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RESONANCES AND GENERICITY IN BIRKHOFF NORMAL FORMS

ERWAN FAOU

Abstract. This paper is based on the presentation done at the seminar Laurent Schwartz in January 2020. It describes and summarizes the results given in *Rational normal forms and stability of small solutions to nonlinear Schrödinger equations*, see [BFG20a], written with Joachim Bernier and Benoît Grébert, and published in Annals of PDE 6, article number: 14 (2020) 65p. We describe here the main arguments of the proof as well as the general strategy used in the Birkhoff normal form for Partial Differential Equations.

1. Context

We consider the following equations:

- The nonlinear Schrödinger equation

$$i\partial_t u = -\Delta u + \varphi(|u|^2)u, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}. \tag{NLS}$$

  where $\varphi : \mathbb{R} \to \mathbb{R}$ is analytic, with $\varphi(0) = 0$ but $\varphi'(0) \neq 0$.
- The Schrödinger-Poisson equation

$$\begin{aligned} &i\partial_t u = -\Delta u + Wu, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}, \\ &-\Delta W = |u|^2 - \frac{1}{2\pi}\int_{\mathbb{T}} |u|^2 \mathrm{d}x \implies W = V \star |u|^2, \end{aligned} \tag{NLSP}$$

  where V is an explicit potential associated with the Laplace operator on the torus.

The main goal is to describe the dynamics of small and smooth solutions to the previous equations. One of the main difficulty is the absence of dispersive estimates used to control the long time behavior of small solutions. Indeed on the torus the Laplace operator has a discrete spectrum producing resonance phenomena.

The case $\varphi = \mathrm{Id}$ corresponds to a very specific case where the equation is *integrable* (which holds true only in dimension 1) and it can be shown that the Sobolev norm of the solutions remain bounded for all times.

The previous equation possess a common Hamiltonian structure ensuring in each case the preservation of an energy:

- Energy of (NLS)

$$H_{\mathrm{NLS}}(u, \overline{u}) = \int_{\mathbb{T}} |\nabla u|^2 + g(|u|^2)\mathrm{d}x \in \mathbb{R} \quad (g' = \varphi),$$

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- Energy of (NLSP)

$$H_{\text{NLSP}}(u,\overline{u}) = \int_{\mathbb{T}} |\nabla u|^2 + \frac{1}{2}(V \star |u|^2)|u|^2 \mathrm{d}x \in \mathbb{R}.$$

A general hamiltonian system associated with such an energy is written

$$i\partial_t u = \partial_{\overline{u}} H(u,\overline{u}),$$

and the Poisson bracket is defined as follows: For $G(u,\overline{u}) \in \mathbb{R}$

$$\{G,H\} = 2\mathrm{Im}\, \partial_u G \partial_{\overline{u}} H = \frac{\mathrm{d}}{\mathrm{d}t} G(u(t),\overline{u}(t)).$$

Concerning the evolution of small solutions, the Hamiltonians can be viewed as perturbations of the linear part:

$$H = H_2 + P, \quad H_2 = \int_{\mathbb{T}} |\nabla u|^2 = \mathcal{O}(u^2), \quad P = \mathcal{O}(u^4).$$

The solution of the linear flow is easily given in Fourier.

$$i\partial_t u = -\Delta\psi \quad \Longleftrightarrow \quad \forall\, k \in \mathbb{Z}, \quad i\partial_t u_k = \omega_k u_k, \quad \begin{cases} \omega_k = k^2 \\ u = \sum_{k\in\mathbb{Z}} u_k e^{ikx}. \end{cases}$$

The solution is thus given by $u_k(t) = e^{-it\omega_k} u_k(0)$, and we have the preservation of the *actions*

$$\forall\, k \in \mathbb{Z} \quad I_k(t) := |u_k(t)|^2 = I_k(0).$$

This preservation property implies the preservation of the Sobolev norm

$$\|\psi\|_{H^s}^2 = \sum_{k\in\mathbb{Z}} (1+|k|^{2s})|u_k|^2.$$

Note that from the dynamical system point of view, this linear Hamiltonian is an integrable system.

$$H_2(u,\overline{u}) = \sum_{k\in\mathbb{Z}} \omega_k I_k, \quad \text{with} \quad \{I_k, I_\ell\} = 0, \quad k,\ell \in \mathbb{Z}^d.$$

Natural questions are therefore the persistence of integrability by nonlinear perturbations, yielding to normal form questions classical in perturbation theory of dynamical systems (Poincaré, Linstedt, Birkhoff, Nekhoroshev, KAM Theory).

2. Birkhoff normal form with external parameters

Many results exists in the situation where some external parameters are present. A typical example is given by the following popular model:

(1) $$i\partial_t u = -\Delta u + W \star u + \varphi(|u|^2)u, \ x \in \mathbb{T}^d, \ d \geq 1.$$

Indeed when the action of the potential is convolutive, it is diagonal in Fourier which simplifies the analysis. A typical result can be stated as follows:

Theorem 2.1 (Bambusi & Grébert 06, [BG06])**.** *For a generic potential W, $\forall\, r$, $\exists\, s, \varepsilon_0$ such that for $\varepsilon < \varepsilon_0$,*

(∗) $$\|u(0)\|_s \leq \varepsilon \Longrightarrow \|u(0)\|_s \leq 2\varepsilon, \quad t \leq \varepsilon^{-r}$$

The expression *generic* here means essentially that for *almost all* potential W satisfying some regularity conditions, the result holds true. This kind of result applies also to one-dimensional wave equations: for almost all m,

$$\partial_{tt} u = \Delta u - mu - \varphi(u)$$

we have a long time preservation of the Sobolev norm.

For the wave in dimension d, a similar result can be obtain with a loss of regularity in the high modes only, see (Bernier, Faou & Grébert 2020, [BFG20b]). Using some optimal truncation technics, Faou and Grébert (2013) proved the stability on exponentially long times for analytic solutions in such situations ($d \geq 1$), [FG13]. More recently, Biasco, Massetti, and Procesi (2018) got stability in Sobolev spaces for exponentially long times ($d = 1$), [BMP20].

In the case of the equation (1) the Hamiltonian is given by

$$H(u, \overline{u}) = H_2(I) + P(u, \overline{u}), \quad H_2 = \sum_{k \in \mathbb{Z}} \omega_k I_k, \quad \omega_k = |k|^2 + \widehat{W}_k.$$

In action-angle variables $u_k = \sqrt{I_k} e^{i\theta_k}$ the nonlinearity is written

$$\dot{I}_k = -\partial_{\theta_k} P(I, \theta), \qquad \dot{\theta}_k = \partial_{I_k} P(I, \theta),$$

and thus we have

$$\begin{aligned} P &= P_4 + P_6 + \cdots = \sum_{k_1 + k_2 - \ell_1 - \ell_2 = 0} a_{\boldsymbol{k\ell}} u_{k_1} u_{k_2} \overline{u}_{\ell_1} \overline{u}_{\ell_2} + \mathcal{O}_{H^s}(u^6) \\ &= \sum_{k_1 + k_2 - \ell_1 - \ell_2 = 0} a_{\boldsymbol{k\ell}} \sqrt{I_{k_1} I_{k_2} I_{\ell_1} I_{\ell_2}} e^{i(\theta_{k_1} + \theta_{k_2} - \theta_{\ell_1} - \theta_{\ell_2})} + \mathcal{O}(u^6). \end{aligned}$$

The general normal form strategy is to transform the system in order to eliminate the angles, and obtain terms depending only on $(I_k)_{k \in \mathbb{Z}^d}$, which means terms depending only on monomials such that $\{k_1, \dots, k_m\} = \{\ell_1, \cdots, \ell_m\}$. Such terms are integrable. In other words, the goal is to find a symplectic transformation τ such that

$$\left| \begin{aligned} &\|\tau(u) - u\|_{H^s} \leq C \|u\|^2_{H^s} \\ &H \circ \tau = H(I) + Z(I) + \mathcal{O}_{H^s}(u^{r+2}). \end{aligned} \right.$$

The main dynamical consequences are as follows: if we define $\psi(t) = \tau(u(t))$, then $\|u(0)\|_{H^s} = \varepsilon$ implies than $\|\psi(0)\|_{H^s} \lesssim \varepsilon$ for ε small enough. Now as we have $\forall\, k \in \mathbb{Z}^d$, $\{I_k, H_0(I) + Z(I)\} = 0$ which is another way to see the integrability of $Z(I)$, we deduce that:

- $|\psi_k(t)|^2 = |\psi_k(0)|^2 + \mathcal{O}_{H^s}(t\varepsilon^{r+1})$ in the new variables.
- $\|\psi(t)\|_{H^s} \lesssim \varepsilon$ for $t \leq \varepsilon^{-r}$.

This implies that

$$\|u(t)\|_{H^s} = \|\tau^{-1}(\psi(t))\|_{H^s} \lesssim \varepsilon \quad \text{for} \quad t \leq \varepsilon^{-r},$$

as well as preservation of the actions over very long times

$$\forall t \leq \varepsilon^{-r}, \quad \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} ||u_k(t)|^2 - |u_k(0)|^2| \lesssim \varepsilon^3.$$

Now to construct this transformation τ, the main tool in perturbation theory is to try to eliminate P_4 by seeking τ of the form $\tau = \varphi_\chi^1$, the flow of a polynomial Hamiltonian

$$\chi = \sum_{k_1+k_2-\ell_1-\ell_2=0} \chi_{\boldsymbol{k}\boldsymbol{\ell}} u_{k_1} u_{k_2} \overline{u}_{\ell_1} \overline{u}_{\ell_2}.$$

We calculate using Taylor expansion that

$$\begin{aligned}(H_2 + P) \circ \varphi_\chi &= H_2 + P + \{H_2 + P, \chi\} + \{\{H_2 + P, \chi\}, \chi\} + \cdots \\ &= H_2 + P_4 + \{H_2, \chi\} + \mathcal{O}(u^6).\end{aligned}$$

This naturally yields to the following *Cohomological equation*

$$\{H_2, \chi\} + P_2 = Z_2(I)$$

to be solved at the level of Hamiltonian functionals. Now for given monomial we calculate that

$$\{H_2, u_{k_1} u_{k_2} \overline{u}_{\ell_1} \overline{u}_{\ell_2}\} = i(\omega_{k_1} + \omega_{k_2} - \omega_{\ell_1} - \omega_{\ell_2}) u_{k_1} u_{k_2} \overline{u}_{\ell_1} \overline{u}_{\ell_2}.$$

Hence the solution of the equation is given by

$$\chi_{\boldsymbol{k}\boldsymbol{\ell}} = \frac{a_{\boldsymbol{k}\boldsymbol{\ell}}}{i(\omega_{k_1} + \omega_{k_2} - \omega_{\ell_1} - \omega_{\ell_2})} \quad \text{when} \quad \{k_1, k_2\} \neq \{\ell_1, \ell_2\}$$

and Z_2 is the part of P_4 corresponding to terms depending on the actions I_k and I_ℓ (when $\{k_1, k_2\} = \{\ell_1, \ell_2\}$). However, to be solvable and also to iterate the process, we require a *diophantine assumption* under the following form: When $\{k_1, \ldots, k_m\} \neq \{\ell_1, \cdots, \ell_m\}$

$$|\omega_{k_1} + \cdots \omega_{k_m} - \omega_{\ell_1} - \cdots - \omega_{\ell_m}| \geq \frac{\gamma}{\mu_3(\boldsymbol{k}, \boldsymbol{\ell})^\alpha}, \tag{2}$$

where $\mu_3(\boldsymbol{k}, \boldsymbol{\ell})$ is the third largest integer amongst $(k_1, \ldots, \ell_m)$.

Under such an assumption, we can proceed as follows: for a truncation index N we can perform the frequency decomposition

$$y = u_{\leq N} \quad \text{and} \quad Y = u_{>N},$$

and the Hamiltonian is given by

$$P = A(y) + B(y) \cdot Y + C(y)(Y, Y) + R(y, Y)$$

where R contains at least three high frequencies. Now it is important to notice that the vector field $X_R = i\partial_{\overline{u}} R$ satisfies

$$\|X_R(u)\|_{H^s} \leq C N^{-s} \|u\|_{H^s}^2,$$

as the Hamiltonian contains at least three terms with frequencies larger than N. In the end, we will thus take $N = \varepsilon^{-\frac{r}{s}}$ which will allow to neglect the effect of R in the analysis.

It is thus enough to consider a Hamiltonian system of the form

$$P = A(y) + B(y) \cdot Y + C(y)(Y, Y)$$

containing at most 2 frequencies larger that N. We can then construct iteratively the transformations as described above on these terms only (quadratic in the high frequencies), and when we estimate the vector field of the transformation using the diophantine condition (2), we obtain a control of the small denominator of the form

$$|\omega_{k_1} + \cdots \omega_{k_m} - \omega_{\ell_1} - \cdots - \omega_{\ell_m}| \geq \frac{\gamma}{N^\alpha} = \gamma \varepsilon^{\alpha r/s}.$$

Hence after each step of the construction, we gain ε and lose $\varepsilon^{\alpha r/s}$. In the end, we obtain a remainder of order $\varepsilon^{r+3-\alpha r^2/s} = \varepsilon^{r+2}$ for s large enough, and the main term remaining in the reduced Hamiltonian is the one depending only of the actions $Z(I)$.

3. Without external parameters

In contrast with (1) the analysis of the behavior of resonant systems of the form

$$i\partial_t u = -\Delta u + \phi(|u|^2)u,\ x \in \mathbb{T}^d,\ d \geq 1$$

has known recently many advances. Grébert and Thomann (2012) found counterexamples to (∗) for NLS quintic (i.e. $\varphi(x) = \pm x^2$), see [GT12]. Similarly, Carles and Faou (2012) obtained counterexamples to (∗) in dimension $d = 2$, see [CF12], and Colliander, Keel, Staffilani, Takaoka and Tao (2010) constructed solutions of (NLS) that are sparse and initially small in dimension $d = 2$, whose Sobolev norms become arbitrarily large, see [CKSTT10].

On the other hand, Kuksin and Pöschel (1996) constructed a Cantor set of quasi periodic solutions for NLS (KAM normal form), see [KP96], and Bourgain (2000) got the expected corollaries for NLS with *typically non-sparse* initial data ($d = 1$). [Bou00].

Note that the problem of finding solutions such that $\limsup_{t\to\infty} \|u(t)\|_{H^s} = +\infty$ is for the moment an open problem.

The main results obtained in [BFG20a] are the following:

Theorem 3.1 (Bernier, Faou & Grébert (2020))**.** *For all $r \geq 1$, for all $s \geq s_0(r)$, for all $\varepsilon \leq \varepsilon_0(r,s)$, there exists an open subset $\mathcal{C}_{\varepsilon,r,s}$ of $B_s(0,\varepsilon)$ on which there exists a canonical change of coordinates that puts (NLS) in normal form at order r:*

$$H_{NLS} \circ \tau(z) = H_2(I) + Z_r(I) + \mathcal{O}(z^{r+1}), \quad z \in \mathcal{C}_{\varepsilon,r,s}.$$

The main dynamical consequence is given by the following result.

Corollary 3.2. *On an open subset $\mathcal{V}_{\varepsilon,r,s}$ of $\mathcal{C}_{\varepsilon,r,s}$ the usual corollary holds:*

$$\left\{\begin{array}{l} u(0) \in \mathcal{V}_{\varepsilon,r,s} \\ \|u(0)\|_s \leq \varepsilon \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} |I_j(t) - I_j(0)| \leq \varepsilon^3 \langle j\rangle^{-2s} \\ \|u(t)\|_s \leq 2\varepsilon \end{array}\right. \quad \textit{for } t \leq \varepsilon^{-r}$$

Note that since the sets are open, the properties are stable, and we are left with the following question: How large is $\mathcal{V}_{\varepsilon,r,s}$?

Note that this set is a cylinder

$$\left\{\begin{array}{l} u \in \mathcal{V}_{\varepsilon,r,s} \\ \widehat{v}_k = e^{i\theta_k}\widehat{u}_k \end{array}\right\} \Longrightarrow v \in \mathcal{V}_{\varepsilon,r,s}.$$

To measure the size of this set, we consider a family of independent positive random variables $I = (I_k)_{k\in\mathbb{Z}}$ such that I_k^2 is uniformly distributed in $(0, \langle k\rangle^{-4s-8})$ and we associate random initial data

$$u^0 = \sum_{k\in\mathbb{Z}} \sqrt{I_k}\, e^{ikx}.$$

We first state the results for (NLS) which differ slightly from the results in the (NLSP) case.

Theorem 3.3 (NLS)**.** *There exists $\varepsilon_0 > 0$ such that*

$$\forall \varepsilon \leq \varepsilon_0,\ \mathbb{P}(\varepsilon u^0 \in \mathcal{V}_{\varepsilon,r,s}) \geq 1 - \varepsilon^{1/3}.$$

Note that in this case, there exist resonances between ε and I. However, they are scarse in the following sense:

Theorem 3.4 (NLS). *For all $0 \leq \varepsilon < \varepsilon_0$ and for all sequence $(x_n)_{n\in\mathbb{N}}$ of random variables uniformly distributed in $(0,1)$ and independent of I, there is a probability larger than $1-\varepsilon^{1/6}$ to realize I such that there is a probability larger than $1-\varepsilon^{1/6}$ to realize $(\varepsilon_n)_{n\in\mathbb{N}} := (\varepsilon 2^{-(n+x_n)})_{n\in\mathbb{N}}$ such that $\varepsilon_n u^0$ is non-resonant for all n. More formally, we have*

$$\mathbb{P}\left(\mathbb{P}\left(\forall n \in \mathbb{N},\ \varepsilon_n u^0 \in \mathcal{V}_{\varepsilon_n,r,s}\ |I\right) \geq 1-\varepsilon^{1/6}\right) \geq 1-\varepsilon^{1/6},$$

Note that the x_n are not necessarily independent. So we can choose $x_n = x_0$.

For (NLSP), the situation is more favourable.

Theorem 3.5 (NLSP). *For NLSP, for all ε_0 small enough*

$$\mathbb{P}(\forall \varepsilon \leq \varepsilon_0, \varepsilon u^0 \in \mathcal{V}_{\varepsilon,r,s}) \geq 1-\varepsilon_0^{1/3}.$$

In this case, contrary to (NLS), no resonance between ε and I can appear.

A consequence of this probabilistic analysis is that for (NLS) and (NLSP) we have that for initial data

$$u^0 = \sum_{k\in\mathbb{Z}} \sqrt{I_k}\, e^{i\theta_k}, \qquad \begin{cases} I_k^2 \text{ i.u.d. } \in (0,\langle k\rangle^{-4s-8}), \\ \theta_k \in (0,2\pi), \end{cases}$$

then for almost all u^0, the usual corollary holds for all $\varepsilon < \varepsilon_0(u^0)$. Here we control $\varepsilon(u^0)$ independently of r and s, and obtain a description of the flow.

4. Strategy of proof

We will now describe the main steps of the proof of our result. The details are given in [BFG20a].

First step: Resonant normal form

The Hamiltonian of (NLS) is written in Fourier

$$H_{NLS}(u) = \sum_{k\in\mathbb{Z}} k^2 I_k + \varphi'(0) \sum_{k_1+k_2=\ell_1+\ell_2} u_{k_1}u_{k_2}\overline{u}_{\ell_1}\overline{u}_{\ell_2} + \mathcal{O}(u^6).$$

First, we want to eliminate the quartic non integrable terms by constructing $\tau = \Phi_\chi^1$ such that

$$H_{NLS} \circ \tau = H_2 + Z_4(I) + \mathcal{O}(u^6).$$

But we have

$$H \circ \Phi_\chi^1 = H_2 + P_4 + \{H_2, \chi_4\} + \mathcal{O}(u^6),$$

so the equation to solve is, as in the non resonant case

$$P_4 + \{H_2, \chi_4\} = Z_4(I).$$

As above, the solution is given by

$$\chi_4 = -i \sum_{\substack{k_1+k_2=\ell_1+\ell_2 \\ k_1^2+k_2^2\neq\ell_1^2+\ell_2^2}} \frac{u_{k_1}u_{k_2}\overline{u}_{\ell_1}\overline{u}_{\ell_2}}{k_1^2+k_2^2-\ell_1^2-\ell_2^2} \quad \text{and} \quad Z_4 = \sum_{\substack{k_1+k_2=\ell_1+\ell_2 \\ k_1^2+k_2^2=\ell_1^2+\ell_2^2}} u_{k_1}u_{k_2}\overline{u}_{\ell_1}\overline{u}_{\ell_2}.$$

Now in dimension 1, there is the following algebraic miracle

$$\left.\begin{array}{r} k_1 + k_2 = \ell_1 + \ell_2 \\ k_1^2 + k_2^2 = \ell_1^2 + \ell_2^2 \end{array}\right\} \Longrightarrow \{k_1, k_2\} = \{\ell_1, \ell_2\},$$

from which we conclude that $Z_4 = Z_4(I)$. Note that in dimension 2 this fact does not hold, and a non trivial normal form term cannot be eliminated, which does depend on the angles and generates non trivial dynamics.

We can then iterate to eliminate all the non resonant terms: there exists τ such that

$$H_{\mathrm{NLS}} \circ \tau = H_2(I) + Z_4(I) + \sum_{n=3}^{r} \sum_{\substack{\sum k_j - \ell_j = 0 \\ \sum k_j^2 - \ell_j^2 = 0}} \alpha_{\boldsymbol{k}\boldsymbol{\ell}} \prod_{j=1}^{n} u_{k_j} \overline{u}_{\ell_j} + \mathcal{O}(u^{2r+1}).$$

that we can write under the form

$$H_{\mathrm{NLS}} \circ \tau = H_2(I) + Z_4(I) + \sum_{n=3}^{r} K_{2n}(u, \overline{u}) + \mathcal{O}(u^{2r+1})$$

where $K_{2n}(u, \overline{u})$ are *resonant* but *non integrable* terms.

In particular, K_6 contains non integrable terms (no miracle at order six!):

$$-1 + 3 + 4 = 0 + 1 + 5 \text{ and } (-1)^2 + 3^2 + 4^2 = 0^2 + 1^2 + 5^2.$$

Note that mimicking the strategy of the non resonant case, we can truncate $\mu_3(\boldsymbol{k}, \boldsymbol{\ell}) < N = \varepsilon^{-\frac{r}{s}}$. But note that in our situation, resonant monomials contain only terms that are bounded:

$$\mu_3(\boldsymbol{k}, \boldsymbol{\ell}) < N \Longrightarrow \mu_1(\boldsymbol{k}, \boldsymbol{\ell}) \leq N^2 \quad \text{(resonant monomials)}.$$

Step 2: Modulated frequencies

We now use an idea that already appears in Kuksin-Pöschel '96: We use the nonlinearity to avoid resonances. Indeed,

$$\begin{aligned} H_2(I) + Z_4(I) &= \sum_{k \in \mathbb{Z}} k^2 I_k + \varphi'(0) \Big(2 \|u\|_{L^2}^2 - \sum_{k \in \mathbb{Z}} I_k^2 \Big) \\ &= \sum_{k \in \mathbb{Z}} (k^2 + \omega_k(I)) I_k \end{aligned}$$

with $\omega_k(I) = 2\|u\|_{L^2}^2 - I_k$. We thus see the natural emergence of new modulated frequencies perturbed by internal parameters. In other words, the solution itself determines the parameters that generically avoid resonances.

Now to eliminate the sixth order terms, we have to solve the equation

$$\{\chi, Z_4\} + \sum_{\substack{k_1+k_2+k_3=\ell_1-\ell_2-\ell_3 \\ k_1^2+k_2^2+k_3^2=\ell_1^2-\ell_2^2-\ell_3^2}} a_{\boldsymbol{k}\boldsymbol{\ell}} u_{k_1} u_{k_2} u_{k_3} \overline{u}_{\ell_1} \overline{u}_{\ell_2} \overline{u}_{\ell_3} = Z_6(I).$$

By linearity, we just have to solve

$$\{\chi, Z_4\} = u_{k_1} u_{k_2} u_{k_3} \overline{u}_{\ell_1} \overline{u}_{\ell_2} \overline{u}_{\ell_3}$$

with $\{k_1, k_2, k_3\} \neq \{\ell_1, \ell_2, \ell_3\}$ and (k, ℓ) resonant, which means that we have the cancellation of the small denominator coming from the ω_k. So we have for this term

$$\{u_{k_1} u_{k_2} u_{k_3} \overline{u}_{\ell_1} \overline{u}_{\ell_2} \overline{u}_{\ell_3}, Z_4\} = i u_{k_1} u_{k_2} u_{k_3} \overline{u}_{\ell_1} \overline{u}_{\ell_2} \overline{u}_{\ell_3} (I_{k_1} + I_{k_2} + I_{k_3} - I_{\ell_1} - I_{\ell_2} - I_{\ell_3}).$$

Hence we set

$$\chi = -i\,\frac{u_{k_1}u_{k_2}u_{k_3}\overline{u}_{\ell_1}\overline{u}_{\ell_2}\overline{u}_{\ell_3}}{I_{k_1}+I_{k_2}+I_{k_3}-I_{\ell_1}-I_{\ell_2}-I_{\ell_3}}.$$

The main issue now is that we have *poles in the denominator*, and the vector field has to be controlled in ℓ^1_s with norm

$$\|u\|_s = \sum_{k\in\mathbb{Z}} \langle k\rangle^s |u_k|.$$

To control these small divisors, we proceed as follows: Let $\|u\|_s = \sum \langle k\rangle^s |I_k|^{1/2} \leq \varepsilon$. Then *generically* (the exact meaning being given by probabilistic arguments) we can expect

$$|I_{k_1}+I_{k_2}+I_{k_3}-I_{\ell_1}-I_{\ell_2}-I_{\ell_3}| \simeq |I_{\mu_{\min}(\boldsymbol{k},\boldsymbol{\ell})}| \simeq \varepsilon^2 \langle \mu_{\min}(\boldsymbol{k},\boldsymbol{\ell})\rangle^{-2s},$$

where $|\mu_{\min}(\boldsymbol{k},\boldsymbol{\ell})| = \min(|k_1|,|k_2|,|k_3|,|\ell_1|,|\ell_2|,|\ell_3|)$. Hence the main problem comes from the fact that we can loose 2s derivatives. In comparison with the classical case [BG06], it means that we loose N^{2s} instead of N^α with $\alpha << s$. This prevents the standard argument to be applied.

Defining $|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_{\min}|$, the worst term is of the form

$$\|X_\chi\|_s \leq |\mu_1|^s \frac{\langle\mu_2\rangle^{-s}\langle\mu_3\rangle^{-s}\langle\mu_4\rangle^{-s}\langle\mu_5\rangle^{-s}\langle\mu_6\rangle^{-s}}{\varepsilon^2\langle\mu_{\min}(k,\ell)\rangle^{-2s}} \prod_{j=2}^{6} |\langle\mu_j\rangle^s u_{\mu_j}|.$$

But this quantity is controlled and of order ε^3, because $|\mu_2| \geq \frac{1}{6}|\mu_1|$, since $\sum k_j - \ell_j = 0$.

With this construction, we can eliminate the non integrable sixth order terms:

$$H\circ\tau = Z_2(I) + Z_4(I) + Z_6(I) + \sum_{m=4}^{r-1} \widetilde{K}_{2m}(z) + \mathcal{O}(z^{2r+1})$$

where $\|X_{R_r}\| \lesssim \varepsilon^{2r-1}$ but $\widetilde{K}_{2m}$ is a rational fraction i.e. a sum of terms of the type

$$\frac{1}{f_{k,\ell}(I)} \prod_j u_{k_j}\overline{u}_{\ell_j}.$$

where $(\boldsymbol{k},\boldsymbol{\ell})$ is resonant. Note that this term is no longer a polynomial, and we have to define a suitable class of rational Hamiltonians for which we control the vector fields in ℓ^1_s.

In theory, we could iterate this construction and in practice, it is possible to solve the $8th$ order term with Z_4 but there are some $10th$ order terms that cannot be solve by this procedure. The Poisson bracket between χ_6 and Z_6 contains terms of the type

$$\frac{u_{k_1}u_{k_2}u_{k_3}\overline{u}_{\ell_1}\overline{u}_{\ell_2}\overline{u}_{\ell_3}I_0^4}{(I_{k_1}+I_{k_2}+I_{k_3}-I_{\ell_1}-I_{\ell_2}-I_{\ell_3})^2}.$$

The elimination of such a term would require the Hamiltonian

$$\frac{u_{k_1}u_{k_2}u_{k_3}\overline{u}_{\ell_1}\overline{u}_{\ell_2}\overline{u}_{\ell_3}I_0^4}{(I_{k_1}+I_{k_2}+I_{k_3}-I_{\ell_1}-I_{\ell_2}-I_{\ell_3})^3},$$

but its vector field cannot be bounded correctly in ℓ^1_s: We have

$$\|X_\chi\|_s \leq \varepsilon^7 |\mu_1|^s \frac{\langle\mu_2\rangle^{-s}\langle\mu_3\rangle^{-s}\langle\mu_4\rangle^{-s}\langle\mu_5\rangle^{-s}\langle\mu_6\rangle^{-s}}{(\mu_{\min}^{-2s})^3}.$$

To overcome this difficulty, we take advantage of the fact that Z_6 has a nice structure.

Step 3: Use of Z_6.

The main idea to finish the construction is to solve the homological equations with $Z_4 + Z_6$ instead of Z_4 alone. Now we calculate explicitly that

$$Z_6(I) = -\frac{1}{2} \sum_{k \neq j \in \mathbb{Z}} \frac{1}{(k-j)^2} \, I_k I_j^2 + \cdots$$

Hence we get that

$$Z_6 = \sum_k \widetilde{\omega}_k(I) I_k \quad \text{with} \quad \widetilde{\omega}_k(I) = -\sum_{j \neq k \in \mathbb{Z}} \frac{1}{(k-j)^2} I_j^2 + \cdots$$

so that

$$\widetilde{\omega}_k(I) \sim -\frac{I_0^2}{k^2} \quad \text{while} \quad \omega_k(I) \sim I_k \sim k^{-2s}.$$

It means that now generically, we have a bound of the type:

$$|\Omega(I)| \geq \left(\prod_{j=1}^{n} \langle \mu_j \rangle^{-6} \right) \max(\varepsilon^2 \langle \mu_{\min} \rangle^{-2s}, \varepsilon^4).$$

Note that a control involves an interaction between I and ε, but this small divisor control is enough to kill all the higher order terms with $Z_4 + Z_6$.

Summing up

We can summarize the construction of [BFG20a] as follows:

- Step 1: Resonant normal form to kill non resonant terms

$$H \circ \tau_0 = H_2(I) + Z_4(I) + \sum_{n=3}^{r} K_{2n}(u, \overline{u}) + \mathcal{O}(u^{2r+1})$$

  with terms K_{2n} that are resonant but not integrable.
- Step 2: Use $Z_4(I)$ to kill the non integrable terms in K_6

$$H \circ \tau_1 = H_2(I) + Z_4(I) + Z_6(I) + \sum_{n=4}^{r} \widetilde{K}_{2n}(u, \overline{u}) + \mathcal{O}(z^{2r+1}).$$

  The price to pay is that the Hamiltonians $\widetilde{K}_{2n}$ are no more polynomials which leads to rational normal forms.
- Step 3: Use $Z_4(I) + Z_6(I)$ to kill all remaining non integrable terms

$$H \circ \tau_2 = H_2(I) + Z_r(I) + \mathcal{O}(u^{2r+1}).$$

We also notice that in the (NLSP) case, the analysis is simpler. In this case we have

$$Z_4(I) = \sum_{k \neq \ell} \frac{I_k I_\ell}{(k-\ell)^2},$$

for the first term in the resonant normal form. But now

$$\omega_k(I) \simeq \frac{I_0}{k^2}$$

which means that Z_4 for NLSP similar to Z_6 for (NLS) and the analysis can be performed directly with this engine of construction.

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