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ELLIPTIC FIBRATIONS OF A CERTAIN $K3$ SURFACE OF THE APÉRY–FERMI PENCIL

by

Marie José Bertin and Odile Lecacheux

Abstract. — We explain how to obtain from Kneser–Nishiyama's method all the elliptic fibrations of the singular (i.e. of Picard number 20) $K3$ surface Y_{10} of discriminant 72 and belonging to the Apéry–Fermi pencil (Y_k). The case of its extremal elliptic fibrations is developped together with Weierstrass equations, noticing that two of them are obtained by 3-isogeny from extremal fibrations of the $K3$ surface Y_2 of discriminant 8.

Résumé. — (*Fibrations elliptiques d'une certaine surface $K3$ du pinceau d'Apéry–Fermi*) On montre comment la méthode de Kneser–Nishiyama permet d'obtenir toutes les fibrations elliptiques de la surface $K3$ singulière (i.e. de nombre de Picard 20) de discriminant 72, notée Y_{10}, appartenant au pinceau (Y_k) de surfaces $K3$ d'Apéry–Fermi. Les fibrations elliptiques extrémales sont en outre données avec des équations de Weierstrass. On remarque que deux d'entre elles sont obtenues par 3-isogénie à partir de fibrations extrémales de la surface Y_2 de discriminant 8.

1. Introduction

The Apéry–Fermi pencil of $K3$ surfaces (Y_k) is obtained by desingularization from the equation

$$(Y_k) \qquad X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} = k.$$

Generically, the $K3$ surfaces (Y_k) have Picard number 19 and for finitely many k the Picard number increases to 20. Such $K3$ surfaces with Picard number 20 are called singular. A list of 8 (resp. 7) singular $K3$ surfaces for k integer (resp. k pure quadratics) has been computed by Boyd [8]. This list has been confirmed independently by Schütt who added 2 more k^2 pure quadratics [23]. The 8 singular $K3$ surfaces Y_0, Y_2, Y_3, Y_6, Y_{10}, Y_{18}, Y_{102}, Y_{198} are cited in [6]. The transcendental lattice $T(Y)$ of a $K3$ surface Y, orthogonal complement of its Néron–Severi lattice $NS(Y)$ in the unimodular lattice $H^2(Y,\mathbb{Z})$ is a birational invariant of the algebraic surface Y. When Y is singular, the Gram matrix of its transcendental lattice $T(Y)$

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has rank 2 and by Livné's modularity theorem the L-series of $T(Y)$ is modular meaning there exists a weight 3 modular form f, with complex multiplication by $\mathbb{Q}(\sqrt{-|\det((T(Y))|})$, satisfying $L(T(Y),s) \doteq L(f,s)$ where $\doteq$ means up to a finite number of primes.
In Bertin [1, 2, 3] it was proved that Y_2 and Y_{10} have respective transcendental lattices $T(Y_2)$ and $T(Y_{10})$ with

$$T(Y_2) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \qquad T(Y_{10}) = \begin{pmatrix} 6 & 0 \\ 0 & 12 \end{pmatrix}$$

and that their transcendental L-series are the same.
In the Apéry–Fermi pencil only Y_2 and Y_{10} share this last property and we guess a geometric link between these two $K3$ surfaces. Notice the relation $T(Y_{10}) = T(Y_2)[3]$ similar to the relation $T(K_2) = T(Y_2)[2]$, where K_2 is the Kummer surface associated to Y_2 in the Shioda–Inose construction and where the link between Y_2 and K_2 is the existence of 2-isogenies between elliptic fibrations (see for example [6]).
Thus to understand the link between Y_2 and Y_{10}, it was tempting to study Y_{10} as Bertin and Lecacheux did for Y_2 [5], that is determine all the non isomorphic elliptic fibrations of Y_{10} together with their Weierstrass equations. However such a study is more tricky.
The difficulty is the application of the Kneser–Nishiyama method to determine all the elliptic fibrations. Concerning Y_2, these fibrations are given by primitive embeddings of a root lattice, namely $A_1 \oplus D_5$, in the various Niemeier lattices, which are only primitive embeddings in the root part of Niemeier lattices. This is no longer the case for Y_{10} since we have to embed the lattice $M = A_1 \oplus A_2 \oplus N$ with $N = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -4 \end{pmatrix}$. Since N is not a root lattice, we have to consider primitive embeddings into Niemeier lattices and not only in their root part. This fact added to the facts that M is composed of three irreducible lattices, A_1 and A_2 embedding primitively in all the other root lattices, give a huge amount of such embeddings, thus of elliptic fibrations. This difficulty is explained in Section 3.
Such a situation has been encountered by Braun, Kimura and Watari [9]. For example they considered the case $M = A_5 \oplus (-4)$. And even in that simpler case, probably Braun, Kimura and Watari found so many cases that they restricted to primitive embeddings containing E_6, E_7 or E_8.
In Section 2, we review some definitions and main known results useful for the paper.
Section 3 is devoted to the determination of elliptic fibrations of Y_{10} obtained with the help of the Kneser–Nishiyama's method.
In [6], Bertin and Lecacheux obtained all elliptic fibrations, called generic, of the Apéry–Fermi pencil together with a Weierstrass equation. However, a simple computation of their specializations for $k = 10$ give only a few elliptic fibrations of positive rank. It is the object of Section 4.
In particuliar, all the 10 extremal fibrations of Y_{10} given in Shimada–Zhang [25] are missing. Following Shimada–Zhang, an extremal fibration of a $K3$ surface is an elliptic fibration whose Mordell–Weil group has rank 0.
Hence, in Section 5 we determine the primitive embeddings into Niemeier lattices giving these extremal fibrations. Their corresponding Weierstrass equations are also exhibited. Most of them are obtained by the 2 or 3-neighbor method initiated by Elkies [12]. We must notice however that two of them are obtained by 3-isogeny from extremal fibrations of the $K3$-surface Y_2, hence revealing the geometric link between Y_2 and Y_{10}.
In the last Section 6 we exhibit elliptic fibrations of high rank on the $K3$ surface Y_{10}.

This famous Apéry–Fermi pencil is identified by Festi and van Straten [13] to be a pencil of $K3$ surfaces $(\mathcal{D}_s)$ appearing in the 2-loop diagrams in Bhabha scattering. The Bhabha scattering process is a specific process of Quantum Electrodynamics occuring in the interaction between an electron and a positron. The complete amplitude for Bhabha scattering appears as a sum of integrals corresponding to Feynmann diagrams with 1-loop or 2-loops diagrams. In solving some integrals for the 2-loops diagrams it was asked if a certain surface defined by the equation

$$z^2(1+xy) = (x+y)(x+y-4xy+xy(x+y))$$

is a rational surface. Festi and van Straten [13] proved that it is not a rational surface but a $K3$ surface with Picard number 20 and transcendental lattice $\left(\begin{smallmatrix}2 & 0\\ 0 & 4\end{smallmatrix}\right)$, hence can be identified with Y_2. Moreover they showed that this surface is a special member of a pencil of $K3$ surfaces $(\mathcal{D}_s)$ (namely the singular $K3$ surface $\mathcal{D}_1$) where the generic member has Picard number 19 and transcendental lattice $U \oplus \langle 12\rangle$. Thus the pencil $(\mathcal{D}_s)$ is the Apéry–Fermi pencil (Y_k). The correspondance is given by the relation $k = 2-4s$. Since Y_k and Y_{-k} are the same surface we observe that $\mathcal{D}_{-2}$ and $\mathcal{D}_3$ are the $K3$ surface Y_{10} studied in the present paper.
Computations were performed using the computer algebra system MAPLE and the Maple Library "Elliptic Surface Calculator" written by Kuwata [16].

2. Definitions and results

We recall briefly what is useful for the understanding of the paper.
A rank r *lattice* is a free $\mathbb{Z}$-module S of rank r together with a symmetric bilinear form b.
A lattice S is called *even* if $x^2 := b(x,x)$ is even for all x from S. For any integer n we denote by $\langle \boldsymbol{n}\rangle$ the lattice $\mathbb{Z}e$ where $e^2 = n$.
If $e = (e_1, \dots, e_r)$ is a $\mathbb{Z}$-basis of a lattice S, then the matrix $G(e) = (b(e_i, e_j))$ is called the *Gram matrix* of S with respect to e. An injective homomorphism of lattices is called an embedding.
An embedding $i : S \to S'$ is called *primitive* if $S'/i(S)$ is a free group. A sublattice is a subgroup equipped with the induced bilinear form. A sublattice S' of a lattice S is called primitive if the identity map $S' \to S$ is a primitive embedding. The *primitive closure* of S inside S' is defined by $\overline{S} = \{x \in S'/mx \in S$ for some positive integer $m\}$. A lattice M is an *overlattice* of S if S is a sublattice of M such that the index $[M : S]$ is finite.
By $S_1 \oplus S_2$ we denote the orthogonal sum of two lattices defined in the standard way. We write S^n for the orthogonal sum of n copies of a lattice S. The *orthogonal complement of a sublattice* S of a lattice S' is denoted $(S)^{\perp}_{S'}$ and defined by $(S)^{\perp}_{S'} = \{x \in S'/b(x,y) = 0$ for all $y \in S\}$.

2.1. Discriminant forms. — Let L be a non-degenerate lattice. The *dual lattice* L^* of L is defined by

$$L^* := \mathrm{Hom}(L, \mathbb{Z}) = \{x \in L \otimes \mathbb{Q} \,|\, b(x,y) \in \mathbb{Z} \text{ for all } y \in L\}$$

and the *discriminant group* G_L by

$$G_L := L^*/L.$$

This group is finite if and only if L is non-degenerate. In the latter case, its order is equal to the absolute value of the lattice determinant $|\det(G(e))|$ for any basis e of L. A lattice L is *unimodular* if G_L is trivial.

Let G_L be the discriminant group of a non-degenerate lattice L. The bilinear form on L extends naturally to a $\mathbb{Q}$-valued symmetric bilinear form on L^* and induces a symmetric bilinear form

$$b_L : G_L \times G_L \to \mathbb{Q}/\mathbb{Z}.$$

If L is even, then b_L is the symmetric bilinear form associated to the quadratic form defined by

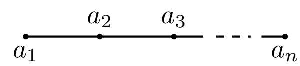
$$\begin{array}{rcl} q_L : G_L & \longrightarrow & \mathbb{Q}/2\mathbb{Z} \\ q_L(x+L) & \longmapsto & x^2 + 2\mathbb{Z}. \end{array}$$

The latter means that $q_L(na) = n^2 q_L(a)$ for all $n \in \mathbb{Z}$, $a \in G_L$ and $b_L(a,a') = \frac{1}{2}(q_L(a+a') - q_L(a) - q_L(a'))$, for all $a, a' \in G_L$, where $\frac{1}{2} : \mathbb{Q}/2\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is the natural isomorphism. The pair $\boldsymbol{(G_L, b_L)}$ (resp. $\boldsymbol{(G_L, q_L)}$) is called the *discriminant bilinear* (resp. *quadratic*) *form* of L.

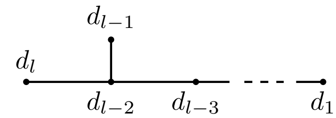
2.2. Root lattices. — In this section we recall only what is needed for the understanding of the paper. For proofs and details one can refer to Bourbaki [7] or Martinet [17].
Let L be a negative-definite even lattice. We call $e \in L$ a *root* if $q_L(e) = -2$. Put $\Delta(L) := \{e \in L/q_L(e) = -2\}$. Then the sublattice of L spanned by $\Delta(L)$ is called the *root type* of L and is denoted by L_{root}.
The *lattices* $A_n = \langle a_1, a_2, \ldots, a_n\rangle$ $(n \geq 1)$, $D_l = \langle d_1, d_2, \ldots, d_l\rangle$ $(l \geq 4)$, $E_p = \langle e_1, e_2, \ldots, e_p\rangle$ $(p = 6, 7, 8)$ defined by the *Dynkin diagrams* listed below are called the *root lattices*. All the vertices a_j, d_k, e_l are roots and two vertices v_j and v_h are joined by a line if and only if $b(v_j, v_h) = 1$. We use Bourbaki's numbering [7] and in brackets Conway–Sloane definitions [11].
Denote ϵ_i the vectors of the canonical basis of $\mathbb{R}^n$ with the usual scalar product.
The lattice A_n can be represented by the set of points in $\mathbb{R}^{n+1}$ with integer coordinates whose sum is zero, and the lattice D_l as the set of points of $\mathbb{R}^l$ with integer coordinates of even sum. We can represent E_8 in the *even coordinate system* [11, p. 120] $E_8 = D_8^+ = D_8 \cup (v + D_8)$ where $v = \frac{1}{2}\sum_{i=1}^{8} \epsilon_i$. Then we represent E_7 as the orthogonal in E_8 of v, and E_6 as the orthogonal of $\langle v, w\rangle \simeq A_2$ in E_8 where $w = \epsilon_1 + \epsilon_8$.
A coset representative (or glue vector) for L^* modulo L is labelled $[j]_L$ where L denotes A_n, D_l or E_p $(p = 6, 7, 8)$ and $0 \leq j \leq |G_L|$.

2.3. Niemeier lattices. — An even unimodular lattice $Ni(L_{\text{root}})$ in dimension 24 is called a Niemeier lattice and is obtained by gluing certain component lattices of L_{root} by means of glue vectors.
If L_{root} has components $L_1, \ldots, L_k$, the glue vectors have the form $y = [y_1, \ldots, y_k]$ where each y_i can be regarded as a coset representative (or glue vector) for L_i^* modulo L_i. These coset representatives, labelled $[0]_{L_i}$, $[1]_{L_i}, \ldots [d-1]_{L_i}$ for a component of determinant $d = |G_{L_i}|$ are listed in the previous subsection. In the sequel the indexes are dropped and for example the glue vector $[[0]_{L_1}, [1]_{L_2}, [1]_{L_3}]$ will be denoted $[011]$.
The set of glue vectors for $Ni(L_{\text{root}})$ forms an additive group called the glue code. The Table 1 below gives generators for the glue code. If a glue vector contains parentheses, this indicates that all vectors obtained by cyclically shifting the part of the vector inside the parentheses are also glue vectors. For example, the glue vectors for the Niemeier lattice $Ni(D_8^3)$ is described

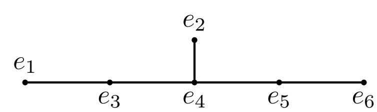
$\mathbf{A_n}$, $\mathbf{G_{A_n}}$
Set
$[1]_{A_n} = a_n^* = -\frac{1}{n+1}\sum_{j=1}^{n}(j)a_j$
then $A_n^* = \langle A_n, [1]_{A_n}\rangle$ and
$G_{A_n} = A_n^*/A_n \simeq \mathbb{Z}/(n+1)\mathbb{Z}$.
$q_{A_n}([1]_{A_n}) = -\frac{n}{n+1}$.
Glue vectors $[i]_{A_n} = a_{n+1-i}^*$
Glue group $[i+j] = [i] + [j]$



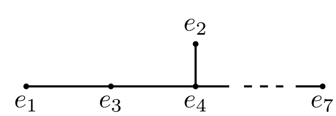
$\mathbf{D_l}, \mathbf{G_{D_l}}$.
Set
$[1]_{D_l} = -d_{l-1}^* =$
$\frac{1}{2}\left(\sum_{i=1}^{l-2} id_i + \frac{1}{2}(l-2)d_l + \frac{1}{2}ld_{l-1}\right)$
$[2]_{D_l} = d_1^* =$
$\sum_{i=1}^{l-2} d_i + \frac{1}{2}(d_{l-1} + d_l)$
$[3]_{D_l} = -d_{l-1}^* + d_1^* =$
$\frac{1}{2}\left(\sum_{i=1}^{l-2} id_i + \frac{1}{2}ld_l + \frac{1}{2}(l-2)d_{l-1}\right)$
then $D_l^* = \langle D_l, [1]_{D_l}, [3]_{D_l}\rangle$,
$G_{D_l} = D_l^*/D_l =< [1]_{D_l} >\simeq \mathbb{Z}/4\mathbb{Z}$ if l is odd,
$G_{D_l} = D_l^*/D_l =< [1]_{D_l}, [3]_{D_l} >\simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if l is even.
$q_{D_l}([1]_{D_l}) = -\frac{l}{4}$, $q_{D_l}([2]_{D_l}) = -1$, $b_{D_l}([1],[2]) = -\frac{1}{2}$.



$\mathbf{E_6}$, $\mathbf{G_{E_6}}$
Set
$[1]_{E_6} = \eta_6 = e_6^* =$
$-\frac{1}{3}(2e_1 + 3e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6)$,
then
$E_6^* = \langle E_6, \eta_6\rangle$ and
$G_{E_6} = E_6^*/E_6 \simeq \mathbb{Z}/3\mathbb{Z}$
$[2]_{E_6} = -[1]_{E_6}$.
$q_{E_6}(\eta_6) = -\frac{4}{3}$.



$\mathbf{E_7}$, $\mathbf{G_{E_7}}$
Set
$[1]_{E_7} = \eta_7 = e_7^* =$
$-\frac{1}{2}(2e_1+3e_2+4e_3+6e_4+5e_5+4e_6+3e_7)$,
then
$E_7^* = \langle E_7, \eta_7\rangle$ and
$G_{E_7} = E_7^*/E_7 \simeq \mathbb{Z}/2\mathbb{Z}$,
$q_{E_7}(\eta_7) = -\frac{3}{2}$.



$\mathbf{E_8}$, $\mathbf{G_{E_8}}$
$E_8^* = E_8$.

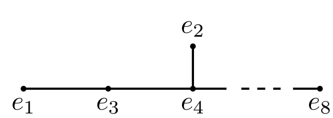


FIGURE 1. Dynkin diagrams

by [(122)], that is the glue words are spanned by

$$[122] = [[1],[2],[2]] = [(1/2)^8,(0^7 1),(0^7 1)]$$
$$[212] = [[2],[1],[2]] = [(0^7 1),(1/2)^8,(0^7 1)]$$
$$[221] = [[2],[2],[1]] = [(0^7 1),(0^7 1),(1/2)^8].$$

The full glue code for this example contains the eight vectors [000], [122], [212], [221], [033], [303], [330], [111].

2.4. Elliptic fibrations. — We recall that all the elliptic fibrations of an elliptic $K3$ surface with Picard number 20 come from primitive embeddings of a specific rank 6 lattice into a Niemeier lattice denoted $Ni(L_{\text{root}})$.

2.5. Nikulin and Niemeier's results. —

Lemma 2.1 **(Nikulin** [20, Proposition 1.4.1]**).** — *Let L be an even lattice. Then, for an even overlattice M of L, we have a subgroup M/L of $G_L = L^*/L$ such that q_L is trivial on M/L. This determines a bijective correspondence between even overlattices of L and subgroups G of G_L such that $q_L\mid_G = 0$.*

Lemma 2.2 **(Nikulin** [20, Proposition 1.6.1]**).** — *Let L be an even unimodular lattice and T a primitive sublattice. Then we have*

$$G_T \simeq G_{T^\perp} \simeq L/(T \oplus T^\perp), \quad q_{T^\perp} = -q_T.$$

In particular, $|\det T| = |\det T^\perp| = [L : T \oplus T^\perp]$.

Theorem 2.3 **(Nikulin** [20, Corollary 1.6.2]**).** — *Let L and M be non-degenerate even integral lattices such that*

$$G_L \simeq G_M, \quad q_L = -q_M.$$

Then there exists an unimodular overlattice N of $L \oplus M$ such that

(1) *the embeddings of L and M in N are primitive*

(2) $L_N^\perp = M$ *and* $M_N^\perp = L$.

Theorem 2.4 **(Niemeier** [19]**).** — *A negative-definite even unimodular lattice L of rank 24 is determined by its root lattice L_{root} up to isometries. There are 24 possibilities for L and L/L_{root} listed in Table 1.*

The lattices L defined in Table 1 are called *Niemeier lattices*.

2.6. The Kneser–Nishiyama's method. — Recall that a $K3$ surface may admit more than one elliptic fibration, but up to isomorphism, there is only a finite number of elliptic fibrations [28]. To establish a complete classification of the elliptic fibrations on an elliptic $K3$ surface, we use Nishiyama's method based on lattice theoretic ideas [21]. The technique builds on a converse of Nikulin's results.

Given a $K3$ surface X with an elliptic fibration $\pi : X \mapsto \mathbb{P}^1$, define the hyperbolic lattice U generated by the class of a fiber F and the class of the zero section. Every elliptic fibration on X is associated to a primitive embedding of the lattice U into the Néron–Severi lattice $NS(X)$ and its orthogonal complement in $NS(X)$ is called the *frame* $W_\pi(X)$ with respect

Table 1. Niemeier lattices

L_{root}	L/L_{root}	Generators for the glue code
E_8^3	(0)	(000)
$D_{16}E_8$	$\mathbb{Z}/2\mathbb{Z}$	[10]
$D_{10}E_7^2$	$(\mathbb{Z}/2\mathbb{Z})^2$	[110],[301]
$A_{17}E_7$	$\mathbb{Z}/6\mathbb{Z}$	[31]
D_{24}	$\mathbb{Z}/2\mathbb{Z}$	[1]
D_{12}^2	$(\mathbb{Z}/2\mathbb{Z})^2$	[12],[21]
D_8^3	$(\mathbb{Z}/2\mathbb{Z})^3$	[(122)]
$A_{15}D_9$	$\mathbb{Z}/8\mathbb{Z}$	[21]
E_6^4	$(\mathbb{Z}/3\mathbb{Z})^2$	[1(012)]
$A_{11}D_7E_6$	$\mathbb{Z}/12\mathbb{Z}$	[111]
D_6^4	$(\mathbb{Z}/2)^4$	[even perms of (0123)]
$A_9^2D_6$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$	[240],[501],[053]
$A_7^2D_5^2$	$\mathbb{Z}/4 \times \mathbb{Z}/8$	[1112],[1721]
A_8^3	$\mathbb{Z}/3 \times \mathbb{Z}/9$	[(114)]
A_{24}	$\mathbb{Z}/5$	[5]
A_{12}^2	$\mathbb{Z}/13\mathbb{Z}$	[2(11211122212)]
D_4^6	$(\mathbb{Z}/2)^6$	[111111],[0,(02332)]
$A_5^4D_4$	$\mathbb{Z}/2 \times (\mathbb{Z}/6)^2$	[2(024)0],[33001],[30302],[30033]
A_6^4	$(\mathbb{Z}/7)^2$	[1(216)]
A_4^6	$(\mathbb{Z}/5)^3$	[1(01441)]
A_3^8	$(\mathbb{Z}/4)^4$	[3(2001011)]
A_2^{12}	$(\mathbb{Z}/3)^6$	[2(11211122212)]
A_1^{24}	$(\mathbb{Z}/2)^{12}$	[1(00000101001100110101111)]

to the elliptic fibration π. The *trivial lattice* $T = U \oplus (W_\pi)_{\text{root}}$ verifies $(W_\pi)_{\text{root}} = \sum_{v \in w} T_v$ where w are the points of $\mathbb{P}^1$ corresponding to the reducible singular fibers and T_v the lattice generated by the fiber components except the zero component (see Schütt–Shioda [24]). Nishiyama aims at embedding the frames of all elliptic fibrations into a negative-definite lattice, more precisely into a Niemeier lattice of rank 24. For this purpose, he first determines an even negative-definite lattice M such that

$$q_M = -q_{NS(X)}, \qquad \operatorname{rank}(M) + \rho(X) = 26.$$

By Theorem 2.3, $M \oplus W_\pi(X)$ has a Niemeier lattice as an overlattice for each frame $W_\pi(X)$ of an elliptic fibration on X. Thus one is bound to determine the (inequivalent) primitive embeddings of M into Niemeier lattices L. To achieve this, it is essential to consider the root lattices involved. In each case, the orthogonal complement of M into L gives the corresponding frame $W(X)$.

2.6.1. *The transcendental lattice and argument from Nishiyama paper.* — Denote by $T(X)$ the transcendental lattice of X, i.e. the orthogonal complement of $NS(X)$ in $H^2(X,\mathbb{Z})$ with respect to the cup-product,

$$T(X) = \mathrm{NS}(X)^{\perp} \subset H^2(X,\mathbb{Z}).$$

In general, $T(X)$ is an even lattice of rank $r = 22 - \rho(X)$ and signature $(2, 20 - \rho(X))$, $\rho(X)$ being the Picard number of the $K3$ surface X. Let $t = r - 2$. By Nikulin's Theorem [20], $T(X)[-1]$ admits a primitive embedding into the following indefinite unimodular lattice:

$$T(X)[-1] \hookrightarrow U^t \oplus E_8[-1].$$

Then define M as the orthogonal complement of $T(X)[-1]$ in $U^t \oplus E_8[-1]$. By construction, M is a negative-definite lattice of rank $2t + 8 - r = r + 4 = 26 - \rho(X)$.
By Lemma 2.2 the discriminant form satisfies

$$q_M = -q_{T(X)[-1]} = q_{T(X)} = -q_{NS(X)}.$$

Hence M takes exactly the shape required for Nishiyama's technique.

3. Elliptic fibrations of Y_{10}

To prove the primitivity of the embeddings we shall use the following.

Lemma 3.1. — *A lattice embedding is primitive if and only if the greatest common divisor of the maximal minors of the embedding matrix with respect to any choice of basis is* 1.

Remark 3.2. — In that case the Smith form of the embedding matrix has only 1 on the diagonal.

3.1. Kneser–Nishiyama technique applied to Y_{10}. — By the Kneser–Nishiyama method we can determine elliptic fibrations of Y_{10}. For further details we refer to [4, 5, 15, 21]. In [4, 5, 21] only singular $K3$ (i.e. of Picard number 20) are considered.
Let us describe how to determine M in the case of the $K3$ surface Y_{10}.
Let $T(Y_{10})$ be the transcendental lattice of Y_{10}. The lattice $T(Y_{10})$ is an even lattice of rank $r = 22 - \rho(Y_{10}) = 2$ and signature $(1,\ 1)$. In that case $t := r - 2 = 0$ and

$$T(Y_{10}) = \begin{pmatrix} 6 & 0 \\ 0 & 12 \end{pmatrix}$$

by Bertin's results [1]. Thus we must determine a primitive embedding of $T(Y_{10})[-1]$ into $E_8[-1]$. If u, v denotes a basis of $T(Y_{10})[-1]$, define

$$\Phi(u) = 2e_1 + e_3$$
$$\Phi(v) = 3e_2 + 2e_1 + 4e_3 + 6e_4 + 5e_5 + 4e_6.$$

By Lemma 3.1, this is a primitive embedding since the greatest common divisor of the maximal minors of the embedding matrix is 1.
Then

$$M = \langle \Phi(u), \Phi(v) \rangle^{\perp}_{E_8} = \langle e_2, e_1 + 3e_4, e_3, e_5, 4e_6 + 3e_7, e_8 \rangle.$$

With a reduced basis using LLL algorithm [10], we get

$$M = \langle e_3 \rangle \oplus \langle e_8, x \rangle \oplus \langle e_2, e_5, y \rangle$$

with

$$x = 2e_1 + 3e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6 + 3e_7 + e_8$$

and

$$y = e_2 + e_1 + 2e_3 + 3e_4 + e_5.$$

Moreover

$$\langle e_8, x\rangle = A_2$$

and the Gram matrix of $N = \langle e_2, e_5, y\rangle$ is

$$\begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -4 \end{pmatrix}$$

with determinant -12. So we get $M = A_1 \oplus A_2 \oplus N$ and all the elliptic fibrations are given by the primitive embeddings of M into the Niemeier lattices $Ni(L_{\text{root}})$.

Remark 3.3. — Since M is not a root lattice, the primitive embeddings of M into a Niemeier lattice $Ni(L_{\text{root}})$ are not necessarily given by the primitive embeddings of M into its root lattice L_{root}. This is the crucial difference with the situation encountered in the previous papers [4, 5].

3.2. Types of primitive embeddings of N into the Niemeier lattices. — There are essentially three types of primitive embeddings of N into a Niemeier lattice $Ni(L_{\text{root}})$.

(1) The embeddings of N into the root lattices A_n, D_n and E_l of L_{root}.

(2) The embeddings of N into a direct sum of two root lattices of L_{root}, following from the fact that a vector of norm -4 is the sum of two roots belonging to two different root lattices.

(3) The embeddings of N into $Ni(L_{\text{root}})$.

The type (2) subdivises in two cases denoted as follows:

(2.a) Type (A_2, A_2). — That means $(-4) = (r_1, r_2)$ and a root v_1 (resp. v_2) is in the same root lattice as r_1 (resp. r_2) and satisfies $r_1.v_1 = 1$ (resp. $r_2.v_2 = 1$). Thus in the first root lattice, the roots r_1 and v_1 (resp. in the second root lattice r_2 and v_2) realize a root lattice A_2. For example the rank 1 elliptic fibration with singular fibers of type $6A_2A_5$ is obtained from an embedding of type (A_2, A_2) into E_6^4.
Another example of embedding of type (A_2, A_2) is the rank 7 fibration $2D_43A_1$ obtained with the following primitive embedding into D_4^6: A_2 in D_4 three times and A_1 in D_4.

(2.b) Type (A_1, A_3). — That means $(-4) = (r_1, r_2)$ and the roots v_1 and v_2 are in the same root lattice as r_2 and satisfies $r_2.v_1 = 1$, $r_2.v_2 = 1$ and $v_1.v_2 = 0$. Thus in the first root lattice, the root r_1 is viewed as A_1 and in the second root lattice, the roots v_1, r_2, v_2 are viewed as A_3. All the fibrations presented in Theorem 6.1, except fibrations (5) and (6) can be obtained with this type of embedding.
Thus, case (2.a) and (2.b) are reduced to find primitive embeddings of direct sums of root lattices, that is of $A_1 \oplus A_2 \oplus A_2 \oplus A_2$, two of the A_2 being embedded in different root lattices, in case (2.a) and $A_1 \oplus A_2 \oplus A_1 \oplus A_3$, the two A_1 and A_3 being embedded in different root lattices, in case (2.b).
As for type (2) the type (1) can be divided in two cases.

(1.a). — From a primitive embedding of $A_2 \oplus A_2$ into a root lattice we get a primitive embedding of N in the same root lattice as follows.
Denoting $A_2^{(1)} = \langle a_1, a_2 \rangle$ and $A_2^{(2)} = \langle b_1, b_2 \rangle$ we can take, for example, one of the four possible embeddings $(-4) = a_1 + b_1$, $v_1 = a_2$, $v_2 = b_2$.

(1.b). — From a primitive embedding of $A_3 \oplus A_1$ into a root lattice we get a primitive embedding of N in the same root lattice as follows.
Denoting $A_3 = \langle a_1, a_2, a_3 \rangle$ and $A_1 = \langle b_1 \rangle$ with $a_i.a_i = -2$, $i = 1, 2, 3$, $a_1.a_2 = a_2.a_3 = 1$ and $a_1.a_3 = 0$, we get the primitive embedding of N as $(-4) = a_2 + b_1$, $v_1 = a_1$, $v_2 = a_3$.

Remark 3.4. — In an irreducible root lattice, contrary to case (2), the decomposition of a norm (-4) vector in two orthogonal roots is not unique. Thus the same embedding can be viewed either of type (A_2, A_2) or of type (A_3, A_1).

The type (3) needs to find representatives of glue vectors with norm -4.
As we can see there is a lot of glue vectors of norm -4, hence added to the other types of embeddings, provide a huge amount of elliptic fibrations for Y_{10}. We give some examples.
A rank 2 fibration with singular fibers of type $4A_1D_4D_8$ follows from an embedding into $Ni(D_{12}^2)$. The glue code is $[1, 2]$ and realizes a vector v of norm 4, with

$$[1] = (1/2, \ldots, 1/2) \qquad\qquad [2] = (0, \ldots, 0, 1).$$

The embedding of N, A_1 and A_2 is given by

$$N = \begin{pmatrix} [1] & [2] \\ 0 & (0^{10}, 1, -1) \\ (0^2, 1, 0^2, 1, 0^6) & 0 \end{pmatrix}$$
$$A_1 = ((0^6, 1, -1, 0^4), 0))$$
$$A_2 = \begin{pmatrix} 0, (1, -1, 0^{10}) \\ 0, (0, 1, -1, 0^9) \end{pmatrix}.$$

A rank 1 fibration with singular fibers of type $2A_13A_5$ follows from an embedding into $Ni(A_{11}D_7E_6)$. We take the glue vector $[6, 2, 0]$ with $[6] = (1/2, 1/2, 1/2, 1/2, 1/2, 1/2, -1/2, \ldots, -1/2)$. We embed N, A_1 and A_2 as

$$N = \begin{pmatrix} [6] & [2] & 0 \\ 0 & (0^5, 1, -1) & 0 \\ 0 & (0^5, 1, 1) & 0 \end{pmatrix}$$
$$A_1 = (0 \quad 0 \quad (-1, 1, 0^6)))$$
$$A_2 = \begin{pmatrix} 0 & (1, -1, 0^5) & 0 \\ 0 & (0, 1, -1, 0^4) & 0 \end{pmatrix}.$$

A rank 4 fibration with singular fibers of type $3A_3 5A_1$ follows from an embedding in $Ni(A_3^8)$ using the norm 4 glue vector $(3,2,0,0,1,0,1,1) = (a_1^*, a_2^*, 0, 0, a_3^*, 0, a_3^*, a_3^*)$ as

$$N = \begin{pmatrix} a_1^* & a_2^* & 0 & 0 & a_2^* & 0 & a_3^* & a_3^* \\ 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For computations in case (3) we use Conway and Sloane notations [11]. For the two first types we use notations and results of Nishiyama [21] expressed in Bourbaki's notations [7].
We proceed in the following way. Once is found a primitive embedding of N into a Niemeier lattice (eventually its root lattice), we compute its orthogonal. If the orthogonal contains A_1 or A_2 or both we test if the previous embedding of N plus A_1 (resp. A_2) (resp.$A_1 \oplus A_2$) is primitive to insure the existence of an elliptic fibration of Y_{10}.

3.3. Primitive embeddings into root lattices. — We give only the examples useful for explaining the Weierstrass equations of the elliptic fibrations given in the paper.

3.3.1. *Embedding of N into D_5.* —

Lemma 3.5. —

(1) *The lattice N embeds primitively in D_5 by $N = \langle d_5, d_4, d_3 + d_1\rangle$ and*

$$(N)^{\perp}_{D_5} = \langle d_1 + d_2, d_5 + d_4 + 2d_3 + d_2\rangle \simeq A_2.$$

(2) *This defines an embedding of $A_2 \oplus N$ in D_5, by*

$$N \oplus A_2 = \langle d_5, d_4, d_3 + d_1\rangle \oplus \langle d_1 + d_2, d_5 + d_4 + 2d_3 + d_2\rangle$$

which is not primitive, since $N \oplus A_2$ is a sublattice of index 3 into D_5.

Proof. — The first assertion is a simple computation and the second follows from the fact that the two lattices $N \oplus A_2$ and D_5 have the same rank and satisfy the relations $\det(A_2 \oplus N) = 4 \times 3^2$ and $\det(D_5) = 4$. □

3.3.2. *Embedding of $N \oplus A_1$ into D_5.* — The primitive embedding of $N \oplus A_1 = \langle d_5, d_4, d_3 + d_1, d_1 + d_2\rangle$ into D_5 satisfies

$$(N \oplus A_1)^{\perp}_{D_5} = \langle d_1 + 2d_2 + 4d_3 + 2d_5 + 2d_4\rangle = (-6).$$

3.3.3. *Embeddings into D_6.* — Exactly as in the case (2) we get essentially two types of embeddings which are not isomorphic.

Lemma 3.6. —

(a) *The following primitive embedding of N into D_6 given by*

$$\langle d_6, d_5, d_4 + d_2\rangle \hookrightarrow D_6$$

has its orthogonal in D_6 isomorphic to $A_2 \oplus (-4)$.

The embedding of $N \oplus A_1$ into D_6 given by

$$\langle d_6, d_5, d_4 + d_2, d_3 + d_2 \rangle \hookrightarrow D_6$$

is primitive.

(b) *But the primitive embedding of N*

$$\langle d_6, d_4 + d_2, d_1 \rangle \hookrightarrow D_6$$

has its orthogonal in D_6 isomorphic to $A_1 \oplus (-4) \oplus (-6)$.

Proof. —

(a). — The orthogonal is $\langle d_6 + d_5 + 2d_4 + 2d_1, d_3 - 2d_1, d_2 + 2d_1 \rangle$. With LLL, we find

$$N^{\perp}_{D_6} = \langle d_6 + d_5 + 2d_4 + d_3, d_3 + d_2, d_6 + d_5 + 2d_4 + 2d_3 + 2d_2 + 2d_1 \rangle = A_2 \oplus (-4).$$

The matrix of the embedding of $N \oplus A_1$ being

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix},$$

it follows that this embedding is primitive since we can extract a matrix of dimension 4 and determinant 1. Its orthogonal is $\langle d_6 + d_5 + 2d_4 + 2d_3 + 2d_2 + 2d_1, d_3 + 3d_2 + d_1 \rangle$, with Gram matrix of determinant 24 and no roots.

(b). — In the second case the orthogonal in D_6 of the embedding

$$\langle d_6, d_4 + d_2, d_1 \rangle \hookrightarrow D_6$$

is

$$\langle d_6 + 3d_5 + 2d_4, -2d_5 + d_3, 3d_5 + 2d_2 + d_1 \rangle.$$

With LLL, we get

$$\langle A_1 = d_6 + d_5 + 2d_4 + d_3 \oplus (-4) \oplus (-6) \rangle.$$

As previously we can prove that the embedding of $N \oplus A_1$

$$\langle d_6, d_4 + d_2, d_1, d_6 + d_5 + 2d_4 + d_3 \rangle \hookrightarrow D_6$$

is primitive. □

Lemma 3.7. — *For all $n \geq 7$, there exists a primitive embedding of N into D_n, such that*

$$((N)^{\perp}_{D_n})_{\text{root}} \simeq A_2 \oplus D_{n-5}.$$

For $n = 6$

$$((N)^{\perp}_{D_6})_{\text{root}} \simeq A_2.$$

Proof. — Suppose first $n > 8$ and consider the following embedding

$$N = \langle d_n, d_{n-1}, d_{n-2} + d_{n-4} \rangle \hookrightarrow D_n.$$

The following roots of D_n are orthogonal to N:

$$d_1, d_2, \ldots, d_{n-6}, x = d_n + d_{n-1} + 2(d_{n-2} + d_{n-3} + d_{n-4} + d_{n-5}) + d_{n-6}.$$

These roots satisfy the relations:

$$d_{n-6}.d_{n-7} = 1, \quad d_{n-7}.d_{n-8} = 1, \quad \dots \quad d_2.d_1 = 1$$
$$x.d_{n-6} = x.d_{n-8} = \dots = x.d_1 = 0$$
$$x.d_{n-7} = 1.$$

We deduce $\langle d_{n-6}, x, d_{n-7}, d_{n-8}, \dots, d_1\rangle \simeq D_{n-5}$. Consider also the following roots y and z of D_n:

$$y = d_n + d_{n-1} + 2(d_{n-2} + d_{n-3}) + d_{n-4}, \qquad z = -(d_{n-3} + d_{n-4}).$$

They satisfy $y.z = 1$, hence $\langle y, z\rangle \simeq A_2$. They are orthogonal to N, d_{n-6}, x, $d_{n-8}, \dots, d_1$.
Finally $(N^{\perp}_{D_n})_{\text{root}} \simeq A_2 \oplus D_{n-5}$.
Notice, if $n = 8$, that $D_{n-5} \simeq A_3$.
If $n = 7$,

$$(N^{\perp}_{D_7})_{\text{root}} \simeq \langle y, z\rangle \oplus \langle x, d_1\rangle \simeq A_2 \oplus 2A_1 \simeq A_2 \oplus D_2.$$

If $n = 6$, a direct computation, see Lemma 3.6 a), gives $N^{\perp}_{D_6} \simeq A_2 \oplus (-4) \simeq A_2 \oplus D_1$. □

Lemma 3.8. —

(1) *There is a primitive embedding of $N \oplus A_1$ into D_7 with orthogonal $A_1 \oplus A_2$.*

(2) *There is no primitive embedding of $N \oplus A_2$ into D_7 with orthogonal $A_1 \oplus A_1$.*

Proof. —

(1). — Consider the embedding:

$$N \oplus A_1 = \langle d_7, d_6, d_5 + d_3, d_1\rangle \hookrightarrow D_7.$$

Take $d = a_1 d_1 + \dots + a_7 d_7$ satisfying $nd = \lambda_7 d_7 + \lambda_6 d_6 + \lambda_5 (d_5 + d_3) + \lambda_1 d_1$. From the relations $na_1 = \lambda_1$, $a_2 = 0$, $na_3 = na_5 = \lambda_5$, $a_4 = 0$, etc., we find $d = a_1 d_1 + a_3(d_3 + d_5) + a_6 d_6 + a_7 d_7$, that is $d \in N \oplus A_1$, proving the primitivity of the embedding.
We can also prove the primitivity using Lemma 3.1.
The orthogonal of the embedding is

$$\langle d_6 + d_7 + 2d_5 + 2d_4 + 2d_3 + 2\ d_2 + d_1\rangle \oplus \langle d_4 + d_3, -d_6 - d_7 - 2d_5 - 2d_4 - d_3\rangle \simeq A_1 \oplus A_2.$$

(2). — The embedding

$$N \oplus A_2 = \langle d_7, d_6, d_5 + d_3, d_4 + d_3, -d_7 - d_6 - 2d_5 - 2d_4 - d_3\rangle \hookrightarrow D_7$$

is not primitive since its matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & -2 & -2 & -1 & 0 & 0 \end{pmatrix}$$

has its extremal minor of determinant 3. □

3.3.4. *Embedding of N into E_8.* —

Lemma 3.9. —

(1) *There is a primitive embedding of N into E_8 whose orthogonal in E_8 is $A_2 \oplus A_3$.*

(2) *There is a primitive embedding of $N \oplus A_1$ into E_8 whose orthogonal in E_8 is $A_3 \oplus (-6)$.*

(3) *There is a primitive embedding of $N \oplus A_1 \oplus A_2$ into E_8 whose orthogonal in E_8 is $(-6) \oplus (-12)$.*

Proof. —

(1). — Embed primitively N in E_8 as $N = \langle e_2, e_7, e_4 + e_6 \rangle$, we get

$$(N)^{\perp}_{E_8} = \langle e_1, e_2 + 3e_3 + 2e_4, e_5 - 2e_3, 2e_3 + e_6 - e_8, -e_3 + e_7 + 2e_8 \rangle.$$

With LLL algorithm [8] we obtain

$$\begin{aligned}(N)^{\perp}_{E_8} = &\langle e_1, e_2 + e_3 + 2e_4 + e_5, 2e_1 + 2e_2 + 3e_3 + 4e_4 + 3e_5 + 2e_6 + e_7 \rangle \\ &\oplus \langle -e_1 - e_2 - 2e_3 - 2e_4 - e_5 - e_6 - e_7 - e_8, \\ &\qquad 2e_1 + 3e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6 + 3e_7 + 2e_8 \rangle\end{aligned}$$

with Gram matrix

$$\begin{pmatrix} -2 & 1 & -1 \\ 1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix} \oplus \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

Thus

$$(N)^{\perp}_{E_8} = A_3 \oplus A_2.$$

(2). — The following embedding

$$N \oplus A_1 = \langle e_2, e_7, e_4 + e_6, -e_1 - e_2 - 2e_3 - 2e_4 - e_5 - e_6 - e_7 - e_8 \rangle \hookrightarrow E_8$$

has for orthogonal in E_8, the lattice $A_3 \oplus (-6)$.

(3). — If moreover we embed A_2 into the previous A_3 we find $(-6) \oplus (-12)$ as orthogonal of $N \oplus A_1 \oplus A_2$ into E_8. □

3.3.5. *Embedding of $N \oplus A_1$ into E_7.* — The following primitive embedding of $N \oplus A_1$ into E_7 given by

$$\langle e_1, e_3 + e_5, e_6, e_2 \rangle \hookrightarrow E_7$$

satisfies

$$((N \oplus A_1)^{\perp}_{E_7})_{\text{root}} = \langle e_7 \rangle = A_1.$$

3.3.6. *Embeddings of N into E_6.* —

Lemma 3.10. — *There are at least two types of non isomorphic primitive embeddings of N into E_6:*

(1) $\phi_0(N) = \langle e_1, e_1 + e_2 + 2e_3 + 2e_4 + e_5, e_2 + e_3 + 2e_4 + 2e_5 + 2e_6 \rangle$, *with* $(\phi_0(N))^{\perp}_{E_6} = \langle e_2, e_4, e_5 \rangle \simeq A_3$.

(2) $\phi_1(N) = \langle e_1, e_3 + e_5, e_6 \rangle$, *with* $((\phi_1(N))^{\perp}_{E_6})_{\text{root}} \simeq A_2$.

Proof. —

(1). — The embedding ϕ_0 is given in Nishiyama [21].

(2). — We get

$$\begin{aligned}(\phi_1(N))^{\perp}_{A_6} &= \langle e_1 + 2e_3 - 2e_5 - e_6, 3e_4 + 4e_5 + 2e_6, e_2 \rangle \\ &= \langle e_1 + 2e_2 + 2e_3 + 2e_5 + e_6, e_2, 3e_4 + 4e_5 + 2e_6 \rangle,\end{aligned}$$

hence the result. □

3.3.7. *Fibrations involving embeddings ϕ_0 and ϕ_1.* —

(a) The embedding ϕ_0 of N into E_6 leads to the rank 0 and 3-torsion fibration $2A_2A_3A_5E_6$.

(b) With the same embeddings but replacing the embedding ϕ_0 by ϕ_1 we get the rank 1 fibration $3A_2\ A_5\ E_6$.

Remark 3.11. — To illustrate the complexity of the determination of elliptic fibrations of Y_{10}, notice the following fibrations all with two fibers of type A_5 coming from primitive embeddings into various Niemeier lattices:

- $2A_12A_5$ $(r = 6)$ resulting from an embedding in $A_5^4D_4$ (type (A_3, A_1) into A_5^2),
- $2A_12A_22A_5$ $(r = 2)$ resulting from an embedding into E_6^4,
- $A_12A_2A_32A_5$ $(r = 0)$ resulting from an embedding into $Ni(A_{11}D_7E_6)$.

Besides, we recall a result obtained by Nishiyama [21].

Lemma 3.12. — *Up to the action of the Weyl group, the unique primitive embeddings of A_1 and A_2 in the following root lattices together with their orthogonals are given in the following list*

- $A_1 = \langle d_l \rangle \subset D_l,\ l \geq 4, \quad (A_1)^{\perp}_{D_l} = A_1 \oplus D_{l-2}$
- $A_1 = \langle d_4 \rangle \subset D_4, \quad (A_1)^{\perp}_{D_4} = A_1^{\oplus 3}$
- $A_1 = \langle e_1 \rangle \subset E_p,\ p = 6, 7, 8$

$$\begin{aligned}(A_1)^{\perp}_{E_6} &= A_5 \\ (A_1)^{\perp}_{E_7} &= D_6 \\ (A_1)^{\perp}_{E_8} &= E_7\end{aligned}$$

– $A_2 = \langle e_1, e_3\rangle \subset E_p,\ p = 6, 7, 8$

$$(A_2)^{\perp}_{E_6} = A_2^{\oplus 2}$$
$$(A_2)^{\perp}_{E_7} = A_5$$
$$(A_2)^{\perp}_{E_8} = E_6$$

4. Specialized elliptic fibrations of Y_{10}

Bertin and Lecacheux gave in [6] all the elliptic fibrations called generic together with Weierstrass equations of the $K3$ surface (Y_k) of the Apéry–Fermi pencil. When specializing k to 10 we obtain certain elliptic fibrations of Y_{10} named *specialized fibrations* and the process is called specialization.

Theorem 4.1. — *The specialized elliptic fibrations of Y_{10} have the same singular fibers and torsion as the generic ones. Their rank is equal to the corresponding generic one plus one, hence is bounded by* 3.
All the embeddings giving such fibrations can be derived from embedding $N \oplus A_1 \oplus A_2$ into the root lattices of the Niemeier lattices.

Proof. — We get first the specialized Weierstrass equations of Y_{10} from the generic ones given in Bertin–Lecacheux [6, Tables 3 and 4].
Then we deduce the primitive embeddings of $N \oplus A_1 \oplus A_2$ into the Niemeier lattices giving the corresponding elliptic fibration in Table 2. Recalling the elliptic fibrations in the generic case [6, Table 2], we derive some observations.
The specialized fibrations are all obtained by replacing a primitive embedding of D_5 in the generic case by a primitive embedding of $A_1 \oplus N$ in the same corresponding root lattice in the Y_{10} case. Moreover the trivial lattices of the elliptic fibrations are the same. Since the Picard number is 19 in the generic case and 20 for Y_{10}, this explains why the rank of the specialized Y_{10} always increases by 1. This is a consequence of the following lemma. □

Lemma 4.2. —

(1) *Denote R any root lattice D_n, $n \geq 5$, E_6, E_7 or E_8. There is a primitive embedding of $N \oplus A_1$ into R such that $((N \oplus A_1)^{\perp}_{R})_{\text{root}} = ((D_5)^{\perp}_{R})_{\text{root}}$.*

(2) *There is a primitive embedding of $N \oplus A_1 \oplus A_2$ into E_8 such that $((N \oplus A_1 \oplus A_2)^{\perp}_{E_8})_{\text{root}} = ((A_2 \oplus D_5)^{\perp}_{E_8})_{\text{root}} = 0$.*

Proof. —

(1). — If $R = D_n$, we embed N into D_n as in Lemma 3.7 and A_1 into the A_2 part of its orthogonal in D_n. That is precisely if $n \geq 7$, the primitive embedding

$$N \oplus A_1 = \langle d_n, d_{n-1}, d_{n-2} + d_{n-4}, d_{n-3} + d_{n-4}\rangle \hookrightarrow D_n.$$

The following roots of D_n,

$$d_{n-6}, d_{n-7}, \dots, d_1, d_n + d_{n-1} + 2(d_{n-2} + d_{n-3} + \cdots + d_{n-5}) + d_{n-6}$$

are orthogonal to $N \oplus A_1$, and generate D_{n-5}. If $n = 6$, take the embedding of Lemma 3.6 (a).

TABLE 2. The specialized elliptic fibrations of Y_{10}

L_{root}	L/L_{root}			type of Fibers	Rk	Tors.
E_8^3	(0)					
	#1	$A_2 \subset E_8$	$(A_1 \oplus N) \subset E_8$	$E_6A_3E_8$	1	(0)
	#2	$A_2 \oplus (A_1 \oplus N) \subset E_8$		E_8E_8	2	(0)
$D_{16}E_8$	$\mathbb{Z}/2\mathbb{Z}$					
	#3	$A_2 \subset E_8$	$(A_1 \oplus N) \subset D_{16}$	E_6D_{11}	1	(0)
	#4	$A_2 \oplus (A_1 \oplus N) \subset E_8$		D_{16}	2	$\mathbb{Z}/2\mathbb{Z}$
	#5	$(A_1 \oplus N) \subset E_8$	$A_2 \subset D_{16}$	A_3D_{13}	2	(0)
	#6	$A_2 \oplus (A_1 \oplus N) \subset D_{16}$		E_8D_8	2	(0)
$D_{10}E_7^2$	$(\mathbb{Z}/2\mathbb{Z})^2$					
	#7	$A_2 \subset E_7$	$(A_1 \oplus N) \subset D_{10}$	$E_7A_5D_5$	1	$\mathbb{Z}/2\mathbb{Z}$
	#8	$A_2 \subset E_7$	$(A_1 \oplus N) \subset E_7$	$A_5A_1D_{10}$	2	$\mathbb{Z}/2\mathbb{Z}$
	#9	$A_2 \oplus (A_1 \oplus N) \subset D_{10}$		$E_7E_7A_1A_1$	2	$\mathbb{Z}/2\mathbb{Z}$
	#10	$(A_1 \oplus N) \subset E_7$	$A_2 \subset D_{10}$	$A_1D_7E_7$	3	(0)
$A_{17}E_7$	$\mathbb{Z}/6\mathbb{Z}$					
	#11	$(A_1 \oplus N) \subset E_7$	$A_2 \subset A_{17}$	A_1A_{14}	3	(0)
D_{24}	$\mathbb{Z}/2\mathbb{Z}$					
	#12	$A_2 \oplus (A_1 \oplus N) \subset D_{24}$		D_{16}	2	(0)
D_{12}^2	$(\mathbb{Z}/2\mathbb{Z})^2$					
	#13	$A_2 \subset D_{12}$	$A_1 \oplus N \subset D_{12}$	D_9D_7	2	(0)
	#14	$A_2 \oplus (A_1 \oplus N) \subset D_{12}$		D_4D_{12}	2	$\mathbb{Z}/2\mathbb{Z}$
D_8^3	$(\mathbb{Z}/2\mathbb{Z})^3$					
	#15	$A_2 \subset D_8$	$(A_1 \oplus N) \subset D_8$	$D_5A_3D_8$	2	$\mathbb{Z}/2\mathbb{Z}$
	#16	$A_2 \oplus (A_1 \oplus N) \subset D_8$		D_8D_8	2	$\mathbb{Z}/2\mathbb{Z}$
$A_{15}D_9$	$\mathbb{Z}/8\mathbb{Z}$					
	#17	$A_2 \oplus (A_1 \oplus N) \subset D_9$		A_{15}	3	$\mathbb{Z}/2\mathbb{Z}$
	#18	$(A_1 \oplus N) \subset D_9$	$A_2 \subset A_{15}$	D_4A_{12}	2	(0)
E_6^4	$(\mathbb{Z}/3\mathbb{Z})^2$					
	#19	$A_2 \subset E_6$	$(A_1 \oplus N) \subset E_6$	$A_2A_2E_6E_6$	2	$\mathbb{Z}/3\mathbb{Z}$
$A_{11}D_7E_6$	$\mathbb{Z}/12\mathbb{Z}$					
	#20	$A_2 \subset E_6$	$(A_1 \oplus N) \subset D_7$	$A_2A_2A_1A_1A_{11}$	1	$\mathbb{Z}/6\mathbb{Z}$
	#21	$A_2 \subset A_{11}$	$(A_1 \oplus N) \subset D_7$	$A_8A_1A_1E_6$	2	(0)
	#22	$A_2 \subset A_{11}$	$(A_1 \oplus N) \subset E_6$	A_8D_7	3	(0)
	#23	$(A_1 \oplus N) \subset E_6$	$A_2 \subset D_7$	$A_{11}D_4$	3	$\mathbb{Z}/2\mathbb{Z}$
D_6^4	$(\mathbb{Z}/2\mathbb{Z})^4$					
	#24	$A_2 \subset D_6$	$(A_1 \oplus N) \subset D_6$	$A_3D_6D_6$	3	$\mathbb{Z}/2\mathbb{Z}$
$A_9^2D_6$	$\mathbb{Z}/2 \times \mathbb{Z}/10$					
	#25	$(A_1 \oplus N) \subset D_6$	$A_2 \subset A_9$	A_6A_9	3	(0)
$A_7^2D_5^2$	$\mathbb{Z}/4 \times \mathbb{Z}/8$					
	#26	$(A_1 \oplus N) \subset D_5$	$A_2 \subset D_5$	$A_1A_1A_7A_7$	2	$\mathbb{Z}/4\mathbb{Z}$
	#27	$(A_1 \oplus N) \subset D_5$	$A_2 \subset A_7$	$D_5A_4A_7$	2	(0)

For example, the rank 0 elliptic fibration #20 in [6, Table 2 $(2A_1 2A_2 A_{11})$] comes from the following primitive embedding

$$D_5 = \langle d_7, d_6, d_5, d_4, d_3 \rangle \hookrightarrow D_7,$$

whose orthogonal into D_7 is $D_2 \simeq 2A_1$ by [21, p. 309, 310, 311].
And the rank 1 specialized fibration #20 in Table 2 $(2A_1 2A_2 A_{11})$ comes from the following primitive embedding

$$N \oplus A_1 = \langle d_7, d_6, d_5 + d_3, d_4 + d_3 \rangle \hookrightarrow D_7,$$

whose orthogonal into D_7 is $\simeq 2A1$.
If $R = E_6$, we embed N as in Lemma 3.10 (2). If $R = E_7$ we embed $N \oplus A_1$ as in 3.3.5. If $R = E_8$, we embed $N \oplus A_1$ as in Lemma 3.9 (2).

(2). — It follows from the embedding given in Lemma 3.9 (3). □

5. The extremal elliptic fibrations of Y_{10}

5.1. Embeddings. — In Bertin–Lecacheux [6] all the generic elliptic fibrations are given with Weierstrass equations. We observe by computation that all their specializations for $k = 10$ give elliptic fibrations of Y_{10} with a positive rank less than 3 but no extremal fibration. The list of the extremal elliptic fibrations of Y_{10} can be found in Shimada–Zhang [25]. We shall keep Shimada–Zhang numbering and give the corresponding primitive embeddings of $M = N \oplus A_2 \oplus A_1$ into Niemeier lattices in the following theorem.

Theorem 5.1. — *The extremal elliptic fibrations of Y_{10} come from primitive embeddings of $M = N \oplus A_2 \oplus A_1$ into the following Niemeier lattices with L_{root}*

$$E_8^3,\ D_{16}E_8,\ D_{10}E_7^2,\ E_6^4,\ A_{11}D_7E_6.$$

Eight of them, namely fibrations number 80, 153, 200, 224, 252, 262, 292, 302 are obtained from primitive embeddings of N into a root lattice while the three remaining come from an embedding of N into $Ni(D_{16}E_8)$, namely number 87 and 241 or into $Ni(A_{11}D_7E_6)$, namely number 8. They are listed below with their Mordell–Weil groups:

Number	*Singular fibers*	*Torsion*	*from*
292	$A_2 + A_3 + E_6 + E_7$	(0)	E_8^3
302	$A_2 + A_3 + A_5 + E_8$	(0)	E_8^3
87	$A_1 + A_1 + A_5 + A_{11}$	$\mathbb{Z}/(2)$	$Ni(D_{16}E_8)$
241	$A_1 + A_{11} + E_6$	(0)	$Ni(D_{16}E_8)$
200	$A_2 + A_5 + D_{11}$	(0)	$D_{16}E_8$
252	$A_1 + A_2 + D_9 + E_6$	(0)	$D_{16}E_8$
153	$A_2 + A_5 + D_5 + D_6$	$\mathbb{Z}/(2)$	$D_{10}E_7^2$
262	$A_1 + A_2 + A_3 + A_5 + E_7$	$\mathbb{Z}/(2)$	$D_{10}E_7^2$
224	$A_2 + A_2 + A_3 + A_5 + E_6$	$\mathbb{Z}/(3)$	E_6^4
80	$A_1 + A_2 + A_2 + A_2 + A_{11}$	$\mathbb{Z}/(3)$	$A_{11}D_7E_6$
8	$A_1 + A_2 + A_2 + A_3 + A_5 + A_5$	$\mathbb{Z}/(6)$	$Ni(A_{11}D_7E_6)$

Proof. —

Fibration 292 ($A_2A_3E_6E_7$). — Embed primitively N into $E_8^{(1)}$ as in Lemma 3.9 (1), A_1 into $E_8^{(2)}$ and A_2 into $E_8^{(3)}$ as in Lemma 3.12. Since for these embeddings, $\det(N \oplus A_1 \oplus A_2)^{\perp}_{E_8^3} = \det(A_2 \oplus A_3 \oplus E_6 \oplus E_7) = 72$, the Mordell–Weil group is equal to (0).

Fibration 302 ($A_2A_3A_5E_8$). — Embed primitively N into $E_8^{(1)}$ as in Lemma 3.9 (1), and $A_1 \oplus A_2$ into $E_8^{(2)}$ as in Nishiyama [21, p. 332]. Since there is a fiber of type E_8, the Mordell–Weil group is equal to (0).

Fibrations 87 and 241. — They follow from a primitive embedding of N into $Ni(D_{16}E_8)$ given in the lemma below.

Lemma 5.2. — *There is a primitive embedding of N into $Ni(D_{16}E_8)$ whose root part of its orthogonal in D_{16} contains $A_{11} \oplus 2A_1$.*

Proof of Lemma 5.2. — The glue code of $Ni(D_{16}E_8)$ is generated by $([1], 0)$ cf. Table 1. The norm -4 vector $v = ([3], 0)$ and the norm -2 vectors $v_1 = (d_{15}, 0)$, $v_2 = (d_1, 0)$ define a primitive embedding of N into $Ni(D_{16}E_8)$. Indeed, $[3] = -d_{15}^* + d_1^*$. The primitivity follows from the fact that in the basis $[3], d_1, d_i, 3 \leq i \leq 16$ of D_{16}^* the matrix of the embedding has for maximal minor the matrix identity. Moreover $(\langle -d_{15}^* + d_1^*, d_{15}, d_1 \rangle^{\perp}_{D_{16}})_{\text{root}}$ contains $\langle d_3, d_4, d_5, d_6, d_7, d_8, d_9, d_{10}, d_{11}, d_{12}, d_{13} \rangle \oplus \langle d_{16} \rangle \oplus \langle d_{16} + d_{15} + 2d_{14} + \cdots + 2d_2 + d_1 \rangle = A_{11} \oplus A_1 \oplus A_1$. □

Fibration 87 ($2A_1A_5A_{11}$). — It is obtained from the embedding of N as in the previous lemma and $A_1 \oplus A_2$ into E_8 as in Nishiyama [21, p. 332]. Since for these embeddings, $\det(N \oplus A_1 \oplus A_2)^{\perp}_{E_8 \oplus D_{16}} = \det(A_1 \oplus A_1 \oplus A_{11} \oplus A_5) = 72 \times 4$, the Mordell–Weil group is equal to $\mathbb{Z}/(2)$.

Fibration 241 ($A_1A_{11}E_6$). — The lattice N being embedded as in Lemma 5.2, we embed $A_1 = d_{16}$ in D_{16} and A_2 in E_8 as in Lemma 3.12.
Since for these embeddings, $\det(N \oplus A_1 \oplus A_2)^{\perp}_{E_8 \oplus D_{16}} = \det(A_1 \oplus A_{11} \oplus E_6) = 72$, the Mordell–Weil group is equal to (0).

Fibration 200 ($A_2A_5D_{11}$). — We embed N into D_{16} as in Lemma 3.7 and $A_1 \oplus A_2$ into E_8 as in Nishiyama [21, p. 332].
Since for these embeddings, $\det(N \oplus A_1 \oplus A_2)^{\perp}_{E_8 \oplus D_{16}} = \det A_2 \oplus A_5 \oplus D_{11} = 72$, the Mordell–Weil group is equal to (0).

Fibration 252 ($A_1A_2D_9E_6$). — We embed $N \oplus A_1$ into D_{16} as $\langle d_{16}, d_{15}, d_{14} + d_{12}, d_{10} \rangle$. A direct computation gives the orthogonal $A_1 \oplus D_9 \oplus A_2$. We complete by embedding A_2 into E_8 with orthogonal E_6 as in Lemma 3.12. Since for these embeddings, $\det(N \oplus A_1 \oplus A_2)^{\perp}_{E_8 \oplus D_{16}} = \det(A_1 \oplus A_2 \oplus D_9 \oplus E_6) = 72$, the Mordell–Weil group is equal to (0).

Fibration 153 ($A_2A_5D_5D_6$). — We embed N in D_{10} as in Lemma 3.7, A_1 in $E_7^{(1)}$ and A_2 in $E_7^{(2)}$ as in Lemma 3.12. Since for these embeddings, $\det(N \oplus A_1 \oplus A_2)^{\perp}_{E_7^2 \oplus D_{10}} = \det(A_2 \oplus A_5 \oplus D_5 \oplus D_6) = 4 \times 72$, the Mordell–Weil group is equal to $\mathbb{Z}/(2)$.

Fibration 262 ($A_1A_2A_3A_5E_7$). — We embed $N\oplus A_1$ in D_{10} as $\langle d_{10}, d_9, d_8+d_6, d_4\rangle$ and A_2 in $E_7^{(1)}$. Since for these embeddings, $\det(N\oplus A_1\oplus A_2)^{\perp}_{E_8\oplus D_{16}} = \det(A_1\oplus A_2\oplus A_3\oplus A_5\oplus E_7) = 4\times 72$, the Mordell–Weil group is equal to $\mathbb{Z}/(2)$.

Fibration 224 ($2A_2A_3A_5E_6$). — We embed N into $E_6^{(1)}$ as in Lemma 3.10 (1), A_1 into $E_6^{(2)}$, A_2 into $E_6^{(3)}$ as in Lemma 3.12.
Since for these embeddings, $\det(N\oplus A_1\oplus A_2)^{\perp}_{E_6^4} = \det(A_2\oplus A_2\oplus A_3\oplus A_5\oplus E_6) = 9\times 72$, the Mordell–Weil group is equal to $\mathbb{Z}/(3)$.

Fibration 80 ($A_13A_2A_{11}$). — We embed $N\oplus A_1$ into D_7 as in Lemma 3.8 (1) and A_2 into E_6 as in Lemma 3.12. Since for these embeddings, $\det(N\oplus A_1\oplus A_2)^{\perp}_{A_{11}\oplus E_6\oplus D_7} = \det(A_1\oplus A_2^3\oplus A_{11}) = 9\times 72$, the Mordell–Weil group is equal to $\mathbb{Z}/(3)$.

Fibration 8 ($A_12A_2A_32A_5$). — This fibration is obtained from a primitive embedding of $N\oplus A_2\oplus A_1$ into $Ni(A_{11}\oplus D_7\oplus E_6)$ giving the rank 0 elliptic fibration $A_12A_2A_32A_5$.
The glue code of $Ni(A_{11}D_7E_6)$ is generated by the class of the glue vector $g=[[1],[1],[1]]$. In the class of $6g=[[6],[2],0]$, take the vector

$$v = (((1/2)^6, (-1/2)^6), (0^6, 1), 0) = (a_6^*, d_1^*, 0)$$

of norm -4 and the vectors

$$v_1 = (0, d_1, 0)$$
$$v_2 = (0, (0,0,0,0,0,-1,-1), 0)$$

of norm -2. These vectors realize a primitive embedding of N into $Ni(A_{11}D_7E_7)$ whose root part of its orthogonal in $Ni(A_{11}D_7E_7)$ is $2A_5\oplus D_5\oplus E_6$ with $D_5=\langle d_7, d_6, d_5, d_4, d_3\rangle$. Moreover, we embed $A_1=\langle d_7\rangle$ into D_5 and A_2 into E_6 as in Lemma 3.12. We get $\det(A_12A_2A_32A_5) = 72\times 6^2$. From Shimada–Zhang [25], an extremal fibration with singular fibers of type $A_12A_2A_32A_5$ is the fibration 8 with 6-torsion. □

5.2. Weierstrass equations of the extremal fibrations of Y_{10}. — Recall first some facts. Classified by Néron [18] and Kodaira [14] the singular fibers are union of irreductible components with multiplicities, each component being a smooth rational curve with self-intersection -2. Their Kodaira types are the following:

- two infinite series $I_n (n>1)$ and $I_n^*(n\geq 0)$
- five types $III, IV, II^*, III^*, IV^*$.

The dual graph of these components (a vertex for each component, an edge for each intersection point of two components) is an *extended Dynkin diagram* of type $\widetilde{A}_n$, $\widetilde{D}_l$, $\widetilde{E}_p$. Deleting the zero component (i.e. the component meeting the zero section) gives the Dynkin diagram graph A_n, D_l, E_p. We draw the most useful diagrams, with the multiplicity of the components, the zero component being represented by a circle (Figure 2).
First we will use the following proposition ([22, p. 559-560] or [24, Proposition 12.10]).

Proposition 5.3. — *Let S be a $K3$ surface and D an effective divisor on S that has the same type as a singular fiber of an elliptic fibration. Then S admits a unique elliptic fibration with D as a singular fiber. Moreover, any irreducible curve C on S with $D.C=1$ induces a section of the elliptic fibration.*

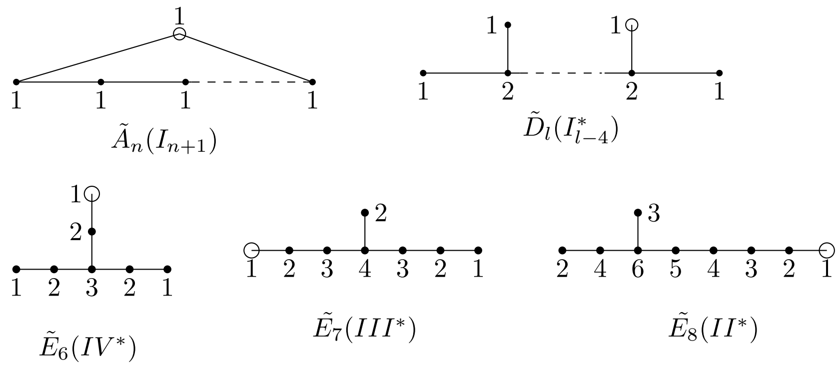


Figure 2. Extended Dynkin diagrams

If S is a $K3$ surface, an elliptic fibration $f : S \to \mathbb{P}^1$ with a section O defines a non constant function t, with $t = f(z)$ for $z \in S$; the function t, called the elliptic parameter is unique (up to an homographic transformation). Then the generic fiber $F/k(t)$ has a Weierstrass equation.

$$y^2 + a_1(t)\,yx + a_3(t)\,y = x^3 + a_2(t)\,x^2 + a_4(t)\,x + a_6(t)$$

with $a_i \in k[t]$ of degree at most $2i$.
An effective divisor as defined in Proposition 5.3 is called an *elliptic divisor*.

5.2.1. *From a fibration to another.* — The method is explained in [12], [15], [29] or [24, §6]. Let S be a $K3$ surface and $f : S \to \mathbb{P}^1$ be an elliptic fibration of elliptic parameter t and D the class of the fiber. Given another effective divisor D' satisfying $D'^2 = 0$, we want to write a Weierstrass equation for this new elliptic fibration.
If $D \cdot D' = r$ we say these elliptic fibrations are *r-neighbor*. We decompose $D' = D'_h + G$ with D'_h horizontal and G made of components $\theta_{t_i,n}$ of singular fibers of f at t_i; then $D \cdot D' = D \cdot D'_h$. In the case $D'_h = 2O$ (resp. $3O$) we are looking for a new elliptic parameter m such that the singular fiber D' is obtained for m equal to ∞. This implies that $m = a(t) + b(t)\,x$ (resp. $a(t) + b(t)\,x + c(t)\,y$) with $a(t), b(t), c(t) \in k(t)$ [15]. If P is a section for the first fibration and $D'_h = O + P$ then a new elliptic parameter has the shape $a(t) + ub(t)$ where $u = \frac{y - y(-P)}{x - x(P)}$. To determine the elements a, b, c we need to look at the order of vanishing along the components $\theta_{t_i,n}$ of the corresponding singular fiber belonging to G. Finally we complete the calculation using the equations of these components cf. [29, §5] and Tate Algorithm [26, IV §9]. An example is given in the next paragraph, with the Fibration 292.
The computation is easier if we know two elliptic divisors D_1 and D_2 of the same new fibration that is satisfying $D_1 \cdot D_2 = 0$. Since D_1 and D_2 represent the same element in the Néron Severi group, writing $D_i = D_{h,i} + G_i$, $1 \le i \le 2$, then the classes of $D_{h,1}$ and $D_{h,2}$ are equal in the quotient $NS(S)/T$ where T is the trivial lattice. Using the isomorphism between $NS(S)/T$ and the Mordell–Weil lattice $E(k(t))$, it follows that $D_{h,1} - D_{h,2}$ is the divisor of a function u_0 on the generic fiber of f. A parameter of the new fibration can be taken as the function $u = u_0 \prod_{t_i \in w}(t - t_i)^{a_i}$ where $a_i \in \mathbb{Z}$, and w is a set of reducible fibers. An example is given in Remark 5.7.
Once we get a new elliptic parameter we have to compute a Weierstrass equation using birational transformations. We can eliminate one variable in function of the new parameter.

Most of time we obtain an equation of bidegree 2 in the other variables. Blowing up the singular points we then get a Weierstrass equation. In case of a cubic equation with a rational point, we can use [27, p. 23] or a software program (Maple [16]).

Notation 5.4. — All these new Weierstrass equations are always written in the final form with variables X, Y and parameter t, even if along the process from the initial fibration to the new one we must use several different notations.
The singular fibers of type $I_n, D_m, IV^*, \ldots$ at $t = t_1, \ldots, t_m$ or at roots of a polynomial $p(t)$ of degree m are denoted $mI_n(t_1, \ldots, t_m)$ or $mI_n(p(t))$. The zero component of a reducible fiber is the component intersecting the zero section and is denoted θ_0 or $\theta_{t_0,0}$. The other components denoted $\theta_{t_0,i}$ satisfy the property $\theta_{t_0,i} \cdot \theta_{t_0,i+1} = 1$.

Recall the following computations. Let Y_k be the surface defined by the Laurent polynomial

$$X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} = k.$$

Considering the elliptic fibration defined with the elliptic parameter $t = X + Y + Z - \frac{k}{2}$, we obtain a Weierstrass equation with the transformation

$$X = \frac{y - x^2}{y(\frac{k}{2} - t)}, \quad Y = \frac{y + x}{x(\frac{k}{2} - t)}, \quad Z = -X - Y + t + \frac{k}{2}$$

of inverse on Y_k

$$x = -\frac{(Z + Y)(Z + X)}{Z^2}, \quad y = -\frac{Y(Z + Y)(Z + X)^2}{XZ^3}, \quad t = X + Y + Z - \frac{k}{2}.$$

Hence we get the elliptic fibration

$$E_{\#20} : y^2 + \left(t^2 + 3 - \frac{k^2}{4}\right) yx + \left(t^2 + 1 - \frac{k^2}{4}\right) y = x^3$$

$$I_{12}(\infty), \quad 2I_3\left(t^2 + 1 - \frac{k^2}{4}\right), \quad 2I_2\left(\pm\frac{k}{2}\right), \quad 2I_1\left(t^2 + 9 - \frac{k^2}{4}\right).$$

The generic rank is 0, the point $(-1, 1)$ is of order 2, and $(0, 0)$ of order 3.
For $k = 10$ this equation becomes

$$y^2 + \left(t^2 - 22\right) yx + \left(t^2 - 24\right) y = x^3,$$

with rank 1 and a generator of the Mordell–Weil group

$$\begin{aligned} P &= \left(-1 - \frac{1}{432}t^2\left(t^2 - 9\right)^2, \frac{-1}{15552}i\sqrt{3}\left(t^2 + i\sqrt{3}t - 12\right)^3\left(t + i\sqrt{3}\right)^3\right) \\ &= \left(-\frac{1}{432}\left(t^2 + 3\right)\left(t^4 - 21t^2 + 144\right), \frac{-1}{15552}i\sqrt{3}\left(t^2 + i\sqrt{3}t - 12\right)^3\left(t + i\sqrt{3}\right)^3\right). \end{aligned}$$

Remark 5.5. — The result $T(Y_{10}) = T(Y_2)[3]$, already known in [2, 3], can be recovered from this fibration, computing the discriminant group of the Néron–Severi lattice.

Fibration 80. — From the Weierstrass equation $E_{\#20}$, using the elliptic parameter n,

$$i\sqrt{3}n = \frac{1}{(t+5)}\frac{y - y_P}{x - x_P} - \frac{\left(11 - 3i\sqrt{3}\right)}{(t+5)} - \frac{i\sqrt{3}}{36}\frac{\left(t - 6 + 3i\sqrt{3}\right)\left(t + 6 + 3i\sqrt{3}\right)}{t + 5}$$

we obtain the Weierstrass equation E_n of the elliptic fibration 80 of rank 0 with 3-torsion.

$$E_n : Y^2 + \left(9t^2 + 6t - 9\right) YX + 9t^4 \left(3t^2 + 6t - 5\right) Y = X^3$$

with singular fibers $I_3(\infty)$, $I_{12}(0)$, $2I_3(3t^2 + 6t - 5)$, $I_2(\frac{3}{5})$, $I_1(-\frac{3}{4})$.

Fibration 262. — From the previous Weierstrass equation E_n of fibration 80 and with the parameter $u = \frac{X}{t^4}$ we obtain

$$E_u : Y^2 - 2\,(t+9)\,YX = X^3 + 9\,(t+3)\,(t+5)\,X^2 - t^3\,(t+5)^2\,X$$
$$III^*(\infty), \quad I_6(0), \quad I_4\,(-5)\,, \quad I_3\,(-9)\,, \quad I_2\,(-4)$$

with rank 0 and a 2-torsion section $(0,0)$.

Remark 5.6. — This fibration can also be obtained from E_n with the parameter $u_1 = \frac{Y}{X(3t^2+6t-5)}$; the singular fiber I_6 is for $u_1 = \infty$ and III^* for $u_1 = 0$.

Fibration 292. — We start from the previous Weierstrass equation of E_u of the fibration 262. With the fibers I_6 at $t = 0$ and I_2 at $t = -4$ we construct an elliptic divisor D of type IV^*

$$D = \theta_{-4,1} + 2O + 3\theta_{0,0} + 2\theta_{0,5} + \theta_{0,4} + 2\theta_{0,1} + \theta_{0,2}.$$

We put at $(0,0)$ all the singular points for $t = 0$ and $t = -4$. For $t = 0$ there is no change but for $t = -4$ the singular point is at $X = -8, Y = -40$. Using the Chinese remainder theorem we make the translation

$$X = -8 + (t+4)\left(2 - \frac{t}{2} + \frac{1}{8}t^2\right) + x \qquad \text{i.e. } X = \frac{t^3}{8} + x$$
$$Y = -40 + (t+4)\left(10 - \frac{5}{2}t + \frac{5}{8}t^2\right) + y \quad \text{i.e. } Y = \frac{5}{8}t^3 + y.$$

This gives the Weierstrass equation

$$y^2 - (2t+18)\,yx - \frac{1}{4}t^3\,(t+4)\,y = x^3 + \left(\frac{3}{8}t^3 + 9t^2 + 72t + 135\right)x^2 + \frac{1}{64}t^3\,(3t+20)\,(t+4)\,(t+16)\,x + \frac{1}{512}t^7\,(t+4)^2\,.$$

At $t = 0$ the components $\theta_{0,5}$ and $\theta_{0,1}$ are defined by $x = tx_1, y = ty_1$ and the two lines $y_1^2 - 18x_1y_1 - 135x_1^2 = 0$. The components $\theta_{0,2}$ and $\theta_{0,4}$ are defined by $x = t^2x_2, y = t^2y_2$ and $y_2^2 - 18x_2y_2 - 135x_2^2 = 0$.

At $t = -4$, the zero component is defined by $x \neq 0 \bmod (t+4)$. So if $w = \frac{8x}{t^3(t+4)} = \frac{X - \frac{t^3}{8}}{t^3(t+4)}$ we obtain the parameter of an elliptic fibration with D at $w = \infty$. It follows a Weierstrass equation

$$E_w : Y^2 = X^3 + 2t^2\,(3t+5)\,X^2 + t^3\,(12t+1)\,(t-1)^2\,X + 8t^5\,(t-1)^4$$

with singular fibers $IV^*(\infty)$, $III^*(0)$, $I_4(1)$, $I_3(-\frac{1}{27})$.

Fibration 252. — From fibration 262 with the Weierstrass equation E_u and the parameter $\frac{X}{t^3}$ we obtain

$$E_t : Y^2 = X^3 + t\,(10t-1)\,X^2 + 10t^4\,(9t-1)\,X + t^7\,(216t-25)$$

with singular fibers $IV^*(\infty)$, $I_5^*(0)$, $I_3(\frac{4}{27})$, $I_2(\frac{1}{8})$.

Fibration 302. — We start from the equation E_u of fibration 262 and parameter $r=\frac{X-2t}{2(t+4)}$ and obtain the Weierstrass equation

$$Y^2=X^3+\left(-14t^2+\frac{63}{2}t+\frac{27}{2}\right)X^2+14t^3\left(3t-19\right)X-2t^6\left(t-7\right)$$

with singular fibers $II^*(\infty)$, $I_6(0)$, $I_4(-5)$, $I_3(9)$, $I_1(\frac{-7}{27})$.

Fibration 200. — We start with the previous fibration 302, do the translation $X=250-\frac{175}{2}\left(t+5\right)+X_1$ and obtain the new Weierstrass equation

$$E_r:Y^2=X_1^3+A\left(t\right)X_1^2+B\left(t\right)\left(t+5\right)^2X_1+C\left(t\right)\left(t+5\right)^4$$

where A,B,C are polynomials with respective degrees $2,2,3$. Then the new parameter $s=\frac{X_1}{(t+5)^2}$ gives the fibration 200. A Weierstrass equation is obtained with $s=\frac{15}{2}+t$

$$E_s:Y^2=X^3+(t^3+\frac{17}{2}t^2+\frac{3}{4}t+\frac{27}{8})X^2-t\left(10t+9\right)\left(2t-3\right)X+2t^2\left(27+50t\right)$$

with singular fibers $I_7^*(\infty)$, $I_6(0)$, $I_3(\frac{-9}{4})$, $2I_1(4t^2+44t-7)$.

Fibration 153. — From fibration 252 we take the parameter $j=-\frac{X+4t^3}{t^2(8t-1)}$ and obtain the fibration 153 with Weierstrass equation

$$Y^2=X^3+t\left(t^2+10t-2\right)X^2+t^2\left(2t+1\right)^3X$$

and singular fibers $I_2^*(\infty)$, $I_1^*(0)$, $I_6(\frac{1}{2})$, $I_3(4)$.

Fibrations 8 and 224. — We consider the two elliptic fibrations of rank 0 of Y_2 [5] with Weierstrass equations

$$E_w:Y^2-\left(t^2+2\right)YX-t^2Y=X^3$$
$$E_j:Y^2-t\left(t+4\right)YX+t^2Y=X^3.$$

In these two cases the section $(0,0)$ is a 3-torsion section.
The two curves H_w and H_j respectively 3-isogenous to E_w and E_j have the following Weierstrass equations and singular fibers

$$\text{Fibration 8}\quad H_w:Y^2+3(t^2+2)YX+\left(t^2+8\right)\left(t-1\right)^2\left(t+1\right)^2Y=X^3$$
$$2I_6\left(1,-1\right),\quad I_4(\infty),\quad 2I_3\left(t^2+8\right),\quad I_2(0).$$
$$\text{Fibration 224}\quad H_j:Y^2+3t\left(t+4\right)YX+t^2\left(t^2+10t+27\right)\left(t+1\right)^2Y=X^3$$
$$VI^*(0),\quad I_6\left(-1\right),\quad 2I_3\left(t^2+10t+27\right),\quad I_4(\infty).$$

The elliptic fibration H_w is an elliptic fibration of an extremal $K3$ surface with 6-torsion and singular fibers of type $I_2,2I_3,I_4,2I_6$ $(A_1,2A_2,A_3,2A_5)$ hence with discriminant 72. Refering to Shimada–Zhang [25, Table 2, entry 8], we can identify H_w as an elliptic fibration of a $K3$ surface with transcendental lattice $[6\ \ 0\ \ 12]$ (Shimada–Zhang notation), hence as a fibration of Y_{10} Fibration 8.
Similarly H_j is an elliptic fibration of an extremal $K3$ surface with 3-torsion, singular fibers of type $2I_3,I_4,I_6,IV^*$ $(2A_2,A_3,A_5,E_6)$ and discriminat 72. Refering to Shimada–Zhang [25,

Table 2, entry 224], we can identify H_j as an elliptic fibration of a $K3$ surface with transcendental lattice [6 0 12] (Shimada–Zhang notation), hence as a fibration of Y_{10} Fibration 224.

Remark 5.7. — The previous results can also be obtained in the following manner:
From a fibration with 3 singular fibers of type $I_{n_1}, I_{n_2}, I_{n_3},\ \ n_i \geq 3$, at respectively t_1, t_2, t_3 we can obtain by 3-neighbour method a fibration with a fiber of type IV^* considering the divisor $D_1 = 3O + 2\sum \theta_{t_i,0} + \sum \theta_{t_i,1}$, where $\theta_{t_i,j}$ are components of I_{n_i} with $O \cdot \theta_{t_i,0} = 1$, $\theta_{t_i,0} \cdot \theta_{t_i,1} = 1$.
For the fibration H_w we choose the 3 fibers $I_6(1), I_3\left(t^2+8\right)$. With the fibers $I_6\left(-1\right), I_4(\infty)$ and two torsion sections we can consider a divisor D_2 of type I_5^* verifying $D_2 \cdot D_1 = 0$, namely

$$D_2 = \theta_{\infty,3} + \theta_{\infty,1} + 2\theta_{\infty,2} + 2\omega_6 + 2\sum_{i=1}^{4} \theta_{-1,i} + \theta_{-1,5} + \omega_3.$$

Then $\left(D_2 - D_1\right)_h = 2\omega_6 + \omega_3 - 3O$. The two torsion sections ω_i are on the line $Y + \left(8+t^2\right)X$. So with the parameter $m = \frac{Y+\left(8+t^2\right)X}{(t-1)^2\left(t^2+8\right)^2}$ we obtain a new fibration which is fibration 252, and we recover the previous result: H_w is a fibration of Y_{10}.
For the fibration H_j we choose the 3 fibers $I_6(-1), I_3(t^2+10t+27)$. With the fibers $IV^*(0)$, $I_4(\infty)$ and a 3-torsion section ω_3 we can construct a divisor D_2 of type II^*, namely

$$D_2 = \theta_{\infty,1} + 2\theta_{\infty,2} + 3\omega_3 + 4\theta_{0,0} + 5\theta_{0,1} + 6\theta_{0,2} + 3\theta_{0,3} + 4\widetilde{\theta_{0,3}} + 2\widetilde{\theta_{0,2}}.$$

Since $\left(D_2 - D_1\right)_h = 3\omega_3 - 3O$ we have as a parameter $m = \frac{Y}{(t+1)^2\left(t^2+10t+27\right)^2}$. Then we can show that the new fibration is obtained by the specialisation of #1 of Table 2 so H_j is a fibration of Y_{10}.

Fibration 87. — We consider the Weierstrass equation

$$Y^2 = x^3 + \left(-wt^4 - bt^3 - ct^2 - et + h\right)x^2 + \left(-rt^2 - tm - a\right)x$$

with singular fibers $I_{12}(\infty), 2I_2\left(\left(rt^2+tm+a\right)\right), 8I_1\left(P\left(t\right)\right)$. The polynomial $P\left(t\right)$ is of degree 8 depending of $w, b, c, e, h\ldots$ We choose these coefficients in such a way to get a singular fiber of type I_6 at $t=0$, that is the $i-$coefficient of P equal to 0 for i from 0 to 5. There is only two solutions, one of them is the fibration $7-w$ of Y_2, the other corresponds to the Weierstrass equation

$$E : Y^2 = X^3 - (9t^4 + 9t^3 + 6t^2 - 6t + 4)X^2 + (+21t^2 - 12t + 4)X.$$

The discriminant of the transcendental lattice of the surface with this elliptic fibration is 72 or 8. The second case occurs if there is a 3-torsion section and therefore this is an elliptic fibration of Y_2. But, using the equation of E and the parameter $m = \frac{Y}{3Xt^2}$, with the change $X = 2 - 3t + t^2W$, we obtain an equation in W and t of bidegree 2, then another elliptic fibration with a Weierstrass equation

$$Y^2 + 3YX - 9\left(t^2+1\right)^2 Y = X\left(X^2 - 6\left(t^2+1\right)X - 9\left(2t^2-1\right)\left(t^2+1\right)^2\right)$$

and singular fibers $I_0^*(\infty)$, $2I_6\left(\pm I\right)$, $I_2(0)$, $4I_1$.

Since the rank is 3, this elliptic fibration cannot come from Y_2, so the discriminant of the transcendental lattice is 72. From the singular fibers and results of Shimada–Zhang [25], this fibration comes from Y_{10} and is fibration 87.

Remark 5.8. — We can also start from E and with the parameter $m = 3\frac{X-\left(2-3t+3t^2\right)}{t^3}$ construct another elliptic fibration with Weierstrass equation

$$F : Y^2 = X^3 - \left(3t^4 - 18t^3 + 15t^2 + 27t + 9\right) X^2 + 3t^4 \left(t^2 - 3t - 2\right) \left(t^2 - 9t + 21\right) X - t^8 \left(t^2 - 9t + 21\right)^2$$

and singular fibers $IV^*(\infty)$, $I_{10}(0)$, $2I_2\left(t^2 - 9t + 21\right)$, $2I_1$.
From F and the parameter $m = \frac{X}{t^4}$ we have another elliptic fibration with Weierstrass equation

$$Y^2 + 18(2-t)XY + 486t^2(t+2)Y = X^3 + 6\left(7t^2 + 76t - 56\right) X^2 + 9t^2\left(t+8\right)\left(64t^2 + 241t - 224\right)X$$

and singular fibers $IV^*(1)$, $III^*(\infty)$, $I_4(0)$, $I_2(-7)$, $I_1(\frac{13}{256})$. This elliptic fibration can also be obtained from fibration 262 with the previous Weierstrass equation and parameter $\frac{X}{t^3(t+9)} - \frac{4}{27}\frac{1}{t+9}$.

Fibration 241. — We start with a general equation

$$Y^2 = X^3 + p\left(t\right) t^2 X^2 + q\left(t\right) t^3 X + r\left(t\right) t^4$$

where p, q, r are general polynomials of degree 2. So there is a singular fiber at $t = 0$ of type IV^*. We choose the coefficients of p, q, r in such a way to get a singular fiber of type I_{12} at $t = \infty$ and I_2 at $t = -1$. Hence we find an extremal fibration with Weierstrass equation

$$\begin{aligned} Y^2 = X^3 &+ 2\left(128t^2 + 8\left(11 + i\sqrt{3}\right)t + \left(17 - i\sqrt{3}\right)\right) t^2 X^2 \\ &+ \frac{1}{3}\left(-3 + i\sqrt{3}\right)\left(384t^2 + 8\left(30 + 4i\sqrt{3}\right)t + 33 - i\sqrt{3}\right) t^3 X \\ &- 2\left(-1 + i\sqrt{3}\right)\left(48t^2 + \left(27 + 5i\sqrt{3}\right)t + 2\right) t^4. \end{aligned}$$

An extremal fibration of discriminant 72 comes from a fibration of Y_{10} without torsion or a fibration of Y_2 with a 3-torsion section. To conclude, we use a 2-neighbor fibration. From the previous Weierstrass equation we have the following fibration with parameter $\frac{X}{t}$ and singular fibers $2I_6\left(0, t_1\right), IV^*(\infty), I_2, 2I_1$. After scaling, a Weierstrass equation of the fibration take the following shape

$$\begin{aligned} Y^2 = X^3 &- \frac{1}{73}\left(-17 + i\sqrt{3}\right)\left(146t^2 + \left(-145 + 43i\sqrt{3}\right)t + 20 - 16i\sqrt{3}\right) X^2 \\ &+ \frac{1}{31}\left(11 + i\sqrt{3}\right)\left(4t - 3 + i\sqrt{3}\right)\left(124t - 85 + 19i\sqrt{3}\right) t^3 X + 16\left(4t - 3 + i\sqrt{3}\right)^2 t^6. \end{aligned}$$

We have the section $P = \left(0, 4t^3(4t - 3 + i\sqrt{3})\right)$. Computing the height of P gives the discriminant of the transcendental lattice equal to 72 if $P \neq 3S$ generates the Mordell–Weil group. If $P = 3S$, then the discriminant of the transcendental lattice should be equal to 8 and this fibration should be a fibration of Y_2, which is impossible since not in the list of all the elliptic fibrations of Y_2 [5]. So this is a fibration of Y_{10} since it is the only one in Shimada–Zhang's table [25].

6. Fibrations of high rank and their corresponding embeddings

A great difference between Y_2 and Y_{10} is revealed in the following theorem.

Theorem 6.1. — *Contrary to Y_2 whose fibrations have rank ≤ 2, the $K3$-surface Y_{10} has elliptic fibrations of high rank, meaning a rank greater than the rank 3 of some specialized fibrations. We exhibit, in the proof below, examples of fibrations of rank 4, 5, 6 and 7 together with their singular fibers and possible primitive embeddings.*

Proof. — The elliptic fibration 262 of rank 0 has the following Weierstrass equation

$$Y^2 = X^3 + \left(10t^2 + 90t + 216\right) X^2 - t^3 (t+5)^2 X.$$

Taking as new parameter $m = \frac{Y}{X(t+4)}$ we obtain the rank 1 fibration

$$E_m \qquad : Y^2 = X^3 + \left(\frac{1}{4}t^4 - 5t^2 + 27\right) X^2 + \left(t^2 - 9\right)\left(2t^2 - 27\right) X + \left(2t^2 - 27\right)^2$$

with singular fibers $I_{10}(\infty)$, $IV(0)$, $2I_3(2t^2-27)$, $2I_2(\pm 4)$ and an infinite order point P_0. Consider the 6 points $P_i = (X_i, Y_i)$.

$$P_0 = \left(27 - 2t^2, -\frac{1}{2}\left(t^2 - 16\right)\left(2t^2 - 27\right)\right)$$

$$P = 2P_0 = \left(0, 2t^2 - 27\right) \qquad P_1 = 3P_0 = \left(-5, \frac{16 - t^2}{2}\right)$$

$$P_2 = 4P_0 = \left(\frac{2t^2 - 27}{4}, -\frac{1}{8}\left(t^2 - 1\right)\left(2t^2 - 27\right)\right)$$

$$P_3 = 5P_0 = \left(\frac{-1}{25}\left(2t^2 - 27\right)\left(t^2 - 7\right), \frac{1}{250}\left(t^2 - 16\right)\left(6t^2 - 1\right)\left(2t^2 - 27\right)\right)$$

$$P_4 = 6P_0 = \left(8\left(t^2 - 1\right), 4t^4 + 22t^2 - 1\right)$$

We use the two singular fibers $I_{10}(\infty)$ and $IV(0)$, the zero section and a section $-P_i$ to obtain a new fibration with a singular fiber of type I_p. The elliptic parameters will be of the shape $m_i = \frac{Y - Y_{P_i}}{(X - X_{P_i})t} + \frac{(a_i t^2 + b_i)}{t}$. Suppose first that P_i does not cut the zero component $\theta_{\infty,0}$ of the singular fiber $I_{10}(\infty)$. The coefficients a_i, b_i are then determined by the two following conditions: let $\theta_{\infty,1}, \theta_{\infty,9}$ the two components then $Y - a_i X$ will define one of the components $\theta_{\infty,1}$ or $\theta_{\infty,9}$ and this gives $a_i = \pm\frac{1}{2}$. The second condition is: the singular point for $t = 0$ will be on the line $Y - Y_{P_i} + b_i\left(X - X_{P_i}\right)$. Only $P_3 = 5P_0$ intersects $\theta_{\infty,0}$, the singular point at $t = \infty$ is on the line $Y - Y_{P_i} + a_{ii}\left(X - X_{P_i}\right)$ and defines the coefficient a_3.

For each example we give in the first column of the following table the elliptic parameter where $P_i\left(a_i, b_i\right)$ means m_i given by the previous formula. In the second column we give a Weierstrass equation and just below the singular fibers with their types. Notice that on this equation appears the x coordinate of some sections with $y = 0$. In the third column is the rank and torsion with pattern $r|tor$.

Also included in this table are possible embeddings in Niemeier lattices with the following form: in the first column the Niemeier root lattice or the Niemeier lattice if we need glue vectors. In the second column is explicited the embeddings of N, A_1, A_2 . Also in the second column at the end of the second line are the roots of orthogonal of the embeddings, for example if the Niemeier root lattice is $A_{15}D_9$ and the roots of the orthogonal is $A_{11}2A_1$ the notation $A_{11}\,(2A_1)$ means that A_{11} is in A_{15} and $(2A_1)$ in D_9. In the third column are indications for the previous computations: here α refers to [21, p. 308, 310, 322], δ refers to [21, p. 309, 311, 323],η refers to [21, p. 326, 327] and $*$ to Lemmas 3.7 or 3.8, a sign $-$ meaning that it is obvious for the corresponding factors.

Param Niemeier lattice	Weierstrass Equations Singular fibers Embeddings	Rk | Tor
(1) $P_0(\frac{1}{2},6)$ $A_{15}D_9$	$y^2-2t\,(7+6t^2)\,yx-72t\,(21+t^2)\,y$ $=(x^2+432\,(t^2+8))\,(x-(36t^6+87t^4+223t^2+87))$ $I_{12}(\infty),\ 2I_2(6t^2-49),\ 8I_1 \qquad A_{11}\,(2A_1)$ $N=\begin{pmatrix} v=(a_1,d_7)\\ v_1=(0,d_9)\\ v_2=(0,d_6)\end{pmatrix} \quad A_2=\begin{pmatrix}(0,d_1)\\(0,d_2)\end{pmatrix} \quad A_1=((a_3,0))$	$5\|0$ $(\alpha\delta)$
(2) $P(\frac{1}{2},3)$ $A_{11}D_7E_6$	$y^2-(t^2+8)\,yx-t^2\,(3t^2+11)\,y=(x^2+12t^2)\,(x-4+3t^2)$ $I_{10}(\infty),\ 5I_2\,(0,\pm 3,3t^2-32)\,,\ 4I_1 \qquad A_9\,(3A_1)\,(2A_1)$ $N=\begin{pmatrix} [c]cv=(0,d_7,e_4)\\ v_1=(0,0,e_2)\\ v_2=(0,0,e_5)\end{pmatrix} \quad A_2=\begin{pmatrix}(0,d_1,0)\\(0,d_2,0)\end{pmatrix} \quad A_1=(a_1,0,0)$	$4\|0$ $(\alpha\delta\eta)$
(3) $P_1(\frac{1}{2},-2)$ $A_{11}D_7E_6$	$y^2-(3+t^2)\,yx+3t^2y=(x+1)\,(x^2+4t^2x+4t^4+27t^2)$ $I_{10}(\infty),\ 2I_3\,(2t^2-3)\,,\ I_2(0),6I_1 \qquad A_9A_1\,(2A_2)$ $N=\begin{pmatrix} v=(a_1,d_5,0)\\ v_1=(0,d_7,0)\\ v_2=(0,d_4,0)\end{pmatrix} \quad A_2=\begin{pmatrix}(0,0,e_1)\\(0,0,e_3)\end{pmatrix} \quad A_1=(0,d_1,0)$	$4\|0$ $(\alpha\delta\delta)$
(4) $P_2(\frac{1}{2},\frac{3}{2})$ $A_5^4D_4$	$y^2-(3+2t^2)yx-t^2(3t^2-1)y=(x+t^2)(x^2+12t^2)$ $2I_6(\infty,0),\ 4I_2(\pm 1,3t^2-2),\ 4I_1 \qquad A_5A_50A_1\,(3A_1)$ $N=\begin{pmatrix} [c]cv=(0,0,0,a_2,d_4)\\ v_1=(0,0,0,a_1,0)\\ v_2=(0,0,0,a_3,0)\end{pmatrix} \quad A_2=\begin{pmatrix}(0,0,a_1,0,0)\\(0,0,a_2,0,0)\end{pmatrix}$ $A_1=(0,0,a_5,0,0)$	$4\|0$ $(-\alpha\alpha\delta)$

Param Niemeier lattice	Weierstrass Equations Singular fibers Embeddings	Rk | Tor
(5) $P_3(\frac{3}{10},\frac{6}{5})$ $N_i(A_9^2D_6)$	$y^2-(3t^2+1)yx-t^2(6t^2-5)y=(x+4t^4-2t^2)(x^2+3t^2)$ $I_4(\infty),\ I_8(0),\ 2I_2(6t^2-1), 8I_1 \qquad (A_7A_1)A_3A_1$ $N=\begin{pmatrix} v=(a_8^*,a_6^*,0)\\ v_1=(0,a_5+a_6,0)\\ v_2=(0,a_6+a_7+a_8,0)\end{pmatrix} \quad A_2=\begin{pmatrix}(0,0,d_1)\\(0,0,d_2)\end{pmatrix}$ $A_1=(0,0,d_6)$	$6\|0$ $(--\delta)$
(6) $P_4(\frac{1}{2},1)$ $A_{11}D_7E_6$	$y^2=x^3+(t^4-44t^2+472)x^2-16(t^2-25)x$ $I_{12}(\infty),\ 2I_3(t^2-24), 2I_2(\pm 5),\ 2I_1 \qquad A_{11}(2A_1A_2)A_2$ $N=\begin{pmatrix} v=(0,d_5+d_3,0)\\ v_1=(0,d_6,0)\\ v_2=(0,d_7,0)\end{pmatrix} \quad A_2=\begin{pmatrix}(0,0,e_1)\\(0,0,e_3)\end{pmatrix}$ $A_1=(0,0,e_2)$	$1\|6$ $(-*\eta)$
(7) $P_0(-\frac{1}{2},6)$ $A_5^4D_4$	$y^2+(1+t^2)yx-t^2(6t^2+7)y=(x-4t^2)(x^2+3t^2)$ $2I_6(\infty,0),\ 2I_2(3t^2-2),\ 8I_1 \qquad A_5A_5A_1A_10$ $N=\begin{pmatrix} v=(0,0,a_1,a_2,0)\\ v_1=(0,0,0,a_1,0)\\ v_2=(0,0,0,a_3,0)\end{pmatrix} \quad A_2=\begin{pmatrix}(0,0,0,0,d_4)\\(0,0,0,0,d_2)\end{pmatrix}$ $A_1=(0,0,a_3,0,0)$	$6\|0$ $(-\alpha\alpha\delta)$
(8) $P(-\frac{1}{2},3)$ $A_7^2D_5^2$	$y^2+(t^2+2)yx-t^2(3t^2+11)y=(x-3t^2)(x^2+12t^2)$ $I_8(\infty),\ I_4(0),\ 4I_2(\pm 1, 6t^2-1),\ 4I_1 \qquad A_7A_3(2A_1)(2A_1)$ $N=\begin{pmatrix} v=(0,a_1,d_3,0)\\ v_1=(0,0,d_5,0)\\ v_2=(0,0,d_4,0)\end{pmatrix} \quad A_2=\begin{pmatrix}(0,0,0,d_5)\\(0,0,0,d_3)\end{pmatrix}$ $A_1=(0,a_3,0,0)$	$4\|2$ $(-\alpha\delta\delta)$
(9) $P_1(-\frac{1}{2},-2)$ $A_{11}D_7E_6$	$y^2+(t^2+7)yx+t^2(9+2t^2)y=(x+2t^2)(x^2+27t^2)$ $I_8(\infty),\ I_4(0),\ 2I_3(t^2-6),\ 6I_1 \qquad A_7A_3(2A_2)$ $N=\begin{pmatrix} v=(a_1,d_5,0)\\ v_1=(0,d_7,0)\\ v_2=(0,d_4,0)\end{pmatrix} \quad A_2=\begin{pmatrix}(0,0,e_1)\\(0,0,e_3)\end{pmatrix} \quad A_1=(a_3,0,0)$	$4\|0$ $(\alpha\delta\eta)$

Param Niemeier lattice	Weierstrass Equations Singular fibers Embeddings	Rk | Tor
(10) $P_2(-\frac{1}{2},\frac{3}{2})$ $A_{11}D_7E_6$	$y^2=x^3-(3t^4+48t^2-264)x^2-(864t^2-3600)x$ $I_{12}(\infty),\ 4I_2(\pm 2, 6t^2-25),\ 4I_1 \qquad A_{11}(3A_1)A_1$ $N=\begin{pmatrix} v=(0,d_1,e_3)\\ v_1=(0,0,e_1)\\ v_2=(0,0,e_4)\end{pmatrix} \quad A_2=\begin{pmatrix}(0,d_7,0)\\(0,d_5,0)\end{pmatrix}$ $A_1=(0,0,e_6)$	$3\|2$ $(-\delta\eta)$
(11) $P_4(-\frac{1}{2},1)$ E_6^4	$y^2+(t^2-4)yx-t^2(t^2-63)y=(x-9t^2)(x^2+108t^2)$ $2I_6(\infty,0),\ 2I_3(2t^2-3),\ 2I_2(\pm1),2I_1 \qquad A_5(2A_1)A_5(2A_2)$ $N=\begin{pmatrix} v=(e_1,e_3,0,0)\\ v_1=(0,e_1,0,0)\\ v_2=(0,e_4,0,0)\end{pmatrix} \quad A_2=\begin{pmatrix}(0,0,0,e_1)\\(0,0,0,e_3)\end{pmatrix}$ $A_1=(0,0,e_1,0)$	$2\|3$ $(\eta\delta\eta\eta)$

Finally the rank 7 elliptic fibration has a Weierstrass equation denoted E_t

$$E_t \qquad Y^2=X^3-5t^2X^2+t^3(t^3+1)^2,$$

and singular fibers $2I_0^*(0,\infty)$, $3I_2(t^3+1)$, $6I_1$. It is obtained in the following way: since $T(Y_{10})=T(Y_2)[3]$ the base change $h=t^3$ of the fibration $13-h$ of Y_2 [5] gives a fibration of Y_{10}. We can take as possible primitive embedding, an embedding of type (A_2,A_2) into D_4^6 (see Section 3.2). □

Remark 6.2. — All the previous embeddings are of type (2.b) (A_1,A_3), except embedding (5) which uses a glue vector of the corresponding Niemeier lattice and embedding (6) which uses embedding 3.7.

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