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ABOUT BOLTZMANN'S H THEOREM FOR THE LANDAU EQUATION
AUTOUR DU THÉORÈME H DE BOLTZMANN

L. DESVILLETTES

ABSTRACT. We propose in this work new (and hopefully close to optimal) variants of entropy production estimates for the Landau equation, in terms of relative weighted Fisher information-like terms. We start by showing how the same kind of estimates can be obtained for a simpler 1D model, sometimes called Kac-Landau equation.

1. INTRODUCTION

Entropy production estimates (sometimes also called entropy/entropy production, or entropy/entropy dissipation estimates) are functional inequalities which relate two quantities $D(f) \geq 0$ (entropy production, or entropy dissipation) and $H(f) - H(M) \geq 0$ (relative entropy), where D and H are functionals which can involve integral and derivatives of their argument f. Those inequalities write in general

$$D(f) \geq C(f)\,(H(f) - H(M)), \tag{1}$$

where $C(f) \geq 0$ also involves quantities related to f (like moments, L^p norms, or norms involving derivatives), and M is a given function.

Those estimates are useful in situations in which one considers an "autonomous" evolution equation

$$\partial_t f = A(f), \tag{2}$$

where A can involve derivatives (case of a PDE) or integrals (case of an integral equation), or at the same time both derivatives and integrals (as in the cases considered in this paper), and when the solutions of this equation satisfy an entropy inequality

$$\partial_t H(f) = -D(f). \tag{3}$$

If the term $C(f)$ which appears in eq. (1) involves only quantities which are conserved (or more generally if $C(f)$ is bounded) in the evolution of eq. (2), then Gronwall's lemma ensures that the solutions of eq. (2) converge exponentially fast to M in terms of entropy, that is $H(f)$ converges to $H(M)$ exponentially fast. Often the (exponential) convergence of f to M in some norm (typically L^1) can then be deduced from the (exponential) convergence in terms of entropy, thanks to specific inequalities (such as Cźiszar-Kullback-Pinsker inequality, cf. [5], [16]).

Typical cases in which the method described above can be used in order to study the large time behavior of a given equation can be found (among others) in the theory of parabolic PDEs (cf. [19] for example), of linear integral equations (cf. [18]

for example), and of coagulation-fragmentation equations (cf. [1] for example). In the framework of kinetic theory (more precisely when collision kernels are involved), estimates like (1) are related to the so-called Cercignani's conjecture (cf. [4], [11]), first proposed in the case of the Boltzmann equation.

This conjecture states that estimate (1) should hold when H and D are the entropy and entropy production associated with the Boltzmann equation, with $C(f)$ involving only the mass, momentum and kinetic energy associated to f, and (an upper bound of) the entropy. The history of the conjecture is recalled in detail in [11]: it holds in a special case (sometimes called "super hard spheres"), while it has to be replaced by a weaker version in the other cases of interest (cf. [22], [20]).

We focus in this work on the Landau equation of plasma physics (cf. [17]), a model introduced for modeling the collisions of charged particles interacting through the Coulomb force. The same name is used for generalizations of this model which are linked (by the so-called grazing collisions limit) to the Boltzmann equation. The equivalent of Cercignani's conjecture was proven for this model in the special case of the so-called Maxwell molecules (cf. [12]), and a weaker version was obtained in [3] in the physical case of the Coulomb interaction. This last paper uses a related entropy production inequality stated in [7].

Our intent in this work is to present (in section 3) variants of the inequalities stated in [12], [3] and [7]. These variants are close to optimal in some sense (this is detailed in the sequel) and enable to obtain a simple proof of (a very slightly weakened version of) Cercignani's conjecture in the case of Landau's equation for hard spheres. For pedagogical purposes, we first show similar estimates on a simpler (1D) model, sometimes called Kac-Landau, in section 2.

2. Kac-Landau equation

2.1. **Kac equation and its entropy structure.** The (spatially homogeneous) Kac equation of kinetic theory describes a 1-dimensional rarefied gas in which collisions conserve mass and kinetic energy (but not momentum), cf. [15]. It writes, for $f := f(t,v) \geq 0$,

$$\frac{\partial f}{\partial t}(t,v) = Q_K(f)(t,v), \qquad f(0,v) = f_{in}(v),$$

where Q_K is the quadratic Kac operator defined by

$$Q_K(f)(v) = \int_{w\in\mathbb{R}} \int_{-\pi}^{\pi} \bigg(f(v\,\cos\theta - w\,\sin\theta)\, f(v\,\sin\theta + w\,\cos\theta) - f(v)\, f(w) \bigg)\, d\theta dw.$$

We consider in this section the generalized operator (cf. [8] for the angular dependence of the cross section):

$$Q_K(f)(v) = \int_{w\in\mathbb{R}} \int_{-\pi}^{\pi} \bigg(f(v\,\cos\theta - w\,\sin\theta)\, f(v\,\sin\theta + w\,\cos\theta) - f(v)\, f(w) \bigg)\, (v^2 + w^2)^{s/2}\, b(|\theta|)\, d\theta dw,$$

where s is a (positive or negative) real number, and the nonnegative function b is (in the language of mathematical kinetic theory) cutoff (that is belonging to $L^1([0,\pi])$) or noncutoff (that is, not belonging to $L^1([0,\pi])$, but such that $\theta \in [0,\pi] \mapsto |\theta|^2\, b(|\theta|)$ belongs to $L^1([0,\pi])$).

The weak formulation of this kernel writes, for suitable test functions φ:

$$\int_{\mathbb{R}} Q_K(f)(v)\,\varphi(v)\,dv = \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{-\pi}^{\pi} \Big(f(v\,\cos\theta - w\,\sin\theta)\, f(v\,\sin\theta + w\,\cos\theta)$$

$$- f(v)\, f(w) \Big) \Big[\varphi(v) + \varphi(w) - \varphi(v\,\cos\theta - w\,\sin\theta) - \varphi(v\,\sin\theta + w\,\cos\theta) \Big]$$

$$\times\, (v^2+w^2)^{s/2}\, b(|\theta|)\, d\theta dw dv. \tag{4}$$

It is obtained by using the changes of variable $(v,w) \mapsto (w,v)$, and $(v,w) \mapsto (v\,\cos\theta - w\,\sin\theta, v\,\sin\theta + w\,\cos\theta)$ (pre/post collisional change of variable), together with $\theta \mapsto -\theta$. Note that this weak formulation holds when $(v^2+w^2)^{s/2}\, b(|\theta|)$ is replaced by a more general cross section $B(v^2+w^2, |\theta|)$.

The conservation of mass and kinetic energy can be obtained by taking $\varphi(v) = 1$ and $\varphi(v) = |v|^2)$ in eq. (4):

$$\int_{\mathbb{R}} Q_K(f)(v) \begin{pmatrix} 1 \\ |v|^2 \end{pmatrix} dv = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The entropy structure of the kernel is obtained by taking $\varphi(v) = \ln f(v)$ in eq. (4): the fact that the resulting formula is clearly nonpositive can be seen as (the first part of) Boltzmann's H theorem for (the generalized) Kac kernel:

$$\int_{\mathbb{R}} Q_K(f)(v)\, \ln f(v)\, dv = \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{-\pi}^{\pi} \Big(f(v\,\cos\theta - w\,\sin\theta)\, f(v\,\sin\theta + w\,\cos\theta)$$

$$- f(v)\, f(w) \Big) \Big[\ln\Big(f(v)\, f(w)\Big) - \ln\Big(f(v\,\cos\theta - w\,\sin\theta)\, f(v\,\sin\theta + w\,\cos\theta)\Big) \Big]$$

$$\times\, (v^2+w^2)^{s/2}\, b(|\theta|)\, d\theta dw dv \le 0.$$

We see that if $b > 0$ (everywhere, or a.e), then the identity

$$D_K(f) := -\int_{\mathbb{R}} Q_K(f)(v)\, \ln f(v)\, dv = 0$$

implies that for all (or a.e.) v, w, θ,

$$f(v\,\cos\theta - w\,\sin\theta)\, f(v\,\sin\theta + w\,\cos\theta) = f(v)\, f(w),$$

that is, for some measurable function $T \ge 0$,

$$f(v)\, f(w) = T(v^2+w^2),$$

that we can rewrite

$$T(v^2)\, T(w^2) = f(0)^2\, T(v^2+w^2).$$

Finally, it is a classical result that (under very weak assumptions on the regularity of T), one can find $a, b \in \mathbb{R}$, such that $T(x) = a\,\exp(b\,x)$. Then, one can find $c \in \mathbb{R}_+$ such that $f(v) = c\,\exp(b\,v^2)$. Finally, under some integrability condition on f,

$$D_K(f) = 0 \quad \Longleftrightarrow \quad \exists \alpha \ge 0, \beta > 0, \quad \forall v \in \mathbb{R}, \quad f(v) = \alpha\,\exp(-\beta\,v^2).$$

This constitutes (the second part of) Boltzmann's H theorem for (the generalized) Kac kernel.

2.2. **The entropy structure of the Kac-Landau equation.** When b concentrates on grazing collisions, that is when one considers $b_\varepsilon(|\theta|) = \frac{cst}{\varepsilon^3}\, b\,(|\theta|/\varepsilon)$, one can see that, at the formal level (and choosing a suitable constant cst), that

$$Q_{K,b_\varepsilon}(f)(v) \to_{\varepsilon\to 0} Q_{KL}(f)(v) := \frac{d}{dv}\bigg[\int_{w\in\mathbb{R}} w\,(v^2+w^2)^{s/2} \times \bigg\{ w\, f(w)\, \frac{df}{dv}(v) - v\, f(v)\, \frac{df}{dw}(w) \bigg\}\, dw \bigg].$$

Under reasonable assumptions on f, s and b, it is possible to show that this convergence rigorously holds (cf. the elementary arguments of [6] in the case of the Boltzmann equation, and the much more elaborated treatment of the renormalized solutions of the spatially inhomogeneous Boltzmann equation in [2]).

Still at the formal level (and choosing a suitable constant cst), it is easy to see that

$$D_{K,b_\varepsilon}(f) \to_{\varepsilon\to 0} D_{KL}(f) := -\int_{\mathbb{R}} Q_{KL}(f)(v)\, \ln f(v)\, dv$$
$$= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} f(v)\, f(w) \left| w\, \frac{f'(v)}{f(v)} - v\, \frac{f'(w)}{f(w)} \right|^2 (v^2+w^2)^{s/2}\, dwdv \geq 0.$$

It is clear that (when $f>0$ is integrable)

$$D_{KL}(f) = 0 \quad \iff \quad \text{for a.e. } v, w \in \mathbb{R}, \quad w\, \frac{f'(v)}{f(v)} = v\, \frac{f'(w)}{f(w)},$$
$$\iff \quad \exists \beta > 0, \quad \text{for a.e. } v\in\mathbb{R}, \quad (\ln f)'(v) = -\beta\, v,$$
$$\iff \quad \exists \alpha \geq 0, \beta > 0, \quad \text{for a.e. } v\in\mathbb{R}, \quad f(v) = \alpha\, \exp(-\beta\, v^2).$$

We recover in this way (the two parts of) Boltzmann's H theorem for Kac-Landau kernel.

2.3. **Entropy production estimates.** In this subsection, we try to find a simple expression (involving only one integral) $I_{KL}(f)\geq 0$, and quantities $C_1(f), C_2(f)>0$ with a very simple dependence w.r.t f such that

$$C_1(f)\, I_{KL}(f) \leq D_{KL}(f) \leq C_2(f)\, I_{KL}(f).$$

Such an estimate from above and below can be seen as close to optimal if $C_1(f)>0$ and $C_2(f)>0$ display a dependence w.r.t. f which is indeed very simple: we will see in the sequel that it can be restricted to moments (provided that the mass and energy of f are fixed).

We start with the following (using the notation $s_+ := s\, 1_{\{s\geq 0\}}$):

Proposition 1. *Let $f\geq 0$ belong to $L^1(\mathbb{R})$ and satisfy*

$$\int_{\mathbb{R}} f(v)\, dv = 1, \qquad \int_{\mathbb{R}} f(v)\, v^2\, dv = 1,$$

and let $s>-2$. Then,

$$D_{KL}(f) \leq C_2(f)\, I_{KL,s}(f),$$

where

$$I_{KL,s}(f) := \int_{\mathbb{R}} (1+v^2)^{s/2} \left| \frac{f'(v)}{f(v)} + v \right|^2 f(v)\, dv,$$

and

$$C_2(f) := 2 \int_{\mathbb{R}} (1+w^2)^{1+s_+/2}\, f(w)\, dw.$$

Proof: Introducing $v\,w - v\,w$ inside the square in the integral, we see that

$$\begin{aligned} D_{KL}(f) &= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} f(v)\, f(w) \left| w\, \frac{f'(v)}{f(v)} - v\, \frac{f'(w)}{f(w)} \right|^2 (v^2+w^2)^{s/2}\, dwdv \\ &\le 2 \int_{\mathbb{R}} \int_{\mathbb{R}} f(v)\, f(w)\, |w|^2 \left| \frac{f'(v)}{f(v)} + v \right|^2 (v^2+w^2)^{s/2}\, dwdv \\ &\le C_2(f)\, I_{KL,s}(f), \end{aligned}$$

since (for $s > -2$)

$$\sup_{v>0} \frac{(v^2+w^2)^{s/2}}{(1+v^2)^{s/2}} \le |w|^{-2}\, (1+w^2)^{1+s_+/2}. \tag{5}$$

Note that following the proof above, the inequality still holds (up to a multiplicative constant) when $s \in\,]-3,-2]$ if $C_2(f)$ is modified in order to include some L^p norm of f.

Then, we turn to the

Proposition 2. *Let $f \ge 0$ belong to $L^1(\mathbb{R})$ and satisfy*

$$\int_{\mathbb{R}} f(v)\, dv = 1, \qquad \int_{\mathbb{R}} f(v)\, v^2\, dv = 1,$$

and let $s \le 2$. Then

$$D_{KL}(f) \ge C_1(f)\, I_{KL,s}(f),$$

where (with the notation of Proposition 1)

$$C_1(f)^{-1} := \int_{\mathbb{R}} (1+w^2)^{1+(-s)_+/2}\, f(w)\, dw.$$

Proof: We define

$$q^f(v,w) := w\, \frac{f'(v)}{f(v)} - v\, \frac{f'(w)}{f(w)},$$

and try to express f in terms of q^f. For this simple model, this is easily done by using $f(w)\, w$ as a multiplicator and by integrating w.r.t w. We see that

$$\int_{\mathbb{R}} f(w)\, w^2\, dw\, \frac{f'(v)}{f(v)} - v \int_{\mathbb{R}} f(w)\, \frac{f'(w)}{f(w)} w\, dw = \int_{\mathbb{R}} f(w)\, q^f(v,w)\, w\, dw,$$

and therefore

$$\frac{f'(v)}{f(v)} + v = \int_{\mathbb{R}} f(w)\, q^f(v,w)\, w\, dw.$$

As a consequence, thanks to Cauchy-Schwarz inequality,

$$\begin{aligned} I_{KL,s}(f) &= \int_{\mathbb{R}} f(v) \left| \frac{f'(v)}{f(v)} + v \right|^2 (1+v^2)^{s/2}\, dv \\ &= \int_{\mathbb{R}} f(v)\, (1+v^2)^{s/2} \left| \int_{\mathbb{R}} f(w)\, q^f(v,w)\, w\, dw \right|^2 dv \\ &\le \int_{\mathbb{R}} f(v)\, (1+v^2)^{s/2} \left(\int_{\mathbb{R}} f(w)\, (q^f)^2(v,w)\, (v^2+w^2)^{s/2}\, dw \right) \end{aligned}$$

$$\times \left(\int_{\mathbb{R}} f(z)\, z^2\, (v^2 + z^2)^{-s/2}\, dz \right) dv$$
$$\le \int_{\mathbb{R}} \int_{\mathbb{R}} f(v)\, f(w) \left| w\, \frac{f'(v)}{f(v)} - v\, \frac{f'(w)}{f(w)} \right|^2 (v^2 + w^2)^{s/2}\, dw dv$$
$$\times \sup_{v \in \mathbb{R}} (1 + v^2)^{s/2} \int_{\mathbb{R}} f(z)\, z^2\, (v^2 + z^2)^{-s/2}\, dz$$
$$\le C_1(f)^{-1}\, D_{KL}(f),$$

thanks to estimate (5) applied to $-s$ instead of s.

As previously, following the proof above, the inequality still holds when $s \in\,]2, 3[$ if $C_1(f)$ is modified in order to include some L^p norm of f.

2.4. **comments and consequences.** We first observe that in both cases $s \in [0, 2[$ and $s \in\,] - 2, 0]$, one of the quantites $C_1(f)$ or $C_2(f)$ is a fixed constant (remember that the mass and the kinetic energy of f is 1), while the other one is a moment of f of order between 2 and 4. The estimates of Propositions 1 and 2 are therefore in some sense close to optimal.

Then we recall that the quantity $I_{KL,s}(f)$ is a weighted relative (to the centered Gaussian) Fisher information. In the case when $s \ge 0$, it is bigger than the standard (non weighted) Fisher information. In that situation (more precisely when $s \in [0, 2]$), the Sobolev logarithmic inequality (cf. [14]) implies Cercignani's conjecture for Kac-Landau model.

Another consequence of Proposition 2 concerns the smoothness of the solutions of Kac-Landau equation. More precisely, any solution (on a time interval $[0, T]$) of Kac-Landau equation (which satisfies the entropy inequality) is such that $\int_0^T D_{KL}(f(t, \cdot))\, dt \le C$ (where C is the relative entropy of the initial datum relative to the centered reduced Gaussian). As a consequence (provided that the considered solution conserves mass and kinetic energy, and that $s \le 2$), $\int_0^T I_{KL,s}(f(t, \cdot))\, dt$ is finite, so that $\sqrt{f} \in L^2([0, T]; H^1_{loc}(\mathbb{R}))$. This regularity a priori estimate can then be used to start a more thorough study of the smoothness/uniqueness of the solutions of Kac-Landau equation (at least when $s \le 2$, and maybe also when $s < 3$).

3. Entropy production estimates for Landau's equation

3.1. **Definition and elementary properties of Landau's operator.** The (spatially homogeneous) Landau equation of plasma physics (cf. [17]) writes $\partial_t f = Q_L(f)$, where the Landau collision operator $Q_L(f)$ is defined by

$$Q_L(f)(v) = \nabla \cdot \left\{ \int_{\mathbb{R}^3} \psi(|v - w|)\, \Pi(v - w) \Big(f(w)\, \nabla f(v) - f(v)\, \nabla f(w) \Big)\, dw \right\},$$

where

$$\Pi_{ij}(z) := \delta_{ij} - \frac{z_i z_j}{|z|^2}$$

is the i, j-component of the orthogonal projection Π on $z^\perp := \{y \,/\, y \cdot z = 0\}$, and, in order to model the Coulomb interaction, $\psi(|z|) = |z|^{-1}$.

For mathematical purposes, we sometimes consider different power laws for ψ, with the following vocabulary, inspired from the theory of the Boltzmann equation (cf. [7]):

- Hard potentials: $\psi(|z|) = |z|^{\gamma+2}$, $\gamma \in]0,1[$,
- Maxwell molecules: $\psi(|z|) = |z|^{\gamma+2}$, $\gamma = 0$,
- Moderetely soft potentials: $\psi(|z|) = |z|^{\gamma+2}$, $\gamma \in]-2,0[$,
- Very soft potentials: $\psi(|z|) = |z|^{\gamma+2}$, $\gamma \in]-4,-2]$.

Note that the very soft potentials include the Coulomb case considered in the study of plasmas $\gamma = -3$. The case when $\gamma = -2$ can also be attributed to moderately soft potentials (cf. [23] for arguments in this direction).

The weak formulation of Landau operator is obtained by performing an integration by parts, and the change of variables $(v,w) \mapsto (w,v)$. It writes

$$\int_{\mathbb{R}^3} Q_L(f,f)(v)\,\varphi(v)\,dv$$

$$= -\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} f(v)\,f(w)\psi(|v-w|)\left(\frac{\nabla f(v)}{f(v)} - \frac{\nabla f(w)}{f(w)}\right)^T \Pi(v-w)$$

$$\Big(\nabla\varphi(v) - \nabla\varphi(w)\Big)\,dvdw.$$

The conservation of mass, momentum and kinetic energy is a consequence of the weak formulation when $\varphi(v) = 1, v$ or $|v|^2/2$:

$$\int_{\mathbb{R}^3} Q_L(f,f)(v)\begin{pmatrix} 1 \\ v \\ |v|^2/2 \end{pmatrix} dv = 0.$$

Finally, the entropy production is defined by

$$D_L(f) := -\int_{\mathbb{R}^3} Q_L(f)(v)\ln f(v)\,dv$$

$$= \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} f(v)\,f(w)\,\psi(|v-w|)$$

$$\left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w)\right)^T \Pi(v-w)\,\left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w)\right)\,dvdw$$

$$= \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} f(v)\,f(w)\,\psi(|v-w|)\,\left|(v-w)\,\times\,\left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w)\right)\right|^2\,dvdw \geq 0.$$

The fact that this quantity is nonnegative constitutes the (first part of) Boltzmann's H-theorem for the Landau operator.

It is clear that if $D_L(f) = 0$ and $f > 0$, then (for a.e. v,w) $\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w)$ is parallel to $v - w$, or equivalently

$$(v-w)\,\times\,\left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w)\right) = 0.$$

We shall see in the sequel that this last equality holds (for $f > 0$ which are integrable) if and only if

$$\exists a, c \in \mathbb{R}_+^*, b \in \mathbb{R}^3, \qquad f(v) = a\,\exp(b\cdot v - c\,|v|^2).$$

This constitutes the (second part of) Boltzmann's H-theorem for the Landau operator.

3.2. **Already existing entropy production estimates for Landau operator.** For the special case of Maxwell molecules, that is $\psi(z) = |z|^2$, it was shown in ([12]) that for all $f \geq 0$ such that

$$\int_{\mathbb{R}^3} f(v)\, dv = 1, \qquad \int_{\mathbb{R}^3} f(v)\, v\, dv = 0 \quad \text{and} \qquad \int_{\mathbb{R}^3} f(v)\, |v|^2\, dv = 3,$$

one has

$$D_L(f) \geq C(f) \int_{\mathbb{R}^3} f(v) \left| \frac{\nabla f}{f}(v) + v \right|^2 dv,$$

where $C(f)$ only depends on (an upper bound) of the entropy $\int_{\mathbb{R}^3} f(v)\, \ln f(v)\, dv$ of f (this dependence can be made more precise, using directional temperatures). A direct use of the monotonicity of the entropy dissipation w.r.t. ψ shows that when $\psi(z) \geq c\,|z|^2$ (for some $c > 0$), the result above still holds (note that this does not include the standard hard potentials, where $\psi(z)\,|z|^{-2} \to 0$ when $z \to 0$).

For soft potentials $\psi(z) = |z|^{2+\gamma}$ (with $\gamma \in\,]-4, 0[$), it is shown in [7] that for all $f \geq 0$ such that $\int_{\mathbb{R}^3} f(v)\, dv = 1$, $\int_{\mathbb{R}^3} f(v)\, v\, dv = 0$, and $\int_{\mathbb{R}^3} f(v)\, |v|^2\, dv = 3$, one has

$$D_L(f) + 1 \geq C(f) \int_{\mathbb{R}^3} f(v) \left| \frac{\nabla f}{f}(v) \right|^2 (1 + |v|^2)^{\gamma/2}\, dv, \tag{6}$$

where, as in the case of Maxwell molecules previously discussed, $C(f)$ only depends on (an upper bound) of the entropy $\int_{\mathbb{R}^3} f(v)\, \ln f(v)\, dv$ of f. In [9], these estimates were extended in the case when ψ decreases strongly (for example like a Maxwellian) at infinity. They were also extended for hard potentials, but with an extra dependence of $C(f)$ (in (6)) w.r.t. some weighted L^p norm of f. Such a dependence will be found again in the estimates presented later in this work.

One significant feature of estimate (6) and its extensions is that the left-hand side is $D_L(f) + 1$ and not $D_L(f)$. This is related to the fact that in the right-hand side appears $\frac{\nabla f}{f}(v)$ and not $\frac{\nabla f}{f}(v) + v$. We refer to [3] for estimates in the case of soft potentials $\psi(z) = |z|^{2+\gamma}$ (with $\gamma \in\,]-4, 0[$) in which it is proven that for all $f \geq 0$ such that $\int_{\mathbb{R}^3} f(v)\, dv = 1$, $\int_{\mathbb{R}^3} f(v)\, v\, dv = 0$, and $\int_{\mathbb{R}^3} f(v)\, |v|^2\, dv = 3$, one has

$$D_L(f) \geq C(f) \left(\int_{\mathbb{R}^3} f(v)\, (1+|v|^2)^{1-\gamma/2}\, dv \right)^{-1} \int_{\mathbb{R}^3} f(v) \left| \frac{\nabla f}{f}(v) + v \right|^2 (1+|v|^2)^{\gamma/2}\, dv, \tag{7}$$

where $C(f)$ only depends on (an upper bound of) the entropy.

3.3. **Examples of applications to the (spatially homogeneous) Landau's equation in the Coulomb case.** Applications (reported here only when they are relevant to the physical case, that is the one corresponding to the Coulomb interaction) can be classified in two categories, those related to the smoothness of the equation, and those related to its large time behavior.

Smoothness. Thanks to estimate (6), it was possible to show that, under reasonable assumptions on the initial data, the solutions (of the spatially homogeneous Landau's equation in the Coulomb case) belong to a weighted $L^1_{loc}(\mathbb{R}, L^3(\mathbb{R}^3))$-type space, and thus are "standard weak solutions", rather than only H-solutions (cf. [21]). Then, it was shown in [13] that those solutions can be singular only on a set whose size is controlled (in terms of Hausdorff dimension). Finally, estimate (6) should also help to produce results of perturbative nature for the (spatially homogeneous) Landau's equation in the Coulomb case (cf. [10])

Large time behavior. Then, thanks to estimate (7), it was shown in [3] that the solutions (of the spatially homogeneous Landau's equation in the Coulomb case) converge to the Maxwellian equilibrium with a controlled rate (like an inverse power or like a stretched exponential, depending on the behavior when $|v| \to \infty$ of the initial datum).

3.4. **Sharper entropy production estimates for Landau equation.** In view of (7), and taking into account the results of Section 2, a naïve guess would be that when $\psi(x) = |x|^{2+\gamma}$, a sharp (close to optimal) estimate would be, for some well-chosen $C(f)$, that

$$D_L(f) \geq C(f) \int_{\mathbb{R}^3} f(v) \left| \frac{\nabla f}{f}(v) + v \right|^2 (1+|v|^2)^{\gamma/2}\, dv, .$$

However, a slightly different estimate from above, which is close to optimal in some sense, appears in the following:

Proposition 3. *For hard or soft potentials $\psi(z) = |z|^{2+\gamma}$ (with $\gamma \in\,]-3,1]$) and f such that $\int_{\mathbb{R}^3} f(v)\, dv = 1$, $\int_{\mathbb{R}^3} f(v)\, v\, dv = 0$ and $\int_{\mathbb{R}^3} f(v)\, |v|^2\, dv = 3$, one has*

$$D_L(f) \leq C_+(f) \int_{\mathbb{R}^3} f(v) \left(\left| \frac{\nabla f}{f}(v) + v \right|^2 + \left| v \times \frac{\nabla f}{f}(v) \right|^2 \right) (1+|v|^2)^{\gamma/2}\, dv,$$

where

$$C_+(f) := 8 \sup_{v \in \mathbb{R}^3} \left\{ \int_{\mathbb{R}^3} f(w)\, (1+|w|^2)\, |v-w|^\gamma\, (1+|v|^2)^{-\frac{\gamma}{2}}\, dw \right\}.$$

Moreover, $C_+(f) \leq cst_\gamma \int_{\mathbb{R}^3} f(v)\, (1+|v|^2)^{1+\gamma/2}\, dv$ if $\gamma \geq 0$, and (for any $\delta > 0$)

$$C_+(f) \leq cst_\gamma \left(\int_{\mathbb{R}^3} f(v)\, (1+|v|^2)^{1+|\gamma|/2}\, dv + ||f\, (1+|\cdot|^2)^{1+|\gamma|/2}||_{L^{(3/(3-|\gamma|))+\delta}(\mathbb{R}^3)} \right)$$

if $\gamma \in\,]-3,0[$, for some constant cst_γ which depends only on γ (and δ).

Proof: We observe that

$$D_L(f) = \frac{1}{2} \sum_{i<j;i,j=1,..,3} \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} f(v)\, f(w) \left| q^f_{ij}(v,w) \right|^2 |v-w|^\gamma\, dvdw,$$

where

$$q^f_{ij}(v,w) = (v_i - w_i) \left(\frac{\partial_j f}{f}(v) - \frac{\partial_j f}{f}(w) \right) - (v_j - w_j) \left(\frac{\partial_i f}{f}(v) - \frac{\partial_i f}{f}(w) \right),$$

which can also be rewritten as

$$\left(v \times \frac{\nabla f}{f}(v) \right)_{ij} - w_i \frac{\partial_j f}{f}(v) + w_j \frac{\partial_i f}{f}(v)$$
$$= v_i \frac{\partial_j f}{f}(w) - v_j \frac{\partial_i f}{f}(w) - \left(w \times \frac{\nabla f}{f}(w) \right)_{ij} + q^f_{ij}(v,w).$$

As a consequence, introducing $v_i\, w_j - v_j\, w_i - v_i\, w_j + v_j\, w_i$ in the formula, we end up with

$$D_L(f) \leq 8 \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} f(v)\, f(w) \left[\left| v \times \frac{\nabla f}{f}(v) \right|^2 + |w|^2 \left| \frac{\nabla f}{f}(v) + v \right|^2 \right] |v-w|^\gamma\, dvdw$$

$$\leq 8 \int_{\mathbb{R}^3} f(v) \left[\left| v \times \frac{\nabla f}{f}(v) \right|^2 + \left| \frac{\nabla f}{f}(v) + v \right|^2 \right] \left\{ \int_{\mathbb{R}^3} f(w)\,(1+|w|^2)\,|v-w|^\gamma dw \right\} dv.$$

This concludes the first part of the estimate.

The second part is easy to get when $\gamma \geq 0$, it relies on the Hölder's inequality when $\gamma < 0$, after cutting the integral between the part when $|v-w| \leq 1$ and the part when $|v-w| \geq 1$.

Proposition 4. *For hard or soft potentials $\psi(z) = |z|^{2+\gamma}$ (with $\gamma \in \,]-4,1]$) and f such that $\int_{\mathbb{R}^3} f(v)\,dv = 1$, $\int_{\mathbb{R}^3} f(v)\,v\,dv = 0$ and $\int_{\mathbb{R}^3} f(v)\,|v|^2\,dv = 3$, one has*

$$D_L(f) \geq C_-(f) \int_{\mathbb{R}^3} f(v) \left(\left| \frac{\nabla f}{f}(v) + v \right|^2 + \left| v \times \frac{\nabla f}{f}(v) \right|^2 \right) (1+|v|^2)^{\gamma/2}\,dv,$$

where

$$C_-(f)^{-1} = 2\,B(f) \left[1 + a_0(f)^{-2} \left(3 + 8\,\frac{\int_{\mathbb{R}^3} |v|^2\,(1+|v|^2)^{\gamma/2}\,f(v)\,dv}{\inf_{e\in S^2} \int_{\mathbb{R}^3} (v\cdot e)^2\,(1+|v|^2)^{\gamma/2}\,f(v)\,dv} \right) \right],$$

and

$$B(f) := \sup_{v\in\mathbb{R}^3} \int_{\mathbb{R}^3} f(z)\,(1+|z|^2)\,|v-z|^{-\gamma}\,(1+|v|^2)^{\gamma/2}\,dz,$$

$$a_0(f) := \inf_{e\in S^2} \int_{\mathbb{R}^3} (v\cdot e)^2\,f(v)\,dv.$$

Moreover, the quantities $a_0(f)$ and $\inf_{e\in S^2} \int_{\mathbb{R}^3} (v\cdot e)^2\,(1+|v|^2)^{\gamma/2}\,f(v)\,dv$ are bounded below by a strictly positive quantity depending only on a bound (above) of the entropy $\int_{\mathbb{R}^3} f(v)\,\ln f(v)\,dv$. Finally,

$$B(f) \leq cst_\gamma \int_{\mathbb{R}^3} (1+|v|^2)^{1+|\gamma|/2}\,f(v)\,dv$$

if $\gamma \in \,]-4,0]$, and (for any $\delta > 0$)

$$B(f) \leq cst_\gamma \left(\int_{\mathbb{R}^3} (1+|v|^2)^{1+\gamma/2}\,f(v)\,dv + ||f\,(1+|\cdot|^2)^{1+\gamma/2}||_{L^{(3/(3-\gamma))+\delta}(\mathbb{R}^3)} \right)$$

if $\gamma \in [0,1]$, for some constant cst_γ which depends only on γ (and δ).

Proof: We first observe that without loss of generality, up to change of (orthogonal) basis, we can suppose that $\int_{\mathbb{R}^3} f(v)\,v_i\,v_j\,dv = a_i\,\delta_{ij}$, where a_i is a directional temperature, that is $a_i := \int_{\mathbb{R}^3} f(v)\,v_i^2\,dv$.

Using the notation q_{ij}^f of the proof of Proposition 3 (and denoting by q^f the vector associated to the antisymmetric matrix q_{ij}^f), we recall that for $i,j \in \{1,2,3\}$, $i \neq j$,

$$\left(v \times \frac{\nabla f}{f}(v) \right)_{ij} - w_i\,\frac{\partial_j f}{f}(v) + w_j\,\frac{\partial_i f}{f}(v)$$

$$= v_i\,\frac{\partial_j f}{f}(w) - v_j\,\frac{\partial_i f}{f}(w) - \left(w \times \frac{\nabla f}{f}(w) \right)_{ij} + q_{ij}^f(v,w).$$

This equation can be inverted in such a way that $v \times \frac{\nabla f}{f}(v)$ and $\frac{\nabla f}{f}(v)$ are written in terms of q^f_{ij}. Indeed, using the multiplicators $f(w)$ and $w_j\, f(w)$ and integrating w.r.t. w, one ends up with the identities:

$$\left(v \times \frac{\nabla f}{f}(v)\right)_{ij} = \int_{\mathbb{R}^3} f(w)\, q^f_{ij}(v,w)\, dw,$$

$$a_j\, \frac{\partial_i f}{f}(v) + v_i = \int_{\mathbb{R}^3} f(w)\, w_j\, q^f_{ij}(v,w)\, dw. \tag{8}$$

Then, thanks to Cauchy-Schwarz inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^3} f(v) \left| v \times \frac{\nabla f}{f}(v)\right|^2 (1+|v|^2)^{\gamma/2}\, dv \\
&\le \int_{\mathbb{R}^3} f(v) \left| \int_{\mathbb{R}^3} f(w)\, q^f(v,w)\, dw\right|^2 (1+|v|^2)^{\gamma/2}\, dv \\
&\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v)\, f(w)\, |q^f(v,w)|^2\, |v-w|^\gamma\, dw \\
&\quad \times \left\{ \int_{\mathbb{R}^3} f(z)\, |v-z|^{-\gamma}\, (1+|v|^2)^{\gamma/2}\, dz \right\} dv \\
&\le 2\, D_L(f) \sup_{v\in\mathbb{R}^3} \int_{\mathbb{R}^3} f(z)\, |v-z|^{-\gamma}\, (1+|v|^2)^{\gamma/2}\, dz \le 2\, D_L(f)\, B(f).
\end{aligned}$$

Finally, for any $i, j \in \{1,2,3\}$, $i \neq j$,

$$\frac{\partial_i f}{f}(v) + \frac{v_i}{a_j} = \frac{1}{a_j} \int_{\mathbb{R}^3} f(w)\, w_j\, q^f_{ij}(v,w)\, dw$$

so that thanks to Cauchy-Schwarz inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^3} f(v) \left| \frac{\partial_i f}{f}(v) + \frac{v_i}{a_j}\right|^2 (1+|v|^2)^{\gamma/2}\, dv \\
&\le a_j^{-2} \int_{\mathbb{R}^3} f(v) \left| \int_{\mathbb{R}^3} f(w)\, w_j\, q^f_{ij}(v,w)\, dw\right|^2 (1+|v|^2)^{\gamma/2}\, dv \\
&\le a_j^{-2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v)\, f(w)\, |q^f_{ij}(v,w)|^2\, |v-w|^\gamma\, dw \\
&\quad \times \left\{ \int_{\mathbb{R}^3} f(z)\, |z|^2\, |v-z|^{-\gamma}\, (1+|v|^2)^{\gamma/2}\, dz \right\} dv \\
&\le a_j^{-2}\, D_L(f) \sup_{v\in\mathbb{R}^3} \int_{\mathbb{R}^3} f(z)\, |z|^2\, |v-z|^{-\gamma}\, (1+|v|^2)^{\gamma/2}\, dz \le a_j^{-2}\, D_L(f)\, B(f).
\end{aligned}$$

Using this last inequality, we see that

$$\begin{aligned}
\int_{\mathbb{R}^3} f(v) \left| \frac{\partial_i f}{f}(v) + v_i\right|^2 (1+|v|^2)^{\gamma/2}\, dv &\le 2 \int_{\mathbb{R}^3} f(v)\, v_i^2\, (1+|v|^2)^{\gamma/2}\, dv\, (1-a_j^{-1})^2 \\
&\quad + 2\, a_j^{-2}\, D_L(f)\, B(f).
\end{aligned} \tag{9}$$

We now use $i, j, k \in \{1,2,3\}$, all different. Thanks to identity (8), we see that

$$v_i\, (a_j^{-1} - a_k^{-1}) = \int_{\mathbb{R}^3} f(w)\, [a_j^{-1}\, w_j\, q^f_{ij}(v,w) - a_k^{-1}\, w_k\, q^f_{ik}(v,w)]\, dw,$$

so that using the multiplicator $(1+|v|^2)^{\gamma/2}\, f(v)$ and integrating w.r.t v the square of this expression, we end up with the estimate

$$\int_{\mathbb{R}^3} v_i^2\,(1+|v|^2)^{\gamma/2}\, f(v)\,dv\;\;(a_j^{-1}-a_k^{-1})^2 \le (a_j^{-2}+a_k^{-2})\, D_L(f)\, B(f),$$

or, more simply,

$$(a_j-a_k)^2 \le (a_j^2+a_k^2)\left(\int_{\mathbb{R}^3} v_i^2\,(1+|v|^2)^{\gamma/2}\, f(v)\,dv\right)^{-1} D_L(f)\, B(f).$$

Noticing that since $a_1+a_2+a_3=1$,

$$(a_j-1)^2 = (a_i+a_k)^2 \le 2\,(a_j-a_i)^2 + 2\,(a_j-a_k)^2,$$

we obtain the estimate

$$(1-a_j^{-1})^2 \le 8\,a_j^{-2} \sup_{l=1,2,3}\left(\int_{\mathbb{R}^3} v_l^2\,(1+|v|^2)^{\gamma/2}\, f(v)\,dv\right)^{-1} D_L(f)\, B(f).$$

Using this last result in estimate (9) and summing for $i=1,2,3$, we end up with the desired estimate.

The estimate of $B(f)$ is obtained exactly as that of $C_+(f)$ in Proposition 3. Finally, the quantities $a_0(f)$ and $\inf_{e\in S^2}\int_{\mathbb{R}^3}(v\cdot e)^2\,(1+|v|^2)^{\gamma/2}\, f(v)\,dv$ are known to be bounded below by a strictly positive constant when (an upper bound of) the entropy of f is assumed to hold (and when $\int_{\mathbb{R}^3} f(v)\,dv = 1$ and $\int_{\mathbb{R}^3} f(v)\,|v|^2 dv = 3$). We refer to [7] for such estimates.

3.5. **Comments.** As can be seen in Propositions 3 and 4, the natural quantity which appears when one studies the entropy production of the Landau equation (with $\psi(|z|)=|z|^{\gamma+2}$) is the weighted relative (with respect to a centered reduced Gaussian) Fisher information with an extra term, that is

$$I_{L,\gamma}(f) := \int_{\mathbb{R}^3} f(v)\left(\left|\frac{\nabla f}{f}(v)+v\right|^2 + \left|v\times\frac{\nabla f}{f}(v)\right|^2\right)(1+|v|^2)^{\gamma/2}\,dv.$$

Note that (in some sense) the weight appearing in $I_{L,\gamma}(f)$ is larger when one considers the gradient in the "orthoradial" direction. This seems coherent with the estimates known in the linear setting.

The terms $C_+(f)$ and $C_-(f)^{-1}$ are bounded above when a sufficient number of moments of f and the entropy of f are bounded, and when moreover some L^p norm of f is bounded when $\gamma<0$ (soft potentials) in the case of $C_+(f)$, and when $\gamma>0$ (hard potentials) in the case of $C_-(f)^{-1}$. This structure is reminiscent of the one exhibited in the case of the Kac-Landau equation, but slightly less favorable. Indeed, only in the case when $\gamma=0$ do the conditions include only the entropy and moments of f. It seems therefore that Propositions 3 and 4 are not completely optimal (also Proposition 3 does not hold when $\gamma=-3$), and that there is still room for some improvement. Note finally that the use of multiplicators decaying quickly at infinity enables to eliminate the need of moments in some situations, as can be seen in estimate (6).

Finally, one can observe that Proposition 4 implies a weak version of Cercignani's conjecture for the Landau equation with true hard potentials, thanks to the use of the Sobolev logarithmic inequality. Here, the term "weak" means that the conjecture only holds when some moments and L^p norms of f are assumed to be bounded.

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