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TRAJECTORIAL HYPOCOERCIVITY AND APPLICATION TO CONTROL THEORY

HELGE DIETERT, FRÉDÉRIC HÉRAU, HARSHA HUTRIDURGA, AND CLÉMENT MOUHOT

Abstract. We present the quantitative method of the recent work [6] in a simple setting, together with a compactness argument that was not included in [6] and has interest per se. We are concerned with the exponential stabilisation (spectral gap) for linear kinetic equations with degenerate thermalisation, i.e. when the collision operator vanishes on parts of the spatial domain. The method in [6] covers both scattering and Fokker-Planck type operators, and deals with external potential and boundary conditions, but in these notes we present only its core argument and restrict ourselves to the kinetic Fokker-Planck in the periodic torus with unit velocities and a thermalisation degeneracy (this equation is not covered by the previous results [2, 9, 7]).

Contents

1. Introduction

1.1. **The setting.** Let us consider the linear kinetic equation

$$\partial_t f + v \cdot \nabla_x f = \sigma \Delta_{\mathrm{LB}} f \tag{1.1}$$

for a time-dependent probability density $f(t,x,v) = f_t(x,v)$ over the phase space $(x,v) \in \mathbb{T}^d \times \mathbb{S}^{d-1}$, where $\mathbb{T}^d$ is the unit torus and $\mathbb{S}^{d-1} = \{v \in \mathbb{R}^d : |v| = 1\}$, modelling massless or nearly massless particles with unit velocities. The right hand side Δ_{LB} is the Laplace-Beltrami operator and corresponds to the classical Fokker-Planck operator when velocities are restricted to the sphere. Finally, $\sigma = \sigma(x) \in L^\infty(\mathbb{T}^d; [0,\infty))$ is a weight that can vanish in part of the spatial domain and models the thermalisation degeneracy.

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The evolution (1.1) has the stationary state $f_\infty = 1$ and conserves the mass $\int f \,\mathrm{d}x\,\mathrm{d}v$. When $\sigma \equiv 1$, it is one of the simplest examples of hypocoercive equation: any solution f_t associated to the initial data $f_{\rm in} \in L^2$ with zero mass $\int f_{\rm in}\,\mathrm{d}x\,\mathrm{d}v = 0$ will converge exponentially to zero. A natural question arises then, inspired from control theory: under what conditions on σ, will the evolution (1.1) yield exponential relaxation to equilibrium?

When the right hand side in (1.1) is a bounded integral scattering operator (linear Boltzmann or relaxation operator), this question has been answered by [2, 9] by compactness arguments when σ satisfies a *geometric control condition* borrowed from control theory of wave equations [1]. However these works crucially rely on the facts that (1) the right hand side operator is bounded, and (2) writes as a non-negative integral operator minus a local part. The case we consider here is conceptually different, and requires new methods. Another direction for bounded operators is given in [7] who answered quantitatively by Harris theorem from probability theory.

1.2. **The geometric control condition.** The transport equation on the left hand side of (1.1) is solved by the characteristics

$$Z_t(x,v) := (X_t(x,v), V_t(x,v)) := (x + tv, v).$$

The (uniform) *geometric control condition* (GCC) intuitively means that, in a given fixed time, all trajectories spends a positive time (bounded below) in a region where $\sigma \gtrsim 1$ (where thermalisation truly occurs), see Figure 1. The non-uniform version of this condition intuitively means that all trajectories eventually enter the support of σ (without restricting the time horizon or asking that the trajectories spend time in a region where σ remains strictly away from zero).

We adopt the following precise definition:

Hypothesis 1 (Geometric control condition)**.** *The (uniform) GCC writes*

$$\exists\, T^*, c > 0 \ \text{ such that } \ \forall\,(x,v) \in \mathbb{T}^d \times \mathbb{S}^{d-1} : \quad \int_0^{T^*} \sigma(X_t(x,v))\,\mathrm{d}t \geq c. \tag{1.2}$$

We also assume $\sigma \in C^1$, and therefore (1.2) *implies that there is $\Sigma \subset \mathbb{T}^d$ open with $\mathcal{C}^1$ boundary and a smooth $\chi : \mathbb{T}^d \to [0,\infty)$, so that $\mathbf{1}_\Sigma \lesssim \sigma$, $\operatorname{supp}\chi \subset \Sigma$ and*

$$\forall\,(x,v) \in \mathbb{T}^d \times \mathbb{S}^{d-1} : \quad \int_0^{T^*} \chi(X_t(x,v))\,\mathrm{d}t \geq 1. \tag{1.3}$$

Remark 1. *In this control condition, we assume slightly more regularity on σ than in the literature:* [2] *only needs $\sigma \in L^\infty$ while* [9] *assumes that σ continuous. The regularity is only used to conclude* (1.3) *and we did not try to optimise this assumption.*

Our main result is:

Theorem 2 (Exponential stabilization)**.** *Assume σ is bounded and satisfies* ***(H1)*** *with Σ having finitely many connected components. Then there are $C \geq 1$ and $\Lambda > 0$ such that for any initial data $f_{\rm in} \in L^2$ the corresponding solution f_t to* (1.1) *satisfies*

$$\left\| f_t - \left(\int_{\mathbb{T}^d \times \mathbb{S}^{d-1}} f_{\rm in} \right) \right\|_{L^2(\mathbb{T}^d \times \mathbb{S}^{d-1})} \leq C \mathrm{e}^{-\Lambda t} \left\| f_{\rm in} - \left(\int_{\mathbb{T}^d \times \mathbb{S}^{d-1}} f_{\rm in} \right) \right\|_{L^2(\mathbb{T}^d \times \mathbb{S}^{d-1})}.$$

Such C, Λ can be computed from the proof and only depend on $T^*, c, \Sigma, \|\chi\|_{W^{1,\infty}}, \|\sigma\|_\infty$.

Remark 3. *The* non-uniform GCC *means that for almost every* $(x,v) \in \mathbb{T}^d \times \mathbb{S}^{d-1}$

$$\exists T = T(x,v) > 0 \quad \text{such that} \quad \int_0^{T(x,v)} \sigma(X_t(x,v))\,\mathrm{d}t > 0. \tag{1.4}$$

When one replaces (1.2) *by the non-uniform condition* (1.4)*, our method can be used to prove the convergence* $f_t \to f_\infty = (\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} f_{\mathrm{in}})$*, although without a rate.*

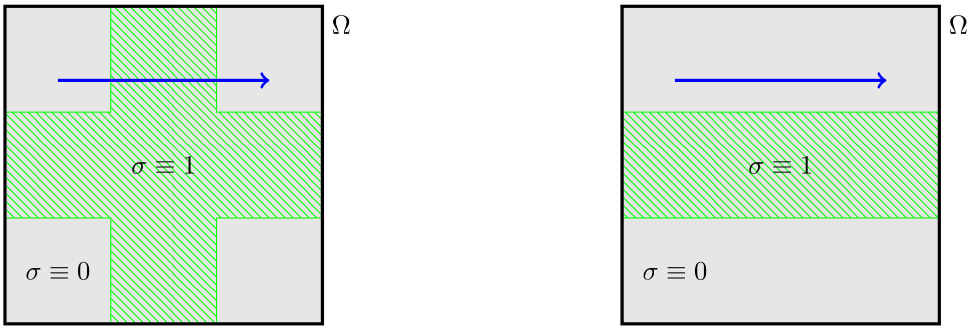

Figure 1. Illustration of **(H1)**. On the left, all lines hit and spend a controlled fraction of time in the thermalisation set $\Sigma = \operatorname{supp}\sigma$ on a time interval $[0,T]$, yielding GCC and exponential convergence. On the right, there is a set of configurations with zero measure whose trajectories never hit Σ, and around this set the time to hit Σ can be arbitrarily large: only the non-uniform GCC holds, and one typically expects polynomial rate of convergence.

2. Trajectorial approach to hypocoercivity

2.1. **Fixing the global average.** Consider $f_{\mathrm{in}} \in L^2$ and its associated solution f_t. Since the mass is conserved and the equation is linear, $g_t := f_t - (\int f_{\mathrm{in}}(x,v)\,\mathrm{d}x\,\mathrm{d}v)$ is solution to (1.1) with zero mass

$$\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{\mathrm{in}}(z)\,\mathrm{d}z = 0, \tag{2.1}$$

and therefore its associated equilibrium is zero.

2.2. **The local projection.** The local equilibrium is $M(v) = |\mathbb{S}^{d-1}|^{-1}$ and we define the spatial density (velocity average)

$$\langle g\rangle(t,x) := \int_{\mathbb{S}^{d-1}} g(t,x,v)\,\mathrm{d}v. \tag{2.2}$$

2.3. **The energy estimate.** The L^2 norm is the natural entropy for this linear model, and the H theorem takes the form of the energy estimate

$$\mathcal{D}(g_t) := -\frac{\mathrm{d}}{\mathrm{d}t}\|g_t\|^2_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})} = \int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} \sigma|\nabla_v g_t|^2\,\mathrm{d}x\,\mathrm{d}v \geq 0 \tag{2.3}$$

where the gradient in v is the differential tangential to the unit sphere.

2.4. **Integral criterion for exponential stabilization.** We first prove a simple sufficient time-integrated entropy production inequality that implies exponential convergence. Such criterion is standard in kinetic theory, and (at least) already appears in a compactness argument in [8]. Exponential decay holds if and only if there are $T > 0$, $\lambda > 1$ such that

$$\|g_{\rm in}\|^2_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})} \le \lambda \int_0^T \mathcal{D}(g_t)\,\mathrm{d}t. \tag{2.4}$$

More precisely: when (2.4) holds, then

$$\|g_t\|_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})} \le \sqrt{\frac{\lambda}{\lambda-1}}\exp\left(-\frac{1}{T}\log\left(\frac{\lambda}{\lambda-1}\right)t\right)\|g_{\rm in}\|_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})}.$$

2.5. **Micro-coercivity.** The Poincaré inequality holds in the compact smooth manifold $\mathbb{S}^{d-1}$: there is $C_P > 0$ so that for any $x \in \mathbb{T}^d$

$$\int_{\mathbb{S}^{d-1}} |g_t - \langle g_t\rangle M|^2\,\mathrm{d}v \le C_P \int_{\mathbb{S}^{d-1}} \sigma|\nabla_v g_t|^2\,\mathrm{d}v. \tag{2.5}$$

This provides control over $(g-\langle g\rangle M)$ on $\operatorname{supp}\sigma\times\mathbb{S}^{d-1}$ hence on the good set $\Sigma\times\mathbb{S}^{d-1}$.

2.6. **Following the characteristics to transfer the control.** The next step is to transfer the control of $(g-\langle g\rangle M)$ on the good set to the whole domain, by following trajectories. Let us prove that there are $C_1, C_2 > 0$ so that

$$\|g_{\rm in}\|^2_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})} \le C_1\int_0^{T^*}\mathcal{D}(g_t)\,\mathrm{d}t + C_2\int_0^{T^*}\int_\Sigma \langle g_t\rangle^2\,\mathrm{d}x\,\mathrm{d}t. \tag{2.6}$$

To prove this first write the evolution equation for g^2:

$$\partial_t\left(g_t^2\right) + v\cdot\nabla_x\left(g_t^2\right) = 2\left(\Delta_{\rm LB} g_t\right) g_t$$

and second write it in Duhamel form along the transport flow (writing $z := (x,v)$)

$$g_t(z)^2 = g_{\rm in}(Z_{-t}(z))^2 + 2\int_0^t \sigma(Z_{t-s}(z))\,(\Delta_{\rm LB} g_s)(Z_{t-s}(z))\,g_s(Z_{t-s}(z))\,\mathrm{d}s$$

and third integrate it against χ from **(H1)** on $[0,T^*]\times\mathbb{T}^d\times\mathbb{S}^{d-1}$:

$$\begin{aligned}\int_0^{T^*}\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_t^2(z)\chi(z)\,\mathrm{d}t\,\mathrm{d}z &= \int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{\rm in}(z)^2\left(\int_0^{T^*}\chi(Z_t(z))\,\mathrm{d}t\right)\mathrm{d}z\\ &\quad + 2\int_0^{T^*}\int_0^t\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}}\sigma(z)\,(\Delta_{\rm LB} g_s)(z)\,g_s(z)\chi(Z_{t-s}(z))\,\mathrm{d}z\,\mathrm{d}s\,\mathrm{d}t\end{aligned}$$

where we have used the unitary change of variables $z\mapsto Z_t(z)$ and $z\mapsto Z_{t-s}(z)$.

Now observe that (1.3) in **(H1)** implies

$$\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{\rm in}(z)^2\left(\int_0^{T^*}\chi(Z_t(z))\,\mathrm{d}t\right)\mathrm{d}z \ge \|g_{\rm in}\|^2_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})}$$

and $\operatorname{supp}\sigma\subset\Sigma$ implies

$$\int_0^{T^*}\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_t^2(z)\chi(z)\,\mathrm{d}t\,\mathrm{d}z \le \|\chi\|_\infty\int_0^{T^*}\|g_t\|^2_{L^2(\Sigma\times\mathbb{S}^{d-1})}\,\mathrm{d}t.$$

As for the last term we perform an integration by parts:

$$
\begin{aligned}
2\int_0^{T^*}\int_0^t\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} \sigma(z)\left(\Delta_{\mathrm{LB}} g_s\right)(z) g_s(z)\chi(Z_{t-s}(z))\,\mathrm{d}z\,\mathrm{d}s\,\mathrm{d}t = \\
-2\int_0^{T^*}\int_0^t\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} \sigma(z)\left|\nabla_v g_s(z)\right|^2\chi(Z_{t-s}(z))\,\mathrm{d}z\,\mathrm{d}s\,\mathrm{d}t \\
-2\int_0^{T^*}\int_0^t (t-s)\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} \sigma(z)\nabla_v g_s(z)\cdot(\nabla\chi)(Z_{t-s}(z)) g_s(z)\,\mathrm{d}z\,\mathrm{d}s\,\mathrm{d}t \\
\geq -T^*\left(2\|\chi\|_\infty + \frac{1}{\varepsilon}\|\nabla\chi\|_\infty^2\|\sigma\|_\infty\right)\int_0^{T^*}\mathcal{D}(g_s)\,\mathrm{d}s - \varepsilon T^*\int_0^{T^*}\|g_s\|^2_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})}\,\mathrm{d}s \\
\geq -T^*\left(2\|\chi\|_\infty + \frac{1}{\varepsilon}\|\nabla\chi\|_\infty^2\|\sigma\|_\infty\right)\int_0^{T^*}\mathcal{D}(g_s)\,\mathrm{d}s - \varepsilon\left(T^*\right)^2\|g_{\mathrm{in}}\|^2_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})}
\end{aligned}
$$

where in the last line we have used that the L^2 norm is non-increasing. Therefore we deduce

$$
\begin{aligned}
\|g_{\mathrm{in}}\|^2_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})} \leq \|\chi\|_\infty\int_0^{T^*}\|g_t\|^2_{L^2(\Sigma\times\mathbb{S}^{d-1})}\,\mathrm{d}t \\
+T^*\left(2\|\chi\|_\infty + \frac{1}{\varepsilon}\|\nabla\chi\|_\infty\right)\int_0^{T^*}\mathcal{D}(g_s)\,\mathrm{d}s + \varepsilon\left(T^*\right)^2\|\nabla\chi\|_\infty\|g_{\mathrm{in}}\|^2_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})}.
\end{aligned}
$$

We now use the micro-coercivity (2.5):

$$
\begin{aligned}
\int_0^{T^*}\|g_t\|^2_{L^2(\Sigma\times\mathbb{S}^{d-1})}\,\mathrm{d}t &\leq \frac{2}{|\mathbb{S}^{d-1}|}\int_0^{T^*}\|\langle g_t\rangle\|^2_{L^2(\Sigma)}\,\mathrm{d}t + 2\int_0^{T^*}\|g_t - \langle g_t\rangle M\|^2_{L^2(\Sigma\times\mathbb{S}^{d-1})}\,\mathrm{d}t \\
&\leq \frac{2}{|\mathbb{S}^{d-1}|}\int_0^{T^*}\|\langle g_t\rangle\|^2_{L^2(\Sigma)}\,\mathrm{d}t + 2C_P\int_0^{T^*}\mathcal{D}(g_t)\,\mathrm{d}t.
\end{aligned}
$$

Taking $\varepsilon = (T^*)^{-2}\|\nabla\chi\|_\infty^{-1}/2$, we finally deduce

$$
\begin{aligned}
\|g_{\mathrm{in}}\|^2_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})} \leq \frac{4\|\chi\|_\infty}{|\mathbb{S}^{d-1}|}\int_0^{T^*}\|\langle g_t\rangle\|^2_{L^2(\Sigma)}\,\mathrm{d}t \\
+\left[4C_P\|\chi\|_\infty + 4T^*\|\chi\|_\infty + 4\left(T^*\right)^3\|\nabla\chi\|_\infty^2\|\sigma\|_\infty\right]\int_0^{T^*}\mathcal{D}(g_t)\,\mathrm{d}t
\end{aligned}
$$

which proves (2.6). We are left with the control of the local projection on the good set, which is the object of the next two sections.

3. The compactness argument

Assume (2.4) to be false with $T = T^*$: there is a contradiction sequence $(g^n)_{n\in\mathbb{N}}$ of solutions with initial data g^n_{in} such that $\|g^n_{\mathrm{in}}\| = 1$ (normalised by linearity) and

$$\int_0^{T^*}\mathcal{D}(g^n_t)\,\mathrm{d}t \to 0.$$

By weak compactness we then find a subsequence such that

$$g^{n'} \rightharpoonup g^* \text{ in } L^2([0,T^*]\times\mathbb{T}^d\times\mathbb{S}^{d-1}) \quad\text{and}\quad g^{n'}_{\mathrm{in}} \rightharpoonup g^*_{\mathrm{in}} \text{ in } L^2(\mathbb{T}^d\times\mathbb{S}^{d-1}).$$

Moreover the velocity averaging lemma ensures that $\langle g^n \rangle$ is relatively compact for the strong topology in $L^2([0,T^*] \times \mathbb{T}^d)$. Therefore, we can furthermore assume that our subsequence satisfies $\langle g^{n'} \rangle \to \langle g^* \rangle$ strongly in $L^2([0,T^*] \times \mathbb{T}^d)$.

The limit then satisfies $\partial_t g^* + v \cdot \nabla_x g^* = 0$ in the weak sense and $\int_0^{T^*} \mathcal{D}(g_t^*) \, \mathrm{d}t = 0$. This implies that g^* is constant in $\Sigma \times \mathbb{S}^{d-1}$ since it has to be constant along the transport flow and equal to its velocity average. By connecting any point to a point in $\Sigma \times \mathbb{S}^{d-1}$ (using the GCC), we deduce that g^* is constant in $\mathbb{T}^d \times \mathbb{S}^{d-1}$. The weak L^2 convergence implies that $\int_{[0,T^*] \times T^d \times \mathbb{S}^{d-1}} g^* = \lim_{n' \to \infty} \int_{[0,T^*] \times T^d \times \mathbb{S}^{d-1}} g^{n'} = 0$ (recall that we have set the total mass of each g_n to zero). Therefore $g^* \equiv 0$.

Using the strong convergence of the velocity average $\langle g \rangle \to \langle g^* \rangle$ in $L^2([0,T^*] \times \mathbb{T}^d \times \mathbb{S}^{d-1})$ and taking the limit in (2.6), we deduce that $\| \langle g^* \rangle \|_{L^2([0,T^* \times \Sigma \times \mathbb{S}^{d-1})} \gtrsim 1$. This contradicts $g^* \equiv 0$ proved in the previous paragraph, and (2.4) is proved.

4. Getting quantitative: the divergence inequality

We now replace the previous non-constructive argument based on compactness and contradiction by a quantitative one. For the sake of readability, we first assume that the Σ from **(H1)** is connected and explain how this can be relaxed at the end.

In view of (2.4) and (2.6) and the fact that the L^2 norm is non-increasing, to close a complete quantitative argument it is enough to prove that for any $\delta > 0$ there is $C_\delta > 0$ so that

$$\int_0^{T^*} \int_\Sigma \langle g_t \rangle^2 \, \mathrm{d}x \, \mathrm{d}t \le C_\delta \int_0^{T^*} \mathcal{D}(g_t) \, \mathrm{d}t + \delta \int_0^{T^*} \|g_t\|^2_{L^2(\mathbb{T}^d \times \mathbb{S}^{d-1})} \, \mathrm{d}t. \tag{4.1}$$

We then define the global average over the good set as

$$\langle\!\langle g \rangle\!\rangle := \frac{1}{m} \int_{[0,T^*] \times \Sigma} \langle g \rangle (t,x) \, \mathrm{d}t \, \mathrm{d}x \qquad \text{with } m := |[0,T^*] \times \Sigma|$$

and then split the term to estimate as

$$\int_0^{T^*} \int_\Sigma \langle g_t \rangle^2 \, \mathrm{d}x \, \mathrm{d}t \le \int_0^{T^*} \int_\Sigma \left(\langle g_t \rangle - \langle\!\langle g \rangle\!\rangle \right)^2 \, \mathrm{d}x \, \mathrm{d}t + m \langle\!\langle g \rangle\!\rangle^2 =: I_1 + I_2.$$

To control I_1 we use the following result that goes back to [3, 4, 5]:

Lemma 4 (Divergence inequality). *Given $\mathsf{U} \subset \mathbb{R}^n$, $n \ge 1$, an open connected bounded C^1 domain, there is $C_\mathsf{D} > 0$ and a linear map D mapping any $h \in L^2(\mathsf{U})$ with $\int_\mathsf{U} h = 0$ to a $\mathbf{F} : \mathsf{U} \to \mathbb{R}^n$ in $H^1(\mathsf{U})$ that satisfies*

$$\begin{cases} \nabla \cdot \mathbf{F} = h \text{ in } \mathsf{U}, \\ \mathbf{F} = 0 \text{ on } \partial \mathsf{U}, \\ \|\mathbf{F}\|_{H^1(\mathsf{U})} \le C_\mathsf{D} \|h\|_{L^2(\mathsf{U})}. \end{cases} \tag{4.2}$$

This is proved constructively in [3, 4, 5], and we refer to our full paper [6] for extensions of this result to general domains with external potentials and boundary conditions.

We apply Lemma 4 to $h := \langle g_t \rangle - \langle\!\langle g \rangle\!\rangle$ on $\mathsf{U} := (0, T^*) \times \Sigma$ (with zero mass): there is $\mathbf{F} \in H^1(\mathsf{U})$ so that (4.2) holds, and we write (using the Dirichlet conditions)

$$\begin{aligned}\int_0^{T^*} \int_\Sigma \left(\langle g_t \rangle - \langle\!\langle g \rangle\!\rangle \right)^2 \, \mathrm{d}x \, \mathrm{d}t &= \int_{\mathsf{U}} \left(\langle g_t \rangle - \langle\!\langle g \rangle\!\rangle \right) \left(\nabla_{t,x} \cdot \mathbf{F} \right) \, \mathrm{d}x \, \mathrm{d}t \\ &= - \int_{\mathsf{U}} \mathbf{F} \cdot \nabla_{t,x} \langle g_t \rangle \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{4.3}$$

Denote $\partial_0 = \partial_t$ and $\partial_i = \partial_{x_i}$ for $i = 1, \ldots, d$. Then there is $C_3 > 0$ so that

$$\forall\, i = 0, \ldots, d, \quad \partial_i \langle g_t \rangle = K_i + \sum_{j=0}^d \partial_j J_{ij} \quad \text{with} \tag{4.4}$$

$$\sum_{i=0}^d \|K_i\|^2_{L^2(\mathsf{U})} + \sum_{i,j=0}^d \|J_{ij}\|^2_{L^2(\mathsf{U})} \le C_3 \int_0^{T^*} \mathcal{D}(g_t) \, \mathrm{d}t. \tag{4.5}$$

Indeed define $\varphi_i \in C^2(\mathbb{S}^{d-1})$, $i = 0, \ldots, d$ so that, denoting $v_0 = 1$,

$$\begin{aligned} &\int_{\mathbb{S}^{d-1}} \varphi_i(v) v_j \, \mathrm{d}v = \delta_{ij}, \ i, j = 0, \ldots, d, \quad \text{and so} \\ &\int_{\mathbb{S}^{d-1}} \Big\{ (\partial_t + v \cdot \nabla_x) \left[\langle g_t \rangle \right] \Big\} \varphi_i \, \mathrm{d}v = \partial_i \langle g_t \rangle. \end{aligned}$$

The evolution equation on g then implies

$$\begin{aligned} \partial_i \langle g_t \rangle &= \sigma \int_{\mathbb{S}^{d-1}} (\Delta_{\mathrm{LB}} g_t) \, \varphi_i \, \mathrm{d}v + \int_{\mathbb{S}^{d-1}} \Big\{ (\partial_t + v \cdot \nabla_x) \left[\langle g_t \rangle M - g_t \right] \Big\} \varphi_i \, \mathrm{d}v \\ &= K_i + \sum_{j=0}^d \partial_j J_{ij} \end{aligned}$$

with

$$\begin{cases} K_i(t,x) := \sigma \displaystyle\int_{\mathbb{S}^{d-1}} \left(g_t - \langle g_t \rangle \right) \left(\Delta_{\mathrm{LB}} \varphi_i \right) \, \mathrm{d}v, \quad i = 0. \ldots, d, \\ J_{ij}(t,x) := \displaystyle\int_{\mathbb{S}^{d-1}} \left[\langle g_t \rangle - g_t \right] v_j \varphi_i \, \mathrm{d}v, \quad i = 0, \ldots, d, \ j = 0, \ldots, d, \end{cases}$$

which proves (4.4)-(4.5). Going back to (4.3) we compute

$$\begin{aligned} \int_0^{T^*} \int_\Sigma \left(\langle g_t \rangle - \langle\!\langle g \rangle\!\rangle \right)^2 \, \mathrm{d}x \, \mathrm{d}t &= - \sum_{i=0}^d \int_{\mathsf{U}} \mathbf{F}_i K_i \, \mathrm{d}x \, \mathrm{d}t - \sum_{i,j=0}^d \int_{\mathsf{U}} \mathbf{F}_i \partial_j J_{ij} \, \mathrm{d}x \, \mathrm{d}t \\ &= - \sum_{i=0}^d \int_{\mathsf{U}} \mathbf{F}_i K_i \, \mathrm{d}x \, \mathrm{d}t + \sum_{i,j=0}^d \int_{\mathsf{U}} \partial_j \mathbf{F}_i J_{ij} \, \mathrm{d}x \, \mathrm{d}t \end{aligned}$$

where we have used the Dirichlet conditions again. Using the $H^1(\mathsf{U})$ bound on $\mathbf{F}$ in (4.2) and (4.5) we deduce

$$\int_0^{T^*}\int_\Sigma \left(\langle g_t\rangle - \langle\!\langle g\rangle\!\rangle\right)^2 \,\mathrm{d}x\,\mathrm{d}t \le \sqrt{d^2+d}\,\|\mathbf{F}\|_{H^1(\mathsf{U})}\left(\sum_{i=0}^d \|K_i\|^2_{L^2(\mathsf{U})} + \sum_{i,j=0}^d \|J_{ij}\|^2_{L^2(\mathsf{U})}\right)^{1/2}$$
$$\le \sqrt{d^2+d}\,C_{\mathsf{D}}\sqrt{C_3}\,\|\langle g_t\rangle - \langle\!\langle g\rangle\!\rangle\|_{L^2(\mathsf{U})}\left(\int_0^{T^*}\mathcal{D}(g_t)\,\mathrm{d}t\right)^{1/2}$$

which implies by splitting the square

$$\int_0^{T^*}\int_\Sigma \left(\langle g_t\rangle - \langle\!\langle g\rangle\!\rangle\right)^2 \,\mathrm{d}x\,\mathrm{d}t \le C_\delta\int_0^{T^*}\mathcal{D}(g_t)\,\mathrm{d}t + \delta\int_0^{T^*}\|g_t\|^2_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})}\,\mathrm{d}t \tag{4.6}$$

for all $\delta > 0$ and some corresponding constant C_δ.

To finish the proof of (4.1), we need to estimate the global average $\langle\!\langle g\rangle\!\rangle$ which we compare to the zero mass condition up to error terms controlled by the dissipation. To relate it to the zero mass condition (2.1) introduce

$$\forall\,(t,z)\in[0,T^*]\times\mathbb{T}^d\times\mathbb{S}^{d-1},\quad \psi(t,z) = \psi_t(z) := \frac{\chi(z)}{\int_0^{T^*}\chi(Z_{s-t}(z))\,\mathrm{d}s}. \tag{4.7}$$

which is well-defined since the denominator is uniformly positive thanks to (1.3). The function ψ is bounded in $C^1([0,T^*]\times\mathbb{T}^d\times\mathbb{S}^{d-1})$, and satisfies $\operatorname{supp}\psi = [0,T^*]\times\operatorname{supp}\sigma$, and, most importantly,

$$\forall\, z\in\mathbb{T}^d\times\mathbb{S}^{d-1},\quad \int_0^{T^*}\psi_t(Z_t(z))\,\mathrm{d}t = 1. \tag{4.8}$$

By the conservation of mass, we find that $h = 1 - \frac{1}{T^*}\langle\psi\rangle$ has mass zero over U. Hence we can apply Lemma 4 to find $\mathbf{F}$ with the properties of the lemma. We then find

$$\langle\!\langle g\rangle\!\rangle - \frac{1}{T^*}\int_{[0,T^*]\times\mathbb{T}^d}\langle g\rangle(t,x)\,\langle\psi\rangle(t,x)\,\mathrm{d}t\,\mathrm{d}x = \int_{\mathsf{U}}\langle g\rangle(t,x)\nabla\cdot\mathbf{F}\,\mathrm{d}t\,\mathrm{d}x.$$

Hence we can use (4.4)-(4.5) as before to find a constant C_4 so that

$$\left|\langle\!\langle g\rangle\!\rangle - \frac{1}{T^*}\int_{[0,T^*]\times\mathbb{T}^d}\langle g\rangle(t,x)\,\langle\psi\rangle(t,x)\,\mathrm{d}t\,\mathrm{d}x\right| \le C_4\left(\int_0^{T^*}\mathcal{D}(g_t)\,\mathrm{d}t\right)^{1/2}. \tag{4.9}$$

We now estimate the ψ-weighted average as

$$\begin{aligned}
&\frac{1}{T^*}\int_{[0,T^*]\times\mathbb{T}^d}\langle g\rangle(t,x)\,\langle\psi\rangle(t,x)\,\mathrm{d}t\,\mathrm{d}x\\
&= \int_{[0,T^*]\times\mathbb{T}^d\times\mathbb{S}^{d-1}}\langle g\rangle(t,x)\psi(t,x,v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v\\
&= \int_{[0,T^*]\times\mathbb{T}^d\times\mathbb{S}^{d-1}}\left[M\langle g_t\rangle(x) - g_t(x,v)\right]\psi(t,x,v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v\\
&\qquad + \int_{[0,T^*]\times\mathbb{T}^d\times\mathbb{S}^{d-1}} g(t,x,v)\psi(t,x,v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v =: J_1 + J_2.
\end{aligned}$$

The first term J_1 is controlled by the micro-coercivity (2.5) and $\psi \lesssim \sigma$ for a constant C_5 as

(4.10)
$$J_1 \le C_5 \left(\int_{[0,T^*]\times\mathbb{T}^d\times\mathbb{S}^{d-1}} \sigma \left[M \langle g_t \rangle - g_t \right]^2 \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \right)^{1/2} \le C_5 C_P \left(\int_0^{T^*} \mathcal{D}(g_t)\,\mathrm{d}t \right)^{1/2}.$$

We rewrite the second term J_2 by Duhamel's principle along the transport flow as

$$\begin{aligned} J_2 = &\int_{[0,T^*]\times\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{\mathrm{in}}(Z_{-t}(z))\psi_t(z)\,\mathrm{d}t\,\mathrm{d}z \\ &\qquad + \int_{[0,T^*]\times\mathbb{T}^d\times\mathbb{S}^{d-1}} \int_0^t \sigma(X_{-(t-s)}(z))\Delta_{\mathrm{LB}} g_s(Z_{-(t-s)}(z))\psi_t(z)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}z \\ = &\int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} g_{\mathrm{in}}(z) \left(\int_0^{T^*} \psi_t(Z_t(z))\,\mathrm{d}t \right) \mathrm{d}z \\ &\qquad + \int_{[0,T^*]\times\mathbb{T}^d\times\mathbb{S}^{d-1}} \int_0^t \sigma(x)\Delta_{\mathrm{LB}} g_s(z)\psi_t(Z_{t-s}(z))\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}z =: J_{21} + J_{22} \end{aligned}$$

and $J_{21} = 0$ because of (4.8) and (2.1), and the second term is estimated by integration by parts:

$$\begin{aligned} |J_{22}| &= \left| \int_{[0,T^*]\times\mathbb{T}^d\times\mathbb{S}^{d-1}} \int_0^t \sigma(x)\nabla_v g_s(z) \cdot \nabla_v \left[\psi_t(x+(t-s)v, v) \right] \mathrm{d}s\,\mathrm{d}t\,\mathrm{d}z \right| \\ &\le C_6 \left(\int_0^{T^*} \mathcal{D}(g_t)\,\mathrm{d}t \right)^{1/2} \end{aligned}$$

for some constant $C_6 > 0$. Together with (4.6) and (4.10) it concludes the proof of (4.1).

Let us finally extend the argument when Σ has finitely many connected components $\Sigma_1, \dots, \Sigma_k$. For each $i = 1, \dots, k$, we define $\mathsf{U}_i = (0,T^*) \times \Sigma_i$ and

$$\langle\langle g \rangle\rangle_i := \frac{1}{m_i} \int_{[0,T^*]\times\Sigma_i} \langle g \rangle(t,x)\,\mathrm{d}t\,\mathrm{d}x \qquad \text{with } m_i := |[0,T^*]\times\Sigma|.$$

Arguing on each component as we did in the estimate (4.6), we get

$$\int_0^{T^*} \int_{\Sigma_i} \left(\langle g_t \rangle - \langle\langle g \rangle\rangle_i \right)^2 \mathrm{d}x\,\mathrm{d}t \le C_\delta \int_0^{T^*} \mathcal{D}(g_t)\,\mathrm{d}t + \delta \int_0^{T^*} \|g_t\|^2_{L^2(\mathbb{T}^d\times\mathbb{S}^{d-1})}\,\mathrm{d}t. \tag{4.11}$$

We then prove, for each pair $i \ne j \in \{1, \dots, k\}$, that $|\langle\langle g \rangle\rangle_i - \langle\langle g \rangle\rangle_j|^2$ is controlled by $\int_0^{T^*} \mathcal{D}(g_t)\,\mathrm{d}t$. Indeed, all components are connected by the transport flow provided T^* is chosen large enough (without loss of generality) so that there are smooth weights $w_i \ge 0$ over $(0,T^*) \times \mathbb{T}^d \times \mathbb{S}^{d-1}$ with unit masses and $\operatorname{supp} w_i \subset \mathsf{U}_i \times \mathbb{S}^{d-1}$, and smooth compactly supported $\psi_{ij} = \psi_{ij}(t,x,v)$ solutions to $\partial_t \psi_{ij} - v \cdot \nabla_x \psi_{ij} = w_i - w_j$. Then integrating the equation on g against ψ_{ij} and using (2.5) shows that

$$\left| \int_{[0,T^*]\times\mathbb{T}^d} \langle g \rangle \langle w_i \rangle \,\mathrm{d}t\,\mathrm{d}x - \int_{[0,T^*]\times\mathbb{T}^d} \langle g \rangle \langle w_j \rangle \,\mathrm{d}t\,\mathrm{d}x \right| \lesssim \left(\int_0^{T^*} \mathcal{D}(g_t)\,\mathrm{d}t \right)^{1/2},$$

and arguing as in the proof of (4.9) we can prove that each $\int_{[0,T^*]\times\mathbb{T}^d} \langle g\rangle \langle w_i\rangle \,\mathrm{d}t\,\mathrm{d}x$ is close to $\langle\!\langle g\rangle\!\rangle_i$ up to an error of order $\left(\int_0^{T^*} \mathcal{D}(g_t)\,\mathrm{d}t\right)^{1/2}$.

We then construct ψ as in (4.7) and, using the zero mass condition (2.1), we can argue as above to prove, for some constants $\alpha_1, \dots, \alpha_k > 0$,

$$\alpha_1 \langle\!\langle g\rangle\!\rangle_1 + \cdots + \alpha_k \langle\!\langle g\rangle\!\rangle_k \lesssim \left(\int_0^{T^*} \mathcal{D}(g_t)\,\mathrm{d}t\right)^{1/2}.$$

Together with the control on the differences between the $\langle\!\langle g\rangle\!\rangle_i$'s this implies the result.

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