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Paul Dario

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Institut des hautes études scientifiques
Le Bois-Marie • Route de Chartres
F-91440 BURES-SUR-YVETTE
http://www.ihes.fr/

Centre de mathématiques Laurent Schwartz
CMLS, École polytechnique, CNRS, Université Paris-Saclay
F-91128 PALAISEAU CEDEX
http://www.math.polytechnique.fr/

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Quantitative hydrodynamic limits of the Langevin dynamics for gradient interface models

Paul Dario

Abstract

We review some recent, quantitative, progress regarding the large-scale behavior of the $\nabla\phi$ (or Ginzburg-Landau) interface model with uniformly convex potential. The arguments rely on a dynamical approach of the problem (following [32] where the Langevin dynamics associated with the model are studied) combined with recent progress in the field of quantitative stochastic homogenization of nonlinear elliptic equations.

1. Introduction

Random surfaces in statistical mechanics are used to model the interface separating two pure thermodynamic phases. A classical way to model an interface is to represent it as a function $\phi : \mathbb{Z}^d \to \mathbb{R}$ encoding its height (see Figure 1.1) to which one associates an energy defined as follows. On a finite set $\Lambda \subset \mathbb{Z}^d$, each surface $\phi : \Lambda \to \mathbb{R}$ satisfying the Dirichlet boundary condition $\phi = 0$ on $\partial\Lambda$ (the external vertex boundary) is assigned the energy

$$H_\Lambda(\phi) = \sum_{\substack{x,y\in\Lambda^+ \\ |x-y|=1}} V(\phi(y) - \phi(x)),$$

where Λ^+ is the set $\Lambda \cup \partial\Lambda$, $|\cdot|$ denotes the Euclidean norm, $V : \mathbb{R} \to \mathbb{R}$ is an interaction potential.

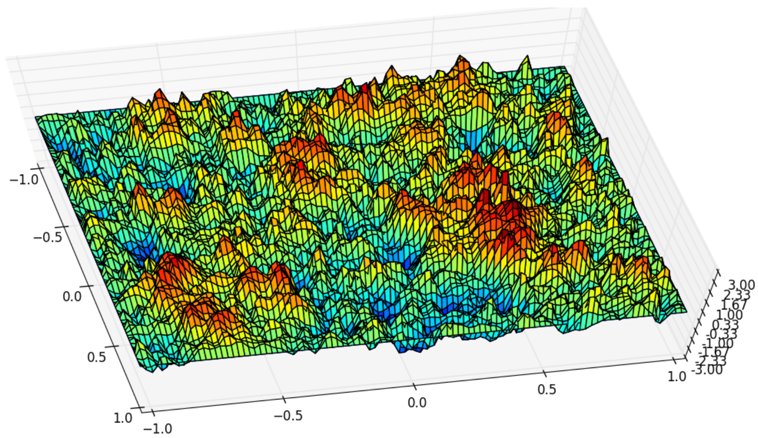

Figure 1.1: A representation of a random surface (by C. Gu).

Different assumptions can be imposed on the potential V. In this report, we focus on the uniformly convex setting, i.e., we assume that:

- V is twice continuously differentiable and symmetric;
- V is uniformly convex, i.e., there exist two constants λ, Λ such that

$$0 < \lambda \leq V''(x) \leq \Lambda < \infty.$$

The law of the random surface is then given by

$$\mu_\Lambda(d\phi) := \frac{1}{Z_\Lambda} \exp\left(-H_\Lambda(\phi)\right) \prod_{v \in \Lambda} d\phi(v), \tag{1.1}$$

where $d\phi(v)$ denotes Lebesgue measure on $\mathbb{R}$ and Z_Λ is the constant which makes μ_Λ a probability measure. The question is then to study the macroscopic behavior of a surface ϕ sampled according to μ_Λ.

The problem admits a dynamical interpretation, and the Gibbs measure (1.1) is naturally associated with Langevin dynamics

$$d\phi(t,x) = \sum_{\substack{y \in \mathbb{Z}^d \\ |y-x|=1}} V'(\phi(t,y) - \phi(t,x))\, dt + \sqrt{2} dB_t(x), \tag{1.2}$$

where $\left\{B_t(x) \,:\, x \in \mathbb{Z}^d, t \geq 0\right\}$ is a family of independent Brownian motions. Specifically, the measure μ_Λ is stationary, reversible and ergodic with respect to the dynamics (1.2).

The qualitative behavior of the $\nabla\phi$-model has been extensively studied under the uniform ellipticity assumption, and we refer to [31, 49, 47] and to Section 3.1 for an overview of its literature.

We are interested in studying quantitatively the large-scale behavior of the Langevin dynamics (1.2) and draw a parallel with the recent progress in quantitative stochastic homogenization of nonlinear equations [8, 4, 3, 29, 21]. The rest of this report presents the framework of (nonlinear) stochastic homogenization as well as some results and contributions, and explains how the techniques developed in this setting can be used to study the Langevin dynamics (1.2).

2. Stochastic homogenization

2.1 The linear setting and historical background

Stochastic homogenization aims at understanding partial differential equations with rapidly varying random coefficients. An important part of the literature on the topic is devoted to the linear, uniformly elliptic equation in divergence form

$$-\nabla \cdot \mathbf{a}(x)\nabla u = f \text{ in } \mathbb{R}^d, \tag{2.1}$$

where the coefficient field $\mathbf{a} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is assumed to be random and to vary on the unit scale. One is then interested in studying the behavior of the solutions of the equation on a length scale much larger than the unit scale.

It is customary to introduce a small parameter $0 < \varepsilon \ll 1$ to represent the ratio between the microscopic and macroscopic scale, and to rescale the equation (2.1) as

$$-\nabla \cdot \mathbf{a}\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon = f \text{ in } \mathbb{R}^d.$$

The question is then to understand the behavior of the function u^ε as $\varepsilon \to 0$.

It was proved in the 80s, under quite general assumption over the coefficient field, that the function u^ε converges in L^2 to the solution of the elliptic equation

$$-\nabla \cdot \bar{\mathbf{a}} \nabla \bar{u} = f \text{ in } \mathbb{R}^d,$$

where $\bar{\mathbf{a}}$ is a constant, deterministic and uniformly elliptic matrix called the *homogenized coefficient* (except in $d = 1$ and in some specific situations in two dimensions, this coefficient is not explicit).

The theory has, since then, been extensively developed, and homogenization has been applied to various class of elliptic and parabolic equations as well as different models of probability and statistical physics. The past ten years have seen the emergence of a quantitative theory that has been developed to the point that the model is now well-understood (at least in the linear setting (2.1)).

To give a more detailed historical background, the theory of stochastic homogenization was developed qualitatively in the 1980s, in the works of Kozlov [42], Papanicolaou and Varadhan [46], Yurinskiĭ [50], Avellaneda and Lin [10, 11], and Dal Maso and Modica [24, 25] (in the nonlinear setting). We refer to [41] for a more detailed bibliographical account of the qualitative theory.

From a quantitative perspective, major progress was achieved by Gloria and Otto in [36, 37], where a satisfactory quantitative theory was developed for the first time. These results were then further developed by Gloria, Marahrens, Neukamm and Otto [38, 39, 34, 35]. Another quantitative approach was initiated by Armstrong and Smart in [8], and further developed by Armstrong, Kuusi and Mourrat [5, 6, 7].

The nonlinear setting, to which Section 2.2 is devoted, was studied qualitatively in [24, 25] and quantitatively in the recent contributions of Armstrong, Smart [8], Armstrong, Ferguson, Kuusi [4, 3], Fischer, Neukamm [29] and Gloria, Clozeau [21].

2.2 The nonlinear setting

The standard problem of stochastic homogenization of nonlinear elliptic equations studies the equations of the form

$$\nabla \cdot D_p L(x, \nabla u) = f \text{ in } \mathbb{R}^d, \tag{2.2}$$

where $L : (x,p) \mapsto L(x,p)$ is called *the Lagrangian* and is assumed to satisfy the following assumptions

- The map L is measurable with respect to the variable x and twice continuously differentiable with respect to the variable p;
- For any $x \in \mathbb{R}^d$, the map $p \mapsto L(x,p)$ is uniformly convex.

The solution of the equation (2.2) can be equivalently characterized as the minimizer of the convex energy functional

$$\int_{\mathbb{R}^d} L(x, \nabla u(x)) - f(x)u(x)\, dx.$$

One then encodes randomness in the model through the Lagrangian as follows: we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a collection of operators $(T_x)_{x\in\mathbb{Z}^d} : \Omega \to \Omega$ satisfying the following properties:

- *The composition rule:* For any pair $x, y \in \mathbb{Z}^d$, $T_x \circ T_y = T_{x+y}$;

- *Translation invariance and ergodicity:* For any $x \in \mathbb{Z}^d$, the measure $\mathbb{P}$ is translation-invariant and ergodic with respect to the operator T_x.

One then considers a (random) Lagrangian to be a map $L : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfying the identity

$$\forall (\omega, x, y, p) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d, \quad L(T_y \omega, x, p) = L(\omega, x + y, p).$$

The objective is then to study the behavior of the solution of the random elliptic equation

$$-\nabla \cdot D_p L \left(\omega, \frac{x}{\varepsilon}, \nabla u^\varepsilon \right) = f \text{ in } \mathbb{R}^d, \tag{2.3}$$

as ε tends to 0. This framework was originally considered by Dal Maso and Modica in [24, 25], where they established (under more general assumptions) that there exists a uniformly convex function $\bar{L} : \mathbb{R}^d \to \mathbb{R}$ such that, with probability one, the solution u^ε of (2.3) converges as ε tends to 0 to the deterministic solution $\bar{u}$ of the nonlinear elliptic equation

$$-\nabla \cdot D_p \bar{L} \left(\nabla \bar{u} \right) = f \;\text{ in }\; \mathbb{R}^d.$$

Under suitable quantitative ergodicity assumptions on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a quantitative version of the homogenization theorem stated above has been obtained in [8, 4, 3, 29]. We record below a finite-volume version of the result stated with Dirichlet boundary conditions (the version below is the one of [29, Th. 7]). The theorem holds under specific quantitative ergodicity assumptions that are not made explicit here; they can be found in [29].

Before stating the result, we introduce the following notation to measure the stochastic integrability of a random variable: given a nonnegative random variable X, an exponent $s > 0$ and a constant $C < \infty$, we write

$$X \leq \mathcal{O}_s(C) \quad \Leftrightarrow \quad \mathbb{E}\left[\exp\left(\left(\frac{X}{C} \right)^s \right) \right] \leq 2.$$

Theorem 2.1 (Quantitative homogenization, Theorem 7 of [29])**.** *For any C^1 or convex Lipschitz bounded domain $D \subseteq \mathbb{R}^d$, and any boundary condition $g \in H^2(D)$, there exist an exponent $s := s(d, \lambda, \Lambda) > 0$ and a constant $C := C(d, \lambda, \Lambda, D, g) < \infty$ such that the solutions of the equations*

$$\begin{cases} -\nabla \cdot D_p L \left(\omega, \frac{x}{\varepsilon}, \nabla u^\varepsilon \right) = 0 & \text{in } D, \\ u^\varepsilon = g & \text{in } \partial D, \end{cases} \tag{2.4}$$

and

$$\begin{cases} -\nabla \cdot D_p \bar{L} \left(\nabla \bar{u} \right) = 0 & \text{in } D, \\ \bar{u} = g & \text{in } \partial D, \end{cases}$$

satisfy

$$\left\| u^\varepsilon - \bar{u} \right\|_{L^2(D)} \leq \mathcal{O}_s \left(C \varepsilon^{1/2} \left(1 + \left| \ln \varepsilon \right|^{1/4} \right) \right).$$

The proof of Theorem 2.1 relies on a standard tool and Ansatz in stochastic homogenization: the first-order corrector and the two-scale expansion. These are explained in more detail in the next section.

2.3 First-order corrector and two-scale expansion

This section is devoted to the first-order corrector and the two-scale expansion. It is split into two parts. We first introduce the first-order corrector and record two of its properties. Section 2.3.2 presents the Ansatz of the two-scale expansion. We mention that the first-order corrector and the two-scale expansion have been studied in various contexts. Different definitions have been used for the first-order corrector and some technicalities are required to make the general Ansatz of the two-scale expansion fully rigorous. In the sections below, the arguments are only presented on a general level to highlight the main ideas.

2.3.1. The first-order corrector. In order to define the first order-corrector, we simplify the setup of Theorem 2.1 and consider a specific case: the situation where the boundary condition g is affine. For $p \in \mathbb{R}^d$, we let l_p be the affine function of slope p, i.e., $l_p(x) = p \cdot x$. In this setting, we have $\bar{u} = l_p$. For each $\varepsilon > 0$, we define the first-order corrector with slope $p \in \mathbb{R}^d$ according to the formula

$$\phi^\varepsilon(x;p) := u^\varepsilon(x) - p \cdot x.$$

One is then interested in establishing two properties on the first-order corrector.

- *Estimates on the corrector:* typically, one wants to show that, for any $x \in D$,

$$\phi^\varepsilon(x;p) \underset{\varepsilon \to 0}{\longrightarrow} 0, \tag{2.5}$$

  and to quantify this result with respect to both the stochastic integrability and the rate of convergence. In this direction, the optimal result should be

$$|\phi^\varepsilon(x;p)| \leq \mathcal{O}_2\left(\varepsilon\big(1 + |\log \varepsilon|^{1/2} \mathbf{1}_{\{d=2\}}\big)\right).$$

  A result of this nature can be found in [38] and [6, Chapter 4] in the linear setting.

- *Estimates on the flux of the corrector:* typically, one wants to show that

$$D_p L\left(\omega, \frac{\cdot}{\varepsilon}, \nabla\phi^\varepsilon\right) \underset{\varepsilon \to 0}{\rightharpoonup} D_p \bar{L}(p), \tag{2.6}$$

  where the arrow refers to the weak convergence in the space $H^1(D)$. Similarly, this convergence can be quantified both over the stochastic integrability and rate of convergence. In this direction, the optimal result should be

$$\left\| D_p L\left(\omega, \frac{\cdot}{\varepsilon}, \nabla\phi^\varepsilon\right) - D_p \bar{L}(p) \right\|_{H^{-1}(D)} \leq \mathcal{O}_2(C\varepsilon),$$

  where the $H^{-1}(D)$-norm is defined by the formula

$$\|f\|_{H^{-1}(D)} := \sup\left\{ \int_D f(x) u(x)\, dx \, : \, u \in H_0^1(D),\ \|\nabla u\|_{L^2(D)} \leq 1 \right\}.$$

  We do not present here the details of the mathematical construction and refer the interested reader to [8, 4, 3, 29].

2.3.2. The two-scale expansion. Once equipped with the first-order corrector, one defines the two-scale expansion w^ε according to the formula

$$w^\varepsilon(x) := \bar{u}(x) + \phi^\varepsilon(x; \nabla \bar{u}(x)).$$

Two properties can then be proved on the two-scale expansion w^ε.

- From the definition of the first-order corrector and the property (2.5), we know that the map w^ε is close to the map $\bar{u}$. Specifically, we have

$$|w^\varepsilon(x) - \bar{u}(x)| = |\phi^\varepsilon(x, \nabla \bar{u}(x))| \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{2.7}$$

- Using the definition of the two-scale expansion w^ε together with an explicit computation (which is not presented here to keep the details light), one can prove that the function w^ε is almost a solution of the heterogeneous equation (2.4). Specifically, one can verify that the convergences (2.5) and (2.6) imply that the term

$$\mathcal{E}^\varepsilon := -\nabla \cdot D_p L \left(\omega, \frac{\cdot}{\varepsilon}, \nabla w^\varepsilon \right)$$

satisfies

$$\|\mathcal{E}^\varepsilon\|_{H^{-1}(D)} \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{2.8}$$

Moreover, the convergence (2.8) can be quantified depending on the rates of convergence in (2.5) and (2.6). We can then use the identity (2.4) to prove that

$$-\nabla \cdot \left(D_p L \left(\omega, \frac{\cdot}{\varepsilon}, \nabla w^\varepsilon \right) - D_p L \left(\omega, \frac{\cdot}{\varepsilon}, \nabla u^\varepsilon \right) \right) = \mathcal{E}^\varepsilon. \tag{2.9}$$

Multiplying the equation (2.9) by the function $w^\varepsilon - u^\varepsilon$, integrating over the domain D and performing an integration by parts, we obtain

$$\begin{aligned} \int_D \left(D_p L \left(\omega, \frac{x}{\varepsilon}, \nabla w^\varepsilon \right) - D_p L \left(\omega, \frac{x}{\varepsilon}, \nabla u^\varepsilon \right) \right) \cdot \left(\nabla w^\varepsilon(x) - \nabla u^\varepsilon(x) \right) \, dx & \\ = \int_D \mathcal{E}^\varepsilon \left(w^\varepsilon(x) - u^\varepsilon(x) \right) & \\ \leq \|\mathcal{E}^\varepsilon\|_{H^{-1}(D)} \|\nabla w^\varepsilon - \nabla u^\varepsilon\|_{L^2(D)}, & \end{aligned}$$

where, in the last inequality, we used that the functions w^ε and u^ε are equal on the boundary of D and thus $w^\varepsilon - u^\varepsilon \in H^1_0(D)$. The term in the left-hand side can be estimated from below using the uniform convexity of the Lagrangian L, and we have

$$\begin{aligned} & \lambda \|\nabla w^\varepsilon - \nabla u^\varepsilon\|^2_{L^2(D)} \\ & \quad \leq \int_D \left(D_p L \left(\omega, \frac{x}{\varepsilon}, \nabla w^\varepsilon \right) - D_p L \left(\omega, \frac{x}{\varepsilon}, \nabla u^\varepsilon \right) \right) \cdot \left(\nabla w^\varepsilon(x) - \nabla u^\varepsilon(x) \right) \, dx. \end{aligned}$$

A combination of the two previous displays shows that

$$\|\nabla w^\varepsilon - \nabla u^\varepsilon\|_{L^2(D)} \leq \lambda^{-1} \|\mathcal{E}^\varepsilon\|_{H^{-1}(D)}.$$

By Poincaré's inequality, we deduce that

$$\|w^\varepsilon - u^\varepsilon\|_{L^2(D)} \leq C \|\nabla w^\varepsilon - \nabla u^\varepsilon\|_{L^2(D)} \leq C \|\mathcal{E}^\varepsilon\|_{H^{-1}(D)}.$$

Consequently

$$\begin{aligned}\|u^\varepsilon - \bar{u}\|_{L^2(D)} &\leq \|w^\varepsilon - u^\varepsilon\|_{L^2(D)} + \|w^\varepsilon - \bar{u}\|_{L^2(D)} \\ &\leq \|\phi^\varepsilon(x, \nabla \bar{u})\|_{L^2(D)} + C \|\mathcal{E}^\varepsilon\|_{H^{-1}(D)}.\end{aligned}$$

Applying (2.7) and (2.8) shows that the right-hand side tends to 0 as ε tends to 0.

3. Random surfaces and Langevin dynamics

In this section, we explain how the techniques of developed in the setting of the homogenization of the nonlinear equation (2.2) can be applied to study the Langevin dynamics (1.2). We first present a short historical account of the study of the model.

3.1 Historical background

The study of random surfaces was initiated in the 1970s by Brascamp, Lieb and Lebowitz [18] who obtained sharp localization and delocalization estimates for uniformly convex potentials.

The question of the scaling limit of the model was first addressed by Brydges and Yau [19] in a perturbative setting based on a renormalization group approach, and settled in the uniformly convex setting by the works of Naddaf, Spencer [44] and Giacomin, Olla, Spohn [33]. The hydrodynnamic limit of the model, discussed in this note, was established by Funaki and Spohn [32], with subsequent refinements of Nishikawa [45]. After the groundbreaking works [32, 44, 33], large deviation estimates and concentration inequalities were established by Deuschel, Giacomin and Ioffe [27], and sharp decorrelation estimates for the discrete gradient of the field were obtained by Delmotte and Deuschel [26]. The fluctuations of the model in a bounded domain were studied by Miller [43]. More recently, Armstrong and Wu applied quantitative homogenization to the Helffer-Sjöstrand PDE of [44] to prove the C^2 regularity of the surface tension.

The case of non-convex potentials was studied in the high temperature regime by Cotar, Deuschel and Müller [23], who established the strict convexity of the surface tension, and by Cotar and Deuschel [22] who proved the uniqueness of ergodic Gibbs measures, obtained sharp estimates on the decay of covariance and identified the scaling limit of the model in this framework (see also [28] where the hydrodynamic limit is established). The strict convexity of the surface tension in the low temperature regime was established by Adams, Kotecký and Müller [1] through a renormalization group argument. In [12], Biskup and Kotecký showed the possible non-uniqueness of infinite-volume, shift-ergodic gradient Gibbs measures for some nonconvex interaction potentials, and Biskup and Spohn [13] proved that, for a general category of nonconvex potentials, the scaling limit of the model is a Gaussian free field.

3.2 The Langevin dynamics as a stochastic homogenization problem

The starting point of the analysis is the observation that the Langevin dynamics (1.2) can be viewed as a (discrete) nonlinear parabolic equation with noise. One should mentally make the replacement

$$\sum_{\substack{y\in\mathbb{Z}^d \\ |y-x|=1}} V'(\phi(t,y)-\phi(t,x)) \quad \leftrightsquigarrow \quad \nabla\cdot D_pL(\nabla\phi).$$

In fact, the difference $\phi(t,y)-\phi(t,x)$ corresponds to the discrete gradient, the sum is the discrete divergence operator, and the map V' plays the role of the gradient of the Lagrangian. With this interpretation of the problem, one can think of the dynamics (1.2) as being similar to the equation

$$\partial_t\phi - \nabla\cdot D_pL(\nabla\phi) = \text{“noise”}, \tag{3.1}$$

where the noise corresponds to the term involving the Brownian motions (1.2) and can be thought of as a discretized version (with respect to the space variable) of a space-time white noise. The equation (3.1) can then be interpreted as a discrete and parabolic version of the equation (2.3) where the randomness is not encoded in the Lagrangian but externally through a random noise. A similar observation was made recently by Cardaliaguet, Dirr and Souganidis [20], who proved a qualitative homogenization result for a continuum version of the Langevin dynamics.

3.3 The quantitative hydrodynamic limit

Once the observation that the Langevin dynmamics can be seen as a nonlinear parabolic equation with an external random noise has been made, one can investigate what is the version of the homogenization theorem discussed in Section 2 for this model. The result is known in the field of random surfaces as the *the hydrodynamic limit* and was originally established by Funaki and Spohn [32]. Their result is stated below, and requires to introduce a few notations. We let $\mathbb{T}^d$ be the d-dimensional torus. For any fixed $\varepsilon>0$, we discretize the torus at scale ε by setting

$$\mathbb{T}^d_\varepsilon := \mathbb{T}^d\cap\varepsilon\mathbb{Z}^d.$$

To ease the notation in the statement of the result, we also denote by

$$\nabla^\varepsilon\cdot V'(\nabla^\varepsilon\phi^\varepsilon)(t,x) = \frac{1}{\varepsilon}\sum_{\substack{y\in\varepsilon\mathbb{Z}^d \\ |y-x|=\varepsilon}} V'\left(\frac{\phi^\varepsilon(t,y)-\phi^\varepsilon(t,x)}{\varepsilon}\right).$$

The hydrodynamic limit for Langevin dynamics of the $\nabla\phi$-model reads as follows.

Theorem 3.1 (Hydrodynamic Limit, Funaki-Spohn [32]). *There exists a uniformly convex function $\bar\sigma:\mathbb{R}^d\to\mathbb{R}$ such that the following holds. For any $g\in L^2(\mathbb{T}^d)$ and any $\varepsilon\in(0,1)$, if we let $\phi^\varepsilon:[0,1]\times\mathbb{T}^d_\varepsilon\to\mathbb{R}$ be the solution of Langevin dynamics*

$$\begin{cases} d\phi^\varepsilon(t,x) = \nabla^\varepsilon\cdot V'(\nabla^\varepsilon\phi^\varepsilon)(t,x)dt + \sqrt{2}\varepsilon dB_{t/\varepsilon^2}\left(\frac{x}{\varepsilon}\right) & \text{for } (t,x)\in[0,1]\times\mathbb{T}^d_\varepsilon, \\ \phi^\varepsilon(0,x) = g(x) & \text{for } x\in\mathbb{T}^d_\varepsilon, \end{cases} \tag{3.2}$$

and $\bar\phi:Q\to\mathbb{R}$ be the solution of the continuous nonlinear parabolic equation

$$\begin{cases} \partial_t\bar\phi - \nabla\cdot D_p\bar\sigma(\nabla\bar\phi) = 0 & \text{in } [0,1]\times\mathbb{T}^d, \\ \bar\phi(0,\cdot) = g & \text{on } \mathbb{T}^d, \end{cases}$$

then, for any time $t>0$,

$$\mathbb{E}\left[\left\|\phi^\varepsilon(t,\cdot)-\bar\phi(t,\cdot)\right\|^2_{L^2(\mathbb{T}^d_\varepsilon)}\right] \underset{\varepsilon\to0}{\longrightarrow} 0.$$

Remark 3.2. 1. The random process $\varepsilon B_{t/\varepsilon^2}$ is a standard Brownian motion, we use this notation to emphasize that the dynamics (1.2) and (3.2) are the same up to suitable rescaling with respect to the space and time variables.

2. The map $\bar{\sigma}$ is called the *surface tension* [32] and corresponds to the effective Lagrangian $\bar{L}$ in stochastic homogenization.

3. The proof of Funaki and Spohn is based on techniques developed in the context of Ginzburg-Landau equations with a conserved order parameter [40], and is different from the approach based on stochastic homogenization developed in this note.

4. The result of [32] is proved on the torus, and their method has been extended to other settings such as the one of a bounded domain with Dirichlet boundary condition by Nishikawa [45].

From a quantitative perspective, it turns out that the model can be treated using the techniques developed in stochastic homogenization of the nonlinear elliptic equation (2.2), and a two-scale expansion can be implemented on the model to prove a quantitative version of the hydrodynamic limit.

In order to state the result, we introduce a few notations. We let D be a bounded C^1-domain, denote by $Q := [0,1] \times D$ the parabolic cylinder and let $\partial_{\mathrm{par}} Q := (\{0\} \times D) \cup ([0,1] \times \partial D)$ be the parabolic boundary of Q. For $\varepsilon > 0$, we discretize the set D by defining $D^\varepsilon := D \cap \varepsilon \mathbb{Z}^d$ and denote by ∂D^ε the external vertex boundary of D^ε. We define $Q^\varepsilon := [0,1] \times D^\varepsilon$ and $\partial_{\mathrm{par}} Q^\varepsilon := (\{0\} \times D^\varepsilon) \cup ([0,1] \times \partial D^\varepsilon)$.

Theorem 3.3 (Quantitative hydrodynamic limit, Theorem 1.1 of [2])**.** *Let $g \in H^2(Q)$. Fix $\varepsilon \in (0,1)$ and let $\phi^\varepsilon : Q^\varepsilon \to \mathbb{R}$ be the solution of the Langevin dynamics*

$$\begin{cases} d\phi^\varepsilon(t,x) = \nabla^\varepsilon \cdot V'(\nabla^\varepsilon \phi^\varepsilon)(t,x)dt + \sqrt{2}\varepsilon dB_{t/\varepsilon^2}\left(\frac{x}{\varepsilon}\right) & \text{for } (t,x) \in Q^\varepsilon, \\ \phi^\varepsilon(t,x) = g(t,x) & \text{for } (t,x) \in \partial_{\mathrm{par}} Q^\varepsilon, \end{cases}$$

and let $\bar{\phi} : Q \to \mathbb{R}$ be the solution of the continuous nonlinear parabolic equation

$$\begin{cases} \partial_t \bar{\phi} - \nabla \cdot D_p \bar{\sigma}(\nabla \bar{\phi}) = 0 & \text{in } Q, \\ \bar{\phi} = g & \text{on } \partial_{\mathrm{par}} Q. \end{cases}$$

Then, there exists a constant $C := C(d, \lambda, \Lambda, D, g) < \infty$ such that

$$\left\| \phi^\varepsilon - \bar{\phi} \right\|_{L^2(Q^\varepsilon)} \leq \mathcal{O}_2\left(C\varepsilon^{1/2}\big(1 + |\log \varepsilon|^{1/2} \mathbf{1}_{\{d=2\}}\big) \right).$$

3.4 Sketch of the proof of Theorem 3.3

As mentioned above the model can be treated using a two-scale expansion, and the sketch of the proof of Theorem 3.1 is essentially the one presented in Section 2.3. We will only focus on the first part of the proof: the definition and bounds of the first-order corrector associated with the Langevin dynamics.

3.4.1. Definition of the first-order corrector for the Langevin dynamics. The first-order corrector for the Langevin dynamics is defined below.

Definition 3.4 (First-order corrector for the Langevin dynamics)**.** For any $\varepsilon > 0$ and any $p \in \mathbb{R}^d$, the first-order corrector $\phi^\varepsilon(\cdot;p) : Q^\varepsilon \to \mathbb{R}$ with slope $p \in \mathbb{R}^d$ is defined to be the solution of the Langevin dynamics

$$\begin{cases} d\phi^\varepsilon(t,x;p) = \nabla^\varepsilon \cdot V'(p + \nabla^\varepsilon \phi^\varepsilon(\cdot;p))(t,x)dt + \sqrt{2}\varepsilon dB_{t/\varepsilon^2}\left(\frac{x}{\varepsilon}\right) & \text{for } (t,x) \in Q^\varepsilon, \\ \phi^\varepsilon(t,x;q) = 0 & \text{for } (t,x) \in \partial_{\mathrm{par}} Q^\varepsilon. \end{cases} \tag{3.3}$$

Bounds on the first-order corrector are then established in the following proposition. The details of the argument outlined below can be found in [2, Prop. 3.3].

Proposition 3.5 (Bounds on the first-order corrector)**.** *There exists a constant $C := C(d,\lambda,\Lambda) < \infty$ such that, for any $(t,x) \in Q^\varepsilon$,*

$$|\phi^\varepsilon(t,x;p)| \leq \mathcal{O}_2\left(C\varepsilon\big(1 + |\log\varepsilon|^{1/2}\mathbf{1}_{\{d=2\}}\big)\right).$$

Remark 3.6.
1. In Proposition 3.5, both the rate of convergence and the stochastic integrability are optimal.
2. In the stationary regime (i.e., when the dynamics are started from a random profile distributed according to the Gibbs measure (1.1)), the result is known as the Brascamp-Lieb inequality [17, 16] (see [30] for the inequality with Gaussian stochastic integrability).

3.4.2. Preliminary results. In order to sketch the proof of Proposition 3.5, we need to collect two preliminary results: the Gaussian concentration inequality and the Nash-Aronson estimate.

The Gaussian concentration inequality. The Gaussian concentration inequality [14, 48] (see also [15, Theorem 5.6]) provides strong concentration properties for Lipschitz functions of Gaussian random variables.

Proposition 3.7 (Gaussian concentration inequality [14, 48])**.** *Let $X_1, \ldots, X_n$ be independent Gaussian random variables with expectation 0 and variance 1. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function and denote by $|F|_{\mathrm{Lip}}$ its Lipschitz norm (i.e., the smallest constant such that $|F(x) - F(y)| \leq |F|_{\mathrm{Lip}} |x - y|$ for all $x, y \in \mathbb{R}^n$ where we used the Euclidian metric on $\mathbb{R}^n$). Then there exists an absolute constant $C < \infty$ such that*

$$|F - \mathbb{E}[F]| \leq \mathcal{O}_2(C\,|F|_{\mathrm{Lip}}).$$

The Nash-Aronson estimate. The second result we collect is the Nash-Aronson estimate which provides an upper bound on the heat kernel for linear parabolic equations with uniformly elliptic environment. To state the result, we fix $\varepsilon > 0$ and introduce two definitions. First, we define a time-dependent uniformly elliptic environment on the cylinder Q^ε to be a map $\mathbf{a} : [0,1] \times E(D^\varepsilon) \to [\lambda, \Lambda]$, where $E(D^\varepsilon)$ denotes the set of undirected edges of D^ε. We next define the discrete elliptic operator $\nabla^\varepsilon \cdot \mathbf{a}\nabla^\varepsilon$ according to the formula, for any function $u : Q^\varepsilon \to \mathbb{R}$,

$$\nabla^\varepsilon \cdot \mathbf{a}\nabla^\varepsilon u(t,x) = \frac{1}{\varepsilon^2} \sum_{\substack{y \in \varepsilon\mathbb{Z}^d \\ |y-x| = \varepsilon}} \mathbf{a}(t,\{x,y\})\,(u(t,y) - u(t,x)).$$

For any $(s, y) \in [0,1] \times D^\varepsilon$, we define the heat-kernel started at time s from the vertex y to be the solution of the linear parabolic equation

$$\begin{cases} \partial_t P_{\mathbf{a}}(\cdot,\cdot,s,y) - \nabla^\varepsilon \cdot \mathbf{a} \nabla^\varepsilon P_{\mathbf{a}}(\cdot,\cdot,s,y) = 0 & \text{in } Q, \\ P_{\mathbf{a}}(s,\cdot,s,y) = \delta_y & \text{in } D^\varepsilon, \\ P_{\mathbf{a}}(\cdot,\cdot,s,y) = 0 & \text{on } [0,1] \times \partial D^\varepsilon, \end{cases} \tag{3.4}$$

where δ_y denotes the discrete version of the Dirac δ function on a lattice with mesh size ε, i.e., the function which takes the value ε^{-d} at the vertex y and 0 everywhere else. The Nash-Aronson estimate, originally established in the continuous setting in [9], provides an upper bound on the heat-kernel $P_{\mathbf{a}}$ which does not depend on the environment $\mathbf{a}$. In the discrete setting considered here, the result can essentially be deduced from [33, Proposition B.3].

Proposition 3.8 (Nash-Aronson estimate)**.** *There exists a constant $C := C(d, \lambda, \Lambda) < \infty$ such that for any $\varepsilon > 0$ and any uniformly elliptic environment $\mathbf{a}$ defined on Q^ε, the heat kernel $P_{\mathbf{a}}$ defined in* (3.4) *satisfies*

$$P_{\mathbf{a}}(t,x,s,y) \le C t^{-d/2} \exp\Big(- \frac{|x-y|}{Ct^{1/2}} \Big).$$

3.4.3. Sketch of the proof of Proposition 3.5. The sketch of the proof proceeds in two steps. We first approximate the first-order corrector by discretizing the Brownian motions in the right-hand side of (3.3). The objective of this step is to obtain a good approximation of the first-order corrector which depends on finitely many independent Gaussian random variables. We then estimate the size of the discretized version of the first-order corrector by combining the Gaussian concentration inequality and the Nash-Aronson estimate.

Discretization of the Brownian motions. We first approximate the map ϕ^ε using a discretization of the Brownian motions. To this end, we fix $n \in \mathbb{N}$, $k \in \{0, \ldots, n\}$ and $x \in D^\varepsilon$, and set $B_t^\varepsilon(x) := \varepsilon B_{t/\varepsilon^2}(x/\varepsilon)$. We then define the increment $X_k^n(x)$ and the piecewise constant function X_n by the formulae

$$X_k^n(x) := B^\varepsilon_{(k+1)/n}(x) - B^\varepsilon_{k/n}(x) \quad \text{and} \quad X^n(t,x) = \sum_{k \in \mathbb{N}} X_k^n(x) \mathbf{1}_{\{k/n \le t \le (k+1)/n\}}.$$

We also denote by $\mathbb{X}^n := (X_k^n(x))_{k \in \{0,\ldots,n\}, x \in D^\varepsilon} \subseteq \mathbb{R}^{n|D^\varepsilon|}$ the collection of all the increments, and let $\phi^{\varepsilon,n}$ be the solution of the parabolic equation

$$\begin{cases} \partial_t \phi^{\varepsilon,n}(t,x) = \nabla^\varepsilon \cdot V'(\nabla^\varepsilon \phi^{\varepsilon,n})(t,x) dt + \sqrt{2} X^n\,(t,x) & \text{for } (t,x) \in Q^\varepsilon, \\ \phi^\varepsilon = 0 & \text{on } \partial_{\mathrm{par}} Q^\varepsilon. \end{cases} \tag{3.5}$$

One then observes that, as n tends to infinity, the map $\phi^{\varepsilon,n}$ converges almost surely (and uniformly over the parabolic cylinder Q^ε) to the first-order corrector ϕ^ε.

Concentration inequalities and regularity estimate. The advantage of the discretization scheme is that the dynamics $\phi^{\varepsilon,n}$ depends only on the collection of increments $\mathbb{X}^n$, that is, on finitely many independent Gaussian random variables.

The strategy is then to consider the map $\phi^{\varepsilon,n}(t,y)$ as a function of the collection of increments $\mathbb{X}^n$, and to apply the Gaussian concentration inequality stated in Proposition 3.7. To this end, we need to prove that the map $\phi^{\varepsilon,n}(t,x)$ is Lipschitz and estimate its Lipschitz norm. By the definition of the Lipschitz norm, we have the identity

$$|\phi^{\varepsilon,n}(t,y)|^2_{\mathrm{Lip}} := \sup_{\mathbb{X}^n \in \mathbb{R}^{n|D^\varepsilon|}} \sum_{k \in \{0,\dots,n\}} \sum_{x \in D^\varepsilon} \left| \frac{\partial \phi^{\varepsilon,n}(t,x)}{\partial X^n_k(x)} \right|^2,$$

where $\partial \phi^{\varepsilon,n}(t,y)/\partial X^n_k(x)$ denotes the partial derivative of the map $\phi^{\varepsilon,n}(t,y)$ with respect to $X^n_k(x)$ and the supremum is considered over all the possible values of the collection of increments $\mathbb{X}^n \in \mathbb{R}^{n|D^\varepsilon|}$.

The right-hand side can then be estimated as follows: we fix $x \in D^\varepsilon$, $k \in \{1,\dots,n\}$ and denote by $w := \partial \phi^{\varepsilon,n}/\partial X^n_k(x)$. Differentiating both sides of (3.5) with respect to the increment $X^n_k(x)$, we obtain that the map w solves the linear parabolic equation

$$\begin{cases} \partial_t w(t,y) = \nabla^\varepsilon \cdot \mathbf{a} \nabla^\varepsilon w(t,y) dt + \sqrt{2} \mathbf{1}_{\{k/n \leq t \leq (k+1)/n\}} \mathbf{1}_{\{y=x\}} & \text{for } (t,x) \in Q^\varepsilon, \\ \qquad w = 0 & \text{on } \partial_{\mathrm{par}} Q^\varepsilon, \end{cases}$$

where the environment $\mathbf{a}$ is given by the formula $\mathbf{a}(t,e) = V''(\nabla^\varepsilon \phi(t,e))$. Duhamel's principle can then be applied to write the solution w in terms of the heat-kernel $P_{\mathbf{a}}$. Specifically, we have

$$w(t,y) = \sqrt{2} \varepsilon^d \int_{k/n}^{(k+1)/n} P_{\mathbf{a}}(t,y;s,x) \, ds.$$

The Nash-Aronson estimate yields an upper bound on the function w which is uniform over the set of increments $\mathbb{X} \in \mathbb{R}^{n|D^\varepsilon|}$. This bound can then be used to obtain an estimate on the Lipschitz norm of $\phi^{\varepsilon,n}(t,y)$. Proposition 3.5 then follows, after appropriate computations carried out in [2, Prop. 3.3], by applying the Gaussian concentration inequality.

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