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THE STABILITY OF THE MINKOWSKI SPACE FOR THE EINSTEIN–VLASOV SYSTEM

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Abstract. This text serves as an introduction to the article [23] written in collaboration with David Fajman and Jérémie Joudioux, and presented at the Laurent Schwarz Seminar in March 2018. In [23], we establish the global stability of the Minkowski space viewed as the trivial solution of the Einstein-Vlasov system. To estimate the Vlasov field, we use the vector field and modified vector field techniques developed in [21, 22]. In particular, the initial support in the velocity variable does not need to be compact. To control the effect of the large velocities, we identify and exploit several structural properties of the Vlasov equation to prove that the worst non-linear terms in the Vlasov equation either enjoy a form of the null condition or can be controlled using the wave coordinate gauge. The basic propagation estimates for the Vlasov field are then obtained using only weak interior decay for the metric components. Since some of the error terms are not time-integrable, several hierarchies in the commuted equations are exploited to close the top order estimates. For the Einstein equations, we use wave coordinates and the main new difficulty arises from the commutation of the energy-momentum tensor, which needs to be rewritten using the modified vector fields.

1. Introduction

This text serves as an introduction to the article [23] written in collaboration with David Fajman and Jérémie Joudioux, and presented at the Laurent Schwarz Seminar in March 2018. In [23], we establish the global stability of the Minkowski space $(\mathbb{R}^{1+3}, \eta)$, where η is the Minkowksi metric given in global Cartesian coordinates by $\mathrm{diag}(-1, 1, 1, 1)$, viewed as the trivial solution of the Einstein-Vlasov system

$$Ric(g) - \frac{1}{2}R(g)g = T[f], \tag{1.1}$$

$$T_g(f) = 0. \tag{1.2}$$

Here g is a Lorentzian metric on a 4-dimensional manifold, $Ric(g)$ and $R(g)$ the Ricci and scalar curvatures of g, f is a massive Vlasov field, $T[f]$ its energy-momentum tensor and T_g is the geodesic spray vector field. The Minkowski space is then the simplest solution to these equations with $f = 0$.

We refer to Section 5.1 below as well as [48, 19, 49] for a presentation of the equations and the terminology used here. We recall that the system (1.1)-(1.2) admits an initial value problem formulation which is, at least when the initial data enjoy sufficient regularity, locally well-posed [11, 49]. We are therefore interested in the global Cauchy problem for this system, that is to say we want to understand the asymptotics of the solutions.

The Einstein-Vlasov system is actively used in astrophysics and cosmology. It describes a statistical ensemble of self-gravitating particles which interact only indirectly through the

Einstein equations. It is in fact the natural fully general relativistic analogue of the Vlasov-Poisson system[1], replacing Newtonian mechanics by general relativity.

2. The vacuum problem

At least from a PDE perspective, there are two fundamental differences between the Poisson and the Einstein equations. The Poisson equation is elliptic and linear in the gravitational potential, while the Einstein equations are (after a suitable gauge choice) hyperbolic and non-linear in the metric components. In particular, in Newtonian mechanics, no matter source implies no gravitational force, while in general relativity, there are plenty of non-trivial, vacuum solutions to the Einstein equations $Ric(g) = 0$. Thus, the stability of the Minkowski space for the *vacuum* Einstein equations is a necessary starting point. This problem was solved in full generality by Christodoulou and Klainerman [13] (see also [41, 8] and [40, 24]).

Let us recall some of the main features of the problem and its proof. The vacuum Einstein equations can be recast in so-called *wave coordinates* as a system of quasilinear wave equations and the stability of the Minkowski space then corresponds to a small data global existence result for this system. For any quasilinear system of wave equations, controlling the non-linearities requires higher order estimates of the solutions, that is to say estimates obtained after commutation of the equations with well chosen vector fields, so as to control a high number of derivatives of the solutions. These vector fields typically arise from the symmetries of the linearised equations. For the wave equation, they are thus the Killing and (some of the) conformal Killing fields of Minkowski space. One then combines these higher order estimates with weighted Sobolev inequalities linked to the equations (this is the vector field method of Klainerman [34]) to prove decay estimates for the non-linear terms[2].

It is well known that the case of three spatial dimensions is critical for this type of questions. In higher spatial dimensions, linear waves enjoy stronger decay properties, so that such small global existence results always hold for general quasilinear wave equations [34], while in dimension 3, small data global existence is linked to structural properties of the equations and blow up is known to occur in some cases [33]. A general criterion that guarantees small data global existence is the null condition of Klainerman [35]. Essentially, for a solution to the free wave equation, it is well known that derivatives tangential to the light-cone decay faster than the transversal ones and the null condition ensures that each non-linear product contains derivatives tangential to the light-cone.

The null condition is however not satisfied in the wave coordinate formulation of the Einstein equations [12]. Thus, the strategy of [13] exploits another formulation of the Einstein equations, where the main energy estimates control the curvature rather than the metric itself. Another key element of [13] is the construction of an optical function, and then vector fields, which are tied to the characteristics of the spacetime, or equivalently, to the null cones of the metric[3].

[1] We refer to the classical [27] for a presentation of the Vlasov-Poisson and related kinetic systems.

[2] There are many other ways to establish decay estimates, though the vector field method is certainly the most robust one. In particular, we stress that a standard strategy for quasilinear wave equations consists in using the basic vector field method and energy estimates to obtain first, rough, decay estimates for the solutions under weak assumptions, and, only in a second step, use another method, for instance, integral estimates and representation formulas, to obtain improved decay estimates.

[3] Interestingly, the null cones of the constructed spacetimes eventually diverge logarithmically compared to the cones of Minkowski space, as a remnant of the failure of the null condition.

Even though the Einstein equations in wave coordinates do not satisfy the null condition, the stability of the Minkowski space was subsequently obtained in this gauge in [40, 41]. The key observation is that the Einstein equations still enjoy a weak version of the null condition, of which a trivial example is provided by the system

$$\begin{aligned} \Box u &= \partial_t v \cdot \partial_t v, \\ \Box v &= 0, \end{aligned} \tag{2.1}$$

where u and v are two scalar functions defined on $\mathbb{R}_t \times \mathbb{R}^3$. The second equation is linear, thus the first is simply a linear inhomogenous wave equation and obviously the solutions of this system do not blow up.

In the case of the Einstein equations, this trivial example is replaced by a hierarchy of wave equations for the metric components. Moreover, in order to control the non-linear terms not satisfying the null condition, the wave gauge condition $\Box_g x^\alpha = 0$ for the coordinates x^α is used extensively.

3. The mass problem

Recall that an initial data set for the vacuum Einstein equation $Ric(g) = 0$ is given by (Σ, g_0, k), where (Σ, g_0) is a smooth Riemannian manifold and k is a symmetric 2-tensor field, such that (Σ, g_0, k) solves the constraint equations

$$\begin{aligned} R(g_0) - |k|^2 + tr_{g_0}(k)^2 &= 0, \\ \mathrm{div} k - d(tr_{g_0} k) &= 0, \end{aligned}$$

where $R(g_0)$ is the scalar curvature of (Σ, g_0), $|k|^2 = k_{ij}k^{ij}$, $tr_{g_0}k = k_{ij}g_0^{ij}$, $[\mathrm{div} k]_j = {}^{(g_0)}\nabla^i k_{ij}$, with ${}^{(g_0)}\nabla$ the Levi-Civita connection of g_0.

In the case of perturbations of the Minkowski space, one considers initial data such that $\Sigma = \mathbb{R}^3$ and the data are asymptotically flat i.e. g_0 tends to the Euclidean metric and k tends to 0 as $|x| = r \to \infty$. The positive mass theorem [51, 60] then implies that $g_0 = \delta_E(1 + 2m/r) + o(r^{-1-\rho})$, where δ_E is the Euclidean metric, $\rho > 0$ and where $m > 0$ unless the initial data correspond to an initial data set induced by the Minkowski space, in which case the solution of the evolution problem must naturally coincide with Minkowski space.

The positive mass theorem limits the possible radial decay of the initial data. In particular, one cannot consider compact initial data, for which the metric perturbations would be all contained in some ball of finite radius. The closest one can get from those are initial data corresponding to the Schwarzschild metric outside from some compact set. We refer to [15, 14, 16] for general methods leading to the construction of such data.

For a solution of the linear wave equation in Minkowski space $\Box\psi = 0$, the interior decay of ψ, i.e. estimates of the form $|\psi(t, x)| \lesssim C(R)\frac{1}{(1+t)^p}$ for $|x| < R$, is directly related to the amount of radial decay of the initial data for ψ. The stronger the decay, the higher the value of p. In view of the r^{-1} behavior of the initial data for the perturbations, this implies that, even at the linearised level, we cannot expect interior decay faster than t^{-1} for the metric perturbations $|g - \eta|$ and t^{-2} for their first derivatives $|\partial g|$.

4. Einstein-matter systems

Consider now a coupled system of the form

$$Ric(g) - \frac{1}{2} g R(g) = T[\psi], \tag{4.1}$$

$$N_g(\psi) = 0, \tag{4.2}$$

where $T[\psi]$ is the energy-momentum tensor of some matter field ψ, itself subject to an evolution equation depending on the metric g, which we write schematically as (4.2).

If ψ solves a wave equation, as in the scalar field case $\Box_g \psi = 0$ or the Maxwell equations, the matter equation (4.2) can be treated by the same methods as the Einstein equations themselves. In particular, one can commute the Einstein equations and the matter equations by the same vector fields and thus extend the vacuum stability results to these cases [41, 8, 43, 53].

One of the simplest models for which this approach does not readily work is the Einstein-Klein-Gordon system, where the matter field ψ is a scalar function solving the Klein-Gordon equation

$$\Box_g \psi - \psi = 0, \tag{4.3}$$

and where the energy momentum tensor is given by

$$T[\psi] = d\psi \otimes d\psi - \frac{1}{2} g \left(g(\nabla \psi, \nabla \psi) + \psi^2 \right).$$

The Klein-Gordon equation shares many properties with the wave equation, but it has less symmetries. In particular, it enjoys poor commutation properties with respect to the scaling vector field $S = x^\alpha \partial_{x^\alpha}$. Moreover, in dimension 3, the interior decay for $\partial \psi$ is limited by $\frac{1}{t^{3/2}}$, which is weaker than the maximal interior decay one can obtain for the first derivatives of the metric components in the vacuum case. It does enjoy on the other hand stronger decay near the light cone than a pure wave. Finally, the classical vector field method of Klainerman for Klein-Gordon fields [36] typically requires the use of a *hyperboloidal foliation*, while the analysis of the vacuum Einstein equations as in [13, 41] uses only a foliation by standard, asymptotically flat, spacelike hypersurfaces as well as a foliation by null cones.

In view of (or despite) the above difficulties, the stability of the Minkowski space for the Einstein-Klein-Gordon system was only recently obtained in [38] (see also [56]). In some sense, this is the first stability result (in three spatial dimensions, without symmetry or cosmological constant) for an Einstein-matter system which cannot be obtained by a direct extension of the methodology of the vacuum case.

5. The Einstein-Vlasov system

The Einstein-Vlasov system (1.1)-(1.2) couples the Einstein equations to kinetic theory. The necessary notions as well as the notation we use here, in particular, the definition of the submanifold $\mathcal{P}$ and the transport operator T_g is reviewed in the next section.

For particles of mass m_p, the Vlasov field f is then a non-negative function defined on the submanifold $\mathcal{P}$ of the tangent bundle[4] corresponding to future-directed causal vectors normalized to $-m_p^2$. The Vlasov field f then is, at each point of $\mathcal{P}$, the density of particles with given position and velocity (or momentum). The Vlasov equation $T_g(f) = 0$ is the conservation of this particle density by the geodesic flow. The local Cauchy theory for the Einstein-Vlasov system was treated in [11] (see also [49], Chapter 6). In particular, to any given appropriate initial data set (Σ, g_0, k, f_0), one can associate a unique (up to diffeomorphism) maximal Cauchy development (M, g, f), where (M, g) is a Lorentzian manifold and f a Vlasov field.

[4] Since we can use the metric to identify the tangent and cotangent bundles, we can also consider f as a function on a submanifold of the cotangent bundle. While this is perhaps less common, we shall actually use this formulation here, see Section 5.1.

5.1. **Vlasov fields in the cotangent bundle formulation.** Let (M, g) be a smooth time-oriented, oriented, 4-dimensional Lorentzian manifold.

We denote by $\mathcal{P}$ the mass-shell. While it is generally considered as a submanifold of the tangent bundle TM, we shall, equivalently, consider here $\mathcal{P}$ as a subset of the cotangent bundle[5] $T^\star M$, defined by

$$\mathcal{P} := \left\{(x, v) \in T^\star M \ : \ g_x^{-1}(v, v) = -1 \text{ and } v \text{ future oriented}\right\}.$$

Given a coordinate system on M, (U, x^α), for any $x \in U \subset M$, any $v \in T_x^\star M$ can be written as

$$v = v_\alpha [dx^\alpha]_x$$

and the functions $v \to v_\alpha$ can be used to define a coordinate system on $T_x^\star M$ called *conjugates* to the coordinates (x^α). In the following, we consider such coordinate systems even if it is not stated explicitly. We denote by π the canonical projection

$$\pi : \mathcal{P} \to M.$$

For $x \in M$, we define a metric on $T_x^\star M$ by

$$g^{-1}_{T_x^\star M} = g^{\alpha\beta} dv_\alpha dv_\beta,$$

where $g^{\alpha\beta}$ are the components of g^{-1} in a local coordinate system (U, x^α) and v_α are conjugate to the x^α. Let $d\mu_{T_x^\star M}$ be the associated volume form, i.e. $d\mu_{T_x^\star M} = \sqrt{-g^{-1}} dv_0 \wedge dv_1 \wedge dv_2 \wedge dv_3$ and let q_x be the map

$$\begin{aligned} q_x : T_x^\star M &\to \mathbb{R}, \\ v &\mapsto g^{-1}_{T_x^\star M}(v, v). \end{aligned}$$

Let dq be its differential (in v). Since $\pi^{-1}(x) = q^{-1}(\{-1\})$ is a level set of q, $dq = 2 dv_\alpha v_\beta g^{\alpha\beta}$ is normal to $\pi^{-1}(x)$ and on $\pi^{-1}(x)$, there is a unique volume form denoted $d\mu_{\pi^{-1}(x)}$ such that

$$d\mu_{T_x^\star M} = \frac{1}{2} dq \wedge d\mu_{\pi^{-1}(x)}.$$

We assume that there exist local coordinates such that $x^0 = t$ is a smooth temporal function, i.e. it is strictly increasing along any future causal curve and its gradient is past directed and timelike[6]. In that case, the algebraic equation

$$v_\alpha v_\beta g^{\alpha\beta} = -1 \text{ and } v_\alpha \text{ future directed}$$

can be solved for v_0 by

$$v_0 = -(g^{00})^{-1} \left(g^{0j} v_j - \sqrt{(g^{0j} v_j)^2 + (-g^{00})(1 + g^{ij} v_i v_j)} \right). \tag{5.1}$$

It follows that (x^α, v_i), $1 \le i \le 3$ are smooth coordinates on $\mathcal{P}$ and for any $x \in M$, (v_i), $1 \le i \le 3$ are smooth coordinates on $\pi^{-1}(x)$. With respect to these coordinates, the volume form $d\mu_{\pi^{-1}(x)}$ reads

$$d\mu_{\pi^{-1}(x)} = \frac{\sqrt{-g^{-1}}}{v_\beta g^{\beta 0}} dv_1 \wedge dv_2 \wedge dv_3.$$

[5] This formulation is linked with the Hamiltonian property of the equations, cf [19].

[6] The fact that the gradient of t is timelike is equivalent to $g^{00} < 0$ and the property of being strictly increasing along any future causal curve implies that the induced metric on each level set of t has to be positive.

For any sufficiently regular[7] distribution function $f : \mathcal{P} \to M$, we define its energy-momentum tensor as the tensor field

$$T_{\alpha\beta}[f](x) = \int_{\pi^{-1}(x)} v_\alpha v_\beta f d\mu_{\pi^{-1}(x)}. \tag{5.2}$$

In the following, to simplify the notation, we write

$$\int_{\pi^{-1}(x)} \text{ as } \int_v \text{ and } d\mu_v \text{ for the measure } d\mu_{\pi^{-1}(x)}.$$

Even on a curved spacetime, we use another reference measure, namely that corresponding to the Minkowski space $\frac{dv}{\sqrt{1+|v|^2}}$. When we do so, we write the measure explicitly.

The Vlasov field f is required to solve the *Vlasov equation*, which can be written in the (x^α, v_i) coordinate system as

$$T_g(f) := g^{\alpha\beta} v_\alpha \partial_{x^\beta} f - \frac{1}{2} v_\alpha v_\beta \partial_{x^i} g^{\alpha\beta} \partial_{v_i} f = 0. \tag{5.3}$$

It follows from the Vlasov equation that the energy-momentum tensor is divergence free for solutions of the Vlasov equation. More generally, for any sufficiently regular distribution function $k : \mathcal{P} \to \mathbb{R}$,

$$\nabla^\alpha T_{\alpha\beta}[k] = \int_v T_g[k] v_\beta d\mu_v.$$

5.2. **Previous results on the stability of the Minkowski space for the Einstein-Vlasov system.** The stability of the Minkowski space for the spherically symmetric Einstein-Vlasov system in dimension $3+1$ has been treated in [45, 47] for the massive case and in [17] for the massless case with compactly supported initial data. A proof of stability for the massless case without spherical symmetry and with compact support in both x and v has been given in [55]. As in [17], the compact support assumptions and the fact that the particles are massless are important as they allow to reduce the proof to that of the vacuum case outside from a strip going to null infinity.

In [23], we prove the stability of the Minkowski space for the Einstein-Vlasov system in the case of a Vlasov field corresponding to massive particles. For simplicity, we assume that all particles have the same mass m_p and we later fix $m_p = 1$.

5.3. **Statement of the results.** The main result of [23] can then be stated as follows.

Theorem 5.1. *Let $(\Sigma = \mathbb{R}^3, g_0, k, f_0)$ be an initial data set for the Einstein-Vlasov system which coincides with a Schwarzschild initial data set of mass $m \geq 0$ outside from a ball of radius $R > 0$.*

Let (M, g, f) be the unique[8] maximal globally hyperbolic development of the given initial data set and denote by $i : \Sigma \to M$ the embedding of Σ into a Cauchy hypersurface of M given by the local existence theorem.

[7] By "sufficiently regular", we mean that f is smooth enough and decays in v sufficiently fast so that $T[f]$ is well-defined and the necessary integration by parts in v can be performed. Later, we also perform integration in x, so we also require the regular distribution function f to obey decay in the x variable along each hyperboloid. In any case, one can assume for simplicity that all distribution functions are smooth and compactly supported for all computations to hold.

[8] As usual, by uniqueness, we mean uniqueness up to diffeomorphism.

Let $N \geq 14$, $q \geq 3$ and $\epsilon > 0$. Assume that

$$\begin{aligned} ||g_0 - \delta_E||_{H^N(|x|<R)} + ||k||_{H^{N-1}(|x|<R)} + m^2 \\ +||(1+|v|^2)^q f_0||_{W^{N+3,1}(T^\star\mathbb{R}^3)} + ||(1+|v|^2)^{q+2} f_0||_{W^{N-2,1}(T^\star\mathbb{R}^3)} \leq \epsilon, \end{aligned}$$

where δ_E is the Euclidean metric and m the mass of the Schwarzschild metric for $|x| \geq R$.

Then, there exists $\epsilon_0(R) > 0$ such that if $\epsilon \leq \epsilon_0(R)$, there exists a global system of wave coordinates (t,x) on $\mathbb{R}^4 \simeq M$ such that, t is a temporal function, $i(\Sigma) = \{t = 2\}$ and with $\mathcal{K} := \{(t,x) \,/\, |x| < t-1\}$, for $(t,x) \in J^+(i(\Sigma)) \setminus \mathcal{K}$, g coincides with the Schwarzschild metric of mass m, while for $(t,x) \in \mathcal{K} \cap J^+(i(\Sigma))$, we have

$$\begin{aligned} \mathcal{E}_N[g](\rho) &\leq D_N \epsilon \rho^{D_N \epsilon^{1/2}}, \\ E_{N-2,q+2}[f](\rho) &\leq D_N \epsilon \rho^{D_N \epsilon^{1/2}}, \\ E_{N,q}[f](\rho) &\leq D_N \epsilon \rho^{D_N \epsilon^{1/2}}, \\ ||g(t,x) - \eta||_{L^\infty_x} &\leq D_N \epsilon^{1/2} (1+t)^{-1+D_N \epsilon^{1/2}}, \end{aligned}$$

where $\rho = \sqrt{t^2 - |x|^2}$ denotes a hyperboloidal time function, D_N is a constant depending only on N and $\mathcal{E}_N[g]$, $E_{N,q}[f]$ are energy norms depending on up to N derivatives of f and g.

In particular, (M,g) is future causal geodesically complete.

Remark 5.2. A similar stability result has been obtained independently by Lindblad and Taylor [42].

Remark 5.3. A similar statement holds for the past of $i(\Sigma)$. Moreover, redefining some of the coordinates, we can shift the slice $\{t = 2\}$ to any other $t = \textit{const}$ slice.

Remark 5.4. We refer to [23], Section 2.9 and 2.10 for a precise definition of the norms $\mathcal{E}_N(g)$ and $E_{N,q}(f)$. Roughly speaking, $\mathcal{E}_N(g)$ is a natural energy norm for g associated to the hyperboloidial foliation, obtained by from a first order energy norm by commutation with Minkowskian vector fields. For the Vlasov fields, we use on the other hand modified vector fields, as explained below in Section 5.4.3.

The index q in $E_{N,q}(f)$ refers to the number of additional v weights, so that $E_{0,0}(f)$ correspond to the natural energy norm of f. The norms $||.||_{H^N}$ and $||.||_{W^{N+3,1}}$ which we use for the initial data are standard Sobolev norms.

Remark 5.5. When $|\alpha| \geq N-2$, we prove L^2 decay estimates for source terms of the form $T[\widehat{K}^\alpha(f)]$ arising in the wave equations. These require more regularity for the Vlasov field, hence three extra derivatives are required for the initial datum of f.

Remark 5.6. We refer to the body of the proof for many extra details concerning the asymptotics of the solutions. For instance, for $q' \geq 0$ sufficiently small (in particular $q' = 0$, corresponding to the basic energy norm), we prove bounds $E_{N-2,q'}(f) \lesssim \epsilon$ without growth. Moreover, we obtain sharp pointwise decay estimates on the components of the energy-momentum tensor $T[f]$ as well as its derivatives.

Remark 5.7. The geodesic completeness is a direct consequence of the asymptotics of the metric and its derivatives. See for instance [40], Section 16.

Remark 5.8. For simplicity, we have considered initial data which coincides with the Schwarzschild data outside of a compact set. Since we use a hyperboloidal foliation, our results do not extend immediately to more general data that would allow the Vlasov field to have non-compact support in the x variable. We note however that the method of this paper are readily applicable (with slightly different asymptotics) for initial data such that f is initially supported in $\mathcal{B}_x \times \mathbb{R}^3_v$, for some compact set $\mathcal{B}_x$, and the data for the metric is such that the analysis of [13] or [41] is applicable. Indeed, in that case, using a standard domain of dependence argument, the solution is vacuum outside from the domain of influence of $\mathcal{B}_x$ and we can repeat the analysis of [13] or [41] in that region. As is clear from the proof of our theorem, the techniques we use do not depend on the exact nature of the asymptotics of the metric at spatial and null infinity. In particular, we prove our main propagation estimates for the Vlasov field using only weak interior decay for the metric coefficients ($|\partial g|(t,x) \lesssim t^{-3/2+\delta}$ for $|x| < \frac{t}{2}$). See also the recent work of Bigorgne [9] for the Vlasov-Maxwell in dimension greater than 4 and those of Wang [58, 59] concerning the Vlasov-Norström and the Vlasov-Norström system in dimension 3, where no compact support in both x and v are assumed.

5.4. **Key elements of the proof and main difficulties.**

5.4.1. *The coupling.* At the linear level, the Vlasov equation is given by the free transport equation on Minkowski space

$$v^\alpha \partial_{x^\alpha} f = 0,$$

for $f := f(t,x,v)$, $(x^\alpha) = (t,x) \in \mathbb{R}^{1+3}$, $(v^\alpha) = \left(v^0, v^i\right)$, with $v^0 = \sqrt{m_p^2 + |v|^2}$, $(v^i) \in \mathbb{R}^3$. In particular, for massive particles, $m_p > 0$, the characteristics of the Vlasov equation are the timelike geodesics, while for massless particles $m_p = 0$, the characteristics are the null geodesics, as for the wave or the Einstein equations.

As in the case of the Einstein-Klein-Gordon equations (4.1)-(4.3), the coupling is non-trivial.

- Kinetic equations such as the Vlasov equation are intrinsically of different nature compared to wave equations. The domain of definition of the unknown f is a different manifold ($\mathcal{P}$) and the coupling through the energy-momentum tensor $T[f]$ takes the form of velocity averages of f, i.e. (weighted) integrals in v of f. In fact, the Einstein-Vlasov system is a system of integro-partial-differential equations and not a pure PDE system.
- For massive particles, the characteristics are different from those of the wave equation. For the free transport operator, they are given by the timelike curves $\left(t, t\frac{v^i}{\sqrt{1+|v|^2}}\right)$. Note that for high velocities $|v| \to +\infty$, these curves approach the null curves $(t, \omega^i t)$, where $\omega^i = \frac{v^i}{|v|} \in \mathbb{S}^2$. For low $|v|$, we do expect to face the difficulty of an equation that does not share the characteristics of the wave equation. On the other hand, for large $|v|$, we expect the difficulties associated with pure wave equations, such as the slow decay of transversal derivatives to the light-cone, to also be an issue. This difficulty naturally disappears for distributions which are of compact v support initially, but we treat here initial data which are merely integrable in v against a measure $(1+|v|^2)^{k/2} dv$.

5.4.2. *Commuting the Vlasov equation using complete lifts.* Another important difficulty arise from commuting the Vlasov equation.

Recall that we cannot expect to control the behaviour of the metric components without commuting the Einstein equations. In view of the coupling, this implies that we must estimate $K^N T[f]$, where K^N is a differential operator of order N. In flat space, where g is the Minkowski metric η, we have

$$T_{\alpha\beta}[f] = \int_{v\in\mathbb{R}^3} f v_\alpha v_\beta \frac{dv}{\sqrt{m_p^2 + |v|^2}},$$

so that, for any vector field $K = K^\alpha \partial_{x^\alpha}$,

$$KT[f] = T[K(f)].$$

The vector fields K are those that commute with the flat wave operator, i.e. the Killing and conformal Killing fields[9] of Minkowski space.

In general, if K is a Killing vector field on a Lorentzian manifold, $K(f)$ does not readily make sense, since f is defined on a different manifold. Thus, one needs first to lift K to $\mathcal{P}$. There are several such possible lifts, but the one we consider here is the *complete lift* of K, denoted $\widehat{K}$. Complete lifts have the following properties.

- The complete lift operation lifts vector fields on M to vector fields on TM.
- If K is Killing, then $\widehat{K}$ is tangent to the submanifold $\mathcal{P}$ of TM. In particular, for any regular distribution f defined on $\mathcal{P}$, $\widehat{K}(f)$ is well-defined.
- If K is Killing, $\widehat{K}$ commutes with the geodesic spray vector field T_g.
- If K is Killing, $\mathcal{L}_K T[f] = T[\widehat{K}(f)]$, where $\mathcal{L}_K$ is the Lie derivative in the direction of K.

In [21], we exploited such a geometric treatment of the commutation properties of the Vlasov equation to extend the traditional vector field method of Klainerman for wave equations to the class of transport equations of Vlasov type[10]. In particular, we established Klainerman-Sobolev inequalities for velocity averages of Vlasov fields and gave an illustration of our method to obtain (almost) sharp asymptotics for the 3-dimensional massless and the $n \geq 4$ massive Vlasov-Nordström systems.

5.4.3. *Non-integrable decay and the modified vector fields.* While it seems that working with complete lifts would thus solve the difficulties involved with commuting the Vlasov equation and the energy-momentum tensor, for a general perturbation of Minkowski space, one should not expect any of the original Killing fields to remain Killing, so that none of the above properties can be directly applied. As a first step, one can write the Vlasov equation in coordinates, and then commutes the Vlasov equation with coordinate equivalents of the original vector fields of Minkowski space. For instance, let us write schematically the Vlasov equation as

$$T_g(f) = v^\alpha \partial_{x^\alpha} f + Q(\partial g, v, v)\partial_v f,$$

for some multi-linear form Q, and consider a Lorentz boost $Z_i = t\partial_{x^i} + x^i \partial_t$. In Minkowski space, the restriction to $\mathcal{P}$ of its complete lift would be given by $\widehat{Z_i} = t\partial_{x^i} + x^i\partial_t + v^0\partial_{v^i}$.

[9] We note that they are many variants of these methods. In particular, one can commute only with a subalgebra of the full algebra of Killing and conformal Killing fields (see for instance [37]), or, in another setting, one can commute with vector fields containing only radial weights, as in [18].

[10] See [62] for an extension of these methods to other dispersive PDEs.

Commuting the above equation, we obtain

$$T_g(\widehat{Z}_i f) = -[\widehat{Z}_i, Q(\partial g, v, v)\partial_v]f.$$

Neglecting the v components, the right-hand side leads to error terms of the form $\partial Z(g) \cdot \partial_v f$. On the other hand, for a solution of the free transport operator, $\partial_v f$ behaves essentially like $t\partial_{x^\alpha} f$. If we expect to prove boundedness for some norms of $|\partial_{x^\alpha} f|$, then $t\partial Z(g)$ needs to be time integrable in order to control $\widehat{Z} f$. Assuming that the interior decay for the metric components[11] can readily be used, in three spatial dimensions, $|t\partial Z(g)| \lesssim \frac{1}{t}$ leads to logarithmic divergences. Any loss in the Vlasov estimates would limit the interior decay for the metric even further, since in order to obtain sharp, or almost sharp, interior decay, one already needs sharp estimates on the source terms $K^N T[f]$ arising in the equations for the metric components.

This interesting issue is in fact already present in the much simpler Vlasov-Poisson system, where it was solved in [52] by modifying the commutation vector fields, replacing the lifted vector fields $\widehat{Z}$ by some $Y = \widehat{Z} + \Phi^i \partial_{x^i}$, where the coefficients Φ^i are functions in the variable (t, x, v), depending on the solution and constructed in order to cancel the worst error terms in the commutator formulas[12]. The method of modified vector fields was adapted to a basic model of wave/kinetic interaction, namely the 3-dimensional Vlasov-Nordström system, in [22]. Many of the difficulties described above are in fact present for this system. In particular, important strutural properties of the system where used in [22] in order to account for difficulties arising for large v.

In [23], we thus also consider commuting the Vlasov equation with modified vector fields. The use of modified vector fields is however not without drawbacks. Since the coefficients of these vector fields depend on the solution itself, they need to be estimated. Moreover, these coefficients depend on (t, x, v) and as a consequence, these modified vector fields cannot be used in return in the wave equations. Thus, an important effort is made to rewrite source terms of the form $K^N T[f]$, that arise after commuting the wave equations, in terms of the modified vector fields. Here, the integration in v present in the definition of $T[f]$ is crucially used. Finally, the Klainerman-Sobolev inequalities must also be rewritten using the modified vector fields.

5.4.4. *Hierarchy of equations and the null structure.* We first prove energy estimates for the Vlasov field assuming weak interior decay for the metric components. With only these weak estimates for the metric coefficients at our disposal, some of the error terms in the commuted Vlasov equation fail to be time-integrable. To close the estimates, we exploit a hierarchy in the commuted equations. More specifically, we first find replacements for the spatial translations that enjoy improved commutation properties with the Vlasov equation. These vector fields[13], denoted X_i, are simply given by $X_i = \partial_{x^i} + \frac{v^i}{\sqrt{1+|v|^2}}\partial_t$ and the improvement results from the

[11] For some metric components, there is already a logarithmic divergence for the interior decay estimates, so that the expected behaviour is in fact worse than the one presented here. Moreover, as the name indicates, the interior decay estimates are only valid in the interior region and the fact that they are not global is another source of difficulty that we neglected in this informal discussion, linked with the null structure of the equations.

[12] See also [31] for previous results concerning sharp asymptotics for solutions of the Vlasov-Poisson system based on the method of characteristics.

[13] We already used a version of these vector fields in [22].

identity

$$X_i = \frac{Z_i}{t} + \frac{\underline{v}_i}{w^0}\partial_t,$$

where $\underline{v}_i = v_i - \frac{x^i}{t}v^0$. Using this identity, one can prove that a product of the form $X_i(\psi) \cdot k$ for ψ a solution to the wave equation and k a solution to the Vlasov equation enjoys better decay properties compared to an arbitrary product $\partial\psi \cdot k$.

Assuming weak bounds on the first order energy, we then prove that commuting with X_i only produces integrable error terms, and thus obtain estimates for $E[X_i(f)]$. We then consider commuting with ∂_t. Only one term is not time-integrable (because of a lack of null structure), but it can be estimated using the bounds on $E[X_i(f)]$ and thus produces only a mild growth ρ^δ, for some small $\delta > 0$. We then commute with the modified vector fields Y and again find that the terms which are not time-integrable only depend on $E[X_i(f)]$ and $E[\partial_t(f)]$, which allows us to close the first order estimates. This hierarchy is then extended to the higher order estimates. It is in fact very reminiscent of similar hierarchies present in the context of the weak null condition, as in the system (2.1).

Once the basic energy estimates for the Vlasov field have been established, one can propagate stronger weighted norms, which then imply, together with the improved decay for the metric coefficients, energy and decay estimates for the Vlasov field without loss.

5.4.5. *The Einstein equations and the top order estimate.* The analysis of the Einstein equations in wave coordinates is now classical and we follow the approach of [40, 41] and its adaptation to the hyperboloidal foliation in [38]. The major new difficulty consists in rewriting and estimating the source terms coming from the Vlasov field in terms of the modified vector fields without any hard[14] loss of decay. However, the key step to avoid loss of decay involves an integration by parts in v, which, in turn, implies a loss of regularity. At top order, we therefore must allow for some hard loss of decay. The worst source term in the top order estimate for the metric coefficients then implies another source of small growth at top order.

5.5. **Related works.** We present here some previous works to put our results in context.

5.5.1. *Stability problems for Vlasov systems without sharp decay.* There is a large number of results concerning small data global existence for various systems of Vlasov type, as in [7, 28, 25]. In these works, the gravitational or electromagnetic fields satisfy a linear, inhomogeneous equation, whose source term is given by velocity averages of the Vlasov field. The linear aspect of the field equations implies that one can control the system at a much lower level of regularity than for a system of quasilinear wave equations. Moreover, these systems typically exhibit a gain of regularity, either because of the elliptic nature of the Poisson equation, or using a non-resonant phenomenon due to the difference between the characteristics of the waves and that of the massive particles. This allows to close the estimates without understanding sharp decays for the velocity averages of the Vlasov field and its derivatives.

5.5.2. *Sharp decay for derivatives.* The first work establishing sharp decay for derivatives of velocity averages of the Vlasov field is [31]. The question was revisited using vector field techniques in [52]. In [21] and [22], we developed and tested a vector field approach to derive sharp asymptotics for the Vlasov-Nordström system. The techniques of [31] have also been extended to the so-called Poisson-Yukawa system in dimension 2 [10]. More recently, they have been further study of these types of problems in order to consider data with more general

[14]We can afford a $\rho^{D\delta}$ for $\delta > 0$ small enough in these estimates and D being a positive constant.

asymptotics (and no compact support assumptions in x or v). In [9], Bigorgne first proved sharp decay for the Vlasov-Maxwell system in dimension greater than 4 using an extension of the vector field method for Vlasov field and an analysis of the null structure of the system. Wang has since obtained similar results for the Vlasov-Poisson [57], Vlasov-Norström [58] and Vlasov-Maxwell systems [59] (all in dimension 3), by combining the vector field and the analysis of the null structure with Fourier analytic techniques insprired by works such as [32].

5.5.3. *Non-trivial stationary states and further stability results.* The strongest results concerning the stability of non-trivial stationary solutions of the gravitational Vlasov-Poisson system have been obtained in [39]. They are not based on decay estimates but on a variational characterisation of the stationary solutions. On the other hand, this type of method does not provide asymptotic stability of the solutions but orbital stability. It is likely that any result addressing the question of asymptotic stability will need to go back to an appropriate linearization of the equations combined with robust decay estimates[15].

There is a large literature concerning the construction of stationary states for the Einstein-Vlasov system [46, 44, 5, 6, 3, 4, 61]. We refer to the living review [2], Section 5, for a detailed discussion of those results. Naturally, it would be interesting to understand the stability properties of any of these stationary solutions.

The vector field method has also been extended to the Kerr background to prove Morawetz estimates for massless Vlasov fields, see [1]. The approach relies on the use of multiplicative symmetries for massless fields.

5.5.4. *The cosmological case.* There is also a large amount of works concerning solutions to the Einstein-Vlasov system arising from initial data given on a compact manifold. Let us mention in particular the work of Ringström [49], concerning the study of expanding solutions with de-Sitter like asymptotics, as well as the stability result [20], where the slower expansion only provides polynomial decay for perturbations.

5.5.5. *Coupled systems.* There are several recent works involving coupled systems of equations for which the coupling is non-trivial, beyond the Einstein-Klein-Gordon system already mentioned. Let us mention in particular [29, 32, 26] concerning coupled systems of equations with different characteristics.

5.5.6. *Introductory materials on kinetic theory in general relativity.* There are many such materials but we would like to mention the classical texts [19, 48, 54] as well as the elegant geometric treatment of the Vlasov equation in [50].

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[15]See for instance [30] for some stability results using the linearization approach in the case of the spherically-symmetric King model.

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