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A probabilistic particle approximation of the “Paveri-Fontana” kinetic model of traffic flow

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Abstract. This paper is devoted to the Paveri-Fontana model and its computation. The master equation of this model has no analytic solution in nonequilibrium case. We develop a stochastic approach to approximate this evolution equation. First, we give a probabilistic interpretation of the equation as a nonlinear Fokker-Planck equation. Replacing the nonlinearity by interaction, we deduce how to approximate its solution thanks to an algorithm based on a fictitious jump simulation of the interacting particle system. This algorithm is improved to obtain a linear complexity regarding the number of particles. Finally, the numerical method is illustrated on one traffic flow scenario and compared with a finite differences deterministic method.

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1. Introduction

The gas-kinetic models describe the traffic flow at a mesoscopic scale. The mesoscopic models can be an alternative offering an additional analytic investigation to microscopic models which are more expensive in terms of computation time. They can also be an alternative capturing the interactions among the vehicles to macroscopic models which provide an average estimation of traffic quantities. In addition, the mesoscopic scale establishes a bridge linking these two resolutions. The gas-kinetic models are governed by a Boltzmann-type equation representing the evolution with respect to time of the phase-space-density ρ which is a generalization of the macroscopic spatial density. This distribution function gives the density of vehicles at specific times, locations and velocities. Among the first authors to propose this kind of models in vehicular traffic flow are Prigogine and Andrews [9], Prigogine and Hermann [10] whose model describes the dynamics of the distribution of velocities in phase-space. The authors suggest that the evolution of the density is caused by: a *convection term* representing the move of vehicles according to their speed which varies their position, an *acceleration term* corresponding to the attempts of the vehicles to achieve their desired speed and a *deceleration term* aimed at avoiding an accident when the vehicle drives too fast. Afterwards, Paveri-Fontana [8] proposed an improved model where the desired speed variable is considered as an independent variable.

Simulation of this kind of models by deterministic methods has been addressed in the literature (see for instance [4]). Furthermore, some stochastic particle methods have been proposed to approximate the solution in the spatially homogeneous case ([3], [5]).

We propose here a new method to simulate the Pavani-Fontana model in a stochastic way. We start interpreting the evolution equation of this model as a nonlinear Fokker-Planck equation. Then, we deduce an approximation based on a system of interacting particles. These particles are only a discretization tool and do not represent true vehicles even if their evolution is reminiscent of the behaviour of vehicles. In particular, particles will be possibly slowed-down (the velocity jumps to a smaller value) by a slower particle ahead even if some other faster particles between them are not slowed-down. Finally, a comparison between our method and a deterministic method is performed on a case study.

2. The Pavani-Fontana model

The Pavani-Fontana model was introduced to include the contributions of the statistical spread of accelerations due to the random fluctuations of driving [8]. The unknown is the density $\rho(t, x, v, v^o)$ of vehicles at time t , position $x \in \mathbb{R}$, with current velocity $v \in \mathbb{R}_+$ and desired velocity $v^o \in \mathbb{R}_+$. Its evolution is given by:

$$\begin{aligned} \partial_t \rho(t, x, v, v^o) + v \partial_x \rho(t, x, v, v^o) = & -\partial_v \left(\rho(t, x, v, v^o) \frac{v^o - v}{\tau} \right) \\ & + (1 - P) \bar{\rho}(t, x, v) \int_v^\infty (w - v) \rho(t, x, w, v^o) dw \\ & - (1 - P) \rho(t, x, v, v^o) \int_0^v (v - w) \bar{\rho}(t, x, w) dw, \end{aligned} \quad (2.1)$$

where the integral

$$\bar{\rho}(t, x, v) = \int_0^\infty \rho(t, x, v, v^o) \nu(dv^o)$$

of ρ with respect to the measure $\nu(dv^o)$ is the density of vehicles at time t , position x with current velocity v . One may choose $\nu(dv^o)$ either equal to the Lebesgue measure dv^o to take into account continuously distributed desired velocities or equal to the sum $\sum_{k=1}^d \delta_{v_k^o}(dv^o)$ of the Dirac masses $\delta_{v_k^o}$ to model d classes of vehicles with v_k^o denoting the common desired velocity among the k -th class for $k \in \{1, \dots, d\}$. The second term in the left-hand side of (2.1) is the *convection term*. The first term in the right-hand side models *acceleration* of the drivers to reach their constant desired velocity. The sum of the two last terms models the *deceleration* of drivers caused by the presence of vehicles driving slower at the same position (or just ahead). The constant $\tau > 0$ is a relaxation time and the parameter $P \in (0, 1)$ represents the overtaking coefficient. The larger P , the easier the fast vehicles may overtake slower ones ahead. When $P = 1$, the *deceleration* term even vanishes. When $P = 0$, overtaking is still possible but will take longer because of a stronger *deceleration* contribution. The desired velocity variable v^o remains constant.

3. Probabilistic Approximation

To construct a probabilistic particle approximation of (2.1), we are first going to derive an equivalent Fokker-Planck equation. We then remark that a linear version of this Fokker-Planck equation describes the evolution of the distribution of a piecewise deterministic Markov process. Because of the nonlinearity in the true equivalent Fokker-Planck equation, the jump rate of the velocity and the Markov kernel giving the new distribution of the velocity at jumps should actually depend on the unknown density. Approximating this density by the empirical measure of N particles, we finally obtain the dynamics of the system with N particles.

3.1. Related Fokker-Planck equation

The evolution equation preserves the mass (or number of vehicles) $\int \rho(t, x, v, v^o) dx dv$ with desired velocity v^o and therefore the total mass $\int \rho(t, x, v, v^o) dx dv \nu(dv^o)$ over time. Dividing (2.1) by $\|\rho_0\|_{L^1} := \int \rho(0, x, v, v^o) dx dv \nu(dv^o)$, we obtain the equation describing the evolution of the probability density $p(t, x, v, v^o) = \frac{1}{\|\rho_0\|_{L^1}} \rho(t, x, v, v^o)$ with respect to the measure $dx dv \nu(dv^o)$.

$$\begin{aligned} \partial_t p(t, x, v, v^o) &= -v \partial_x p(t, x, v, v^o) - \partial_v \left(p(t, x, v, v^o) \frac{v^o - v}{\tau} \right) \\ &\quad + \int \lambda(t, x, w) \mu(t, x, w, v) p(t, x, w, v^o) dw - \lambda(t, x, v) p(t, x, v, v^o), \end{aligned} \quad (3.1)$$

where, for $\gamma = (1 - P)\|\rho_0\|_{L^1}$,

$$\lambda(t, x, v) = \gamma \int \int_0^v (v - w) p(t, x, w, v^o) dw \nu(dv^o), \quad (3.2)$$

$$\mu(t, x, w, v) = \mathbb{I}_{\{\lambda(t, x, w) > 0\}} \frac{\gamma(w - v) \int p(t, x, v, v^o) \nu(dv^o) \mathbb{I}_{[0, w]}(v)}{\lambda(t, x, w)}. \quad (3.3)$$

When $\lambda(t, x, w) > 0$, $v \mapsto \mu(t, x, w, v)$ is a probability density on \mathbb{R}_+ . For given functions λ and μ (which do not depend on the unknown p as above), we are now going to check that (3.1) is the Fokker-Planck equation associated with a piecewise deterministic Markov process.

3.2. The piecewise deterministic Markov process

Let (X_0, V_0, V_{des}) be distributed according to $p(0, x, v, v^o) dx dv \nu(dv^o)$ and $\{(X_t, V_t, V_{des}), t \in [0, T]\}$ be the piecewise deterministic Markov process evolving according to the ODE

$$(S) : \begin{cases} V_{des} & \text{remains constant,} \\ \frac{dV_t}{dt} & = \frac{V_{des} - V_t}{\tau}, \\ \frac{dX_t}{dt} & = V_t, \end{cases}$$

between the jumps of the velocity component V_t (the two other components do not jump and the desired velocity component is even constant over time). On the time interval $[t, t + dt]$, the velocity V_t jumps with probability $\lambda(t, X_t, V_t) dt + o(dt)$ (the jump rate λ is supposed bounded to avoid the accumulation of jumps) and then is redistributed according to the probability measure $\mu(t, X_t, V_t, w) dw$.

The infinitesimal generator L_t at time t of this Markov process is obtained for a smooth test function φ by

$$L_t \varphi(x, v, v^o) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left\{ \mathbb{E} \left[\varphi(X_{t+\varepsilon}, V_{t+\varepsilon}, V_{des}) - \varphi(X_t, V_t, V_{des}) \middle| (X_t, V_t, V_{des}) = (x, v, v^o) \right] \right\}.$$

Decomposing on the number of jumps of the velocity on the time interval $[t, t + \varepsilon]$, one has

$$\begin{aligned} &\varphi(X_{t+\varepsilon}, V_{t+\varepsilon}, V_{des}) \\ &= \left(\varphi(X_t, V_t, V_{des}) + \varepsilon V_t \partial_x \varphi(X_t, V_t, V_{des}) + \varepsilon \frac{V_{des} - V_t}{\tau} \partial_v \varphi(X_t, V_t, V_{des}) + o(\varepsilon) \right) \times \mathbb{I}_{\{\text{no jump on } [t, t+\varepsilon]\}} \\ &\quad + \left(\int \varphi(X_t, w, V_{des}) \mu(t, X_t, V_t, w) dw + o(1) \right) \times \mathbb{I}_{\{\text{one jump on } [t, t+\varepsilon]\}} \\ &\quad + \varphi(X_{t+\varepsilon}, V_{t+\varepsilon}, V_{des}) \times \mathbb{I}_{\{\text{at least 2 jumps on } [t, t+\varepsilon]\}}. \end{aligned}$$

Given $(X_t, V_t, V_{des}) = (x, v, v^o)$, the probability for the velocity to jump once on the time interval $[t, t + \varepsilon]$ is $\lambda(t, x, v) \varepsilon + o(\varepsilon)$. The observation of at least two jumps on $[t, t + \varepsilon]$ is an event of probability

of order ε^2 , thus negligible in the limit considered to compute the infinitesimal generator. Hence,

$$\begin{aligned} L_t\varphi(x, v, v^o) &= v\partial_x\varphi(x, v, v^o) + \frac{v^o - v}{\tau}\partial_v\varphi(x, v, v^o) \\ &\quad + \lambda(t, x, v) \left(\int \varphi(x, w, v^o)\mu(t, x, v, w)dw - \varphi(x, v, v^o) \right). \end{aligned}$$

3.3. Associated linear Fokker-Planck equation

If (X_0, V_0, V_{des}) admit a density $p(0, x, v, v^o)$, according to the measure $dx dv \nu(dv^o)$, then (X_t, V_t, V_{des}) also has a density $p(t, x, v, v^o)$ and for φ a smooth and compactly supported test function,

$$\begin{aligned} \int \varphi(x, v, v^o)\partial_t p_t(x, v, v^o)dx dv \nu(dv^o) &= \frac{d}{dt}\mathbb{E}[\varphi(X_t, V_t, V_{des})] = \mathbb{E}[L_t\varphi(X_t, V_t, V_{des})], \\ &= \int L_t\varphi(x, v, v^o)p(t, x, v, v^o)dx dv \nu(dv^o), \\ &= \int \left[v\partial_x\varphi(x, v, v^o) + \frac{v^o - v}{\tau}\partial_v\varphi(x, v, v^o) \right. \\ &\quad \left. + \lambda(t, x, v) \left(\int \varphi(x, w, v^o)\mu(t, x, v, w)dw - \varphi(x, v, v^o) \right) \right] p(t, x, v, v^o)dx dv \nu(dv^o). \end{aligned}$$

Applying an integration by parts with respect to x (resp. v) for the contribution of the first (resp. second) term between square brackets in the right-hand side and exchanging the dummy variables v and w in the integral $\int \lambda(t, x, v)p(t, x, v, v^o) \left(\int \varphi(x, w, v^o)\mu(t, x, v, w)dw \right) dx dv$, we obtain:

$$\begin{aligned} &\int \varphi(x, v, v^o)\partial_t p(t, x, v, v^o)dx dv \nu(dv^o) \\ &= \int \varphi(x, v, v^o) \left[-\partial_x(vp(t, x, v, v^o)) + \partial_v\left(\frac{v^o - v}{\tau}p(t, x, v, v^o)\right) \right. \\ &\quad \left. - \lambda(t, x, v)p(t, x, v, v^o) + \int \lambda(t, x, w)\mu(t, x, w, v)p(t, x, w, v^o)dw \right] dx dv \nu(dv^o). \end{aligned}$$

By identification, we recover (3.1).

3.4. Particle Approximation of (3.1) combined with (3.2) and (3.3)

Because of the dependence of λ and μ on the unknown density p , the Fokker-Planck equation (3.1) derived from the Pavani-Fontana evolution equation (2.1) is nonlinear. We propose a suitable Monte-Carlo approximation of these unknown functions λ and μ . For each time t , we are going to approximate the measure $p(t, x, v, v^o)dx dv \nu(dv^o)$ by the empirical measure $\frac{1}{N} \sum_{j=1}^N \delta_{(X_t^j, V_t^j, V_{des}^j)}$ of N interacting particles $((X_t^j, V_t^j, V_{des}^j)_{1 \leq j \leq N})_{t \geq 0}$.

The Monte-Carlo approximation of $\lambda(t, x, v)$ by the corresponding empirical mean is :

$$\begin{aligned} \lambda^N(t, x, v) &= \gamma \int_{v^o} \int_{w=0}^v (v - w) \left(\frac{1}{N} \sum_{j=1}^N \delta_{(X_t^j, V_t^j, V_{des}^j)} \right) (dx, dw, dv^o), \\ &= \frac{\gamma}{N} \sum_{j=1}^N (v - V_t^j)^+ \delta_{X_t^j}(dx). \end{aligned}$$

To obtain a function, we need to delocalize the positions. For $\varepsilon > 0$, let $\varphi_\varepsilon(y) = \frac{1}{\varepsilon}\varphi\left(\frac{y}{\varepsilon}\right)$, where φ is a probability density with support in \mathbb{R}_- (to model that only the particles/vehicles ahead can slow down

a given particle by interaction). In what follows, we choose $\varphi(x) = \frac{2}{\sqrt{2\pi}}e^{-x^2/2}\mathbb{I}_{\{x<0\}}$. The delocalized empirical approximation of λ is

$$\begin{aligned}\lambda_\varepsilon^N(t, x, v) &= \gamma \int_{v^o} \int_y \int_{w=0}^v (v-w)\varphi_\varepsilon(x-y) \left(\frac{1}{N} \sum_{j=1}^N \delta_{(X_t^j, V_t^j, V_{des}^j)}\right)(dydw dv^o), \\ &= \frac{\gamma}{N} \sum_{j=1}^N (v - V_t^j)^+ \varphi_\varepsilon(x - X_t^j).\end{aligned}$$

In the same way, the probability measure $\mu(t, x, v, w)dw$ according to which the velocity is redistributed after a jump is approximated by :

$$\mu_\varepsilon^N(t, x, v, dw) = \frac{\sum_{j=1}^N (v - V_t^j)^+ \varphi_\varepsilon(x - X_t^j) \delta_{V_t^j}(dw)}{\sum_{j=1}^N (v - V_t^j)^+ \varphi_\varepsilon(x - X_t^j)}.$$

We can now complete the description of the particles dynamics. The triplet $(X_t^j, V_t^j, V_{des}^j)_{t \geq 0}$ of the j -th particle evolves according to the ODE (S) :

$$\begin{cases} V_{des}^j & \text{remains constant,} \\ \frac{dV_t^j}{dt} & = \frac{V_{des}^j - V_t^j}{\tau}, \\ \frac{dX_t^j}{dt} & = V_t^j. \end{cases}$$

between the jumps of the velocity component V_t^j (the two other components do not jump and the desired velocity component is even constant over time). On the time interval $[t, t + dt]$, the velocity V_t^j jumps with probability $\lambda_\varepsilon^N(t, X_t^j, V_t^j)dt + o(dt)$ and is then redistributed according to the probability measure $\mu_\varepsilon^N(t, X_t^j, V_t^j, dw)$. This jump procedure leads to interaction between the particles.

3.5. Convergence of the particle approximation as the number N of particles tends to infinity

The Pavri-Fontana model has some common features with the Boltzmann equation : evolution of the position according to the current velocity which may jump according to the nonlinear term in those equations. The term accounting for the acceleration towards the desired velocity in the Pavri-Fontana model does not appear in the standard Boltzmann equation. Still, the sum of this acceleration term and the convection term corresponds to a deterministic and therefore Markovian evolution which is part of the free evolution operators considered by Graham and Méléard in [2]. This paper deals with the convergence in total variation distance of particle approximations of generalized Boltzmann equations with general motion between collisions and bounded nonlinear jump operator. For a fixed value $\varepsilon > 0$ of the delocalization parameter, the jump rate $\lambda_\varepsilon^N(t, x, v)$ associated in the previous paragraph with the Pavri-Fontana equation is bounded by $\frac{2\gamma v}{\varepsilon\sqrt{2\pi}}$. Moreover the jump measure

$$\begin{aligned}\lambda_\varepsilon^N(t, x, v)\mu_\varepsilon^N(t, x, v, dw) &= \frac{\gamma}{N} \sum_{j=1}^N (v - V_t^j)^+ \varphi_\varepsilon(x - X_t^j) \delta_{V_t^j}(dw), \\ &= \gamma \int_{y, u, v^o} (v - u)^+ \varphi_\varepsilon(x - y) \delta_u(dw) \frac{1}{N} \sum_{j=1}^N \delta_{(X_t^j, V_t^j, V_{des}^j)}(dydudv^o)\end{aligned}$$

depends linearly on the empirical measure $\frac{1}{N} \sum_{j=1}^N \delta_{(X_t^j, V_t^j, V_{des}^j)}$ at time t . If one assumes the existence of a deterministic constant $\bar{v} > 0$ bounding all the desired and initial velocities ($\nu((\bar{v}, \infty)) = 0 =$

$\int_{x,v} \mathbb{I}_{\{v > \bar{v}\}} \bar{\rho}(0, x, v) dx dv$, the jump rate is bounded by $\Lambda = \frac{2\gamma\bar{v}}{\varepsilon\sqrt{2\pi}}$. The existence of such a bound is not restrictive from the point of view of traffic modeling.

According to Theorem 3.1 [2], if the initial triplets $(X_0^j, V_0^j, V_{des}^j)$ of the N particles are i.i.d. to the density $p(0, x, v, v^o)$, then the first particle $(X_t^1, V_t^1, V_{des}^1)$ may be coupled with probability

$$\mathbb{P}\left(\forall t \in [0, T], (X_t^1, V_t^1, V_{des}^1) = (X_t^\varepsilon, V_t^\varepsilon, V_{des}^\varepsilon)\right) \geq 1 - 3\frac{e^{\Lambda T} - 1}{N + 1} \quad (3.4)$$

to a piecewise deterministic Markov process nonlinear in the sense of McKean $(X_t^\varepsilon, V_t^\varepsilon, V_{des}^\varepsilon)_{t \in [0, T]}$ with time marginals $p_\varepsilon(t, x, v, v^o) dx dv \nu(dv^o)$ where p_ε solves the spatially delocalized Pavri-Fontana equation

$$\begin{aligned} \partial_t p_\varepsilon(t, x, v, v^o) + v \partial_x p_\varepsilon(t, x, v, v^o) &= -\partial_v \left(p_\varepsilon(t, x, v, v^o) \frac{v^o - v}{\tau} \right) \\ &+ \gamma \int_{y=x}^\infty \varphi_\varepsilon(x - y) \bar{p}_\varepsilon(t, y, v) dy \int_v^\infty (w - v) p_\varepsilon(t, x, w, v^o) dw \\ &- \gamma p_\varepsilon(t, x, v, v^o) \int_{y=x}^\infty \int_{w=0}^v (v - w) \varphi_\varepsilon(x - y) \bar{p}_\varepsilon(t, y, w) dw dy, \end{aligned}$$

with $\bar{p}_\varepsilon(t, y, v) = \int p_\varepsilon(t, y, v, v^o) \nu(dv^o)$ and initial condition $p_\varepsilon(0, x, v, v^o) = p(0, x, v, v^o)$. Even if it could lead to diagonal convergence estimates with ε depending on the number N of particles, the study of the convergence as $\varepsilon \rightarrow 0$ of $\|\rho_0\|_{L^1} p_\varepsilon$ to a solution of the Pavri-Fontana equation is beyond the scope of the present paper. Notice that the results in [2] also apply to the spatially homogeneous Pavri-Fontana equation where there is no spatial variable x so that no delocalization of the jump term is needed. Of course the existence of the deterministic bound \bar{v} on the velocity variable is still needed.

In fact, it is possible to simulate exactly the piecewise deterministic Markov process nonlinear in the sense of McKean $(X_t^\varepsilon, V_t^\varepsilon, V_{des}^\varepsilon)_{t \in [0, T]}$ by following the deterministic evolution of the position according to the velocity which relaxes to the desired velocity at rate $1/\tau$ between the jump times of a Poisson process with intensity Λ and generating independent copies of the process that may slow it down at each of these jump times. If $(\tilde{X}_\eta^\varepsilon, \tilde{V}_{\eta-}^\varepsilon)$ denotes the independent copy of $(X_\eta^\varepsilon, V_{\eta-}^\varepsilon)$ associated with the jump time $\eta \leq T$ ($V_{\eta-}^\varepsilon$ and $\tilde{V}_{\eta-}^\varepsilon$ denote left-hand limits of the velocities at time η) and \tilde{U} is an independent random variable uniformly distributed on $[0, 1]$, then when $\tilde{U} \leq \frac{\gamma(V_{\eta-}^\varepsilon - \tilde{V}_{\eta-}^\varepsilon)^+ \varphi_\varepsilon(X_\eta^\varepsilon - \tilde{X}_\eta^\varepsilon)}{\Lambda} = \frac{\varepsilon\sqrt{2\pi}(V_{\eta-}^\varepsilon - \tilde{V}_{\eta-}^\varepsilon)^+ \varphi_\varepsilon(X_\eta^\varepsilon - \tilde{X}_\eta^\varepsilon)}{2\bar{v}}$, the nonlinear process is slowed-down : $V_\eta = \tilde{V}_{\eta-}$ and otherwise it keeps its velocity : $V_\eta = V_{\eta-}$. Of course, the independent copies needed at the jump times of the Poisson process associated with the nonlinear process themselves possibly jump before at times given by their own (independent) Poisson processes with intensity Λ and more independent copies are needed to compute their evolution. In fact, looking backward in time from time T and remarking that the time-reversal of a Poisson process is still a Poisson process with the same parameter, one sees that the process counting for $t \in [0, T]$ the number N_t of copies needed at time $T - t$ plus one for the original process is a Yule process with intensity Λ : the rate at which each of the N_t copies needs a new copy is Λ so that N_t jumps to $N_t + 1$ at rate ΛN_t and remains constant otherwise. Therefore for each $t \in [0, T]$, N_t is a geometric random variable with parameter $e^{-\Lambda t}$ (see for instance [1] p:109). The expectation $e^{\Lambda T}$ of the initial number N_T of copies grows exponentially with T which explains why this exact simulation technique is not used in practice. Nevertheless, we are now going to take advantage of this construction to derive the estimation of the coupling probability

$$\mathbb{P}\left(\leq \forall t \in [0, T], (X_t^1, V_t^1, V_{des}^1) = (X_t^\varepsilon, V_t^\varepsilon, V_{des}^\varepsilon)\right) \geq 1 - \frac{e^{2\Lambda T} - e^{\Lambda T}}{N}.$$

slightly less precise than the one (3.4) stated in [2] but relying on more elementary arguments. Indeed, one may set $(X_0^1, V_0^1, V_{des}^1) = (X_0^\epsilon, V_0^\epsilon, V_{des}^\epsilon)$ and let X_t^1 evolve according to the speed V_t^1 which relaxes at rate $1/\tau$ to the desired speed V_{des}^1 between the jump times of the Poisson process with intensity Λ which drives the jumps of V^ϵ . At each jump time η smaller than T of this Poisson process, an independent index J uniformly distributed on $\{1, 2, \dots, N\}$ is chosen. If $(X_\eta^J, V_{\eta-}^J)$ denotes the position and the left-hand limit of the velocity of the J -th particle at time η , when the uniform random variable \tilde{U} associated to this jump time is smaller than $\frac{\gamma(V_{\eta-}^1 - V_{\eta-}^J)^+ \varphi_\epsilon(X_\eta^1 - X_\eta^J)}{\Lambda} = \frac{\epsilon\sqrt{2\pi}(V_{\eta-}^1 - V_{\eta-}^J)^+ \varphi_\epsilon(X_\eta^1 - X_\eta^J)}{2\bar{v}}$, the first particle is slowed-down by the J -th particle : $V_\eta^1 = V_{\eta-}^J$ and otherwise it keeps its velocity : $V_\eta^1 = V_{\eta-}^1$. Of course, one needs to construct the J -th particle up to time η . If J is different from one and from the random indices associated with the previous jump times, the J -th particle is assigned the initial condition, the Poisson process and the uniform random variables driving the independent copie of (X^ϵ, V^ϵ) involved at the same jump time η . When $J \neq 1$, at each jump time before η of its just assigned Poisson process, the J -th particle may be slowed-down by a particle with an independent index uniformly distributed on $\{1, \dots, N\}$. When an index takes a value different from one and from the previously considered indices, the particle with this index is assigned the initial condition, the Poisson process and the uniform random variables driving the independent copy involved at the same jump time. Going on, we generate random indices associated to up to the $N_T - 1$ jump times of the Poisson processes involved in the construction of the nonlinear process on the time interval $[0, T]$. When one of the random indices is equal to one or to a previously considered index, then possibly less Poisson processes and less jump times are needed to construct the first particle up to time T . Let \mathcal{A}_{N_T} denote the complementary event that all the indices up to the $N_T - 1$ jump times are pairwise distinct and different from 1. On \mathcal{A}_{N_T} , the first particle coincides on $[0, T]$ with the nonlinear process and we bound the coupling probability from below by $\mathbb{P}(\mathcal{A}_{N_T})$. The probability for $k - 1$ uniform random variables on $\{1, 2, \dots, N\}$ to be pairwise distinct and all different from 1 is

$$\prod_{j=1}^{k-1} \left(1 - \frac{j}{N}\right) \geq 1 - \frac{1}{N} \sum_{j=1}^{k-1} j = 1 - \frac{k(k-1)}{2N}.$$

Therefore

$$\begin{aligned} \mathbb{P}(\mathcal{A}_{N_T}) &= e^{-\Lambda T} \sum_{k \geq 1} (1 - e^{-\Lambda T})^{k-1} \prod_{j=1}^{k-1} \left(1 - \frac{j}{N}\right) \geq 1 - \frac{e^{-\Lambda T}}{2N} (1 - e^{-\Lambda T}) \sum_{k \in \mathbb{N}} (1 - e^{-\Lambda T})^{k-2} k(k-1), \\ &= 1 - \frac{e^{2\Lambda T} - e^{\Lambda T}}{N}. \end{aligned}$$

3.6. Description of the simulation method

To simulate the particle system that we have just built, we use a fictitious jump technique. The fictitious jumps method [6] consists in simulating a Poisson process of intensity α (> 0) given as an upper-bound of the sum of the jump rates outside of each possible state of the particle system. At each time of the jump of the Poisson process, we realize a random selection independently of the Poisson process and the other previous selections to decide if one particle is slowed down by another one and then to choose the two interacting particles. The instant of jumps of the Poisson process when the system does not jump are called the fictitious jumps.

Between of the jump times of the Poisson process, the speeds and positions of the particles evolve according to the O.D.E. (S). This method is applied here in the following manner.

- Firstly, we initialize the N -particles by choosing $((X_0^i, V_0^i, V_{des}^i))_{1 \leq i \leq N}$ independent and identically distributed according to the probability measure $p(0, x, v, v^o) dx dv \nu(dv^o)$ on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$.

We calculate the upper bound

$$M_V = \max_{1 \leq i \leq N} \max(V_0^i, V_{des}^i)$$

of the initial and desired speeds of the N particles.

- We calculate an upper bound α of the sum of the jump rates of the particles system out of each feasible state. The jump rate at time t of the particle i is given by $\lambda_\varepsilon^N(t, X_t^i, V_t^i) = \frac{\gamma}{N} \sum_{j=1}^N (V_t^i - V_t^j)^+ \varphi_\varepsilon(X_t^i - X_t^j)$. Hence, the sum of the jump rates at time t is

$$\sum_{i=1}^N \lambda_\varepsilon^N(t, X_t^i, V_t^i) = \frac{\gamma}{N} \sum_{i,j=1}^N (V_t^i - V_t^j)^+ \varphi_\varepsilon(X_t^i - X_t^j). \quad (3.5)$$

Between the jumps, the speed of each particle evolves by the dynamics (S) between its value at the last jump and the desired speed. Furthermore, the speed jumps always corresponding to a decrease (slowing down by a leading car that drives slower). Therefore, the speed of each particle i stays lower than the maximum between its initial speed and its desired speed : $\sup_t V_t^i \leq \max(V_0^i, V_{des}^i)$.

Let (i, j) be a couple of distinct indices in $\{1, \dots, N\}$.

If $V_t^i \geq V_t^j$, then $0 \leq (V_t^i - V_t^j)^+ \leq M_V$ and $(V_t^j - V_t^i)^+ = 0$ so that

$$(V_t^i - V_t^j)^+ + (V_t^j - V_t^i)^+ \leq M_V. \quad (3.6)$$

If $V_t^i \leq V_t^j$ the inequality (3.6) still holds by symetry.

Moreover, since $\varphi_\varepsilon(\cdot) = \frac{1}{\varepsilon} \varphi(\frac{\cdot}{\varepsilon})$ with $\varphi(x) = \frac{2}{\varepsilon\sqrt{2\pi}} e^{-x^2/2} \mathbb{I}_{\{x < 0\}}$, $\varphi_\varepsilon(X_t^i - X_t^j) \leq \|\varphi_\varepsilon\|_\infty = \frac{1}{\varepsilon} \|\varphi\|_\infty = \frac{2}{\varepsilon\sqrt{2\pi}}$. Taking successively back this upper-bound, and then (3.6) into (3.5), we obtain

$$\begin{aligned} \sum_{i=1}^N \lambda_\varepsilon^N(t, X_t^i, V_t^i) &= \frac{\gamma}{N} \sum_{i=2}^N \sum_{j=1}^{i-1} \left((V_t^i - V_t^j)^+ \varphi_\varepsilon(X_t^i - X_t^j) + (V_t^j - V_t^i)^+ \varphi_\varepsilon(X_t^j - X_t^i) \right), \\ &\leq \frac{2\gamma}{N\varepsilon\sqrt{2\pi}} \sum_{i=2}^N \sum_{j=1}^{i-1} \left((V_t^i - V_t^j)^+ + (V_t^j - V_t^i)^+ \right), \\ &\leq \frac{2\gamma}{N\varepsilon\sqrt{2\pi}} \times \frac{N(N-1)M_V}{2} = \frac{\gamma(N-1)M_V}{\varepsilon\sqrt{2\pi}}. \end{aligned}$$

Thus, we choose

$$\alpha = \frac{\gamma \times (N-1) \times M_V}{\varepsilon\sqrt{2\pi}}.$$

- Let $(\xi_k)_{k \geq 1}$ be a sequence of independent random variables uniformly distributed on $[0, 1]$. The times

$$T_k = -\frac{1}{\alpha} \sum_{l=1}^k \ln \xi_l, \quad k \geq 1$$

are the jump times of the Poisson process with intensity α

$$\left(N_t = \sum_{k \in \mathbb{N}} \mathbb{I}_{\{T_k \leq t\}} \right)_{t \geq 0}. \quad (3.7)$$

We also use an independent sequence $(U_k)_{k \geq 1}$ of independent random variables uniformly distributed on $[0, 1]$ to choose which particles will be slowed-down at the times $(T_k)_{k \geq 1}$.

- Setting $T_0 = 0$, we simulate the evolution of the particle system by induction on $k \in \mathbb{N}$.
 - On the interval $[T_k, T_{k+1}[$, we solve the O.D.E. (S) using the variation of the constant method :

$$\forall t \in [T_k, T_{k+1}[, \begin{cases} V_t &= (V_{T_k} - V_{des}) \times \exp(-\frac{t-T_k}{\tau}) + V_{des}, \\ X_t &= X_{T_k} + \tau \times (V_{T_k} - V_{des}) \times (1 - \exp(-\frac{t-T_k}{\tau})) + V_{des} \times (t - T_k), \end{cases}$$

where V_t, X_t respectively denote the vectors of the speeds and positions of the particles $1, \dots, N$ at time t . V_{des} is the constant vector of the desired speed of all the particles. We set $X_{T_{k+1}} = \lim_{t \rightarrow T_{k+1}-} X_t$ because the positions evolve continuously in time. In contrast, the speeds may jump, thus we denote $V_{T_{k+1}-} = \lim_{t \rightarrow T_{k+1}-} V_t$.

- We use the random variable U_{k+1} to decide if a particle is slowed down at T_{k+1} and, when appropriate, which particle is slowed-down by which other particle :
 - * if

$$\alpha U_{k+1} > \sum_{i=1}^N \lambda_\varepsilon^N(T_{k+1}-, X_{T_{k+1}}^i, V_{T_{k+1}-}^i) = \frac{\gamma}{N} \sum_{i,j=1}^N (V_{T_{k+1}-}^i - V_{T_{k+1}-}^j)^+ \varphi_\varepsilon(X_{T_{k+1}}^i - X_{T_{k+1}}^j), \quad (3.8)$$

then no speed change occurs : $V_{T_{k+1}} = V_{T_{k+1}-}$,

- * otherwise, the index I_{k+1} of the slowed down particle at T_{k+1} is given by :

$$I_{k+1} = \min \left\{ i \in \{1, \dots, N\} : \alpha U_{k+1} \leq \sum_{l=1}^i \lambda_\varepsilon^N(T_{k+1}-, X_{T_{k+1}}^l, V_{T_{k+1}-}^l) \right\}.$$

If we set $S_{k+1} = \sum_{l=1}^{I_{k+1}-1} \lambda_\varepsilon^N(T_{k+1}-, X_{T_{k+1}}^l, V_{T_{k+1}-}^l)$, the index J_{k+1} of particle which slows it down, is given by

$$J_{k+1} = \min \left\{ j \in \{1, \dots, N\} : \alpha U_{k+1} \leq S_{k+1} + \frac{\gamma}{N} \sum_{m=1}^j (V_{T_{k+1}-}^{I_{k+1}} - V_{T_{k+1}-}^m)^+ \varphi_\varepsilon(X_{T_{k+1}}^{I_{k+1}} - X_{T_{k+1}}^m) \right\}.$$

To take into account the slow-down of the particle I_{k+1} by the particle J_{k+1} and the preservation of the speeds of all the other particles, we set

$$V_{T_{k+1}}^{I_{k+1}} = V_{T_{k+1}-}^{J_{k+1}} \quad \text{and} \quad \forall i \neq I_{k+1}, V_{T_{k+1}}^i = V_{T_{k+1}-}^i.$$

Since the average number αT of jump times of the Poisson process up to the simulation horizon T is proportional to $N - 1$, and a double loop is needed to choose the couple of particles interacting at each jump time, this method leads to an algorithmic complexity of $O(N^3)$ (see [7]), which is costly if the number N of particles is large.

We propose an alternative algorithm where the selection of the pair of particles interacting at each jump time of the Poisson process is carried out randomly and uniformly.

- Let $(\{I_k, J_k\})_{k \geq 1}$ be a sequence of random variables independent and uniformly distributed on

$$\{\{i, j\}, 1 \leq i, j \leq N, i \neq j\},$$

independent of the Poisson process of parameter $\alpha = \frac{\gamma \times (N-1) \times M_V}{\varepsilon \sqrt{2\pi}}$ and of the sequence $(U_k)_{k \geq 1}$ of independent random variables uniformly distributed on $[0, 1]$.

- Between the jump times $(T_k)_{k \geq 1}$ of the Poisson process, the positions and speeds evolve following the O.D.E. (S).

- For $k \geq 1$, at time T_k , if

$$U_k \leq \frac{1}{M_V} \left((V_{T_k-}^{I_k} - V_{T_k-}^{J_k})^+ \mathbb{I}_{\{X_{T_k}^{J_k} > X_{T_k}^{I_k}\}} + (V_{T_k-}^{J_k} - V_{T_k-}^{I_k})^+ \mathbb{I}_{\{X_{T_k}^{I_k} > X_{T_k}^{J_k}\}} \right) \exp \left(-\frac{(X_{T_k}^{I_k} - X_{T_k}^{J_k})^2}{2\varepsilon^2} \right),$$

the slowest of the two particles, necessarily ahead, slows down the other particle and the velocities of the other particles are preserved :

$$V_{T_k}^{I_k} = V_{T_k}^{J_k} = \min(V_{T_k-}^{I_k}, V_{T_k-}^{J_k}) \quad \text{et} \quad \forall i \notin \{I_k, J_k\}, \quad V_{T_k}^i = V_{T_k-}^i.$$

Let us check that this algorithm actually simulates the particle approximation introduced above. For $1 \leq i, j \leq N$ with $i \neq j$, the probability that the particle j slows down the particle i between t and $t + dt$ is equal, up to a $o(dt)$ term, to

$$\begin{aligned} & \mathbb{P}(N_{t+dt} - N_t = 1) \times \mathbb{P}(\{I_1, J_1\} = \{i, j\}) \times \mathbb{P} \left(U_1 \leq \frac{1}{M_V} (V_t^i - V_t^j)^+ \mathbb{I}_{\{X_t^j > X_t^i\}} \exp \left(-\frac{(X_t^i - X_t^j)^2}{2\varepsilon^2} \right) \right), \\ &= \alpha dt \times \frac{1}{\binom{N}{2}} \times \frac{1}{M_V} (V_t^i - V_t^j)^+ \mathbb{I}_{\{X_t^j > X_t^i\}} \exp \left(-\frac{(X_t^i - X_t^j)^2}{2\varepsilon^2} \right), \\ &= \frac{2\gamma}{N\varepsilon\sqrt{2\pi}} (V_t^i - V_t^j)^+ \mathbb{I}_{\{X_t^j > X_t^i\}} \exp \left(-\frac{(X_t^i - X_t^j)^2}{2\varepsilon^2} \right) = \frac{\gamma}{N} (V_t^i - V_t^j)^+ \varphi_\varepsilon(X_t^i - X_t^j). \end{aligned}$$

Hence the total probability that particle i is slowed down between t and $t + dt$ is equal to

$$\sum_{j=1}^N \frac{\gamma}{N} (V_t^i - V_t^j)^+ \varphi_\varepsilon(X_t^i - X_t^j) + o(dt) = \lambda_\varepsilon^N(t, X_t^i, V_t^i) + o(dt)$$

and then its new speed is drawn according to

$$\frac{\sum_{j=1}^N \frac{\gamma}{N} (V_t^i - V_t^j)^+ \varphi_\varepsilon(X_t^i - X_t^j) \delta_{V_t^j}(dw)}{\sum_{j=1}^N \frac{\gamma}{N} (V_t^i - V_t^j)^+ \varphi_\varepsilon(X_t^i - X_t^j)} = \mu_\varepsilon^N(t, X_t^i, V_t^i, dw).$$

We remark that it is not necessary to know the couples $(X_{T_k}^i, V_{T_k-}^i)_{i \notin \{I_k, J_k\}}$ to implement this algorithm. Thus, to save computation time, only the positions and speeds of the particles I_k and J_k are updated to decide whether one is slowed-down by the other. For this purpose, we need to associate with each particle i a new variable S^i giving the last time when its couple position/speed (X^i, V^i) has been updated. The cost of the algorithm is linear with respect to the number N of particles since only two particles are updated at each jump time of the Poisson process.

Algorithm 1 Simulation of the Pavveri-Fontana model with linear cost in N

Input: $N, T, \tau, P, \rho(0, x, v, v^o), \varepsilon$.

Initialization:

- Computation of $\|\rho_0\|_{L^1}$, $\gamma = (1 - P)\|\rho_0\|_{L^1}$ and $p(0, x, v, v^o) = \frac{1}{\|\rho_0\|_{L^1}}\rho(0, x, v, v^o)$.
- Initialization of the particles : generation of $((X_0^i, V_0^i, V_{des}^i))_{1 \leq i \leq N}$ independent and identically distributed according to the probability measure $p(0, x, v, v^o)dx dv v^o$. **Initialize the last update time S^i of the couple position/speed of the particle i to 0 for each $i \in \{1, \dots, N\}$.**
- Computation of $M_V = \max_{1 \leq i \leq N} \max(V_0^i, V_{des}^i)$ and $\alpha = \frac{\gamma \times (N-1) \times M_V}{\varepsilon \sqrt{2\pi}}$.
- Generate $U \hookrightarrow \mathbb{U}([0, 1])$ and set $t = -\frac{\log(U)}{\alpha}$.

Iteration:

while $t < T$ **do**

Uniform choice of a couple of indices $\{i, j\}$:

$$i = \lceil N \times \mathbb{U}[0, 1], \lceil \quad j = \lceil (N - 1) \times \mathbb{U}[0, 1] \rceil$$

if $j > i - 1$ **then**

$j = j + 1$

end if

The positions, speed and last update time of particles i and j are updated following :

$$X^i = X^i + \tau \times (V^i - V_{des}^i) \times (1 - \exp(-\frac{t - S^i}{\tau})) + V_{des}^i \times (t - S^i)$$

$$V^i = (V^i - V_{des}^i) \times \exp(-\frac{t - S^i}{\tau}) + V_{des}^i$$

$$S^i = t$$

$$V^j = (V^j - V_{des}^j) \times \exp(-\frac{t - S^j}{\tau}) + V_{des}^j$$

$$X^j = X^j + \tau \times (V^j - V_{des}^j) \times (1 - \exp(-\frac{t - S^j}{\tau})) + V_{des}^j \times (t - S^j)$$

$$S^j = t$$

Interaction with the right probability

if $M_V \times \mathbb{U}([0, 1]) < \left\{ (V^i - V^j)(V^i > V^j)\mathbb{I}_{\{X^j > X^i\}} + (V^j - V^i)(V^j > V^i)\mathbb{I}_{\{X^i > X^j\}} \right\} \times \exp\left(-\frac{(X^i - X^j)^2}{2\varepsilon^2}\right)$

then

$$V^i = \min(V^i, V^j); \quad V^j = V^i$$

end if

Generate $U \hookrightarrow \mathbb{U}([0, 1])$, $s = -\frac{\log(U)}{\alpha}$, and increase the current time : $t = t + s$.

end while

for $i = 1 \rightarrow N$ **do**

$$X^i = X^i + \tau \times (V^i - V_{des}^i) \times (1 - \exp(-\frac{T - S^i}{\tau})) + V_{des}^i \times (T - S^i)$$

$$V^i = (V^i - V_{des}^i) \times \exp(-\frac{T - S^i}{\tau}) + V_{des}^i$$

end for

Output: $(X_T^i, V_T^i, V_{des}^i)_{1 \leq i \leq N}$.

4. Comparison to a deterministic finite differences method

In order to reduce the dimension in which the Pavveri-Fontana equation is posed, we consider the case where there are d classes of vehicles and for $k \in \{1, \dots, d\}$, all the vehicles in the k -th share the same desired speed v_k^o . Setting $\rho^k(t, x, v) = \rho(t, x, v, v_k^o)$, with $(x, v) \in \mathbb{R} \times \mathbb{R}_+$, the Pavveri-Fontana equation (2.1) rewrites :

$$\begin{aligned} \partial_t \rho^k(t, x, v) + v \partial_x \rho^k(t, x, v) &= -\partial_v \left(\rho^k(t, x, v) \frac{v_k^o - v}{\tau} \right), \\ &+ (1 - P) \sum_{l=1}^d \rho^l(t, x, v) \int_v^\infty (w - v) \rho^k(t, x, w) dw, \\ &- (1 - P) \rho^k(t, x, v) \int_0^v (v - w) \sum_{l=1}^d \rho^l(t, x, w) dw, \quad 1 \leq k \leq d. \end{aligned} \quad (4.1)$$

We consider this equation under the initial condition $\rho^k(t, x, v) = \rho^{0,k}(x, v)$, for $(x, v) \in \mathbb{R} \times \mathbb{R}_+$. The different classes interact through the two last terms on the right-hand side of equation (4.1) which model the slow down of the vehicles by other vehicles at the same position, but with lower speed. The finite differences method consists in approximating the partial derivatives in a P.D.E. using a Taylor expansion. It first necessitates to build a mesh for the time, space and speed variables, where the approximate solution is computed.

Discretization of the computational domain:

We truncate the domain $\mathbb{R} \times \mathbb{R}_+$ into a bounded rectangular subdomain $[x_{min}, x_{max}] \times [v_{min}, v_{max}]$ and we solve the system on the time interval $[0, T_{max}]$. The interval $[v_{min}, v_{max}]$ should be chosen to contain the desired speed v_k^o of the class k for $k \in \{1, \dots, d\}$.

- *Discretization of time* : We choose a uniform grid with n_T steps and step size $\Delta t = \frac{T_{max}}{n_T}$. The approximate solution is initialized at time $t^0 = 0$ by discretizing the initial condition $(\rho^{0,1}, \dots, \rho^{0,d})$. At each step $n \in \{1, \dots, n_T\}$, the approximate solution at time $t^n = t^0 + n\Delta t$ is obtained from the approximate solution at t^{n-1} .
- *Spatial discretization* : For $i \in \{0, \dots, I + 1\}$, we set $x_i = x_{min} + i\Delta x$ where $\Delta x = \frac{x_{max} - x_{min}}{I + 1}$.
- *Discretization of the set of velocities* : The nodes of the velocity domain $[v_{min}, v_{max}]$ are given by $v_j = v_{min} + j\Delta v$, for $0 \leq j \leq J + 1$ where $\Delta v = \frac{v_{max} - v_{min}}{J + 1}$ and the interfaces by $v_{j+\frac{1}{2}} = v_{min} + (j + \frac{1}{2}) \times \Delta v$ for $j \in \{0, \dots, J\}$.

For $(n, i, j) \in \{0, \dots, n_T\} \times \{0, \dots, I + 1\} \times \{0, \dots, J + 1\}$, the approximate density of class $k \in \{1, \dots, d\}$ at time t^n , in position x_i with speed v_j is denoted $\rho_{i,j}^{n,k} \simeq \rho^k(t^n, x_i, v_j)$.

Initialization :

Set $\rho_{i,j}^{0,k} = \rho^{0,k}(x_i, v_j)$ for $k \in \{1, \dots, d\}$, $i \in \{1, \dots, I\}$ and $j \in \{1, \dots, J\}$.

Boundary conditions in space and speed :

We use the homogenous Dirichlet conditions :

$$\rho_{i,j}^{n,k} = 0 \text{ for } n \in \{0, \dots, n_T\} \text{ and } (i, j) : i = 0 \text{ or } j \in \{0, J + 1\}$$

on all edges of the domain $[x_{min}, x_{max}] \times [v_{min}, v_{max}]$ but the edge $x = x_{max}$ where there is no need to require a condition because the vehicles have a positive speed and move from x_{min} to x_{max} . The value

of $(\rho_{I+1,j}^{n,k})_{1 \leq j \leq J}$ is initialized to $(0, 0, \dots, 0)$ at time step $n = 0$ and will evolve with n to ensure the conservation of the total number of vehicles: the outgoing vehicles from the spatial interval $[x_{\min}, x_{\max}]$ are artificially stored in x_{\max} .

Discretization of the equation :

Using a splitting approach, the equation (4.1) can be rewritten in the form

$$\partial_t \rho^k(t, x, v) = T_1 + T_2 + T_3 + T_4$$

with

- $T_1 = -v \partial_x \rho^k(t, x, v)$,
- $T_2 = -\partial_v \left(\rho^k(t, x, v) \frac{(v_k^0 - v)}{\tau} \right)$,
- $T_3 = (1 - P) \rho^k(t, x, v) \int_0^\infty (w - v) \rho^k(t, x, w) dw$,
- $T_4 = (1 - P) \sum_{l \neq k} \rho^l(t, x, v) \int_v^\infty (w - v) \rho^k(t, x, w) dw + (1 - P) \rho^k(t, x, v) \int_0^v (w - v) \sum_{l \neq k} \rho^l(t, x, w) dw$,

where the interaction term is splitted into the auto-interaction term T_3 between the vehicles of the class k and the interaction between the vehicles of the class k and the vehicles of a different class $l \neq k$ (term T_4).

To get the approximate solution $(\rho_{i,j}^{n+1,k})_{k,i,j}$ at time t^{n+1} from the approximate solution $(\rho_{i,j}^{n,k})_{k,i,j}$ at time t^n , one successively approximates the evolution equations $\partial_t \rho^k(t, x, v) = T_l$ for $l \in \{1, 2, 3, 4\}$ and one denotes $(\rho_{i,j}^{n+m/4,k})_{k,i,j}$ the approximate solution after m of these steps. Note that when the four steps are performed, we obtain the approximate solution at time t^{n+1} . The discretization of every equation is made in order to preserve the total quantity $\sum_{i=0}^{I+1} \sum_{j=0}^{J+1} \rho_{i,j}^{n,k} \Delta x \Delta v = \sum_{i=1}^{I+1} \sum_{j=1}^J \rho_{i,j}^{n,k} \Delta x \Delta v$ of the vehicles of each class $k \in \{1, \dots, d\}$ over time.

The homogeneous Dirichlet boundary conditions are extended to the intermediate times

$$\rho_{i,j}^{n+m/4,k} = 0 \text{ for } (n, m) \in \{0, \dots, n_T - 1\} \times \{1, 2, 3\} \text{ and } (i, j) \text{ such as } i = 0 \text{ or } j \in \{0, J + 1\}.$$

Discretization of $\partial_t \rho^k(t, x, v) = T_1$:

For a given initial condition, the solution at time t of this equation is the translation $\rho^k(t, x, v) = \rho^{0,k}(x - vt, v)$ of the initial condition. The numerical discretization is carried out by the upstream spacially-decentered scheme :

$$\partial_t \rho^k(t^n, x_i, v_j) + v_j \partial_{x_i} \rho^k(t^n, x_i, v_j) \approx \frac{\rho_{i,j}^{n+1/4,k} - \rho_{i,j}^{n,k}}{\Delta t} + v_j \times \frac{\rho_{i,j}^{n,k} - \rho_{i-1,j}^{n,k}}{\Delta x} = 0$$

\Rightarrow

$$\rho_{i,j}^{n+1/4,k} = \rho_{i,j}^{n,k} - \frac{\Delta t}{\Delta x} \times v_j \times (\rho_{i,j}^{n,k} - \rho_{i-1,j}^{n,k}) \text{ for } (k, i, j) \in \{1, \dots, d\} \times \{1, \dots, I\} \times \{1, \dots, J\}.$$

As

$$\frac{\rho_{i,j}^{n,k} - \rho_{i-1,j}^{n,k}}{\Delta x} = \frac{\rho_{i+1,j}^{n,k} - \rho_{i-1,j}^{n,k}}{2\Delta x} - \frac{\rho_{i+1,j}^{n,k} + \rho_{i-1,j}^{n,k} - 2\rho_{i,j}^{n,k}}{2\Delta x}$$

is multiplied by $-\Delta t \times v_j < 0$, the choice of an upstream decentered scheme ensures that the second order spatial derivative term added in comparison with a centered scheme has a positive coefficient in the second member. This stabilizes the calculations at the cost of introducing numerical viscosity.

For $(k, j) \in \{1, \dots, d\} \times \{1, \dots, J\}$, the boundary condition $\rho_{0,j}^{n,k} = 0$ implies that the telescoping sum $\sum_{i=1}^I (\rho_{i,j}^{n,k} - \rho_{i-1,j}^{n,k})$ is equal to $\rho_{I,j}^{n,k}$. In order to ensure the conservation of the total quantity of vehicles of the class k , one then sets

$$\rho_{I+1,j}^{n+1/4,k} = \rho_{I+1,j}^{n,k} + \frac{\Delta t}{\Delta x} \times v_j \times \rho_{I,j}^{n,k}.$$

Actually, the discretization of the other evolution equations $\partial_t \rho^k(t, x, v) = T_l$ for $l \in \{2, 3, 4\}$ will be conservative into the domain and does not require to modify the values of the approximate solution in $x = x_{\max}$ so that we set

$$\rho_{I+1,j}^{n+1,k} = \rho_{I+1,j}^{n,k} + \frac{\Delta t}{\Delta x} \times v_j \times \rho_{I,j}^{n,k}.$$

Discretization of $\partial_t \rho^k(t, x, v) = T_2$:

We apply a scheme based on an upstream decentered discretization of the operator $\partial_v \left(\rho^k(t, x, v) \frac{v_k^o - v}{\tau} \right)$. As previously, this ensures that the second order speed derivative term added in comparison with a centered scheme, has a positive coefficient to second member.

We choose the direction of discretization according to the sign of $v_k^o - v$. Let $j_k^o \in \{1, \dots, J\}$ be the index of the speed grid point closest to the desired speed v_k^o of the class k . The approximation is performed on each interface $v_{j+\frac{1}{2}}$.

Let $i \in \{1, \dots, I\}$.

If $j \in \{1, \dots, j_k^o - 1\}$,

$$\rho_{i,j}^{n+1/2,k} = \rho_{i,j}^{n+1/4,k} - \frac{\Delta t}{\tau \Delta v} \left[(v_k^o - v_{j+\frac{1}{2}}) \times \rho_{i,j}^{n+1/4,k} - (v_k^o - v_{j-\frac{1}{2}}) \times \rho_{i,j-1}^{n+1/4,k} \right].$$

If $j = j_k^o$,

$$\rho_{i,j_k^o}^{n+1/2,k} = \rho_{i,j_k^o}^{n+1/4,k} - \frac{\Delta t}{\tau \Delta v} \left[(v_k^o - v_{j+\frac{1}{2}}) \times \rho_{i,j_k^o+1}^{n+1/4,k} - (v_k^o - v_{j-\frac{1}{2}}) \times \rho_{i,j_k^o-1}^{n+1/4,k} \right].$$

If $j \in \{j_k^o + 1, \dots, J\}$,

$$\rho_{i,j}^{n+1/2,k} = \rho_{i,j}^{n+1/4,k} - \frac{\Delta t}{\tau \Delta v} \left[(v_k^o - v_{j+\frac{1}{2}}) \times \rho_{i,j+1}^{n+1/4,k} - (v_k^o - v_{j-\frac{1}{2}}) \times \rho_{i,j}^{n+1/4,k} \right].$$

To prove the conservativity of this scheme, we compute the sum on j for fixed i :

$$\begin{aligned} \sum_{j=1}^J (\rho_{i,j}^{n+1/2,k} - \rho_{i,j}^{n+1/4,k}) &= -\frac{\Delta t}{\tau \Delta v} \times \sum_{j=1}^{j_k^o-1} \left[(v_k^o - v_{j+\frac{1}{2}}) \times \rho_{i,j}^{n+1/4,k} - (v_k^o - v_{j-\frac{1}{2}}) \times \rho_{i,j-1}^{n+1/4,k} \right] \\ &\quad - \frac{\Delta t}{\tau \Delta v} \times \left((v_k^o - v_{j+\frac{1}{2}}) \times \rho_{i,j_k^o+1}^{n+1/4,k} - (v_k^o - v_{j-\frac{1}{2}}) \times \rho_{i,j_k^o-1}^{n+1/4,k} \right) \\ &\quad - \frac{\Delta t}{\tau \Delta v} \times \sum_{j=j_k^o+1}^J \left[(v_k^o - v_{j+\frac{1}{2}}) \times \rho_{i,j+1}^{n+1/4,k} - (v_k^o - v_{j-\frac{1}{2}}) \times \rho_{i,j}^{n+1/4,k} \right]. \end{aligned}$$

Using the two telescoping sums, one gets

$$\sum_{j=1}^J (\rho_{i,j}^{n+1/2,k} - \rho_{i,j}^{n+1/4,k}) = -\frac{\Delta t}{\tau \Delta v} \times \left(-(v_k^o - v_{\frac{1}{2}}) \times \rho_{i,0}^{n+1/4,k} + (v_k^o - v_{J+\frac{1}{2}}) \times \rho_{i,J+1}^{n+1/4,k} \right).$$

As $\rho_{i,0}^{n+1/4,k} = \rho_{i,J+1}^{n+1/4,k} = 0, \forall i \in \{1, \dots, I\}$ (Dirichlet boundary conditions), one concludes that

$$\sum_{j=1}^J (\rho_{i,j}^{n+1/2,k} - \rho_{i,j}^{n+1/4,k}) = 0.$$

Hence, one has $\sum_{i=1}^I \sum_{j=1}^J \rho_{i,j}^{n+1/2,k} = \sum_{i=1}^I \sum_{j=1}^J \rho_{i,j}^{n+1/4,k}$.

Discretization of $\partial_t \rho^k(t, x, v) = T_3$:

For $k \in \{1, \dots, d\}$ and $(i, j) \in \{1, \dots, I\} \times \{1, \dots, J\}$, we set

$$\rho_{i,j}^{n+3/4,k} = \rho_{i,j}^{n+1/2,k} + (1 - P) \times \Delta t \times \rho_{i,j}^{n+1/2,k} \times \left[\sum_{m=1}^J (v_m - v_j) \rho_{i,m}^{n+1/2,k} \right] \times \Delta v.$$

The conservativity is satisfied, because exchanging the dummy indices j and m , one gets

$$\sum_{j=1}^J \rho_{i,j}^{n+1/2,k} \sum_{m=1}^J v_m \rho_{i,m}^{n+1/2,k} = \sum_{j=1}^J \rho_{i,j}^{n+1/2,k} \sum_{m=1}^J v_j \rho_{i,m}^{n+1/2,k}.$$

Discretization of $\partial_t \rho^k(t, x, v) = T_4$:

For $k \in \{1, \dots, d\}$ and $(i, j) \in \{1, \dots, I\} \times \{1, \dots, J\}$, we set

$$\begin{aligned} \rho_{i,j}^{n+1,k} = \rho_{i,j}^{n+3/4,k} + (1 - P) \Delta v \times & \left(\rho_{i,j}^{n+3/4,k} \times \sum_{m=1}^{j-1} (v_m - v_j) \times \sum_{l \neq k} \rho_{i,m}^{n+3/4,l} \right. \\ & \left. + \sum_{l \neq k} \rho_{i,j}^{n+3/4,l} \times \sum_{m=j+1}^J (v_m - v_j) \times \rho_{i,m}^{n+3/4,k} \right). \quad (4.2) \end{aligned}$$

Summing by parts, we obtain the equality

$$\sum_{j=1}^J \rho_{i,j}^{n+3/4,k} \sum_{m=1}^{j-1} (v_m - v_j) \times \sum_{l \neq k} \rho_{i,m}^{n+3/4,l} = \sum_{m=1}^J \sum_{l \neq k} \rho_{i,m}^{n+3/4,l} \times \sum_{j=m+1}^J (v_m - v_j) \rho_{i,j}^{n+3/4,k}$$

that ensures the conservativity (we can exchange the dummy indices j and m in the right-hand side).

5. Numerical Investigations

In order to test the ability of the Pavveri-Fontana model implemented by our schemes to reproduce some typical scenarios observed in a traffic flow, we propose to investigate a test case concerning two groups of vehicles.

5.1. Description of the test case

This scenario examines the interaction between two groups of vehicles. The ahead group I is supposed to be slower than the rear one, group II . Namely, when initializing the algorithm, we assign smaller speeds to the ahead vehicles and smaller positions and larger speeds to the rear vehicles. Each group has its own desired speed.

Firstly we present the numerical values of the parameters used in the two approximating methods. We investigate convergence properties of each method and then compare their numerical results.

We initialize the density with a piecewise constant density function whose analytical expression is:

$$\rho_0(x, v) = \begin{cases} \rho^I & \text{if } x \in [X_{min}^I, X_{max}^I] \text{ and } v \in [V_{min}^I, V_{max}^I], \\ \rho^{II} & \text{if } x \in [X_{min}^{II}, X_{max}^{II}] \text{ and } v \in [V_{min}^{II}, V_{max}^{II}], \\ 0 & \text{otherwise.} \end{cases}$$

The L^1 -norm of this density is

$$\|\rho_0\|_{L^1} = \rho_0^I \times (X_{max}^I - X_{min}^I) \times (V_{max}^I - V_{min}^I) + \rho_0^{II} \times (X_{max}^{II} - X_{min}^{II}) \times (V_{max}^{II} - V_{min}^{II}).$$

The Pavri-Fontana model parameters are presented in Table 5.1. Table 5.2 contains the parameters of the initial density and of the discretization domain used in the finite differences method.

Parameters	Values
v_I^o	25m/s
v_{II}^o	30m/s
τ	15s and 30s
P	$\frac{1}{2}$

TABLE 5.1. Parameters of P-F model

Parameters	Values
ρ^I, ρ^{II}	0.02, 0.01 veh.s/m ²
X_{min}^I, X_{max}^I	500m , 1000m
$X_{min}^{II}, X_{max}^{II}$	0m , 300m
V_{min}^I, V_{max}^I	[17 , 25] m/s ie [61 , 90] km/h
$V_{min}^{II}, V_{max}^{II}$	[25 , 30] m/s ie [90 , 108] km/h
$[x_{min}, x_{max}]$	[-20, 5980] m
$[v_{min}, v_{max}]$	[15.5, 30.5] m/s

TABLE 5.2. Initial density and discretization domain

Notice that, at the beginning of the discussion of the numerical results, we will quickly investigate the choices $P = 0$ and $P = 1$ for the overtaking coefficient. But most of the simulations are performed with $P = \frac{1}{2}$.

5.1.1. Initialization of the particle method

The numbers of particles N_I in class I and N_{II} in class II are selected in a deterministic way :

$$N^I = \lceil N \times (X_{max}^I - X_{min}^I) \times (V_{max}^I - V_{min}^I) \times \frac{\rho^I}{\|\rho_0\|_{L^1}} \rceil,$$

$$N^{II} = N - N^I,$$

with $N \in \{10000, 100000\}$.

For class I, the initial positions and velocities are chosen *i.i.d.* : one has $\forall 1 \leq i \leq N_I, X_0^i \hookrightarrow \mathcal{U}[X_{min}^I; X_{max}^I]$ and $V_0^i \hookrightarrow \mathcal{U}[V_{min}^I; V_{max}^I]$. The desired speed of each particle of this class is fixed to $V_{des}^i = v_I^o$.

Likewise for class II, $\forall N_I + 1 \leq j \leq N, X_0^j \hookrightarrow \mathcal{U}[X_{min}^{II}; X_{max}^{II}]$ and $V_0^j \hookrightarrow \mathcal{U}[V_{min}^{II}; V_{max}^{II}]$. The corresponding desired speed is $V_0^j = v_{II}^o$.

Let us now derive some theoretical properties of the solution to the Pavri-Fontana equation. We will later investigate the ability of the two numerical methods to reproduce these properties.

5.2. Minimal position and velocity

When there is only one class i.e. all vehicles have the same desired speed v^o , setting $v(t) = v^o + (v - v^o)e^{-\frac{t}{\tau}}$ and $x(t) = x + v^o t + \tau(v - v^o)(1 - e^{-\frac{t}{\tau}})$ so that $v'(t) = \frac{v^o - v(t)}{\tau}$ and $x'(t) = v(t)$, and using the Pavéri-Fontana equation describing the evolution of the density $\bar{\rho}(t, x, v)$ of vehicles, one gets that

$$\begin{aligned} & \frac{d}{dt} \left(e^{-\frac{t}{\tau}} \bar{\rho}(t, x(t)v(t)) \right) \\ &= e^{-t/\tau} \left(\partial_t \bar{\rho}(t, x(t), v(t)) + v(t) \partial_x \bar{\rho}(t, x(t), v(t)) + \frac{v^o - v(t)}{\tau} \partial_v \bar{\rho}(t, x(t), v(t)) - \frac{1}{\tau} \bar{\rho}(t, x(t), v(t)) \right), \\ &= e^{-\frac{t}{\tau}} \left(\partial_t \bar{\rho}(t, x(t), v(t)) + v(t) \partial_x \bar{\rho}(t, x(t), v(t)) + \partial_v \left(\bar{\rho}(t, x(t), v) \frac{v^o - v}{\tau} \right) \Big|_{v=v(t)} \right) \\ &= e^{-\frac{t}{\tau}} (1 - P) \bar{\rho}(t, x(t), v(t)) \int_0^{+\infty} (w - v(t)) \bar{\rho}(t, x(t), w) dw. \end{aligned}$$

Hence

$$e^{-\frac{t}{\tau}} \bar{\rho}(t, x(t), v(t)) = \bar{\rho}_0(x, v) \exp \left((1 - P) \int_0^t e^{-\frac{s}{\tau}} \int_0^{+\infty} (w - v(s)) \bar{\rho}(s, x(s), w) dw ds \right).$$

If $\bar{\rho}_0(x, v)$ vanishes when $v \leq v_{min}$ for any position x , we deduce that $\bar{\rho}(t, x, v)$ vanishes when $v \leq v^o + (v_{min} - v^o)e^{-\frac{t}{\tau}}$ for any position x .

In the same way when the initial density $\bar{\rho}_0(x, v)$ vanishes for $x < x_{min}$ or $v < v_{min}$, we deduce that $\bar{\rho}(t, x, v)$ vanishing for $x < x_{min} + v^o \times t + \tau \times (v_{min} - v^o)(1 - e^{-\frac{t}{\tau}})$ or $v < v^o + (v_{min} - v^o) \times e^{-\frac{t}{\tau}}$.

In the test case introduced above, as the vehicles of class I are not slowed-down by those of class II (the maximum speed $V_{max}^I = v_I^o = 25m/s$ over class I equals the minimum speed $V_{min}^{II} = 25m/s$ over class II), we can generalize this reasoning to assert that:

$$\rho^I(t, x, v) = 0 \text{ if } x < X_{min}^I(t) = X_{min}^I + v_I^o \times t + \tau \times (V_{min}^I - v_I^o)(1 - e^{-\frac{t}{\tau}}), \quad (5.1)$$

$$\text{or } v < V_{min}^I(t) = v_I^o + (V_{min}^I - v_I^o)e^{-\frac{t}{\tau}}. \quad (5.2)$$

Similar arguments can be applied to assert that the density of class II is null below some minimal positions and velocities until the interaction with class I begins.

5.3. The interaction time between both classes

The interaction time t_{rec} is defined as the time required by the leading vehicle of class II to catch up the last vehicle of class I.

By the previous paragraph, the minimal position where the spatial density of class I is non-zero is :

$$X_{min}^I(t) = X_{min}^I + v_I^0 \times t + \tau \times (1 - e^{-\frac{t}{\tau}}) \times (V_{min}^I - v_I^0).$$

Until the vehicles of class II catch up those of class I, a similar reasoning ensures that the maximal position where the spatial density of class II is non-zero is :

$$X_{max}^{II}(t) = X_{max}^{II} + v_{II}^0 \times t.$$

Therefore the interacting time t_{rec} solves $f(t_{rec}) = 0$ where

$$f(t) = X_{max}^{II} - X_{min}^I + (v_{II}^0 - v_I^0) \times t_{rec} + \tau \times (1 - e^{-\frac{t_{rec}}{\tau}})(v_I^0 - V_{min}^I).$$

Approximating t_{rec} by Newton's method which consists in computing inductively

$$t_{k+1} = t_k - \frac{f(t_k)}{f'(t_k)}, \text{ at } k\text{-th time step}, \quad (5.3)$$

we obtain $t_{rec}^{theo} \approx 21.7s$ if the relaxation time is $\tau = 15s$, and $t_{rec}^{theo} \approx 18.2s$ if $\tau = 30s$.

5.4. Discussion of the results

Influence of the overtaking coefficient P

Figure 5.1 shows the positions and speeds of the particles (in red for class I and blue for class II) as a function of time for various choices of the overtaking coefficient P .

The left part deals with the case $P = 1$. The interaction term vanishes in the Pavri-Fontana equation. Thus the speeds do not jump and all the particles of class II overtake those of class I without interacting with them. The middle part corresponds to $P = \frac{1}{2}$. We observe speed jumps for particles of class II which are slowed-down. The rightmost figure shows the case $P = 0$ where the deceleration term is the strongest. The particles of class II stay behind and adjust their speed to those of the particles of the ahead class until the end of the simulation time considered here. However, for a longer simulation time, the particles of class II end up overtaking.

From now on, we fix the overtaking coefficient to $P = \frac{1}{2}$.

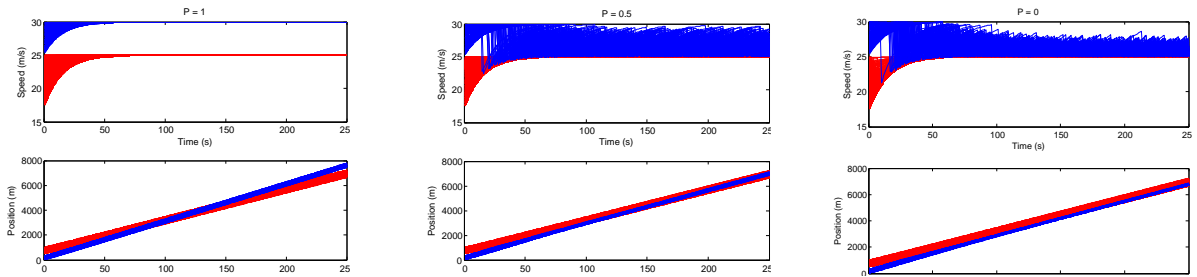


FIGURE 5.1. Speeds and positions for the overtaking probability $P = 1, \frac{1}{2}$ and 0 ($\tau = 15s$).

Convergence of the probabilistic method

The convergence of this method is illustrated by Figures 5.2 to 5.5 representing the spatial and speed densities of the two classes with $N = 10000$ (blue curves) and $N = 100000$ particles (red curves).

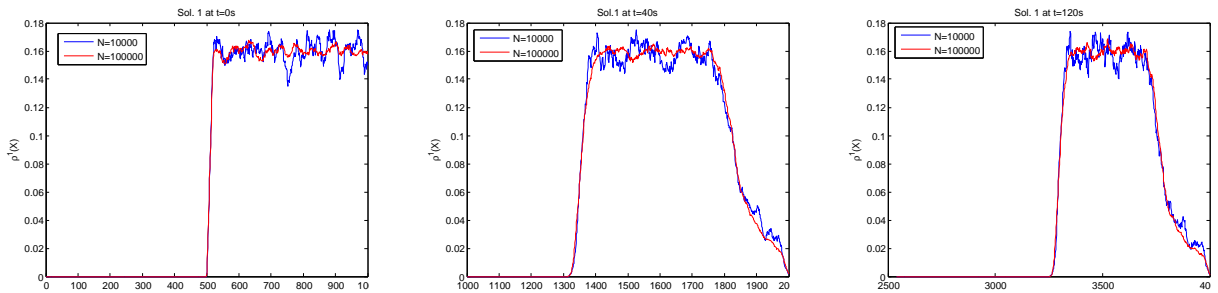


FIGURE 5.2. Spatial density of class I obtained by the stochastic method with $N = 10000$ (blue) and $N = 100000$ (red) particles at times $t = 0$ (left), $t = 40s$ (middle) and $t = 120s$ (right). $\tau = 30s$.

We find the same profile with random fluctuations whose amplitude is divided by approximately $\sqrt{10}$ when the number N of particles is multiplied by 10, which is consistent with the decay of the variance of the empirical mean in an I.I.D. setting.

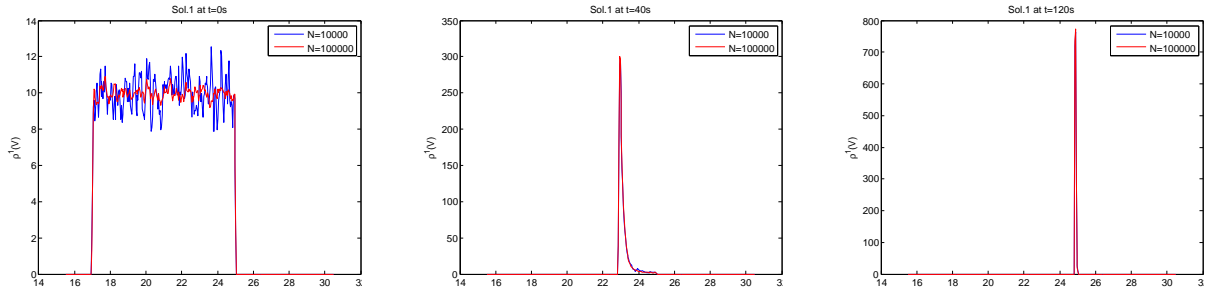


FIGURE 5.3. Speed density of class I obtained by the stochastic method with $N = 10000$ (blue) and $N = 100000$ (red) particles at times $t = 0$ (left), $t = 40s$ (middle) and $t = 120s$ (right). $\tau = 30s$.

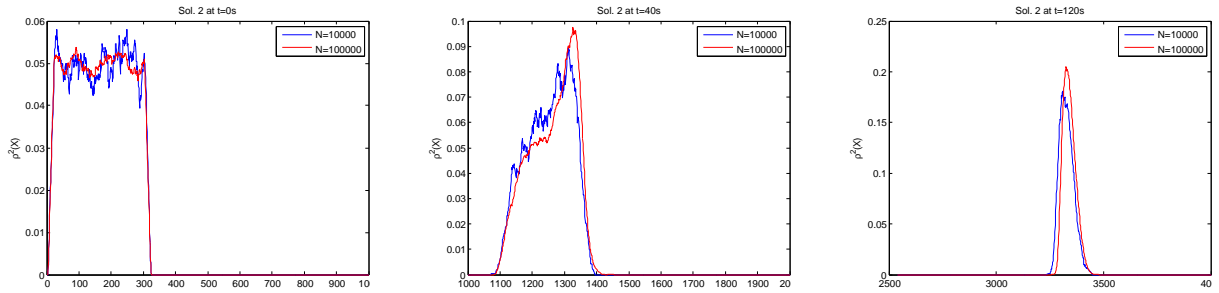


FIGURE 5.4. Spatial density of class II obtained by the stochastic method with $N = 10000$ (blue) and $N = 100000$ (red) particles at times $t = 0$ (left), $t = 40s$ (middle) and $t = 120s$ (right). $\tau = 30s$.

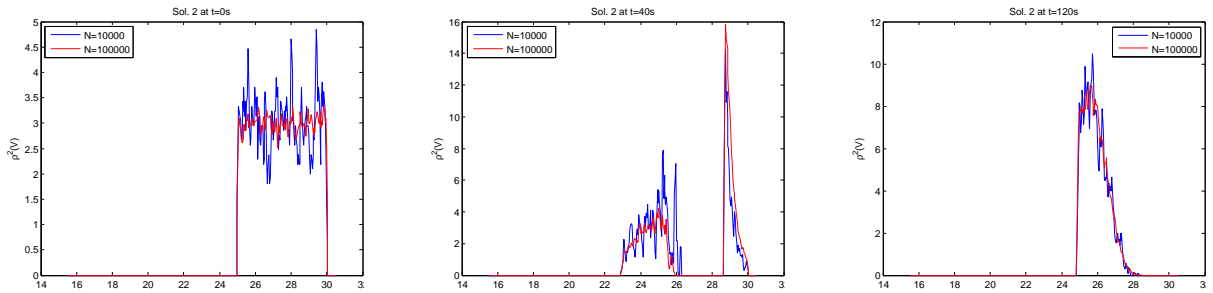


FIGURE 5.5. Speed density of class II obtained by the stochastic method with $N = 10000$ (blue) and $N = 100000$ (red) particles at times $t = 0$ (left), $t = 40s$ (middle) and $t = 120s$ (right). $\tau = 30s$.

Convergence of the deterministic method

We compare the results of the upwind scheme on two meshes with different thinness. The stepsizes Δt , Δx and Δv are chosen to satisfy the CFL condition. Figures 5.6 to 5.9 compare the spatial and speed densities of the two classes for these two discretizations.

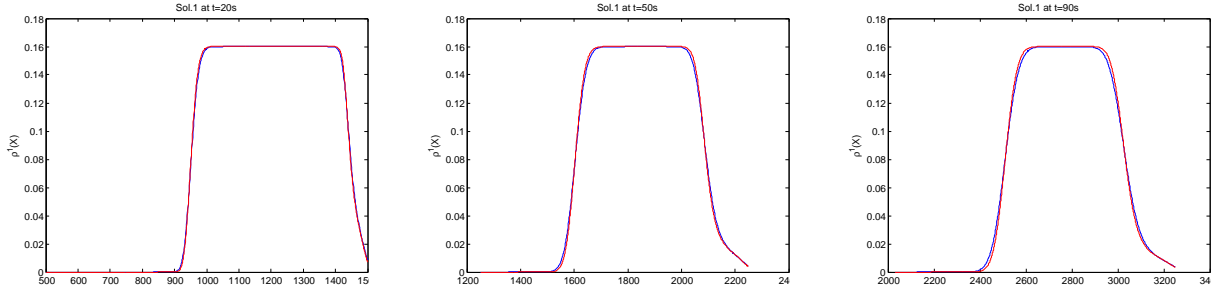


FIGURE 5.6. Spatial density of class *I* obtained by the finite differences method at times $t = 20s$ (left), $t = 50s$ (middle) and $t = 90s$ (right). In blue: $\Delta t = 1.25 \cdot 10^{-2}s$, $\Delta x = 1.25m$ and $\Delta v = 2.10 \cdot 10^{-2}m/s$. In red: $\Delta t = 8.3 \cdot 10^{-3}s$, $\Delta x = 0.83m$ and $\Delta v = 1.3 \cdot 10^{-2}m/s$. $\tau = 30s$.

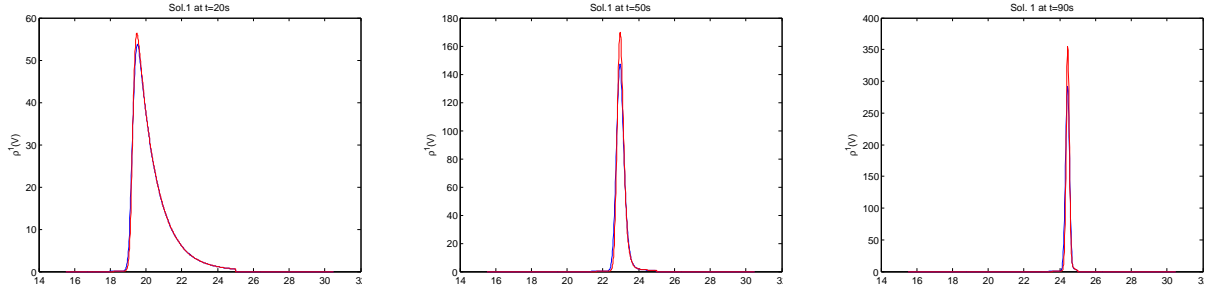


FIGURE 5.7. Speed density of class *I* obtained by the finite differences method at times $t = 20s$ (left), $t = 50s$ (middle) and $t = 90s$ (right). In blue: $\Delta t = 1.25 \cdot 10^{-2}s$, $\Delta x = 1.25m$ and $\Delta v = 2.10 \cdot 10^{-2}m/s$. In red: $\Delta t = 8.3 \cdot 10^{-3}s$, $\Delta x = 0.83m$ and $\Delta v = 1.3 \cdot 10^{-2}m/s$. $\tau = 30s$.

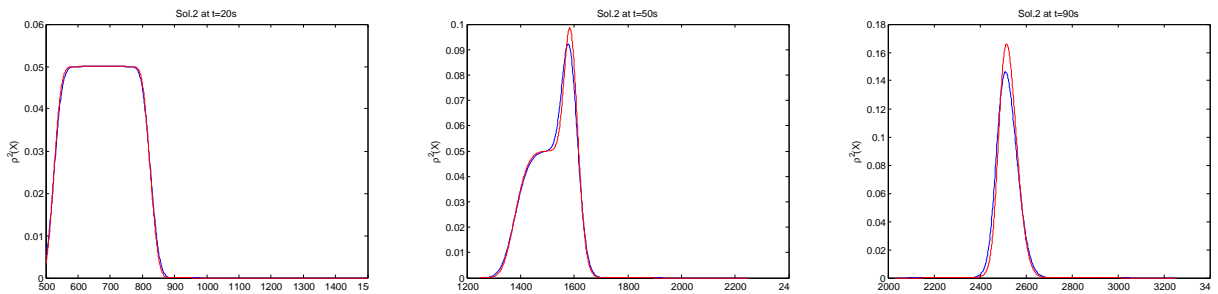


FIGURE 5.8. Spatial density of class *II* obtained by the finite differences method at times $t = 20s$ (left), $t = 50s$ (middle) and $t = 90s$ (right). In blue: $\Delta t = 1.25 \cdot 10^{-2}s$, $\Delta x = 1.25m$ and $\Delta v = 2.10 \cdot 10^{-2}m/s$. In red: $\Delta t = 8.3 \cdot 10^{-3}s$, $\Delta x = 0.83m$ and $\Delta v = 1.3 \cdot 10^{-2}m/s$. $\tau = 30s$.

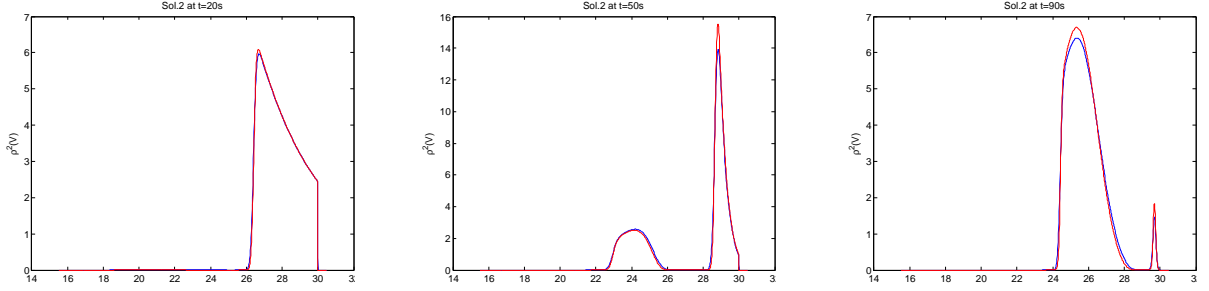


FIGURE 5.9. Speed density of class II obtained by the finite differences method at times $t = 20s$ (left), $t = 50s$ (middle) and $t = 90s$ (right). In blue: $\Delta t = 1.25 \cdot 10^{-2}s$, $\Delta x = 1.25m$ and $\Delta v = 2.10 \cdot 10^{-2}m/s$. In red: $\Delta t = 8.3 \cdot 10^{-3}s$, $\Delta x = 0.83m$ and $\Delta v = 1.3 \cdot 10^{-2}m/s$. $\tau = 30s$.

The stability of the finite differences scheme is observed during the numerical iterations.

We observe the convergence of the spatial density of class I . After a long time, the speed density of this class tends to the quantity $\rho^I \times (X_{max}^I - X_{min}^I) \times (V_{max}^I - V_{min}^I)$ multiplied by the Dirac mass in $v_I^0 = 25m/s$. Approximating this measure on a discrete mesh is problematic, which explains the observed differences between the two approximate solutions on Figure 5.7 at $t = 90s$. The approximate spatial density of class II at time $t = 90s$ is slightly more concentrated for the thin discretization than the rough discretization (see Figure 5.8). This is due to the numerical diffusion.

Comparison of the two methods

Figures 5.10 to 5.13 show that the spatial and speed densities obtained from the two methods with the finest discretization ($N = 100000$ in the particle method, $\Delta t = 8.3 \cdot 10^{-3}s$, $\Delta x = 0.83m$ and $\Delta v = 1.3 \cdot 10^{-2}m/s$ in the finite differences method) match well. Figures 5.10 and 5.11 show that the marginal spatial and speed densities for class I obtained by the stochastic method respectively vanish below the theoretical minimal position $X_{min}^I(t)$ and speed $V_{min}^I(t)$. In contrast, the deterministic densities are still positive slightly below the minimal position and speed. Figures 5.14 and 5.15 respectively present the bidimensional densities obtained from the deterministic method and the stochastic method.

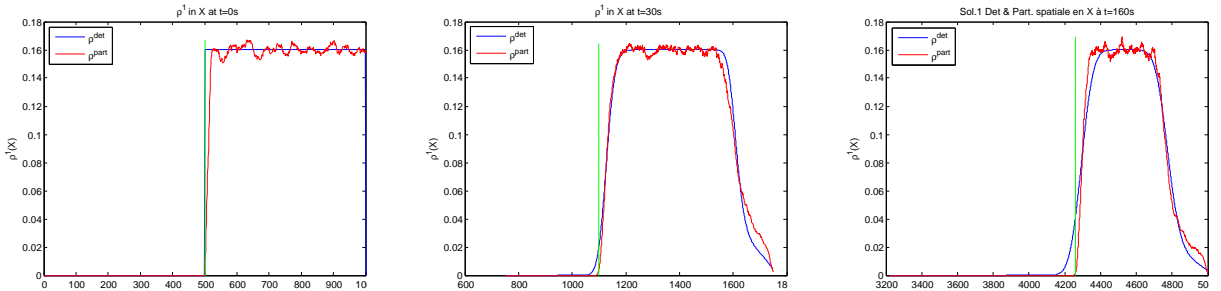


FIGURE 5.10. Spatial density of Class I at times $t = 0$ (left), $t = 30s$ (middle), $t = 160s$ (right) computed by the finite differences method in blue — and by the stochastic method in red —. The green line | delineates the minimal position $X_{min}^I(t)$ given by (5.1). $\tau = 30s$.

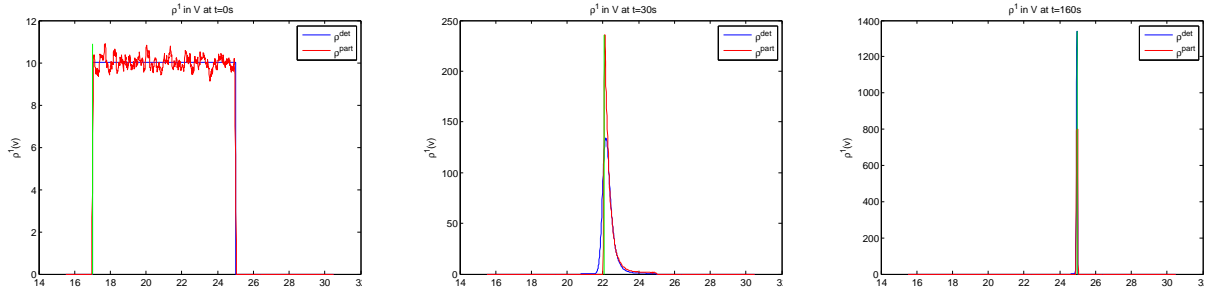


FIGURE 5.11. Speed density of Class *I* at times $t = 0$ (left), $t = 30$ (middle), $t = 160$ (right) computed by the finite differences method in blue – and by the stochastic method in red –. The green line | delineates the minimal speed $V_{min}^I(t)$ given by (5.2). $\tau = 30s$.

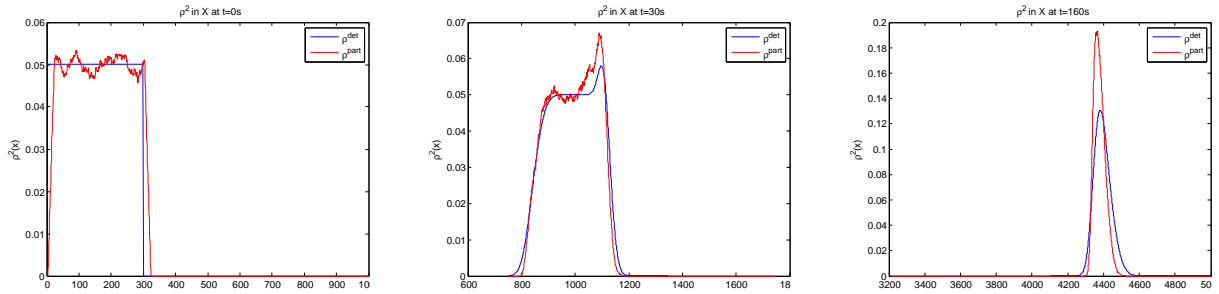


FIGURE 5.12. Spatial density of Class *II* at times $t = 0$ (left), $t = 30s$ (middle), $t = 160s$ (right) computed by the finite differences method in blue – and by the stochastic method in red –. $\tau = 30s$.

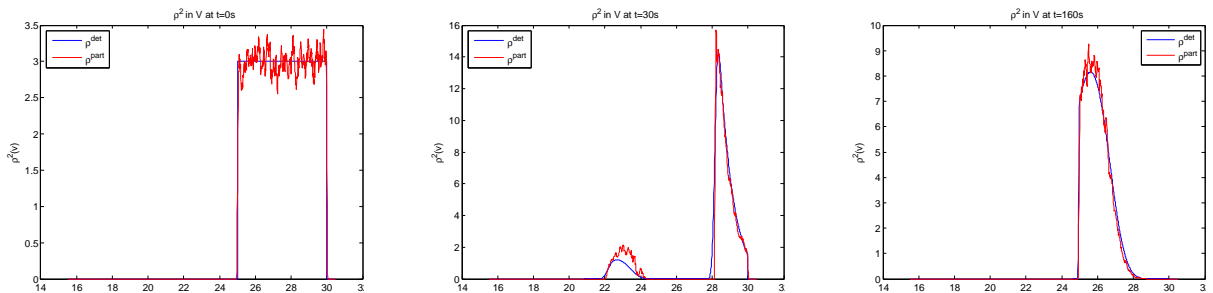


FIGURE 5.13. Speed density of Class *II* at times $t = 0$ (left), $t = 30s$ (middle), $t = 160s$ (right) computed by the finite differences method in blue – and by the stochastic method in red –. $\tau = 30s$.

The deterministic method detects the interaction at time $t_{rec}^{det} \in [18s, 19s]$ when $\tau = 30s$ and $t_{rec}^{det} \in [20s, 21s]$ when $\tau = 15s$. The stochastic method detects the interaction at $t_{rec}^{stoch} \approx 18.5s$, whereas the Newton algorithm predicts it at $t_{rec}^{theo} \approx 18.2s$ when $\tau = 30s$ and at $t_{rec}^{stoch} \approx 21.8s$ while $t_{rec}^{theo} \approx 21.7s$ for $\tau = 15s$. The slight anticipation of this time by the deterministic method can be explained by the numerical diffusion introduced

A PROBABILISTIC PARTICLE APPROXIMATION OF THE “PAVERI-FONTANA” KINETIC MODEL

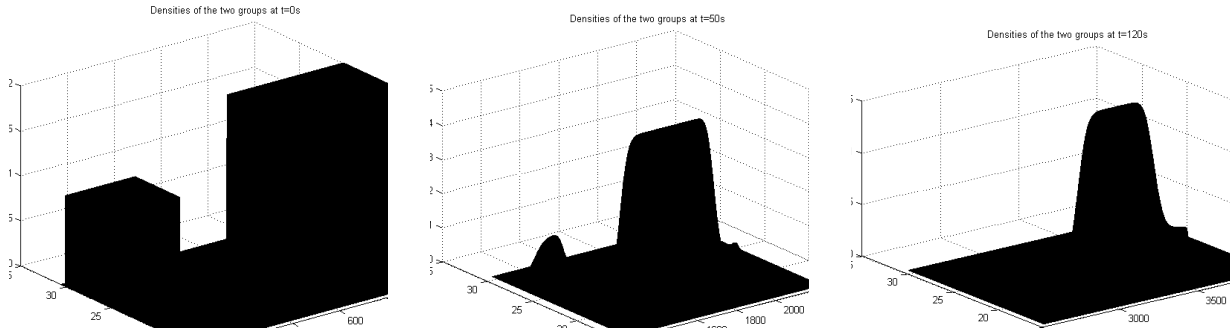


FIGURE 5.14. Densities of the two classes computed from the deterministic method at times $t = 0$ (left), $t = 50s$ (middle) and $t = 120s$ (right). $\tau = 30s$.

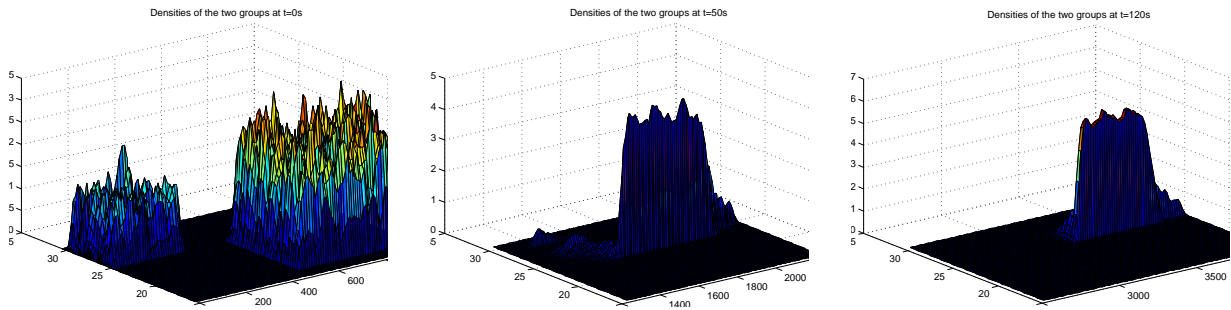


FIGURE 5.15. Densities of the two classes computed from stochastic method at times $t = 0$ (left), $t = 50s$ (middle) and $t = 120s$ (right). $\tau = 30s$.

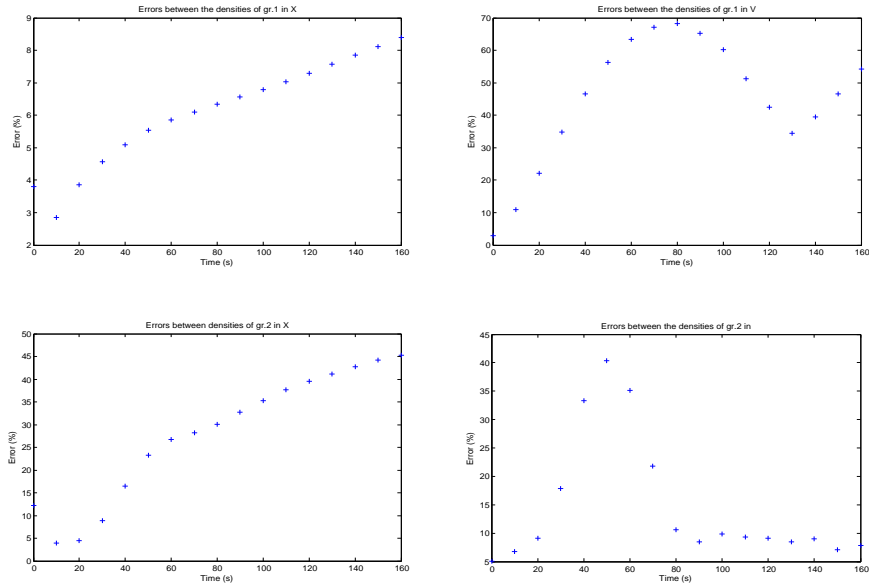


FIGURE 5.16. Errors between the densities computed for the two methods

by the second order term present in the upwind scheme.

The numerical experiments show that the two methods reproduce well the expected properties of the Pavari-Fontana model.

Figure 5.16 depicts the evolution of the relative L^1 -errors between the two methods through time. For example, the top left picture shows the evolution of the quantity $\frac{\int_x \left| \int_v \rho_I^{det}(t,x,v) - \rho_I^{part}(t,x,v) dv \right| dx}{\rho^I \times (X_{max}^I - X_{min}^I) \times (V_{max}^I - V_{min}^I)}$.

The error on the spatial density of class I stays small (below 10). The deterioration of the error on the speed density for this class is probably related to the difficulty for the deterministic method to account for the convergence of this density to $\rho^I \times (X_{max}^I - X_{min}^I) \times (V_{max}^I - V_{min}^I)$ times the Dirac mass in v_I^0 . Moreover, the L^1 -error is not a relevant criterion to measure the discrepancy between two Dirac distributions.

6. Conclusion

In this work, we have developed a new stochastic particle method to approximate the Pavari-Fontana model. We built a deterministic upwind scheme to have a reference solution and validate the benefits of the particle method on a test case.

In the scenario discussed here, a first class of vehicles with lower desired speed was caught up by a class of vehicles with a higher desired speed.

The numerical experiments showed :

- the ability of the both methods to reproduce the model properties;
- that the deterministic method slightly anticipates the first interacting time t_{rec} owing to the numerical diffusion generated in the upwind scheme;
- that the stochastic density solution is null outside the expected support, but that the support is larger than expected for the deterministic solution, which is probably due to the numerical diffusion;
- that the computation time of the particle method is much smaller than the one of the deterministic scheme.

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