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REGULARITY OF FREE BOUNDARIES IN OBSTACLE PROBLEMS FOR INTEGRO-DIFFERENTIAL OPERATORS

XAVIER ROS-OTON

1. The classical obstacle problem

The obstacle problem is probably the most classic and motivating example in the study of variational inequalities and free boundary problems. Its simplest mathematical formulation is to seek for minimizers of the Dirichlet energy functional

$$\mathcal{E}(u) = \int_D |\nabla u|^2 dx \tag{1.1}$$

among all functions u satisfying $u \geq \varphi$ in D, for a given smooth obstacle $\varphi \in C^\infty$. Here, $D \subset \mathbb{R}^n$, and one usually has Dirichlet boundary conditions $u = g$ on ∂D.

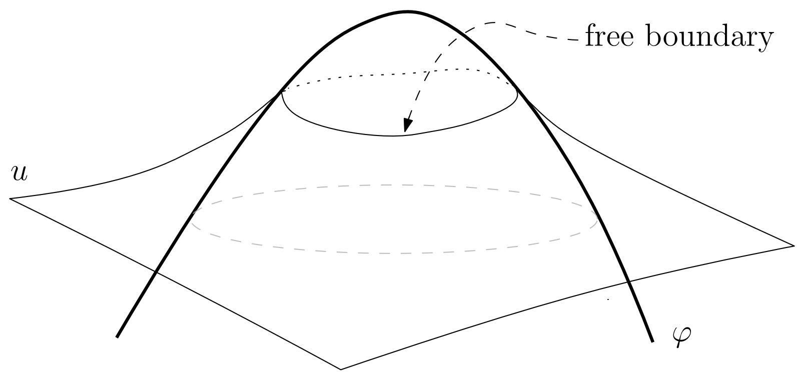

Figure 1. The obstacle φ and the solution u.

The Euler-Lagrange equations of such minimization problem are

$$\begin{array}{rcll} u & \geq & \varphi & \text{in } D \\ \Delta u & = & 0 & \text{in } \{u > \varphi\} \\ -\Delta u & \geq & 0 & \text{in } D. \end{array} \tag{1.2}$$

In other words, the solution u is above the obstacle φ, it is harmonic whenever it does not touch the obstacle, and moreover it is superharmonic everywhere.

The domain D will be split into two regions: one in which the solution u is harmonic, and one in which the solution equals the obstacle. The latter region is known as the *contact set* $\{u = \varphi\}$. The interface that separates these two regions is the *free boundary*.

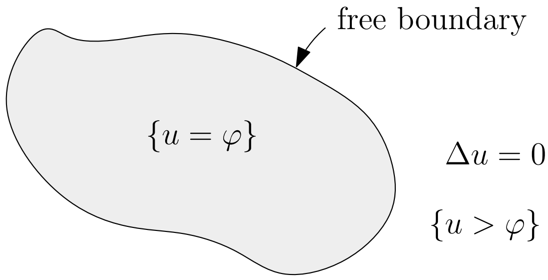

Figure 2. The contact set and the free boundary in the classical obstacle problem.

From the mathematical point of view, the most challenging question in these problems is to understand the *regularity of free boundaries*. Such type of questions are usually very hard, and even in the simplest cases almost nothing was known before the 1970s. The development of the regularity theory for free boundaries started in the late seventies, with the groundbreaking paper of L. Caffarelli [5]. Since then, it has been a very active area of research.

1.1. **Regularity theory: known results.** Let us next describe the main known mathematical results on the classical obstacle problem.

Notice that the existence and uniqueness of solutions follows by standard techniques: the solution can be constructed either by minimizing an energy functional among all functions satisfying $u \geq \varphi$, or by using the theory of viscosity solutions (see e.g. [30, 16]).

The central mathematical challenge in obstacle problems is to understand the geometry and regularity of the free boundary, i.e., of the interface $\partial\{u > \varphi\}$. A priori such interface could be a very irregular object, even a fractal set with infinite perimeter. As explained next, it turns out that this cannot happen, and that free boundaries are smooth (maybe outside a certain set of singular points).

The first results for this problem established the optimal $C^{1,1}$ regularity of solutions (i.e., second derivatives of u are bounded but not continuous). Then, the first general result for free boundaries was proved by Kinderlehrer and Nirenberg [27], who showed that, if the free boundary is C^1, then it is C^∞. This is a perturbative result that is proved by flattening the (free) boundary —via a hodograph transform— and then using a bootstrap argument. The main open problem was still open: to understand what happens in general with the regularity of the free boundary. As said before, a priori it could be a very irregular set with infinite perimeter, while in order to apply the results of [27] one needs the free boundary to be at least C^1. The breakthrough came with the work [5], where Caffarelli developed a regularity theory for free boundaries in the obstacle problem, and established the regularity of free boundaries near regular points.

The main known results from [5, 27, 8, 6, 32] can be summarized as follows:

- At every free boundary point x_0 one has
$$0 < cr^2 \leq \sup_{B_r(x_0)} (u-\varphi) \leq Cr^2 \qquad r \in (0,1) \tag{1.3}$$
- If the free boundary is C^1, then it is C^∞ [27].
- The free boundary is C^1 (and thus C^∞), possibly outside a certain set of singular points [5].
- Singular points are those at which the contact set $\{u = \varphi\}$ has density zero, and these points (if any) are locally contained in a $(n-1)$-dimensional C^1 manifold [8, 6, 32].

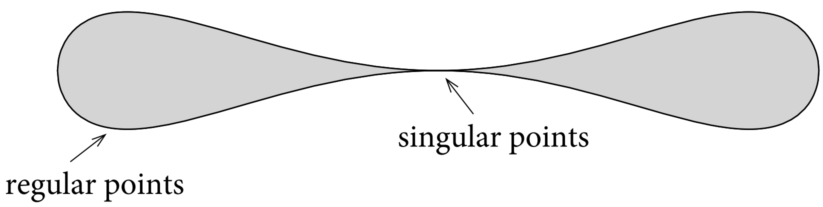

FIGURE 3. A free boundary with a singular point. The contact set $\{u = \varphi\}$ (colored gray) has zero density at the singular point.

- Moreover, similar results hold for the parabolic obstacle problem (Stefan problem).

To prove such regularity results, one considers *blow-ups*. Namely, given a free boundary point x_0 one shows that
$$u_r(x) := \frac{(u-\varphi)(x_0+rx)}{r^2} \longrightarrow u_0(x) \qquad \text{in } C^1_{\text{loc}}(\mathbb{R}^n),$$
for some function u_0 which is a global solution of the obstacle problem. Notice that the rescaling parameter r^2 comes from the non-degeneracy condition (1.3).

Then, the main difficulty is to *classify blow-ups*, i.e., show that
$$\text{regular point} \quad \Longrightarrow \quad u_0(x) = (x\cdot e)_+^2 \quad \text{(1D solution)}$$
$$\text{singular point} \quad \Longrightarrow \quad u_0(x) = \sum_i \lambda_i x_i^2 \quad \text{(paraboloid);}$$
see Figure 4. Notice that, after the blow-up, the contact set $\{u_0 = 0\}$ becomes a half-space in case of regular points, while it has zero measure in case of singular points.

Finally, once the classification of blow-ups is well understood, then one has to transfer the information from u_0 to u, and show that if x_0 was a regular point, then

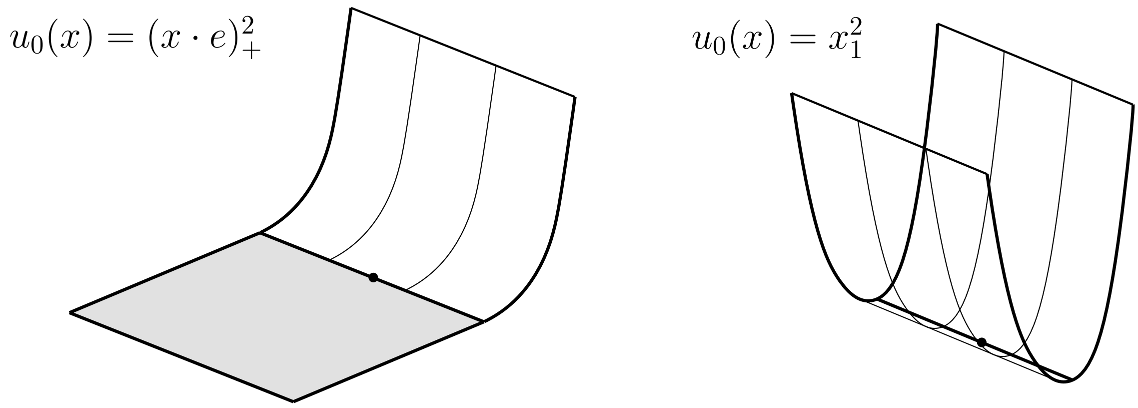

Figure 4. The blow-up profile u_0 at a regular point (left) and at a singular point (right).

the free boundary is C^1 in a neighborhood of x_0. We refer to [6] and [33] for more details.

2. More general obstacle problems

The previous discussion was for the obstacle problem for the Laplacian Δ. Next, we want to answer the following.

Question: What happens in more general situations, or in other types of obstacle problems?

We would like to understand the following:

(a) The *thin* obstacle problem
(b) Obstacle problems for *integro-differential* operators

Problem (a) arises in Elasticity (the Signorini problem), and in Fluid Mechanics (semipermeable membranes). Problem (b) arises in Probability and Finance (optimal stopping for jump processes, pricing of options), as well as in Interacting energies in physical, biological, or material sciences. For a detailed description of these motivations/applications, we refer to [19], [22], [17], and [15]; see also [35].

3. The thin obstacle problem

The thin obstacle problem (also called the boundary obstacle problem) arises when minimizing the Dirichlet energy

$$\mathcal{E}(u) = \int_{D^+} |\nabla u|^2 dx$$

among all functions u satisfying

$$u \geq \varphi \qquad on \quad \{x_n = 0\} \cap D^+.$$

Here, $D^+ \subset \{x_n \geq 0\}$, and usually one would take either $D^+ = B_1^+$ or $D^+ = \mathbb{R}^n_+$. When D^+ is bounded, then the Dirichlet boundary conditions are $u = g$ on $\partial D^+ \cap \{x_n > 0\}$, while when $D^+ = \mathbb{R}^n_+$ one simply prescribes $u \to 0$ at ∞.

A simple variational argument shows that the Euler-Lagrange equations of such minimization problem are

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D^+ \cap \{x_n > 0\} \\ u &\geq \varphi \quad \text{in } D^+ \cap \{x_n = 0\} \\ \partial_{x_n} u &\leq 0 \quad \text{in } D^+ \cap \{x_n = 0\} \\ \partial_{x_n} u &= 0 \quad \text{in } D^+ \cap \{x_n = 0\} \cap \{u > \varphi\}. \end{aligned} \tag{3.1}$$

As in the classical obstacle problem, the existence and uniqueness of solutions for such problem is standard.

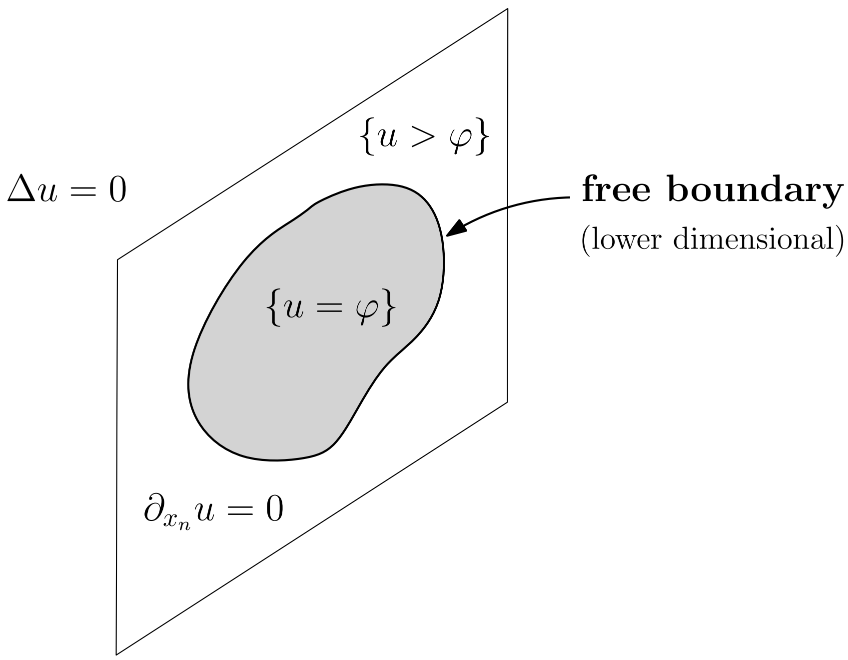

FIGURE 5. The contact set and the free boundary in the thin obstacle problem.

The set $D^+ \cap \{x_n = 0\}$ will be split into two regions: one in which $\partial_{x_n} u$ is zero, and one in which u equals the obstacle. The latter region is the *contact set*. The interface that separates these two regions is the *free boundary*.

The equations (3.1) can be written as

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D^+ \cap \{x_n > 0\} \\ \min\bigl\{-\partial_{x_n} u,\, u - \varphi\bigr\} &= 0 \quad \text{on } D^+ \cap \{x_n = 0\}. \end{aligned} \tag{3.2}$$

After an even reflection with respect to the hyperplane $\{x_n = 0\}$, the solution u will be harmonic across such hyperplane wherever $\partial_{x_n} u = 0$, and it will be super-harmonic wherever $\partial_{x_n} u < 0$. Thus, such reflected function would formally solve the classical obstacle problem, but with the restriction $u \geq \varphi$ only on $\{x_n = 0\}$ (the obstacle is thin).

3.1. **The thin obstacle problem: regularity theory.** The regularity theory for the free boundary differs substantially if we consider the *thin* obstacle problem instead of the classical one.

In the classical obstacle problem, all blow-ups are homogeneous of degree 2, and the full structure of the free boundary is completely understood, as explained above. In the thin obstacle problem, instead, understanding the regularity of free boundaries is much harder. An important difficulty comes from the fact that in thin obstacle problems there is no a priori analogous of (1.3), and thus blow-ups may have different homogeneities.

The first results for the thin obstacle problem were obtained in the 1960's and 1970's. However, even if the regularity of free boundaries in the classical obstacle problem had been established in 1977 [5], nothing was known for the thin obstacle problem. Such question remained open for 30 years, and was finally answered by Athanasopoulos, Caffarelli, and Salsa in [1].

The main result in [1] establishes that if u solves the thin obstacle problem (3.2) with $\varphi \equiv 0$, then for every free boundary point x_0 we have

(i) either
$$0 < cr^{3/2} \leq \sup_{B_r(x_0)} u \leq Cr^{3/2} \qquad \text{(regular points)}$$

(ii) or
$$0 \leq \sup_{B_r(x_0)} u \leq Cr^2.$$

Moreover, they proved that set of regular points (i) is an open subset of the free boundary, and it is $C^{1,\alpha}$ for some small $\alpha > 0$.

The proof of this result is strongly related to the theory of *minimal surfaces*; see the survey paper [12]. Namely, to study the regularity of the free boundary they found a quantity that is monotone as we zoom in a solution at a given free boundary point. In the theory of minimal surfaces, the corresponding monotonicity formula implies that the blow-ups of a minimal surface at any point are *cones* [25]. In case of harmonic functions or in free boundary problems, the corresponding formula implies that blow-ups are always *homogeneous* [10, 33].

In the thin obstacle problem, Athanasopoulos, Caffarelli, and Salsa found that the Almgren frequency formula, a known monotonicity formula for harmonic functions, is still valid for solutions to the thin obstacle problem. Such monotonicity formula states that
$$r \;\mapsto\; N(r) := \frac{r\int_{B_r(x_0)} |\nabla u|^2}{\int_{\partial B_r(x_0)} u^2} \qquad \text{is monotone.}$$

Thanks to this powerful tool, the blow-up sequence

$$u_r(x) := \frac{u(x_0 + rx)}{\left(\fint_{\partial B_r(x_0)} u^2 \right)^{1/2}}$$

converges to a *homogeneous* global solution u_0 of degree $\mu = N(0^+)$.

Therefore, the characterization of blow-up profiles in the thin obstacle problem reduces to the characterization of homogeneous blow-up profiles. Analyzing an eigenvalue problem on the sphere $\mathbb{S}^{n-1}$, and using the semi-convexity of solutions, they proved that

$$\mu < 2 \qquad \Longrightarrow \qquad \mu = \frac{3}{2},$$

and for $\mu = 3/2$ they classified blow-ups. Finally, using again the monotonicity property of solutions, and an appropriate boundary Harnack inequality, they established the result.

After the results of [1], the regularity of the set of regular points (i) was improved to C^∞ in [28, 18] by using higher order boundary regularity estimates. See Garofalo-Petrosyan [23], our work [2] in collaboration with Barrios and Figalli, and the recent work of Focardi and Spadaro [21] for a precise description and regularity results on the set of (non-regular) free boundary points satisfying (ii).

4. Obstacle problems for integro-differential operators

A more general class of obstacle problems is obtained when minimizing *nonlocal* energy functionals of the form

$$\mathcal{E}(u) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \big|u(x) - u(y)\big|^2 K(x-y)\, dx\, dy$$

among all functions $u \geq \varphi$ in $\mathbb{R}^n$ —or with $u = g$ in D^c and $u \geq \varphi$ in D. Here, K is a nonnegative and even kernel ($K \geq 0$ and $K(z) = K(-z)$), and the minimal integrability assumption is

$$\int_{\mathbb{R}^n} \min\big\{1,\, |z|^2\big\} K(z) dz < \infty.$$

The most simple and canonical example is

$$K(z) = \frac{c}{|z|^{n+2s}}, \qquad\qquad s \in (0,1), \tag{4.1}$$

while a typical "uniform ellipticity" assumption is

$$\frac{\lambda}{|z|^{n+2s}} \leq K(z) \leq \frac{\Lambda}{|z|^{n+2s}}, \tag{4.2}$$

with $s \in (0,1)$ and $0 < \lambda \leq \Lambda$; see for example [4, 14, 34].

The Euler-Lagrange equations of such minimization problem are

$$\begin{aligned} u &\geq \varphi && \text{in} \quad \mathbb{R}^n \\ Lu &= 0 && \text{in} \quad \{u > \varphi\} \\ -Lu &\geq 0 && \text{in} \quad \mathbb{R}^n, \end{aligned} \tag{4.3}$$

where L is an integro-differential operator of the form

$$\begin{aligned} Lu(x) &= \text{p.v.} \int_{\mathbb{R}^n} \big(u(y) - u(x)\big) K(x-y)\,dy \\ &= \text{p.v.} \int_{\mathbb{R}^n} \big(u(x+z) - u(x)\big) K(z)\,dz. \end{aligned} \tag{4.4}$$

In other words, u solves the obstacle problem (1.2) but with the Laplacian Δ replaced by the integro-differential operator L in (4.4).

Remark 4.1 (Relation to the thin obstacle problem). When K is given by (4.1), then L is a multiple of the *fractional Laplacian* $-(-\Delta)^s$, defined by

$$(-\Delta)^s u(x) = c_{n,s}\text{p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\,dy.$$

When $s = 1/2$ the half-Laplacian $(-\Delta)^{1/2}$ can be written as a Dirichlet-to-Neumann operator in $\mathbb{R}^{n+1}_+$: for any function $w(x)$ in $\mathbb{R}^n$, if we consider its harmonic extension $\tilde{w}(x, x_{n+1})$ in $\mathbb{R}^{n+1}_+$ then the Neumann derivative $\partial_{x_{n+1}}\tilde{w}$ on $\{x_{n+1} = 0\}$ is exactly the half-Laplacian of $w(x)$ as a function on $\mathbb{R}^n$.

Therefore, the thin obstacle problem (3.2) (with $D^+ = \mathbb{R}^n_+$) is the same as

$$\min\big\{(-\Delta)^{1/2}u,\, u - \varphi\big\} = 0 \quad \text{in} \quad \mathbb{R}^{n-1}.$$

Notice that here we just consider the function u on $\{x_n = 0\}$, and this is why the problem is in one dimension less, $\mathbb{R}^{n-1}$. Note also that with this alternative formulation of the thin obstacle problem the free boundary is not lower-dimensional anymore, but the operator has changed and it is now $(-\Delta)^{1/2}$.

4.1. **Regularity theory: known results.** In the last decade, there have been considerable efforts to extend the classical regularity theory for free boundaries of [5, 6] to the case of integro-differential operators. In the simplest case, L would be the fractional Laplacian $(-\Delta)^s$, $s \in (0, 1)$. On the one hand, this operator serves as a model case to study the regularity of the free boundary for general integro-differential operators (4.4). On the other hand, the obstacle problem for the fractional Laplacian extends at the same time the classical obstacle problem (which corresponds to the limiting case $s \to 1$) and the thin obstacle problem (which corresponds to the case $s = 1/2$).

The first results in this direction were obtained by Silvestre in [38], who established the almost-optimal regularity of solutions, $u \in C^{1,s-\varepsilon}$ for all $\varepsilon > 0$. The optimal $C^{1,s}$ regularity of solutions, as well as the regularity of the free boundary, were established later by Caffarelli, Salsa, and Silvestre [11].

The main result of [11] establishes that if x_0 is a regular point then the free boundary is $C^{1,\alpha}$ in a neighborhood of x_0. More precisely, they proved that if u solves the obstacle problem for the fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$, then $u \in C^{1,s}$, and for every free boundary point $x_0 \in \partial\{u > \varphi\}$ we have

(i) either
$$0 < cr^{1+s} \leq \sup_{B_r(x_0)} (u - \varphi) \leq Cr^{1+s} \qquad \text{(regular points)}$$

(ii) or
$$0 \leq \sup_{B_r(x_0)} (u - \varphi) \leq Cr^2.$$

Moreover, the set of regular points (i) is an open subset of the free boundary, and it is $C^{1,\alpha}$ for some small $\alpha > 0$.

Notice that the result is completely analogous to the one for the thin obstacle problem (recall that these two problems coincide if $s = 1/2$!).

To establish such result they found a new equivalence between the obstacle problem for the fractional Laplacian in $\mathbb{R}^n$ —for *every* $s \in (0,1)$—, and an appropriate *thin* obstacle problem in $\mathbb{R}^{n+1}$. Namely, it turns out that the fractional Laplacian $(-\Delta)^s$ can be written as a Dirichlet-to-Neumann map in $\mathbb{R}^{n+1}_+$ for a *local* operator with a weight,

$$\operatorname{div}\big(y^{1-2s}\nabla_{x,y}\tilde{u}\big) \quad \text{for} \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}_+;$$

see [13] for more details. When $s = 1/2$, such Dirichlet-to-Neumann map is exactly the one in Remark 4.1.

Thanks to such new equivalence between the obstacle problem for the fractional Laplacian and a (weighted) thin obstacle problem, they found an Almgren-type frequency formula for the obstacle problem for the fractional Laplacian in terms of such extension problem in $\mathbb{R}^{n+1}_+$. Using such new monotonicity formula, they extended the regularity theory of [1] to all $s \in (0,1)$, and also to non-zero obstacles φ, as stated above.

After the results of [11], several new results were established concerning the structure of singular points, the higher regularity of the free boundary near regular points, or the case of operators with drift; see [2, 21, 28, 26, 24].

4.2. **Open questions.** Despite all these developments in the last decade, two important problems remained open.

Open question 1: A very important problem that remained open after these results was the understanding of obstacle problems for more general integro-differential operators (4.2).

For the fractional Laplacian, the proofs of all known results relied very strongly on certain particular properties of such operator. Indeed, the obstacle problem for this (nonlocal) operator is equivalent to a thin obstacle problem in $\mathbb{R}^{n+1}$ for a local operator, for which monotonicity formulas are available.

For more general nonlocal operators these tools are not available, and nothing was known about the regularity of free boundaries. The understanding of free boundaries for more general integro-differential operators was an important problem that was completely open.

Open question 2: Another key question that remained open was the regularity of free boundaries in *parabolic* obstacle problems, even in case of the fractional Laplacian. The main difficulty in this context was that no parabolic Almgren's frequency formula seems to exist for $s < 1$, and thus the methods of [11] cannot be used in the parabolic setting.

5. Regularity theory: new results

5.1. **Obstacle problems for general integro-differential operators.** One of our main contributions in this context is the understanding of free boundaries in obstacle problems for general integro-differential operators [9]. In this work we extend the results of [11] to a much more general context, solving a long-standing open problem in the field.

Our paper [9], in collaboration with Caffarelli and Serra, introduces a new approach to the regularity of free boundaries in obstacle problems, and extends the results of [11] to a general class of integro-differential operators (4.4). The main difficulty to do so was that for more general nonlocal operators L there are no monotonicity formulas, while the proofs of [11] relied strongly on such type of formulas.

Our main result in [9] studies obstacle problems for operators (4.4) satisfying

$$\frac{\lambda}{|z|^{n+2s}} \leq K(z) \leq \frac{\Lambda}{|z|^{n+2s}}, \qquad \text{with } K(z) \text{ homogeneous},$$

and establishes that the set of regular points is open and the free boundary is $C^{1,\alpha}$ near such points. The first assumption on K is a natural uniform ellipticity assumption, and the homogeneity of K is equivalent to the fact that L has certain scale invariance.

More precisely, our result in [9] establishes that, under such assumptions on L, if u solves the obstacle problem then for every free boundary point $x_0 \in \partial\{u > \varphi\}$ we have:

(i) either $\qquad 0 < cr^{1+s} \leq \sup_{B_r(x_0)} (u - \varphi) \leq Cr^{1+s} \qquad$ (regular points)

(ii) or $\qquad 0 \leq \sup_{B_r(x_0)} (u - \varphi) \leq Cr^{1+s+\alpha},$

where $\alpha > 0$ is such that $1 + s + \alpha < 2$. Moreover, we proved that set of regular points (i) is an open subset of the free boundary, and it is $C^{1,\alpha}$ for all $\alpha < s$. Furthermore, we gave a fine description of solutions near all regular free boundary points in terms of the distance function to the free boundary.

As said before, all this was only known for the fractional Laplacian. For more general integro-differential operators new techniques had to be developed, since one does not have any monotonicity formula. Our proofs in [9] are based only on very general Liouville and Harnack's type techniques, completely independent from those in [11].

Let us briefly explain the global strategy of the proof. Recall that an important difficulty is that we have no monotonicity formula, and therefore a priori blow-ups could be non-homogeneous. As we will see, the only property we can use on blow-ups is that they are *convex*. Another difficulty is that the nonlocal operator (4.4) makes no sense for functions that grow too much at infinity. Thus, we need to be very careful with the growth of functions at infinity, and the meaning of the equation as we rescale the solution and consider blow-ups.

Sketch of the proof: The general argument goes as follows. Initially, we say that a free boundary point x_0 is regular whenever (ii) does *not* hold. Then, we have to prove that all regular points satisfy (i), that such set is open, and that the free boundary is $C^{1,\alpha}$ near these points.

Thus, we start with a free boundary point x_0, and assume that (ii) does not hold —otherwise there is nothing to prove. Then, the idea is to take a blow-up sequence of the type

$$v_r(x) = \frac{(u-\varphi)(x_0+rx)}{\|u-\varphi\|_{L^\infty(B_r(x_0))}}.$$

However, we need to do it along an appropriate subsequence $r_k \to 0$ so that the blow-up sequence v_{r_k} (and their gradients) have a certain good growth at infinity (uniform in k). We do not want the rescaled functions v_{r_k} to grow too much. Once we do this, in the limit $r_k \to 0$ we get a global solution v_0 to the obstacle problem, which is convex and has the following growth at infinity

$$|\nabla v_0(x)| \le C(1+|x|^{s+\alpha}).$$

Such growth condition is very important in order to take limits $r_k \to 0$ and to show that v_0 solves the obstacle problem. (The fact that v_0 solves the obstacle problem needs to be understood in a certain generalized sense; see [9] for more details.)

Notice that the convexity of v_0 comes from the initial assumption that (ii) does not hold, and is essential in our proof. The idea that blow-ups in the obstacle problem are convex is first seen in the celebrated article of Caffarelli [5].

The next step is to classify global *convex* solutions v_0 to the obstacle problem with such growth. We need to prove that the contact set $\{v_0 = 0\}$ is a half-space (a priori we only know that it is convex). For this, the first idea is to do a blow-down argument to get a new solution $\tilde{v}_0$, with the same growth as v_0, and for which the contact set is a convex *cone* Σ. Then, we separate into two cases, depending on the size of Σ. If Σ has zero measure, by a Liouville theorem we show that $\tilde{v}_0$ would be a paraboloid, which is incompatible with the growth of $\tilde{v}_0$ (here we use that $s+\alpha < 1$). On the other hand, if Σ has nonempty interior, then we prove by a dimension reduction argument (doing a blow-up at a lateral point on the cone) that Σ must be C^1 outside the origin. After that, we notice that thanks to the convexity of $\tilde{v}_0$ there is a cone of directional derivatives satisfying $\partial_e \tilde{v}_0 \ge 0$ in $\mathbb{R}^n$. Then, using a boundary Harnack estimate in C^1 domains (which we prove in a separate paper [36]), we show that all such derivatives have to be equal (up to multiplicative constant) in $\mathbb{R}^n$, and thus that Σ must be a half-space. Since Σ was the blow-down of the original contact set $\{v_0 = 0\}$, and this set is convex, this implies that $\{v_0 = 0\}$ was itself a half-space. Once we know that $\{v_0 = 0\}$ is a half-space, it follows that v_0 is a 1D solution, which can be completely classified.

Once we have the classification of such blow-ups, we show that the free boundary is Lipschitz in a neighborhood of x_0, and C^1 at that point. This is done by adapting techniques from the classical obstacle problem to the present context of nonlocal operators. Finally, by an appropriate barrier argument we show that the regular set is open, i.e., that all points in a neighborhood of x_0 do *not* satisfy (ii). From this, we deduce that the free boundary is C^1 at every point in a neighborhood of x_0, and we show that this happens with a uniform modulus of continuity around x_0. Finally, using again the boundary Harnack in C^1 domains [36], we deduce that the free boundary is $C^{1,\alpha}$ near x_0.

5.2. **Parabolic obstacle problems for integro-differential operators.** In collaboration with Barrios and Figalli, we studied the parabolic obstacle problem for the fractional Laplacian in [3].

Despite all the developments for the elliptic problem in the last decade (described above), much less was known in the parabolic setting. The only result was due to Caffarelli and Figalli [7], who showed the optimal C^{1+s}_x regularity of solutions in space. However, nothing was known about the *regularity of the free boundary* in the parabolic setting.

Our main theorem in [3] extends the results of [11] to the parabolic setting when $s > 1/2$, and establishes the $C^{1,\alpha}$ regularity of the free boundary in x and t near regular points. The result is new even in dimension $n = 1$, and reads as follows. Let us denote $Q_r(x_0, t_0)$ parabolic cylinders of size r around (x_0, t_0). Then, for each free boundary point (x_0, t_0), we have:

(i) either
$$0 < c\, r^{1+s} \le \sup_{Q_r(x_0,t_0)} (u - \varphi) \le C\, r^{1+s},$$

(ii) or
$$0 \le \sup_{Q_r(x_0,t_0)} (u - \varphi) \le C_\varepsilon\, r^{2-\varepsilon} \qquad \text{for all } \varepsilon > 0.$$

Moreover, the set of points (x_0, t_0) satisfying (i) is an open subset of the free boundary and it is locally a $C^{1,\alpha}$ graph in x and t, for some small $\alpha > 0$.

Furthermore, for any point (x_0, t_0) satisfying (i) there is $r > 0$ such that $u \in C^{1+s}_{x,t}(Q_r(x_0, t_0))$, and we have the expansion

$$u(x,t) - \varphi(x) = c_0\big((x-x_0)\cdot e + \kappa(t-t_0)\big)_+^{1+s} + o\big(|x-x_0|^{1+s+\alpha} + |t-t_0|^{1+s+\alpha}\big), \quad (5.1)$$

for some $c_0 > 0$, $e \in \mathbb{S}^{n-1}$, and $\kappa > 0$.

Remark 5.1 (On the assumption $s > 1/2$). It is important to notice that the assumption $s > 1/2$ is necessary for this result to hold. Indeed, the scaling of the equation in (x,t) is completely different in the regimes $s > 1/2$, $s = 1/2$, and $s < 1/2$. Since the analysis of the free boundary is always based on blow-ups, the free boundary is expected to be quite different in these three regimes.

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