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Transverse linear stability of line periodic traveling waves for water-wave models

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Abstract

We review some recent results on transverse linear stability of line periodic traveling waves for the water-wave problem. A common feature of these results is that they can be obtained from two, rather simple, abstract stability criteria. While the first criterion gives sufficient conditions for linear instability, the second one, which is a counting result for unstable eigenvalues, leads to sufficient conditions for spectral stability. We restrict to waves of small amplitude bifurcating in four different parameter regimes. We focus on the simplest model equations, the Kadomtsev-Petviashvili I and II equations, and refer to existing works for other models, including the full Euler equations.

1 Introduction

The classical water-wave problem concerns the irrotational flow of a perfect fluid of constant density in a three-dimensional domain bounded below by a flat bottom and above by a free surface under the influence of gravity and surface tension. The governing equations are the Euler equations for the velocity potential and the free surface (see (3.1)-(3.4)). In dimensionless variables, the different physical parameters reduce to two dimensionless numbers, α and β which are the inverse square of the Froude number and the Weber number, respectively. The Weber number being proportional to the coefficient of surface tension it is either positive, in the presence of surface tension (gravity-capillary waves), or it vanishes, in the absence of surface tension (pure gravity waves). The main difficulties in the mathematical study of these Euler equations are due to the unknown free surface and to the nonlinear boundary conditions at this free surface. We point out that these equations possess several important and useful symmetries, as for instance, temporal and spatial reversibilities and Hamiltonian structures.

Of particular interest for the hydrodynamic problem is the dynamical behavior of traveling waves such as solitary or periodic waves. The underlying mathematical questions concern, in particular, their existence and stability. In the frame of the full Euler equations, the existence of traveling waves has been extensively studied (see for instance [5, 8, 30] and the references therein). In contrast, there are few stability results, most of them being obtained for solitary waves (see for instance the works on transverse stability [28, 15, 12] and the references therein). Here we focus on line periodic waves and the particular question of their linear transverse stability.

Line periodic waves are solutions of the three-dimensional hydrodynamic problem which are periodic one horizontal coordinate and do not depend on a second, transverse, horizontal coordinate

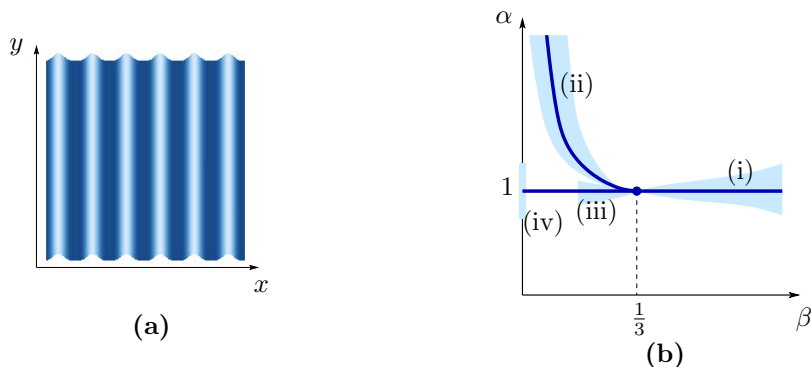


Figure 1.1: (a) In Cartesian coordinates (x, y, z) , schematic plot of a line periodic wave which is periodic in the horizontal coordinate x and constant in the horizontal coordinate y . (b) In the (β, α) -parameter plane, the shaded regions show the parameter regimes (i)-(iv).

(see Figure 1.1a). The transverse stability question is concerned with their stability with respect to three-dimensional perturbations, hence also depending on the horizontal coordinate in which the line periodic waves are constant. We consider the line periodic traveling water waves which bifurcate in four different parameter regimes:

- (i) large surface tension, when $\beta > 1/3$ and $\alpha \approx 1$;
- (ii) weak surface tension, when $0 < \beta < 1/3$ and α is close to a critical value $\alpha_*(\beta) > 1$;
- (iii) critical surface tension, when $\beta \approx 1/3$, $\beta < 1/3$, and $\alpha \approx 1$;
- (iv) zero surface tension, when $\beta = 0$ and $\alpha \approx 1$;

(see also Figure 1.1b). In each of these regimes, the dynamics of the full water-wave problem is rather well predicted by simpler model equations: the Kadomtsev-Petviashvili (KP) I and II equations in the cases (i) and (iv), respectively, a Davey-Stewartson system in the case (ii), and a fifth order Kadomtsev-Petviashvili equation in the case (iii). We review the transverse stability results for these model equations and also the Euler equations in Section 3. Without going into the details of proofs, we focus on the KP-I and II equations. We find that the periodic gravity-capillary waves are linearly transversely unstable (Section 3.2), whereas the periodic gravity waves are spectrally transversely stable (Section 3.3). In all these cases, the question of nonlinear transverse stability, or instability, of periodic waves is widely open. It turns out that these linear transverse instability and spectral transverse stability results can be obtained from two abstract results for reversible and Hamiltonian systems, respectively. We briefly present these results in Section 2.

The first abstract result, due to Godey [6], gives a linear instability criterion for a rather general class of partial differential equations which are reversible in one unbounded spatial coordinate. Considering line nonlinear waves, as for instance periodic but also solitary waves, which are constant in this coordinate, the criterion can be used for the study of their transverse stability [7, 11, 12]. In particular, it allows to show linear transverse instability of line periodic waves for the equations mentioned above. We point out that some of these results can be obtained by applying a different transverse instability criterion due to Rousset and Tzvetkov [27] (see [13]). Instead of spatial reversibility, this criterion requires a temporal Hamiltonian structure of the system. This temporal structure makes it suitable

for a further study of the nonlinear transverse instability [28, 29], whereas the criterion in [6] is more convenient for a further study of the bifurcations induced by this transverse instability [12].

The second abstract result is a counting result for unstable eigenvalues in Hamiltonian systems [17]. In such systems, the question of spectral stability consists in studying the unstable spectrum of a linear operator of the particular form JL , in which J and L are skew- and self-adjoint operators, respectively. It is well known that, under suitable assumptions, the number of unstable eigenvalues of JL , counted with algebraic multiplicities, is less or equal to the number of nonpositive eigenvalues of the self-adjoint operator L (see for instance [3, 16, 22] and the references therein). In particular, if the operator L is positive, the operator JL does not have unstable eigenvalues, which then implies spectral stability. While extremely efficient for solitary waves, this type of result does not always work well for periodic waves. Among the water-wave models above, spectral stability is expected to hold for the line periodic waves of the KP-II equation, a situation in which this classical result does not seem to be applicable [14], the negative spectrum of the self-adjoint operator L being unbounded. An extension of this classical counting result has been recently obtained in [17], showing that the operator L can be replaced by another self-adjoint operator K , provided the operators JL and JK commute. More precisely, under suitable assumptions, the number of unstable eigenvalues of the operator JL is bounded by the number of nonpositive eigenvalues of the self-adjoint operator K . This abstract result is summarized in Section 2.2, and it is applied to the KP-II equation in Section 3.3.

We conclude with a brief discussion of the similar transverse stability question for solitary waves in Section 4.

2 General stability criteria

In this section, we briefly present the abstract linear instability result from [6] and the counting result for unstable eigenvalues of linear Hamiltonian systems from [17]. We refer to [6] and [17] for the details of proofs.

2.1 Linear instability criterion

The starting point of the linear instability criterion in [6] is a formulation of the evolution problem as an abstract dynamical system of the form

$$u_y = Du_t + F(u), \tag{2.1}$$

in which the unknown u depends upon the time variable t , a space variable y , and takes values in some Banach space X (typically a space of functions defined on a domain $\Omega \subset \mathbb{R}^n$). We assume that D is a linear operator with domain $Z \subset X$ and F a nonlinear map defined on a subspace $Y \subset X$. In transverse stability problems for line traveling waves, y represents the spatial coordinate in which the traveling wave is constant and this is the system written in a coordinate frame moving with the wave, so that the traveling wave is a time-independent equilibrium of the dynamical system (2.1). Notice that the evolutionary variable in this dynamical system is the space variable y , instead of the usual time variable t . We point out that such ‘spatial dynamics’ formulations go back to the work of Kirchgässner [23] and have been extensively used for the analysis of steady systems arising in many different applications, and in particular for the water-wave problem (see for instance [24, 5, 8] and the references therein).

For a time-independent equilibrium u_* of (2.1), hence satisfying $F(u_*) = 0$, the question of transverse linear instability concerns the existence of growing-in-time solutions to the linear equation

$$u_y = Du_t + Lu, \tag{2.2}$$

in which $L = DF(u_*)$ is the differential of F at u_* . We say that u_* is *transversely linearly unstable* if the equation (2.2) has a solution u of the form $u(y, t) = e^{\lambda t}v(y)$, with λ a complex number with positive real part, $v(y) \in Y$, for any $y \in \mathbb{R}$, and the map $y \mapsto v(y)$ is bounded on \mathbb{R} . Equivalently, this means that the linear equation

$$v_y = \lambda Dv + Lv, \tag{2.3}$$

possesses a bounded solution $y \mapsto v(y)$ defined on \mathbb{R} .

The key observation in the proof of the linear instability criterion is that assuming that the linear operator $\lambda D + L$ in the right hand side of the equation (2.3) has a pair of complex conjugated eigenvalues $\pm i\omega_\lambda$ with associated complex conjugates eigenvectors V_λ and \bar{V}_λ , then the linear equation (2.3) has the periodic solution

$$v(y) = e^{i\omega_\lambda y}V_\lambda + e^{-i\omega_\lambda y}\bar{V}_\lambda.$$

Clearly this solution is bounded on \mathbb{R} , and its existence implies the linear transverse instability of u_* . The hypotheses below, together with a rather standard perturbation argument, then lead to the instability criterion. More precisely, we make the following hypotheses.

Hypothesis 2.1. *Assume that*

- (i) *D and L are closed real operators in X with domains Z and Y , respectively, and $Y \subset Z$;*
- (ii) *the spectrum of the linear operator L contains a pair of complex conjugated purely imaginary eigenvalues $\pm i\omega$, which are isolated and have odd algebraic multiplicities;*
- (iii) *the linear equation (2.3) is reversible, i.e., there exists a linear map R acting in X which anti-commutes with both operators D and L .*

The main result proved in [6] is the following theorem which shows that these hypotheses are sufficient to conclude on transverse instability.

Theorem 1 ([6]). *Assume that the partial differential equation (2.1) possesses a time-independent equilibrium u_* satisfying $F(u_*) = 0$ and such that Hypothesis 2.1 holds. Then the linear equation (2.2) possesses a solution of the form $u(y, t) = e^{\lambda t}v(y)$, with λ a positive real number, $v(y) \in Y$, for any $y \in \mathbb{R}$, and the map $y \mapsto v(y)$ is periodic on \mathbb{R} . Consequently, u_* is transversely linearly unstable.*

2.2 Count of unstable eigenvalues for linear Hamiltonian systems

Following [17], we consider a Hamiltonian linear operator of the form JL with J and L being skew- and self-adjoint operators, respectively, both acting in a Hilbert space \mathcal{H} . We denote by $\langle \cdot, \cdot \rangle$ the scalar product in \mathcal{H} .

Notation 2.2. *For a linear operator A acting in \mathcal{H} , we denote by $\sigma_s(A)$, $\sigma_c(A)$, and $\sigma_u(A)$, the subsets of the spectrum $\sigma(A)$ of A lying in the open left-half complex plane, on the imaginary axis, and in the*

open right-half complex plane, respectively,

$$\begin{aligned}\sigma_s(A) &= \{\lambda \in \sigma(A) ; \operatorname{Re} \lambda < 0\}, \\ \sigma_c(A) &= \{\lambda \in \sigma(A) ; \operatorname{Re} \lambda = 0\}, \\ \sigma_u(A) &= \{\lambda \in \sigma(A) ; \operatorname{Re} \lambda > 0\},\end{aligned}$$

We refer to these sets as the *stable*, *central*, and *unstable spectra* of A , respectively. Further, we denote by $n_s(A)$, $n_c(A)$, and $n_u(A)$, the dimension of the spectral subspaces associated to $\sigma_s(A)$, $\sigma_c(A)$, and $\sigma_u(A)$, respectively, if these exist.

We make the following hypothesis.

Hypothesis 2.3. *Assume that J , L , and K are closed linear operators acting in \mathcal{H} with the following properties.*

- (i) J is a skew-adjoint operator ($J^* = -J$) with bounded inverse.
- (ii) L and K are self-adjoint operators ($L^* = L$ and $K^* = K$) such that the operators JL and JK commute, i.e., the operators $(JL)(JK)$ and $(JK)(JL)$ have the same domain $\mathcal{D} \subset \mathcal{H}$, and

$$(JL)(JK)u = (JK)(JL)u, \quad \forall u \in \mathcal{D}. \quad (2.4)$$

- (iii) The nonpositive spectrum $\sigma_s(K) \cup \sigma_c(K)$ of the self-adjoint operator K consists, at most, of a finite number of isolated eigenvalues with finite multiplicities.
- (iv) The unstable spectrum $\sigma_u(JL)$ of the operator JL consists, at most, of isolated eigenvalues with finite algebraic multiplicities, and the generalized eigenvectors associated to these eigenvalues belong to the domain of the operator JK .

The key step in the proof of the counting result in [17] is the property

$$\langle Ku, u \rangle = 0,$$

which holds, under the assumptions in Hypothesis 2.3, for any u in the spectral subspace E_u associated to the unstable spectrum $\sigma_u(JL)$ of JL . The abstract counting result is the following theorem.

Theorem 2 ([17]). *Under the assumptions in Hypothesis 2.3 the following properties hold.*

- (i) The number $n_u(JL)$ of unstable eigenvalues of the operator JL (counted with algebraic multiplicities) and the number $n_{sc}(K) = n_s(K) + n_c(K)$ of nonpositive eigenvalues of the self-adjoint operator K (counted with multiplicities) satisfy

$$n_u(JL) \leq n_{sc}(K).$$

- (ii) If, in addition, the kernel of the operator K is contained in the kernel of the operator JL , then

$$n_u(JL) \leq n_s(K). \quad (2.5)$$

The following immediate consequence of this general result gives a sufficient condition for stability.

Corollary 2.4. *Under the assumptions of Hypothesis 2.3, further assume that K is a nonnegative operator. Then $n_u(JL) \leq n_c(K)$. If in addition the kernel of K is contained in the kernel of JL , then $n_u(JL) = 0$, and the spectrum of JL is purely imaginary.*

Notice that the particular case of Theorem 2 with $K = L$ recovers the classical counting result showing that $n_u(JL) \leq n_s(L)$. More refined versions of this result are available in the literature in which, under different additional assumptions, the inequality is replaced by an equality (see for instance [3, 16, 22]). The difference $n_s(L) - n_u(JL)$ is shown to be given by the number of purely imaginary eigenvalues of JL which have a negative Krein signature. We expect that such results can be extended to the present setting by introducing for the purely imaginary eigenvalues of JL a Krein signature relative to the operator K .

3 Transverse stability of periodic water waves

In this section, we review the transverse stability results for periodic waves for different water-wave models. We focus on the simplest model equations KP-I and II, and refer to existing works for other equations.

3.1 The water-wave problem

The governing equations for the water-wave problem are the Euler equations that we briefly recall below.

Consider a three-dimensional inviscid fluid layer of mean depth h and constant density ρ . In usual Cartesian coordinates (x, y, z) , the fluid occupies the domain

$$D_\eta = \{(x, y, z) : x, y \in \mathbb{R}, z \in (0, h + \eta(x, y, t))\},$$

where $\eta > -h$ is a function of the horizontal spatial coordinates x, y and of the time t , and $z = h + \eta(x, y, t)$ describes the free surface. Assume that the forces of gravity and surface tension are present, and denote by g the acceleration due to gravity and by T the coefficient of surface tension. The flow is supposed to be irrotational and is therefore described by an Eulerian velocity potential ϕ . In a coordinate system moving from left to right with constant velocity $c > 0$, the mathematical problem consists in solving Laplace's equation

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \text{for } 0 < z < h + \eta, \tag{3.1}$$

with boundary conditions

$$\phi_z = 0 \quad \text{on } z = 0, \tag{3.2}$$

$$\phi_z = \eta_t - c\eta_x + \eta_x\phi_x + \eta_y\phi_y \quad \text{on } z = h + \eta, \tag{3.3}$$

$$\phi_t - c\phi_x + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\eta - \frac{T}{\rho}\mathcal{K} = 0 \quad \text{on } z = h + \eta, \tag{3.4}$$

in which

$$\mathcal{K} = \left(\frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_y^2}} \right)_x + \left(\frac{\eta_y}{\sqrt{1 + \eta_x^2 + \eta_y^2}} \right)_y$$

is twice the mean curvature of the free surface. We introduce dimensionless variables by choosing the characteristic length to be h and the characteristic velocity to be c . Then the system (3.1)–(3.4) becomes

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \text{for } 0 < z < 1 + \eta, \quad (3.5)$$

with boundary conditions

$$\phi_z = 0 \quad \text{on } z = 0, \quad (3.6)$$

$$\phi_z = \eta_t - \eta_x + \eta_x \phi_x + \eta_y \phi_y \quad \text{on } z = 1 + \eta, \quad (3.7)$$

$$\phi_t - \phi_x + \frac{1}{2} (\phi_x^2 + \phi_y^2 + \phi_z^2) + \alpha \eta - \beta \mathcal{K} = 0 \quad \text{on } z = 1 + \eta, \quad (3.8)$$

in which the dimensionless numbers

$$\alpha = \frac{gh}{c^2}, \quad \beta = \frac{T}{\rho h c^2}$$

are respectively the inverse square of the Froude number and the Weber number.

The mathematical analysis of these equations is complicated and technically involved. The main difficulties are due to the presence of the free surface $z = 1 + \eta$ and to the nonlinear boundary conditions (3.7) and (3.8). However, the water-wave dynamics is often well predicted by model equations, much simpler than these Euler equations. While several existence results for line traveling periodic water waves are available for the Euler equations, most of the results on their transverse (in)stability are only obtained for model equations. It turns out, that all these results show transverse instability in the presence of surface tension ($\beta > 0$, gravity-capillary waves) and transverse stability in the absence of surface tension ($\beta = 0$, gravity waves).

3.2 Transverse instability of gravity-capillary water waves

We consider the three parameter regimes (i)–(iii) mentioned in the introduction. In all these regimes the existence of line periodic waves for the Euler equations is known [24, 19, 2], but their transverse instability has only been studied in the regime (i) of large surface tension [15]. We recall below the results on transverse instability for the model equations, and then briefly comment on the Euler equations.

Large surface tension ($\beta > 1/3$). Fixing the Weber number $\beta > 1/3$, Kirchgässner [24] proved that line periodic waves for the system (3.5)–(3.8) bifurcate at $\alpha = 1$. In this parameter regime, the model equation which approximately describes the dynamics of the system (3.5)–(3.8) is the KP-I equation

$$(u_t + cu_x + 6uu_x + u_{xxx})_x - u_{yy} = 0, \quad (3.9)$$

written here in a reference frame moving with speed c and normalized form. Line periodic waves of this equation are steady periodic solutions of the well-known Korteweg-de Vries (KdV) equation

$$u_t + cu_x + 6uu_x + u_{xxx} = 0. \quad (3.10)$$

A complete characterization of steady periodic waves is available in terms of Jacobi elliptic functions [4] showing that, up to scaling and translation invariances, for any $c > 1$ the equation (3.10) possesses

a unique 2π -periodic even steady solution ϕ_c (see also [17]). The linear transverse instability of this solution, i.e., its instability as solution of the KP-I equation (3.9) has been proved, by different methods in [20, 14, 13, 7]. The proof in [7] uses the abstract result in Section 2.1.

Following [7], we set $v = u_y$ and write the equation (3.9) in the form (2.1), for the unknown $U = (u, v)$ and

$$D = \begin{pmatrix} 0 & 0 \\ \partial_x & 0 \end{pmatrix}, \quad F(U) = \begin{pmatrix} v \\ (cu + 3u^2 + u_{xx})_{xx} \end{pmatrix}.$$

The periodic wave ϕ_c gives a time-independent equilibrium $U_c = (\phi_c, 0)$ such that $F(U_c) = 0$, and we find differential

$$L = DF(U_c) = \begin{pmatrix} 0 & 1 \\ \partial_{xx}(c + 6\phi_c + \partial_{xx}) & 0 \end{pmatrix}.$$

Restricting to 2π -periodic perturbations of the line periodic wave ϕ_c , i.e., co-periodic perturbations, we choose the function space $X = H^1(0, 2\pi) \times L^2(0, 2\pi)$. Then the operator D is bounded in X and the operator L is closed with domain $X = H^4(0, 2\pi) \times H^1(0, 2\pi)$. The properties in Hypothesis 2.1 are easily checked [7], and then the criterion in Theorem 1 implies the linear transverse instability of ϕ_c .

For the Euler equations, the linear transverse instability of the line periodic waves constructed in [24] has been shown in [15]. Though the abstract result in Theorem 1 is not explicitly used, its proof is part of the analysis done in [15]. We point out that the most difficult part of this analysis consists in showing that the operator L possesses a pair of complex conjugated simple purely imaginary eigenvalues.

Weak surface tension ($\beta < 1/3$). In contrast to the case of large surface tension, when $\beta < 1/3$, line periodic waves for the Euler equations (3.5)-(3.8) bifurcate at two different values of the parameter α , for $\alpha = 1$ and $\alpha = \alpha_*(\beta) > 1$, and their existence has been shown in [19]. For $\alpha \approx 1$ some of these line periodic waves are well-approximated by the KP-II equation, but this equation does not capture the full bifurcation picture in this case. So far, it is not clear what would be a good model equation in this regime. For $\alpha \approx \alpha_*(\beta)$, the dynamics can be approximated by an elliptic-elliptic focusing Davey-Stewartson system,

$$\begin{aligned} iA_t + A_{xx} + A_{yy} + (\gamma_1|A|^2 + \gamma_2\phi_x)A &= 0, \\ \gamma_3\phi_{xx} + \phi_{yy} - \gamma_3(|A|^2)_x &= 0, \end{aligned}$$

in which $\gamma_1 + \gamma_2 = 2$ and γ_2, γ_3 are positive real numbers (see [11] and the references therein). Using the abstract criterion in Theorem 1 the transverse linear instability of line periodic waves of this equation has been proved in [6]. For the Euler equations, this is an open problem. However, one would expect that transverse instability can be proved by combining the arguments in [12, 15], and using the result in Theorem 1.

Critical surface tension ($\beta = 1/3$). For the Euler equations (3.1)-(3.4), the existence of line periodic waves in this regime follows from the analysis in [2]. They are found for $\alpha \approx 1$, $\beta \approx 1/3$, and $\beta < 1/3$. In this regime, the simplified model is a 5th order KP equation,

$$(u_t + cu_x + 6uu_x + u_{xxx} + u_{xxxxx})_x + u_{yy} = 0, \tag{3.11}$$

written here in a reference frame moving with speed c and normalized form. The transverse instability of line periodic waves of this equation has been shown in [18]. The proof does not make use of the

general criterion in Theorem 1, but this abstract result can also be applied, in the way explained before for the KP-I equation. The necessary spectral result on the operator L can be obtained from the results in [18]. As in the case of weak surface tension, for the Euler equations the question is open.

3.3 Transverse stability of gravity water waves

In the absence of surface tension, $\beta = 0$, line periodic traveling waves of the Euler equations have been constructed in [24]. They bifurcate at $\alpha = 1$ and are approximately described by the KdV equation (3.10), just as the periodic waves found in the large surface tension regime. However, in the absence of surface tension one expects these waves to be transversely stable for general bounded perturbations [21]. In this regime, the model equation is the KP-II equation

$$(u_t + cu_x + 6uu_x + u_{xxx})_x + u_{yy} = 0, \quad (3.12)$$

having the same 2π -periodic steady solutions ϕ_c , for $c > 1$, as the KP-I equation (3.9). Relying upon the counting result in Theorem 2, the transverse stability of these periodic solutions have been shown in [17]. We present the main steps of the proof and refer to [17] for further details.

The first step of the analysis consists in suitably formulating the spectral stability problem in order to apply the general result in the Corollary 2.4 of Theorem 2. The linearization of the KP-II equation (3.12) at ϕ_c is given by

$$(w_t + w_{xxx} + cw_x + 6(\phi_c(x)w)_x)_x + w_{yy} = 0. \quad (3.13)$$

Following [14], we consider solutions of the form

$$w(x, y, t) = e^{\lambda t + ipy} W(x),$$

with W satisfying the differential equation

$$\lambda W_x + W_{xxx} + cW_{xx} + 6(\phi_c(x)W)_{xx} - p^2 W = 0.$$

The left hand side of this equation defines a linear differential operator with 2π -periodic coefficients

$$\mathcal{A}_{c,p}(\lambda) = \lambda \partial_x + \partial_x^4 + c \partial_x^2 + 6 \partial_x^2 (\phi_c(x) \cdot) - p^2,$$

and the spectral stability problem is concerned with the invertibility of this operator, for certain values of p and in a suitable function space. The periodic wave ϕ_c is spectrally stable if $\mathcal{A}_{c,p}(\lambda)$ is invertible for any $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$, and unstable otherwise. The type of the perturbations determines the choice of the underlying function space and the values of p . For general bounded two-dimensional perturbations of the periodic wave, we assume that $\mathcal{A}_{c,p}(\lambda)$ acts in $C_b(\mathbb{R})$, the Banach space of uniformly bounded continuous functions on \mathbb{R} , and consider any real number p .

A standard Bloch decomposition shows that the operator $\mathcal{A}_{c,p}(\lambda)$ is invertible in $C_b(\mathbb{R})$ if and only if the operators

$$\mathcal{A}_{c,p}(\lambda, \gamma) = \lambda(\partial_x + i\gamma) + (\partial_x + i\gamma)^4 + c(\partial_x + i\gamma)^2 + 6(\partial_x + i\gamma)^2(\phi_c(x) \cdot) - p^2,$$

are invertible in the space $L_{per}^2(0, 2\pi)$ of square-integrable 2π -periodic functions, for any $\gamma \in [0, 1)$. Restricting to $\gamma \in (0, 1)$ (see [17] for the case $\gamma = 0$), the operator $\partial_x + i\gamma$ has a bounded inverse in $L_{per}^2(0, 2\pi)$, so that $\mathcal{A}_{c,p}(\lambda, \gamma)$ is invertible if and only if λ belongs to the resolvent set of the operator

$$\mathcal{B}_{c,p}(\gamma) = -(\partial_x + i\gamma)^3 - c(\partial_x + i\gamma) - 6(\partial_x + i\gamma)(\phi_c(x) \cdot) + p^2(\partial_x + i\gamma)^{-1}, \quad (3.14)$$

which is a closed operator in $L_{per}^2(0, 2\pi)$ with domain $H_{per}^3(0, 2\pi)$. The stability problem consists now in the study of the spectrum of $\mathcal{B}_{c,p}(\gamma)$. This operator has compact resolvent, hence point spectrum consisting of isolated eigenvalues with finite algebraic multiplicities, only, and more importantly, it has the JL product structure in Section 2.2,

$$\mathcal{B}_{c,p}(\gamma) = J(\gamma)L_{c,p}(\gamma), \quad (3.15)$$

with

$$J(\gamma) = (\partial_x + i\gamma), \quad L_{c,p}(\gamma) = -(\partial_x + i\gamma)^2 - c - 6\phi_c(x) + p^2(\partial_x + i\gamma)^{-2}. \quad (3.16)$$

It is not difficult to check that the operators $J(\gamma)$ and $L_{c,p}(\gamma)$ satisfy the properties required by the Hypothesis 2.3.

The key step consists in finding nonnegative operators $K = K_{c,p}(\gamma)$ such that Hypothesis 2.3 holds. The construction of these operators is based on the approach developed for the KdV equation in [4]. First, we construct the operators

$$M_{c,p}(\gamma) = (\partial_x + i\gamma)^4 + 10(\partial_x + i\gamma)\phi_c(x)(\partial_x + i\gamma) - 10c\phi_c(x) - c^2 + \frac{5}{3}p^2(1 + c(\partial_x + i\gamma)^{-2}), \quad (3.17)$$

which satisfy the commutativity property (2.4), and then we consider a linear combination of the operators $M_{c,p}(\gamma)$ and $L_{c,p}(\gamma)$,

$$K_{c,p}(\gamma) = M_{c,p}(\gamma) - bL_{c,p}(\gamma), \quad (3.18)$$

for some real number b . We prove that for suitably chosen b , the operator $K_{c,p}(\gamma)$ is nonnegative, which allows then to apply the result in Corollary 2.4 and conclude that the periodic waves ϕ_c are spectrally transversely stable. In addition, it has been shown in [17] that ϕ_c is linearly transversely stable with respect to perturbations which are $2N\pi$ -periodic in x (subharmonic perturbations).

For the Euler equations, the question of spectral stability of these line periodic waves is a particularly challenging open problem.

4 Discussion

In this presentation we focused on the transverse stability question for line periodic waves in several models for water waves. All these results concern either the linear instability or the spectral stability of periodic waves, the questions of nonlinear orbital stability or instability being widely open.

Besides periodic waves, the Euler equations possess several classes of solitary waves. Being much more studied, their dynamics is better understood. As for the periodic waves, it turns out that gravity-capillary solitary waves are transversely unstable, whereas pure gravity solitary waves are transversely stable. For large surface tension, the transverse instability of solitary waves goes back to the work of Kadomtsev and Petviashvili [21] and their derivation of the KP equations (see also [1]). The nonlinear transverse instability of these waves have been studied in [29], where it is also shown that the solitary waves can be stabilized by suitably choosing the set of perturbations. For the Euler equations, the linear transverse instability of solitary waves has been first proved in [9], and their nonlinear transverse instability has been recently shown in [28]. The case of weak surface tension has been considered more recently in [11] for the Davey-Stewartson system and in [12] for the Euler equations. Both results prove linear transverse instability. While the case of critical surface tension remains open, there are several results on transverse stability of solitary waves in the absence of surface tension, but for the KP-II equation, only. We refer to the recent works [26] and [25] showing the transverse nonlinear stability of solitary waves for periodic transverse perturbations and for fully localized perturbations, respectively.

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