{k+1}\|f - f_h^l\|_{H^1(\Omega)},$$

where C depends on $\|u\|_{H^{j+1}(\Omega;\Gamma)}$, $\|f\|_{H^1(\Omega)}$ and $\|g\|_{L^2(\Gamma)}$.
This completes the proof of Theorem 5.3. ∎

Remark 5.6. Compared with Theorem 5.1, we need for $j = 1$ the additional assumption that $f \in H^1(\Omega)$. This is due to the first two estimates of Proposition 4.6, which only give a h^k-error bound for $f \in L^2(\Omega)$ in (5.7). Alternatively, since $f_h^l \in H^1(\Omega)$, we could simply estimate

$$m^\Omega(f_h^l, I_h z) - m_h^\Omega(f_h, \widetilde{I_h z}) \leq ch^{k+1}\|f_h^l\|_{H^1(\Omega)}\|e\|_{L^2(\Omega;\Gamma)}$$

in (5.7) without using the triangle inequality and then make the reasonable assumption that f_h can be chosen such that $\|f_h^l\|_{H^1(\Omega)} \leq c\|f\|_{L^2(\Omega)}$ with a constant independent of h. Keeping in mind that we need $f \in H^{k+1}(\Omega)$ anyway to obtain the full order, the assumption $f \in H^1(\Omega)$ becomes redundant in this case.

Corollary 5.7. *Consider the standard Robin problem*

$$\begin{cases} -\Delta u + \kappa u = f & in\ \Omega, \\ \dfrac{\partial u}{\partial \nu} + \alpha u = g & on\ \Gamma = \partial\Omega, \end{cases}$$

Here, the weak solution u is in $H^1(\Omega)$, and with minor modifications to the above convergence proof, we obtain under suitable assumptions the error estimate

$$\|u - u_h^l\|_{L^2(\Omega)} + h\|u - u_h^l\|_{H^1(\Omega)} \leq Ch^{k+1}$$

for the isoparametric finite element method. The same result holds for the Neumann boundary condition, i.e. $\alpha = 0$ and $\kappa > 0$.

6. Numerical examples

We illustrate the theoretical results with some numerical examples. We use isoparametric finite elements of degree one and two to solve a generalized Robin problem in two and three space dimensions. Polyhedral approximations are obtained with *distmesh* [17]. For quadratic finite elements, we add new nodes and project the boundary nodes on the boundary. All functions are implemented in MATLAB, the isoparametric elements are implemented based on the ideas of [1].

Example 6.1. *(Two-dimensional)*
We solve the generalized Robin boundary value problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial \nu} + u - \Delta_\Gamma u = g & \text{on } \Gamma = \partial\Omega\,, \end{cases} \tag{6.1}$$

where $\Omega = \left\{ x \in \mathbb{R}^2 : |x| < 1 \right\}$ is the unit circle, with isoparametric finite elements of degree one and two. As exact solution, we chose

$$u(x,y) = xy(x^2 + y^2)^2$$

from which we compute the right-hand side functions f and g. We compute numerical solutions for different mesh sizes. The finest mesh we used for linear finite elements has around 18000 nodes and the refined version used for quadratic finite elements has around 73500 nodes. The error between the lifted numerical solution and the exact solution corresponding to each mesh size h is reported in Figure 6.1 with a double logarithmic scale for finite elements of polynomial degree 1 and 2. The error err(h) is expected to converge with order k and $k+1$ ($k = 1, 2$) as stated in Theorem 5.1 for the $H^1(\Omega; \Gamma)$-norm and in Theorem 5.3 for the $L^2(\Omega; \Gamma)$-norm, respectively. The numerical results therefore illustrate the expected rates of convergence.

Example 6.2. *(Three-dimensional)*
We solve the generalized Robin boundary value problem (6.1) where $\Omega = \left\{ x \in \mathbb{R}^3 : |x| < 1 \right\}$ is the unit ball, with isoparametric finite elements of degree one and two. As exact solution, we chose

$$u(x,y) = x^2 + y^2 - x^2 z^2$$

from which we compute the right-hand side functions f and g. The finest mesh we used for linear finite elements has around 7000 nodes, and the refined version used for quadratic finite elements has around 55000 nodes. The error between the lifted numerical solution and the exact solution is reported in Figure 6.2 for elements of polynomial degree 1 and 2. The numerical results illustrate the expected rates of convergence as stated in Theorem 5.1 and Theorem 5.3.

Acknowledgments
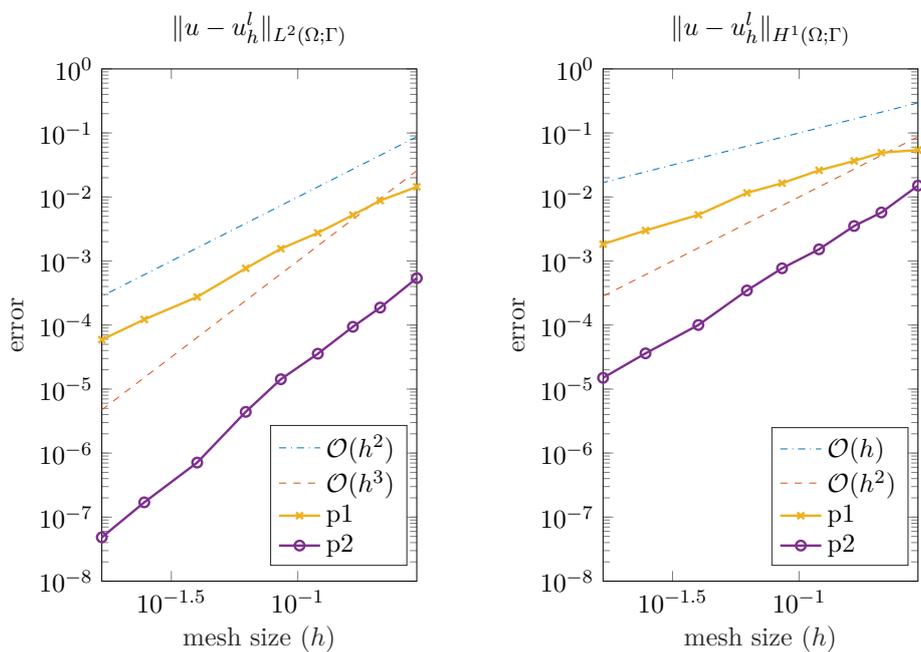
Figure 6.1. Convergence rate of the GRP discretization of Example 6.1 with isoparametric finite elements of degree 1 and 2 in two dimensions.
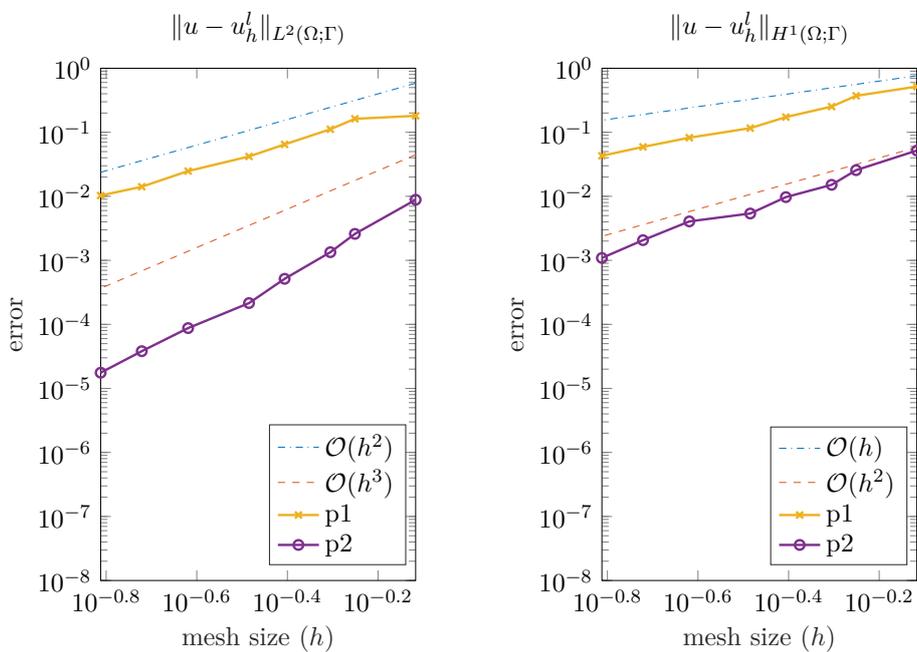
Figure 6.2. Convergence rate of the GRP discretization of Example 6.2 with isoparametric finite elements of degree 1 and 2 in three dimensions.

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