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SOME TOPICS IN THE THEORY OF TANNAKIAN CATEGORIES AND APPLICATIONS TO MOTIVES AND MOTIVIC GALOIS GROUPS

by

Florian Ivorra

Abstract. — These notes are taken from a series of lectures given at the conference "Fundamental Groups in Arithmetic Geometry 2016" in Paris. They cover the basics of the theory of Tannakian categories and provide an introduction to more recent developments and their applications to motivic Galois groups.

Résumé. — (*Quelques aspects autour de la théorie des catégories tannakiennes et applications aux motifs et groupes de Galois motiviques*) Ces notes sont tirées d'une série de cours donnés à la conférence « Fundamental Groups in Arithmetic Geometry » à Paris en 2016. Elles couvrent les bases de la théorie des catégories tannakiennes et fournissent une introduction aux développements récents et leurs applications aux groupes de Galois motiviques.

These notes are a revision of a series of three lectures on Tannakian categories I gave at the conference "Fundamental Groups in Arithmetic Geometry 2016" in Paris.
Together with a course on Galois categories and fundamental groups, they were designed (especially with younger participants in mind) to set the scene for the main body of the conference the week after. The purpose of these lectures was two-fold: I was asked to provide an introduction to the classical Tannaka duality and then to give an idea of some of the more recent developments obtained independently by Nori and Ayoub which have led to important results in motivic Galois theory.
In preparing these notes, I kept the overall structure of the oral exposition but added some details, I would have liked to give, but could not fit in the lectures for lack of time.
The result is neither an historical survey nor a comprehensive survey. Lots of choices had to be made. I also did not seek the greatest generality or exhibited the weakest hypothesis. For example, I considered only the theory in the neutral case and in the applications of Tannaka theory to motives, I chose to restrict to fields of characteristic zero and to rational coefficients. In using category theory, I also chose, as in the oral lectures, to simply forget about set theoretic problems and avoid the (necessary) use of 2-categories in the formulation of universal properties.

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1. Introduction

Let Λ be a field.

Tannaka duality is a duality between affine group schemes over Λ and their categories of representations. It was developed by Grothendieck and his student Saavedra Rivano [48] (see [18, 23, 25, 50] for expositions and further developments of the theory). The so-called Tannakian categories were named in [48] after an earlier work by Tannaka [52]. The contribution of Tannaka was not isolated at the time but was part of a rich current of researches in which groups (abstract, topological or Lie) were put in duality with other algebraic structures related to representations of the group. Let us mention, for example, the block algebras appearing in Kreĭn's work [42], or the algebras of (continous) representative functions (see [41, §1] for an exposition of the Tannaka–Kreĭn duality). A precursor of Tannaka duality was Pontryagin's duality for Abelian locally compact topological groups. Those works led utimately to the problem of reconstructing a group from its category of representations.

One of the main contributions of Grothendieck was the understanding that the process could be reverted (see for example [19] for a historical survey). Not only could representations be used to reconstruct a group but many categories could also be used to produce groups, a process by which many properties of the given category could be translated into group theoretic statements. This idea was the cornerstone of Grothendieck's approach to fundamental groups developed in [47]. There, he introduced the notion of Galois categories, showed that a fiber functor on a given Galois category defines a profinite group and established a duality between profinite groups and (pointed) Galois categories. These foundations, via the category of finite étale covers, led to the construction by Grothendieck of the étale fundamental group of a scheme, the analogue in algebraic geometry of the fundamental group introduced by Poincaré in topology.

Tannakian categories were meant by Grothendieck, with applications to motives and periods in mind, as an additive variant of the Galois categories he had earlier introduced. The theory developed in [48] answers the following questions:

Problem 1.1 **(The group reconstruction problem).** — *Can an affine group scheme G over Λ be reconstructed from its category $\mathbf{rep}(G)$ of representations in finite dimensional Λ-vector spaces?*

Problem 1.2 **(The categorical recognition problem).** — *Which categories $\mathcal{A}$ are equivalent to the category $\mathbf{rep}(G)$ of representations in finite dimensional Λ-vector spaces of an affine group scheme G over Λ.*

The main objects of the theory are the so-called Tannakian categories:

Definition 1.3. — A neutral Tannakian category is a Λ-linear Abelian rigid symmetric monoidal category $\mathcal{A}$ such that there exists a Λ-linear faithful exact symmetric monoidal functor

$$\omega : \mathcal{A} \to \mathbf{vec}(\Lambda).$$

Such a functor is called a fiber functor. The pair $(\mathcal{A}, \omega)$ is called a neutralized Tannakian category.

The basic example of a neutralized Tannakian category is the category $\mathbf{rep}(G)$ of representations on finite dimensional Λ-vector spaces of an affine group scheme G over Λ. The functor

$$\mathbf{rep}(G) \to \mathbf{vec}(\Lambda)$$

that forgets the action of the group G provides a canonical fiber functor. The cornerstone of the theory, proved by Saavedra in [48], is that every neutralized Tannakian category is canonically of this form:

Theorem 1.4. — *Let $(\mathcal{A}, \omega)$ be a neutralized Tannakian category. Let $\mathbf{Alg}_\Lambda$ be the category of commutative (unitary) Λ-algebras and $\mathbf{Grp}$ the category of groups. Let $\mathcal{A}\mathrm{ut}^{\otimes}(\omega)$ be the functor*

$$\begin{aligned} \mathcal{A}\mathrm{ut}^{\otimes}(\omega) : \mathbf{Alg}_\Lambda &\to \mathbf{Grp} \\ R &\mapsto \mathrm{Aut}^{\otimes}(\omega_R) \end{aligned}$$

which associates with R the group of tensor automorphisms of the functor

$$\omega_R : \mathcal{A} \xrightarrow{\omega} \mathbf{vec}(\Lambda) \xrightarrow{-\otimes_\Lambda R} \mathbf{Mod}(R).$$

The functor $\mathcal{A}\mathrm{ut}^{\otimes}(\omega)$ is representable by an affine group scheme $G(\omega)$ over Λ and the fiber functor ω enriches into an equivalence of tensor categories between $\mathcal{A}$ and the category of finite dimensional representations of $G(\omega)$.

This theorem establishes a duality between affine group schemes over Λ and neutralized Tannakian categories: the so-called "Tannaka duality". One of the main interest of this duality lies in reversing the process.
Starting with a given Tannakian category, it produces, from a fiber functor, an affine algebraic group that encapsulates the properties of the initial categories and translates them into group theory.
In the form of Theorem 1.4, the theory comes with some limitations such as the fact that it requires an Abelian category to start with. This has proven to be a challenging difficulty in the application of Tannaka duality to the theory of motives and periods, one of Grothendieck's main motivations for developing the theory (see Section 4.1).
To bypass these problems, Nori and Ayoub have broaden independently the scope of the classical Tannaka duality. By doing so they were able to obtain a construction of motivic Galois groups and applications to the theory of periods (see [11, 12, 26] or [3, 13, 33] for surveys).
To describe these more recent developements, it is useful to keep in mind that if G is an affine group scheme over Λ, then its Λ-algebra of global functions $\mathscr{O}(G)$ is a Hopf algebra and that the category $\mathbf{rep}(G)$ of finite dimensional representations of G is equivalent to the category $\mathbf{comod}(\mathscr{O}(G))$ counitary right $\mathscr{O}(G)$-comodules of finite dimension. In particular, a neutralized Tannakian category is canonically equivalent to the category of finite dimensional counitary right comodules over the Hopf algebra

$$C(\omega) := \mathscr{O}(G(\omega)).$$

The approach of Nori, which appears in his unpublished work [26], relies on quiver representations. To construct a coalgebra, which in favorable cases turns out to be a biunitary bialgebra or even a Hopf algebra, Nori does not even need to start with a category, less a neutralized Tannakian category: he only needs a quiver $\mathcal{Q}$ and a representation $T : \mathcal{Q} \to \mathbf{vec}(\Lambda)$. This approach fits nicely with the classical Tannaka duality which may be recovered from it, but it also produces coalgebras from very general quiver representations. Using this extended scope of application, Nori has been able to get at the same time a motivic Galois group and a construction of an Abelian category of mixed motives over subfields of the field of complex numbers.

More precisely, with a quiver representation $T : \mathcal{Q} \to \mathbf{vec}(\Lambda)$, Nori associates a Λ-coalgebra $H(T)$ (see Section 5.1 for its construction) and proves that it enjoys the following properties:

Theorem 1.5. —

1. *The quiver representation $T : \mathcal{Q} \to \mathbf{vec}(\Lambda)$ admits a factorization*

$$\begin{tikzcd} \mathcal{Q} \arrow[r, "\overline{T}"] \arrow[dr, bend right, "T"'] & \mathbf{comod}(H(T)) \arrow[d, "\text{forgetful}"] \\ & \mathbf{vec}(\Lambda) \end{tikzcd}$$

moreover this factorization is universal among all factorizations

$$\begin{tikzcd} \mathcal{Q} \arrow[r, "R"] \arrow[dr, bend right, "T"'] & \mathcal{A} \arrow[d, "F"] \\ & \mathbf{vec}(\Lambda) \end{tikzcd}$$

where $\mathcal{A}$ is a Λ-linear Abelian category, R is a representation and F a Λ-linear faithful exact functor.

2. *Let $\mathcal{A}$ be a Λ-linear Abelian category and $\omega : \mathcal{A} \to \mathbf{vec}(\Lambda)$ be a Λ-linear exact faithful functor. Then, the representation*

$$\overline{\omega} : \mathcal{A} \to \mathbf{comod}(H(\omega))$$

is a Λ-linear functor and an equivalence of categories.

Recall that a quiver is simply an oriented graph or informally a category without composition. The category of counitary comodules over $H(\omega)$ of finite dimension can be seen as an Abelian category generated from the quiver $\mathcal{Q}$ and the representation T. Vertices in $\mathcal{Q}$ define objects and edges define morphisms between the corresponding objects. Nori's Abelian category of mixed motives is obtained by considering a quiver whose definition is geometric and closely related to the ring of abstract periods of Kontsevich–Zagier (see e.g. [13, 33] and Section 5.4). The weak Tannakian formalism introduced by Ayoub in [11, 12] is designed to be able to deal with "fiber functor" such as the Betti realization

$$\mathsf{Bti}^* : \mathbf{DA}^{\text{ét}}(k, \mathbb{Q}) \to \mathrm{D}(\mathbb{Q})$$

on the category of étale motives $\mathbf{DA}^{\text{ét}}(k, \mathbb{Q})$ (see [8]). This category being triangulated and not Abelian, the weak Tannakian formalism has to be flexible enough to bypass, for example,

the classical restrictions of Tannaka duality (and above all the assumption that the source of the fiber functor has to be an Abelian category). Contrary to Nori's approach or the classical Tannaka duality, its application to the Betti realization does not yield immediately a Hopf algebra in the classical sense. What it produces is a derived Hopf algebra. To actually get a classical Hopf algebra from it requires some further computations.
The requirements of this formalism are very weak compared to those of the classical Tannaka duality (hence the name) and it can be formulated in the general context of symmetric monoidal categories (see e.g. [6, Definition 2.1.85]):

Theorem 1.6. — *Let $(\mathcal{M}, \otimes, \mathbb{1})$ and $(\mathcal{E}, \otimes, \mathbb{1})$ be symmetric monoidal categories. Let $f : \mathcal{M} \to \mathcal{E}$ be a symmetric monoidal functor. If the following assumptions are satisfied:*

1. *the functor f admits a right adjoint g;*

2. *there exists a* monoidal *functor $e : \mathcal{E} \to \mathcal{M}$ and an isomorphism of monoidal functors $fe \simeq \mathrm{Id}_{\mathcal{E}}$;*

3. *the morphism of coprojection*
$$c_d : g(A) \otimes e(B) \to g(A \otimes fe(B))$$
is an isomorphism for every objects $A, B \in \mathcal{E}$;

then, the object $\mathrm{H} = fg(\mathbb{1})$ *in $\mathcal{E}$ has a canonical structure of commutative biunitary bialgebra and the functor $f : \mathcal{M} \to \mathcal{E}$ has a canonical factorization*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\overline{f}"] \arrow[dr, bend right, "f"'] & \mathbf{coMod}(\mathrm{H}) \arrow[d, "\text{forgetful}"] \\ & \mathcal{E} \end{tikzcd}$$

where $\mathbf{coMod}(\mathrm{H})$ *is the category of unitary left comodules over* H*. Moreover for every commutative biunitary bialgebra K and every factorization*

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[dr, bend right, "f"'] & \mathbf{coMod}(K) \arrow[d, "\text{forgetful}"] \\ & \mathcal{E} \end{tikzcd}$$

there exists a unique morphism of commutative biunitary bialgebras $\mathrm{H} \to K$ *such that for every $M \in \mathcal{M}$ the coaction of K on $f(M)$ is obtained by corestriction of the coaction of* H *on $f(M)$.*

Under the very general hypothesis of Theorem 1.6 is not reasonable to expect duality. Nevertheless, as in the first part of Theorem 1.5, Ayoub's result still produces a biunitary bialgebra whose category of counitary left comodules best approximates the original symmetric monoidal category $\mathcal{M}$. Under additional assumptions, the bialgebra $\mathrm{H} = fg(\mathbb{1})$ turns out to be a Hopf algebra in the symmetric monoidal category $\mathcal{E}$.

2. Brief reminder on coalgebras and comodules

The weak Tannakian formalism introduced by Ayoub in [11, 12] works under very mild assumption for monoidal functors between symmetric monoidal categories. The notions of coalgebra, bialgebra, Hopf algebra or comodules over them make sense in this very general context. In this section, we review the basic definitions. The reader may skip this section if he so desires and only refers to it when needed.

2.1. Symmetric monoidal categories and monoidal functors. — Let us start by briefly recalling the definition of a symmetric monoidal category and that of a symmetric (pseudo)-monoidal functor.

Definition 2.1. — A monoidal category is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ called the tensor product, an object $\mathbb{1} \in \mathcal{C}$ called the unit object, a natural isomorphism (called the associator)

$$(A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

natural in $A, B, C \in \mathcal{C}$ and two isomorphisms (called the left unitor and the right unitor)

$$\mathbb{1} \otimes A \to A \qquad A \otimes \mathbb{1} \to A$$

natural in $A \in \mathcal{C}$ which make the following two diagrams commute.

– (Pentagonal identity)

$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur] \arrow[d] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[r] & A \otimes ((B \otimes C) \otimes D). \arrow[ur] &
\end{tikzcd}$$

– (Triangle identity)

$$\begin{tikzcd}
(A \otimes \mathbb{1}) \otimes B \arrow[rr] \arrow[dr] & & A \otimes (\mathbb{1} \otimes B) \arrow[dl] \\
 & A \otimes B. &
\end{tikzcd}$$

Note that the role of the pentagonal identity and the triangle identity is to ensure the coherence of the data and as a consequence the peace of mind of the user. Indeed, as shown in [43, VII, §2], they ensure that every diagram built out of the structural morphisms is commutative: any two ways of going from an iterated tensor product of objects of $\mathcal{C}$ to another iterated tensor product are the same (see *loc. cit.* for a precise statement).
A symmetric monoidal category is a monoidal category equipped with an isomorphism

$$\tau : A \otimes B \to B \otimes A$$

natural in $A, B \in \mathcal{C}$ such that $\tau^2 = \mathrm{Id}$ and the following diagram

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r] \arrow[d] & A \otimes (B \otimes C) \arrow[r] & (B \otimes C) \otimes A \arrow[d] \\
(B \otimes A) \otimes C \arrow[r] & B \otimes (A \otimes C) \arrow[r] & B \otimes (C \otimes A)
\end{tikzcd}$$

commute.

Remark 2.2. — Note that this implies the commutativity of the triangle

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[rr, "\tau"] \arrow[dr] & & A \otimes \mathbb{1} \arrow[dl] \\
& A. &
\end{tikzcd}$$

Definition 2.3. — Let $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{D}, \otimes, \mathbb{1})$ be monoidal categories. A pseudo-monoidal functor is a functor $f : \mathcal{C} \to \mathcal{D}$ equipped with a morphism

$$f(A) \otimes f(B) \to f(A \otimes B) \tag{1}$$

natural in $A, B \in \mathcal{C}$ and a morphism $\mathbb{1} \to f(\mathbb{1})$ such that

$$\begin{tikzcd}
(f(A) \otimes f(B)) \otimes f(C) \arrow[r] \arrow[d] & f(A \otimes B) \otimes f(C) \arrow[r] & f((A \otimes B) \otimes C) \arrow[d] \\
f(A) \otimes (f(B) \otimes f(C)) \arrow[r] & f(A) \otimes f(B \otimes C) \arrow[r] & f(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \otimes f(A) \arrow[r] \arrow[d] & f(\mathbb{1}) \otimes f(A) \arrow[d] \\
f(A) & f(\mathbb{1} \otimes A) \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
f(A) \otimes \mathbb{1} \arrow[r] \arrow[d] & f(A) \otimes f(\mathbb{1}) \arrow[d] \\
f(A) & f(A \otimes \mathbb{1}) \arrow[l]
\end{tikzcd}$$

are commutative.

If f,g are two pseudo-monoidal functors, a natural transformation $\theta : f \to g$ is said to be pseudo-monoidal if the diagrams

$$\begin{tikzcd}
f(A) \otimes f(B) \arrow[r, "\theta_A \otimes \theta_B"] \arrow[d] & g(A) \otimes g(B) \arrow[d] \\
f(A \otimes B) \arrow[r, "\theta_{A \otimes B}"] & g(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \arrow[r] \arrow[dr] & f(\mathbb{1}) \arrow[d, "\theta_{\mathbb{1}}"] \\
& g(\mathbb{1})
\end{tikzcd}$$

commute for every objects $A, B \in \mathcal{C}$.
If $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{D}, \otimes, \mathbb{1})$ are symmetric monoidal categories, then a pseudo-monoidal functor is said to be symmetric if the square

$$\begin{tikzcd}
f(A) \otimes f(B) \arrow[r] \arrow[d] & f(A \otimes B) \arrow[d] \\
f(B) \otimes f(A) \arrow[r] & f(B \otimes A)
\end{tikzcd}$$

is commutative for every $A, B \in \mathcal{C}$.
A pseudo-monoidal functor is said to be monoidal if the morphism (1) and the morphism $\mathbb{1} \to f(\mathbb{1})$ are isomorphisms in $\mathcal{D}$.
A variant is useful. A pseudo-comonoidal functor is a functor $f : \mathcal{C} \to \mathcal{D}$ equipped with a morphism

$$f(A \otimes B) \to f(A) \otimes f(B)$$

and a morphism $f(\mathbb{1}) \to \mathbb{1}$ which are compatible with the associator, the left unitor and the right unitor. Again if $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{D}, \otimes, \mathbb{1})$ are symmetric monoidal categories, the notion of symmetric pseudo-comonoidal functor makes sense similarly.
This variant is useful via the following simple observation.

Lemma 2.4. — *Let $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{D}, \otimes, \mathbb{1})$ be (symmetric) monoidal categories and $f : \mathcal{C} \to \mathcal{D}$ be a (symmetric) pseudo-comonoidal functor. If f admits a right adjoint g then the functor $g : \mathcal{D} \to \mathcal{C}$ is a (symmetric) pseudo-monoidal functor, the morphism $g(A) \otimes g(B) \to g(A \otimes B)$ being given by*

$$g(A) \otimes g(B) \xrightarrow{\eta} gf(g(A) \otimes g(B)) \to g(fg(A) \otimes fg(B)) \xrightarrow{\delta \otimes \delta} g(A \otimes B).$$

and the pseudo-unit morphism $\mathbb{1} \to g(\mathbb{1})$ by the composition $\mathbb{1} \xrightarrow{\eta} gf(\mathbb{1}) \to g(\mathbb{1})$.

In particular the right adjoint of a (symmetric) monoidal functor is a (symmetric) pseudo-monoidal functor.

2.2. Algebras, coalgebras, bialgebras and Hopf algebras. — Fix a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$. We refer to [11] and to [50] for the classical case (that is the case where the symmetric monoidal category $\mathcal{C}$ is the category of modules over a commutative ring with unit).

Definition 2.5. — An algebra in $\mathcal{C}$ is a pair (A, m) where A is an object in $\mathcal{C}$ and $m : A \otimes A \to A$ is a morphism such that the diagram

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id} \otimes m"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

commutes. The algebra is said to be unitary if there exists a morphism $u : \mathbb{1} \to A$ such that

(2)
$$\begin{tikzcd} \mathbb{1} \otimes A \arrow[r, "u \otimes \mathrm{Id}"] \arrow[dr, dash, "\simeq"'] & A \otimes A \arrow[d, "m"] & A \otimes \mathbb{1} \arrow[l, "\mathrm{Id} \otimes u"'] \arrow[dl, dash, "\simeq"] \\ & A & \end{tikzcd}$$

commutes.

Remark 2.6. — Note that if there exists a morphism $u : \mathbb{1} \to A$ such that (2) is commutative, then it is unique. Indeed, if $v : \mathbb{1} \to A$ is also such a morphism, it easy to see that the composition

$$\mathbb{1} \xrightarrow{\simeq} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\mathrm{Id} \otimes v} \mathbb{1} \otimes A \xrightarrow{u \otimes \mathrm{Id}} A \otimes A \xrightarrow{m} A$$

is equal to both u and v.

Dually we have the notions of coalgebra and counitary coalgebra.

Definition 2.7. — A coalgebra in $\mathcal{C}$ is a pair (C, cm) where C is an object in $\mathcal{C}$ and $cm : C \to C \otimes C$ is a morphism such that the diagram

$$\begin{tikzcd} C \arrow[r, "cm"] \arrow[d, "cm"] & C \otimes C \arrow[d, "cm\otimes\mathrm{Id}"] \\ C \otimes C \arrow[r, "\mathrm{Id}\otimes cm"] & C \otimes C \otimes C \end{tikzcd}$$

commutes. The coalgebra is said to be counitary if there exists a morphism $cu : C \to \mathbb{1}$ such that

(3)
$$\begin{tikzcd} \mathbb{1} \otimes C & C \otimes C \arrow[l, "cu\otimes\mathrm{Id}"'] \arrow[r, "\mathrm{Id}\otimes cu"] & C \otimes \mathbb{1} \\ & C \arrow[u, "cm"] \arrow[ul, dash, "\simeq"] \arrow[ur, dash, "\simeq"'] & \end{tikzcd}$$

commutes.

As in the case of an algebra, the counit, if it exists, is uniquely determined.

Lemma 2.8. —

1. *Let $f : (\mathcal{C}, \otimes, \mathbb{1}) \to (\mathcal{D}, \otimes, \mathbb{1})$ be a pseudo-monoidal functor. If (A, m) is an algebra in $\mathcal{C}$, then $f(A)$ is an algebra for the multiplication*
$$f(A) \otimes f(A) \to f(A \otimes A) \xrightarrow{f(m)} f(A).$$
If A is unitary, so is $f(A)$ and its unit is given by the composition
$$\mathbb{1} \to f(\mathbb{1}) \xrightarrow{f(u)} f(A).$$

2. *Let $f : (\mathcal{C}, \otimes, \mathbb{1}) \to (\mathcal{D}, \otimes, \mathbb{1})$ be a pseudo-comonoidal functor. If (C, cm) is a coalgebra in $\mathcal{C}$, then $f(C)$ is a coalgebra for the comultiplication*
$$f(C) \to f(C \otimes C) \to f(C) \otimes f(C).$$
If C is counitary, so is $f(C)$ and its counit is given by the composition
$$f(C) \xrightarrow{f(cu)} f(\mathbb{1}) \to \mathbb{1}.$$

Definition 2.9. — A bialgebra in $\mathcal{C}$ is a triple (H, m, cm) such that (H, m) is an algebra (H, cm) is a coalgebra and the diagram

$$\begin{tikzcd} H \otimes H \arrow[r, "m"] \arrow[d, "cm\otimes cm"'] & H \arrow[r, "cm"] & H \otimes H \\ (H \otimes H) \otimes (H \otimes H) \arrow[rr, "\mathrm{Id}\otimes\tau\otimes\mathrm{Id}"] & & (H \otimes H) \otimes (H \otimes H) \arrow[u, "m\otimes m"'] \end{tikzcd}$$

is commutative. It is said to be biunitary if (H, m) is unitary, (H, cm) is counitary and the diagrams

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\simeq"] \arrow[d, "u"'] & \mathbb{1}\otimes\mathbb{1} \arrow[d, "u\otimes u"] \\ H \arrow[r, "cm"] & H\otimes H \end{tikzcd} \qquad \begin{tikzcd} H\otimes H \arrow[r, "m"] \arrow[d, "cu\otimes cu"'] & H \arrow[d, "cu"] \\ \mathbb{1}\otimes\mathbb{1} \arrow[r, "\simeq"] & \mathbb{1} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "u"] \arrow[dr, equal, bend right] & H \arrow[d, "cu"] \\ & \mathbb{1} \end{tikzcd}$$

are commutative.

Definition 2.10. — Let H be a biunitary algebra in $\mathcal{C}$. A morphism $\iota : H \to H$, such that the diagrams

$$\begin{tikzcd} H\otimes H \arrow[rr, "\iota\otimes\mathrm{Id}"] & & H\otimes H \arrow[d, "m"] \\ H \arrow[u, "cm"] \arrow[r, "cu"] & \mathbb{1} \arrow[r, "u"] & H \end{tikzcd} \qquad \begin{tikzcd} H\otimes H \arrow[rr, "\mathrm{Id}\otimes\iota"] & & H\otimes H \arrow[d, "m"] \\ H \arrow[u, "cm"] \arrow[r, "cu"] & \mathbb{1} \arrow[r, "u"] & H \end{tikzcd}$$

are commutative, is a called an antipode. A Hopf algebra is a biunitary bialgebra in $\mathcal{C}$ that admits an antipode.

Remark 2.11. — Let (A, m) be a unitary algebra in $\mathcal{C}$. Since $\mathcal{C}$ is symmetric monoidal, the pair $(A^{\mathrm{op}}, m^{\mathrm{op}})$ where $A^{\mathrm{op}} = A$ and $m^{\mathrm{op}} = m \circ \tau$ is a unitary algebra in $\mathcal{C}$. The algebra $(A^{\mathrm{op}}, m^{\mathrm{op}})$ is called the opposite of (A, m).
Similarly if (C, cm) is a counitary coalgebra in $\mathcal{C}$, then $(C^{\mathrm{op}}, cm^{\mathrm{op}})$ where $C^{\mathrm{op}} = C$ and $cm^{\mathrm{op}} = \tau \circ cm$ is a counitary coalgebra in $\mathcal{C}$. The coalgebra $(C^{\mathrm{op}}, cm^{\mathrm{op}})$ is likewise called the opposite of (C, cm).
An algebra (resp. coalgebra) equal to its opposite is said to be commutative (resp. cocommutative).

If a biunitary bialgebra is a Hopf algebra, then the antipode is unique. This fact is best understood via the following proposition.

Proposition 2.12. — *Let A be a unitary algebra in $\mathcal{C}$.*

1. *Let C be a counitary coalgebra in $\mathcal{C}$. The set $\mathrm{Hom}_{\mathcal{C}}(C, A)$ is a monoid for the operation*
$$\mathrm{Hom}_{\mathcal{C}}(C, A) \times \mathrm{Hom}_{\mathcal{C}}(C, A) \to \mathrm{Hom}_{\mathcal{C}}(C, A)$$
that maps a pair (a, b) to the composition
$$C \xrightarrow{cm} C \otimes C \xrightarrow{a\otimes b} A \otimes A \xrightarrow{m} A.$$
The unit of this monoid is the morphism $C \xrightarrow{cu} \mathbb{1} \xrightarrow{u} A$.

2. *Let H be a Hopf algebra in $\mathcal{C}$. Let ι be an antipode of H. Every morphism $a : H \to A$ of unitary algebras is invertible in the monoid $\mathrm{Hom}_{\mathcal{C}}(H, A)$ and its inverse is given by $a \circ \iota$.*

Remark 2.13. — If H is a biunitary bialgebra and A is a commutative unitary algebra in $\mathcal{C}$, then the subset $\mathrm{Hom}_{\mathbf{Alg}}(H, A)$ of morphisms of unitary algebras is a submonoid of $\mathrm{Hom}_{\mathcal{C}}(H, A)$. By 2., if H is a commutative Hopf algebra in $\mathcal{C}$, then $\mathrm{Hom}_{\mathbf{Alg}}(H, A)$ is a group.

Corollary 2.14. —

1. *Let H be a biunitary algebra. Then there exists at most one antipode in H i.e. an Hopf algebra in $\mathcal{C}$ has a unique antipode.*

2. *Let $h : H \to H'$ be a morphism of biunitary algebras. If H and H' are Hopf algebras with respective antipodes ι and ι' then $h \circ \iota = \iota' \circ h$.*

Proof. — *1.* — By Proposition 2.12, the antipode ι if it exists must be the inverse of the identity $\mathrm{Id} : H \to H$ in the monoid $\mathrm{Hom}_{\mathcal{C}}(H, H)$. Therefore, the unicity of the antipode follows from the unicity of an inverse in a monoid.

2. — Let $h : H \to H'$ a morphism of biunitary bialgebras. Then, h is invertible in the monoid $\mathrm{Hom}_{\mathcal{C}}(H, H')$ with inverse $h \circ \iota$. On the other hand h is also invertible in the monoid $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(H', H)$ with inverse given this time by $\iota' \circ h$. Since the two monoids $\mathrm{Hom}_{\mathcal{C}}(H, H')$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(H', H)$ are the same, we must have $h \circ \iota = \iota' \circ h$. □

Note that Proposition 2.12 has also the following consequence.

Proposition 2.15. — *Let H be a Hopf algebra in $\mathcal{C}$. Then its antipode $\iota : H \to H^{\mathrm{op}}$ is a morphism of Hopf algebras.*

As a consequence, we get the following property of the antipode.

Corollary 2.16. — *Let H be a Hopf algebra in $\mathcal{C}$ and ι be its antipode. If H is commutative (resp. cocommutative) then $\iota^2 = \mathrm{Id}$.*

Proof. — Let us assume that H is commutative (the proof is dual if H is cocommutative). Then, the antipode $\iota : H \to H$ is a morphism of unitary algebras. It is therefore invertible in the monoid $\mathrm{Hom}_{\mathcal{C}}(H, H)$ with inverse $\iota^2 = \iota \circ \iota$. On the other hand, we have seen that ι as the identity for inverse. This implies that ι^2 must be the identity. □

2.3. Comodules over a coalgebra. — Again we assume that $(\mathcal{C}, \otimes, \mathbb{1})$ is a symmetric monoidal category.

Definition 2.17. — Let (C, cm) be a coalgebra in $\mathcal{C}$. A right comodule over $\mathcal{C}$ is pair (X, ca) where X is an object in $\mathcal{C}$ and $ca : X \to X \otimes C$ is a morphism, called the coaction, such that the diagram

$$\begin{tikzcd} X \arrow[r, "ca"] \arrow[d, "ca"'] & X \otimes C \arrow[d, "\mathrm{Id} \otimes cm"] \\ X \otimes C \arrow[r, "ca \otimes \mathrm{Id}"] & X \otimes C \otimes C \end{tikzcd}$$

is commutative. If (C, cm) is counitary, then a comodule (X, ca) is said to be counitary if moreover the diagram

$$\begin{tikzcd} X \arrow[r, "ca"] & X \otimes C \arrow[dl, "\mathrm{Id} \otimes cu"] \\ X \otimes \mathbb{1} \arrow[u, "\simeq"] & \end{tikzcd}$$

is commutative.

One defines similarly the notions of left comodules and counitary left comodules.

Lemma 2.18. — *Let C_1 and C_2 be two coalgebras in $\mathcal{C}$. If X_1 is a right C_1-comodule and X_2 is a right C_2-comodule, then $X_1 \otimes X_2$ is a right $C_1 \otimes C_2$-comodule with coaction given by the morphism*

$$X_1 \otimes X_2 \xrightarrow{ca \otimes ca} C_1 \otimes X_1 \otimes C_2 \otimes X_2 \xrightarrow{\mathrm{Id} \otimes \tau \otimes \mathrm{Id}} (C_1 \otimes C_2) \otimes (X_1 \otimes X_2)$$

If X_1 and X_2 are counitary, so is $X_1 \otimes X_2$.

In the above lemma, $C_1 \otimes C_2$ is a coalgebra for the comultiplication given by the composition

$$C_1 \otimes C_2 \xrightarrow{cm \otimes cm} C_1 \otimes C_1 \otimes C_2 \otimes C_2 \xrightarrow{\mathrm{Id} \otimes \tau \otimes \mathrm{Id}} C_1 \otimes C_2 \otimes C_1 \otimes C_2.$$

If both C_1 and C_2 are counitary, then so is $C_1 \otimes C_2$ with counit given by

$$C_1 \otimes C_2 \xrightarrow{cu \otimes cu} \mathbb{1}.$$

Assume now that (H, m, cm) is a biunitary bialgebra. Then, the multiplication $m : H \otimes H \to H$ is a morphism of counitary coalgebras. In particular, if X and Y are counitary right H-comodules, the counitary right $H \otimes H$-comodule $X \otimes Y$, obtained in Lemma 2.18, defines by corestriction along m a counitary right H-comodule.

Proposition 2.19. — *Let H be a biunitary bialgebra. Let X, Y be two counitary right H-comodules. Then $X \otimes Y$ is a counitary right H-comodule with coaction given by the morphism*

$$X \otimes Y \xrightarrow{ca \otimes ca} X \otimes H \otimes Y \otimes H \xrightarrow{\mathrm{Id} \otimes \tau \otimes \mathrm{Id}} X \otimes Y \otimes H \otimes H \xrightarrow{\mathrm{Id} \otimes \mathrm{Id} \otimes m} X \otimes Y \otimes H.$$

If H is commutative, then the isomorphism $\tau : X \otimes Y \xrightarrow{\simeq} Y \otimes X$ is an isomorphism of counitary right H-comodules.

In particular, if H is a biunitary bialgebra, the category $\mathbf{coMod}(H)$ is a monoidal category and the forgetful functor

$$\mathbf{coMod}(H) \to \mathcal{C}$$

is a monoidal functor. If H is moreover commutative, then $\mathbf{coMod}(H)$ is a symmetric monoidal category and the forgetful functor is symmetric monoidal.
Let H be a Hopf algebra in $\mathcal{C}$ and ι be the antipode of H. We can turn counitary left H-comodules into counitary right H-comodules and vice-versa. Indeed, if (X, ca) is a (counitary) left H-comodule, then the morphism

$$ca^r : X \xrightarrow{ca} H \otimes X \xrightarrow{\tau} X \otimes H \xrightarrow{\mathrm{Id} \otimes \iota} X \otimes H$$

defines a structure of (counitary) right H-comodule on X. Similarly, if (X, ca) is a (counitary) right H-comodule, then the morphism

$$ca^\ell : X \xrightarrow{ca} X \otimes H \xrightarrow{\tau} H \otimes X \xrightarrow{\iota \otimes \mathrm{Id}} H \otimes X$$

is a (counitary) left coaction of H on X. The comodule obtained by going back and forth, is the corestriction of the initial comodule along the morphism of counitary coalgebras $\iota^2 : H \to H$. In particular, if H is commutative or cocommutative, by Corollary 2.16, the categories of (counitary) left and right comodules are equivalent.

Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category. Recall that an object $X \in \mathcal{C}$ has a right dual if there exist an object $X^\vee$ in $\mathcal{C}$ and morphisms

$$\mathbb{1} \xrightarrow{\text{coev}} X^\vee \otimes X \qquad X \otimes X^\vee \xrightarrow{\text{ev}} \mathbb{1} \tag{4}$$

such that the morphisms

$$X \xrightarrow{\text{Id}\otimes\text{coev}} X \otimes X^\vee \otimes X \xrightarrow{\text{ev}\otimes\text{Id}} X, \quad X^\vee \xrightarrow{\text{coev}\otimes\text{Id}} X^\vee \otimes X \otimes X^\vee \xrightarrow{\text{Id}\otimes\text{ev}} X^\vee$$

are equal to the identity maps. Similarly X is said to admit a left dual if there exist an object ${}^\vee X$ in $\mathcal{C}$ and morphisms

$$\mathbb{1} \xrightarrow{\text{coev}'} X \otimes {}^\vee X \qquad {}^\vee X \otimes X \xrightarrow{\text{ev}'} \mathbb{1} \tag{5}$$

such that the morphisms

$$X \xrightarrow{\text{coev}'\otimes\text{Id}} X \otimes {}^\vee X \otimes X \xrightarrow{\text{Id}\otimes\text{ev}'} X, \quad {}^\vee X \xrightarrow{\text{Id}\otimes\text{coev}'} {}^\vee X \otimes X \otimes {}^\vee X \xrightarrow{\text{ev}'\otimes\text{Id}} {}^\vee X$$

are equal to the identity maps.
Note that the triples $(X^\vee, \text{coev}, \text{ev})$ and $({}^\vee X, \text{coev}', \text{ev}')$ are unique up to a unique isomorphism.
If $(\mathcal{C}, \otimes, \mathbb{1})$ is symmetric, then a right dual is also a left (and vice-versa) with $\text{coev}' = \tau \circ \text{coev}$ and $\text{ev}' = \text{ev} \circ \tau$. In that case we simply say that X admits a dual or that X is dualizable.

Definition 2.20. — A symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ is said to be rigid if and only if every object in $\mathcal{C}$ admits a dual.

Assume that $\mathcal{C}$ is rigid and H is a coalgebra in $\mathcal{C}$. Then, one has a canonical isomorphism

$$\text{Hom}_{\mathcal{C}}(X, X \otimes H) \simeq \text{Hom}_{\mathcal{C}}(X^\vee, H \otimes X^\vee) \tag{6}$$

Explicitely, a morphism $\gamma : X \to X \otimes H$ in $\mathcal{C}$ is mapped to the morphism

$$X^\vee \xrightarrow{\text{Id}\otimes\text{coev}'} X^\vee \otimes X \otimes X^\vee \xrightarrow{\text{Id}\otimes\gamma\otimes\text{Id}} X^\vee \otimes X \otimes H \otimes X^\vee \xrightarrow{\text{ev}'\otimes\text{Id}} H \otimes X^\vee$$

and conversely a morphism $\delta : X^\vee \to H \otimes X^\vee$ in $\mathcal{C}$ is mapped to the morphism

$$X \xrightarrow{\text{coev}'\otimes\text{Id}} X \otimes X^\vee \otimes X \xrightarrow{\text{Id}\otimes\delta\otimes\text{Id}} X \otimes H \otimes X^\vee \otimes X \xrightarrow{\text{Id}\otimes\text{ev}'} X \otimes H.$$

Under (6), a right coaction is mapped to a left coaction and vice versa. In particular, if (X, ca) is a right comodule, then $X^\vee$ is canonically a left comodule. Assume that H is a commutative Hopf algebra in $\mathcal{C}$. Then, as explained above, the left comodule structure on $X^\vee$ can be turned into a right comodule structure. In this way, the duality functor $(-)^\vee$ becomes a contravariant endofunctor of the category of counitary right H-comodules in such a way that the evaluation and coevaluation morphisms (4) and (5) become morphisms of counitary right H-comodules.
This shows that the symmetric monoidal category of counitary right H-comodules is also rigid.

2.4. Representations and comodules. — Recall that an affine Λ-monoid scheme is a pair (G, m) where G is an affine Λ-scheme and $m : G \times_\Lambda G \to G$ is a morphism of Λ-schemes satisfying the following conditions:

1. the square
$$\begin{tikzcd} G \times_\Lambda G \times_\Lambda G \arrow[r, "m\times\mathrm{Id}"] \arrow[d, "\mathrm{Id}\times m"'] & G \times_\Lambda G \arrow[d, "m"] \\ G \times_\Lambda G \arrow[r, "m"] & G \end{tikzcd}$$
is commutative;

2. there exists a morphism $1_G : \mathrm{Spec}(\Lambda) \to G$ which makes the diagram
$$\begin{tikzcd} \mathrm{Spec}(\Lambda) \times_\Lambda G \arrow[r, "1_G\times\mathrm{Id}"] \arrow[dr, dash, "\simeq"'] & G \times_\Lambda G \arrow[d, "m"] & G \times_\Lambda \mathrm{Spec}(\Lambda) \arrow[l, "\mathrm{Id}\times 1_G"'] \arrow[dl, dash, "\simeq"] \\ & G & \end{tikzcd}$$
commute.

Note that if a morphism as the one in 2. exists then it is unique.
Let $\mathbf{Alg}_\Lambda$ be the category of commutative unitary Λ-algebras. If G is an affine Λ-monoid scheme, then, for every R in $\mathbf{Alg}_\Lambda$,
$$G(R) := \mathrm{Hom}_\Lambda(\mathrm{Spec}(R), G)$$
is a monoid. The product of two elements $g, h \in G(R)$ is given by $g \cdot h = m \circ (g, h)$ and the neutral element $1_{G(R)}$ of $G(R)$ given by the composition $\mathrm{Spec}(R) \to \mathrm{Spec}(\Lambda) \xrightarrow{1_G} G$
Let **Mon** be the category of monoids. Let V be a Λ-vector space and consider the functor
$$\begin{aligned} \mathcal{E}\mathrm{nd}(V) : \mathbf{Alg}_\Lambda &\to \mathbf{Mon} \\ R &\mapsto \mathrm{End}_R(V \otimes_\Lambda R). \end{aligned}$$

Definition 2.21. — Let G be an affine Λ-monoid scheme. Let V be a Λ-vector space. A left G-module structure on V is a morphism of functors on $\mathbf{Alg}_\Lambda$ with values in **Mon**
$$\rho : G \to \mathcal{E}\mathrm{nd}(V)$$

In other words a left G-module structure on V is the data for every R in $\mathbf{Alg}_\Lambda$ of a morphism of monoids
$$\rho_R : G(R) \to \mathrm{End}_R(V \otimes_\Lambda R)$$
such that the squares
$$\begin{tikzcd} G(R) \arrow[r] \arrow[d] & \mathrm{End}_R(V \otimes_\Lambda R) \arrow[d] \\ G(S) \arrow[r] & \mathrm{End}_S(V \otimes_\Lambda S) \end{tikzcd}$$
are commutative for every morphism of commutative (unitary) Λ-algebras $R \to S$.

Remark 2.22. — One can similarly define a right G-module structure on a Λ-vector space V. Let $G^\circ(R) = G(R)^\circ$ where $G(R)^\circ$ is the set $G(R)$ with the opposite monoid structure. A right G-module structure on V is simply a morphism of functors

$$\rho : G^\circ \to \mathscr{E}\mathrm{nd}(V)$$

on $\mathbf{Alg}_\Lambda$ with values in $\mathbf{Mon}$.

A left (resp. right) representation of G is a pair (V, ρ) where V is a Λ-vector space and ρ is a left (resp. right) G-module structure on V. Morphisms of representations are defined as morphisms of Λ-vector spaces that commute with the action of G. A representation is said to be finite dimensional if the underlying vector space is finite dimensional over Λ.
We denote by $\mathbf{Rep}(G, \Lambda)$ the Abelian category of representations of G and by $\mathbf{rep}(G, \Lambda)$ the full Abelian subcategory whose objects are the finite dimensional representations.
Let G be an affine Λ-monoid scheme. Let $H := \mathscr{O}(G)$ be the affine ring of G. This is a commutative unitary Λ-algebra. Via the classical equivalence of categories

$$\begin{array}{rcl} \begin{array}{c}\textbf{commutative}\\ \textbf{unitary } \boldsymbol{\Lambda}\textbf{-algebras}\end{array} & \longleftrightarrow & \textbf{affine } \boldsymbol{\Lambda}\textbf{-schemes} \\ A & \mapsto & \mathrm{Spec}(A) \\ \mathscr{O}(X) & \leftarrow\!\shortmid & X \end{array}$$

affine Λ-monoid schemes correspond to commutative biunitary Λ-bialgebras (see Definition 2.9). Moreover an affine Λ-monoid scheme G is a Λ-group scheme if and only if $\mathscr{O}(G)$ is a Λ-Hopf algebra (see Definition 2.10).
We can reformulate the definition of (left) representations of an affine Λ-group scheme G in terms of right comodules over the associated Λ-Hopf algebra $H := \mathscr{O}(G)$ (see e.g. [50, §3.2]). Let V be a Λ-vector space. Let us first remark that there is a canonical bijection

$$\mathrm{Hom}(G, \mathscr{E}\mathrm{nd}(V)) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(V, V \otimes_\Lambda H) \tag{7}$$

where on the left hand-side we consider morphisms of functors on $\mathbf{Alg}_\Lambda$ with values in $\mathbf{Sets}$ and on the right hand-side morphisms of Λ-vector spaces. The morphism (7) is defined as follows. With a morphism $\rho : G \to \mathscr{E}\mathrm{nd}(V)$, it associates the morphism

$$ca : V \xrightarrow{\mathrm{Id}\otimes u} V \otimes_\Lambda H \xrightarrow{\rho_H(\mathrm{Id})} V \otimes_\Lambda H$$

where Id denotes the element of $G(H) = \mathrm{Hom}_{\mathbf{Alg}_\Lambda}(H, H)$ given by the identity of H.
To see that (7) is indeed a bijection, it is enough to see that, given a morphism of Λ-vector spaces $ca : V \to V \otimes_\Lambda H$, we can reconstruct ρ as follows. If R is a commutative unitary Λ-algebra and $g \in G(R)$, then $\rho_R(g)$ is the R-linear extension of the morphism

$$V \xrightarrow{ca} V \otimes_\Lambda H \xrightarrow{\mathrm{Id}\otimes g} V \otimes_\Lambda R.$$

Lemma 2.23. — *Let G be an affine Λ-group scheme and $H := \mathscr{O}(G)$ its associated Λ-Hopf algebra.*

1. *Let $\rho : G \to \mathscr{E}\mathrm{nd}(V)$ be a morphism of functors from on $\mathbf{Alg}_\Lambda$ with values in $\mathbf{Sets}$. Then, the morphism $ca : V \to V \otimes_\Lambda H$ obtained via (7) defines a counitary right comodule structure on V if and only if ρ is left G-module structure on V.*

2. *The functor*

$$\mathbf{Rep}(G,\Lambda) \to \mathbf{coMod}(H) \qquad (8)$$
$$(V,\rho) \mapsto (V, ca)$$

is an equivalence of categories and induces an equivalence of categories between $\mathbf{rep}(G,\Lambda)$ *and* $\mathbf{comod}(H)$.

Note that the equivalence (8) commutes with the forgetful functors to the category $\mathbf{Vec}(\Lambda)$.

Proof. — The morphism $\rho_R(gh)$ is obtained from the morphism $ca : V \to V \otimes_\Lambda H$ as the R-linear extension of the composition

$$\begin{tikzcd} V \arrow[r, "ca"] & V \otimes_\Lambda H \arrow[r, "\mathrm{Id}\otimes gh"] \arrow[d, "\mathrm{Id}\otimes cm"'] & V \otimes_\Lambda R \\ & V \otimes_\Lambda H \otimes_\Lambda H \arrow[r, "\mathrm{Id}\otimes g\otimes h"] & V \otimes_\Lambda R \otimes_\Lambda R. \arrow[u, "\mathrm{Id}\otimes m"'] \end{tikzcd} \qquad (9)$$

On the other hand, the composition $\rho_R(g)\rho_R(h)$ is the R-linear extension of the morphism given by the commutative diagram

$$\begin{tikzcd} V \arrow[r, "ca"] & V \otimes_\Lambda H \arrow[r, "\mathrm{Id}\otimes h"] \arrow[d, "ca\otimes\mathrm{Id}"] & V \otimes_\Lambda R \arrow[d, "ca\otimes\mathrm{Id}"] \\ & V \otimes_\Lambda H \otimes_\Lambda H \arrow[r, "\mathrm{Id}\otimes\mathrm{Id}\otimes h"] \arrow[dr, bend right, "\mathrm{Id}\otimes g\otimes h"'] & V \otimes_\Lambda H \otimes_\Lambda R \arrow[d, "\mathrm{Id}\otimes g\otimes\mathrm{Id}"] \\ & & V \otimes_\Lambda R \otimes_\Lambda R \arrow[d, "\mathrm{Id}\otimes m"] \\ & & V \otimes_\Lambda R. \end{tikzcd} \qquad (10)$$

From this, it is clear that if $ca : V \to V \otimes_\Lambda H$ is a right comodule structure on V, then $\rho_R(gh) = \rho_R(g)\rho_R(h)$.
Conversely, assume that $\rho_R(gh) = \rho_R(g)\rho_R(h)$, for every $R \in \mathbf{Alg}_\Lambda$ and every $g, h \in G(R)$. Then, the morphisms (9) and (10) are the same. By taking $R = H \otimes_\Lambda H$ and $g = \mathrm{Id} \otimes u$, $h = u \otimes \mathrm{Id}$, we have then

$$m \circ (g \otimes h) = \mathrm{Id}$$

and we see that the equality of (9) and (10) amounts in that case to saying that ca is a right comodule structure on V.
It remains to prove that $(\mathrm{Id} \otimes cu) \circ ca = \mathrm{Id}$ if and only if $\rho_R(1) = \mathrm{Id}$, for every commutative unitary Λ-algebra R. But this follows easily from the fact that the neutral element in $G(R)$ being given by the composition

$$H \xrightarrow{cu} \Lambda \xrightarrow{u} R,$$

the endomorphism $\rho_R(1)$ is the R-linear extension of the composition

$$V \xrightarrow{ca} V \otimes_\Lambda H \xrightarrow{\mathrm{Id}\otimes cu} V \otimes_\Lambda \Lambda \xrightarrow{V\otimes u} V \otimes_\Lambda R. \qquad \square$$

Remark 2.24. — Similarly counitary left $\mathscr{O}(G)$-comodules correspond to right representations of G.

Remark 2.25. — If H is a Hopf Λ-algebra, then the category of counitary right C-comodules is an Abelian category. This remains true if for every counitary coalgebra C over the field Λ. Moreover every counitary right comodule of C is the union of its finite dimensional subcomodules. In categorical terms, the category of all counitary right comodules $\mathbf{coMod}(C)$ is equivalent to the category of Ind-objects of $\mathbf{comod}(C)$.

3. Classical Tannaka duality

Let $\mathcal{A}$ be a Λ-linear Abelian category. If X is an object of $\mathcal{A}$, we denote by $\langle X\rangle$ the strictly full Abelian subcategory of $\mathcal{A}$ generated by X. The objects in $\langle X\rangle$ are the subquotients of finite direct sums of copies of X. Note that the category $\langle X\rangle$ depends only on the isomorphism class of X. Therefore, we can consider $\mathrm{ob}(\mathcal{A})$ as a poset for the relation $X \leqslant Y$ if and only if $\langle X\rangle \subseteq \langle Y\rangle$. This poset is directed since, for every object X, Y in $\mathcal{A}$, the category $\langle X \oplus Y\rangle$ contains both $\langle X\rangle$ and $\langle Y\rangle$.

3.1. The group reconstruction problem. — We first consider the problem of reconstructing an affine group scheme from its category of finite dimensional representations (see Problem 1.1 in the introduction). The fundamental question here is the following: how can an affine Λ-group scheme be reconstructed from its category of finite dimensional representations?

As we shall see, there are different ways of answering this question. These different approaches or formulations are obviously closely related but have nonetheless different flavors. Before going into details, let us state a reconstruction process based on the description of the functor of points of an affine groupe scheme in terms of its category of finite dimensional representations (Theorem 3.1).

Let G an affine group scheme over Λ. The category $\mathbf{rep}(G)$ of finite dimensional representations is a Λ-linear Abelian category, even a rigid symmetric monoidal category, endowed with a forgetful functor

$$\omega : \mathbf{rep}(G, \Lambda) \to \mathbf{vec}(\Lambda)$$

which is Λ-linear faithful exact and also symmetric monoidal. Let R be a commutative unitary Λ-algebra. Consider the group

$$\mathcal{A}\mathrm{ut}^{\otimes}(\omega)(R) := \mathrm{Aut}^{\otimes}(\omega_R)$$

where ω_R is the composition of the functor $\omega : \mathbf{rep}(G, \Lambda) \to \mathbf{vec}(\Lambda)$ and the base change functor

$$\begin{aligned} \mathbf{vec}(\Lambda) &\to \mathbf{mod}(R) \\ V &\mapsto V \otimes_\Lambda R \end{aligned}$$

and $\mathrm{Aut}^{\otimes}(\omega_R)$ is the group of invertible monoidal natural transformations of ω_R. We may then consider the morphism of groups

$$G(R) \to \mathrm{Aut}^{\otimes}(\omega_R)$$

which associates with a fixed element $g \in G(R)$ and with varying objects (V, ρ) of $\mathbf{rep}(G, \Lambda)$ the family of linear maps $\rho(g) : V \otimes_\Lambda R \to V \otimes_\Lambda R$.

Theorem 3.1. — *Let G be an affine group scheme over Λ and R be a commutative unitary Λ-algebra. Then, the canonical morphism*

$$G(R) \to \mathrm{Aut}^{\otimes}(\omega_R)$$

is an isomorphism of groups.

The proof of Theorem 3.1 will be given at the end of the subsection. It answers completely the reconstruction problem: the affine group scheme G represents the functor

$$\begin{aligned} \mathcal{A}\mathrm{ut}^{\otimes}(\omega) : \mathbf{Alg}_{\Lambda} &\to \mathbf{Grp} \\ R &\mapsto \mathrm{Aut}^{\otimes}(\omega_R) \end{aligned}$$

whose definition only involves the forgetful functor on the category $\mathbf{rep}(G, \Lambda)$ of finite dimensional representations of G. Theorem 3.1 highlights a general phenomenon: the reconstruction process involves the forgetful functor more than the category itself.

Using the duality between affine group schemes and Hopf algebras, the problem can be equivalently reformulated in terms of Hopf algebras and comodules: how can a commutative Λ-Hopf algebra be reconstructed from its category of counitary right comodules of finite dimension?

The reconstruction theorem in terms of Hopf algebras is a finer version of a more general statement which only involves coalgebras (see Theorem 3.7).

We start by giving a canonical description of a counitary Λ-coalgebra as a union (colimit) of finite dimensional counitary subcoalgebras associated with finite dimensional counitary right C-comodules (see Lemma 3.3).

Let V be a finite dimensional Λ-vector space and $ca : V \to V \otimes_{\Lambda} C$ be a Λ-linear morphism. Let $\mathscr{B} = (e_1, \ldots, e_n)$ be a basis of V and let c_{ij}, for $i, j \in [\![1, n]\!]$, be the elements of C defined by the equality

$$ca(e_j) = \sum_{i=1}^{n} e_i \otimes c_{ij}.$$

Then, (V, ca) is a counitary right comodule if and only if the elements c_{ij} satisfy the following relations for every $i, j \in [\![1, n]\!]$

$$\begin{cases} cm(c_{ij}) &= \sum_{k=1}^{n} c_{ik} \otimes c_{kj} \\ cu(c_{ij}) &= \delta_{ij} \end{cases}$$

where δ_{ij} is the Kronecker symbol. In particular, if (V, ca) is a counitary right C-comodule, then the Λ-linear subspace C_V of C spanned by elements c_{ij} for $i, j \in [\![1, n]\!]$ is counitary subcoalgebra of C which is finite dimensional over Λ. By construction V is not only a C-comodule but a C_V-comodule. Note that the definition of C_V does not depend upon the choice of a basis of V since it can also be described as the image of the morphism $V \otimes V^{\vee} \to C$ associated with the coaction. Moreover it is easy to see that if V and W are isomorphic counitary right C-comodules then the two counitary subcoalgebras C_V and C_W are the same.

Remark 3.2. — Note that if W is in $\langle V \rangle$, then W is a subquotient of a finite direct sum of copies of V, and C_W is contained in C_V (hence C_V is the union of all the C_W for W in $\langle V \rangle$). In particular, if $\langle W \rangle \subseteq \langle V \rangle$, then C_W is contained in C_V. The coalgebras C_V form a directed system indexed by the directed poset of isomorphism classes of counitary right C-comodules.

Lemma 3.3. — *Let (C, cm) be a counitary coalgebra. Then, C is the union of its (finite dimensional) subcoalgebras C_V as V runs among the finite dimensional counitary right comodules:*

$$C = \operatorname{colim}_{V \in \mathrm{ob}(\mathbf{comod}(C))} C_V. \tag{11}$$

Proof. — As C is the union of its finite dimensional counitary subcomodules (see Remark 2.25), it is enough to show that for such a subcomodule V, one has $V \subseteq C_V$. The counit $cu : C \to \Lambda$ by restriction to V defines an element in $V^\vee$. Let $\mathscr{B} = (e_1, \ldots, e_n)$ be a basis of V. Then, for every $j \in [\![1, n]\!]$, the image of $e_j \otimes cu$ by the morphism $V \otimes V^\vee \to C$ associated with cm is given by

$$\begin{aligned}\sum_{i=1}^{n} cu(e_i)c_{ij} &= (cu \otimes \mathrm{Id})\left(\sum_{i=1} e_i \otimes c_{ij}\right) = (cu \otimes \mathrm{Id})(cm(e_j)) \\ &= (\mathrm{Id} \otimes cu)(cm(e_j)) = (\mathrm{Id} \otimes cu)\left(\sum_{i=1} e_i \otimes c_{ij}\right) \\ &= \sum_{i=1}^{n} cu(c_{ij})e_i = \sum_{i=1}^{n} \delta_{ij} e_i = e_j.\end{aligned}$$

This shows that e_j belongs to C_V for every $j \in [\![1, n]\!]$ and therefore that V is contained in C_V. □

Though the next lemma is not needed to obtain the main theorems, it clarifies the strategy of the proof of the recognition theorem (see Theorem 3.9).

Lemma 3.4. — *Let C be a counitary coalgebra and V a counitary right comodule. The corestriction functor*

$$\mathbf{comod}(C_V) \to \mathbf{comod}(C)$$

is fully faithful and its essential image is the strictly full Abelian subcategory $\langle V \rangle$ generated by V.

Proof. — The fully faithfulness of the functor follows immediately from the fact that C_V is a subcoalgebra of C. Let W be a comodule which belongs to $\langle V \rangle$. One has then the inclusion $C_W \subseteq C_V$ which implies that W is also a C_V-comodule. Hence W belongs to the essential image which therefore contains all of $\langle V \rangle$.
Conversely, let W be a C_V-comodule. Let $A = C_V^\vee$ be the corresponding finite Λ-algebra and $M = V^\vee$ and $N = W^\vee$ be the corresponding right A-modules. To see that W is in $\langle V \rangle$, it suffices to show that N is a subquotient of a direct sum of copies of M.
Observe that M is a faithful right A-module. Indeed, let $\mathcal{B} = (e_1, \ldots, e_n)$ be a basis of V over Λ and $(e_1^\vee, \ldots, e_n^\vee)$ the corresponding basis of M. Let $a \in A$ be an element such that $ma = 0$ for all $m \in M$. Then, for every $j \in [\![1, n]\!]$

$$\begin{aligned}0 = ma(e_j) = (m \otimes a)(ca(e_j)) &= (m \otimes a)\left(\sum_{k=1}^{n} e_k \otimes c_{kj}\right) \\ &= \sum_{k=1}^{n} m(e_k)a(c_{kj}).\end{aligned}$$

In particular, by taking $m = e_i^\vee$, we get that $a(c_{ij}) = 0$ for every $i, j \in [\![1, n]\!]$. Since the c_{ij}'s span C_V, we must have $a = 0$.
As a right A-module N is a quotient of a free module A^m for some integer $m \geqslant 1$, hence it is enough to show that A is a right A-submodule of a direct sum of copies of M. Since M is a faithful right A-module, the morphism of right A-modules

$$\begin{aligned} A &\to M^n \\ a &\mapsto (e_1^\vee a, \ldots, e_n^\vee a) \end{aligned}$$

is injective. This concludes the proof. □

Let us get back to the reconstruction problem (see [50, §3.4]). Let C be a counitary Λ-coalgebra. Let R be a commutative unitary Λ-algebra. Let $\phi \in \mathrm{Hom}_\Lambda(C, R)$. If (V, ca) is a right C-comodule, then we can define a R-linear endomorphism $\theta_\phi(V, ca)$ of $V \otimes_\Lambda R$ has the scalar extension of the Λ-linear morphism

$$V \xrightarrow{ca} V \otimes_\Lambda C \xrightarrow{\mathrm{Id}\otimes\phi} V \otimes_\Lambda R. \tag{12}$$

It is easy to see that the collection of the $\theta_\phi(V, ca)$'s for all comodules (V, ca) define an endomorphism θ_ϕ of the functor

$$\omega_R \colon \mathbf{coMod}(C) \xrightarrow{\omega} \mathbf{Vec}(\Lambda) \xrightarrow{-\otimes_\Lambda R} \mathbf{Mod}(R). \tag{13}$$

In this way, we obtain a Λ-linear morphism

$$\begin{aligned} \mathrm{Hom}_\Lambda(C, R) &\to \mathrm{End}(\omega_R) \\ \phi &\mapsto \theta_\phi. \end{aligned} \tag{14}$$

Since every counitary right C-comodule is a directed colimit of a system of counitary right C-comodules of finite dimension over Λ (see Remark 2.25), the functor (13) and its restriction to finite dimensional counitary right C-comodules have the same Λ-algebras of endomorphisms. In the proof of Proposition 3.5, it is useful to be able to consider infinite dimensional counitary right C-comodules.

Proposition 3.5. — *Let C be a Λ-coalgebra and R be a unitary commutative Λ-algebra. Then, the morphism* (14) *is an isomorphism of Λ-algebras.*

Remark 3.6. — Let (V, ca) be a counitary right C-comodule. Note that $V \otimes_\Lambda C$ is a counitary right C-comodule with

$$V \otimes_\Lambda C \xrightarrow{\mathrm{Id}\otimes cm} V \otimes_\Lambda C \otimes_\Lambda C$$

as coaction. The definition of a comodule implies that the coaction $ca : V \to V \otimes_\Lambda C$ is a morphism of counitary right C-comodules. Moreover for a counitary comodule the coaction $ca : V \to V \otimes_\Lambda C$ is an injective morphism. In particular, every comodule (V, ca) in $\mathbf{comod}(C)$ is a subobject of a finite direct sum of copies of the comodule (C, cm).

Proof. — Let θ be an endomorphism of the functor ω_R. Since (C, cm) is right comodule over C, we may consider the Λ-linear morphism $\phi(\theta)$, obtained as the composition

$$\phi(\theta) \colon C \xrightarrow{\mathrm{Id}\otimes u} C \otimes_\Lambda R \xrightarrow{\theta(C,cm)} C \otimes_\Lambda R \xrightarrow{cu\otimes\mathrm{Id}} R.$$

In this way, one defines a Λ-linear morphism

$$\begin{aligned} \operatorname{End}(\omega_R) &\to \operatorname{Hom}_\Lambda(C, R) \\ \theta &\mapsto \phi(\theta). \end{aligned}$$

Let us show that (14) and this morphism are inverse one to the other. Let $\phi \in \operatorname{Hom}_\Lambda(C, R)$. Then, the diagram

$$\begin{tikzcd}
C \arrow[r, "\mathrm{Id}\otimes u"'] \arrow[rrr, bend left, "\phi(\theta_\phi)"] \arrow[rrd, bend right, "cm"] \arrow[rrdd, bend right, equal, "\mathrm{Id}"'] & C \otimes_\Lambda R \arrow[r, "\theta_\phi"] & C \otimes_\Lambda R \arrow[r, "cu\otimes\mathrm{Id}"'] & R \\
 & & C \otimes_\Lambda C \arrow[u, "\mathrm{Id}\otimes\phi"] \arrow[ur, "cu\otimes\phi"'] \arrow[d, "cu\otimes\mathrm{Id}"'] & \\
 & & C \arrow[uur, bend right, "\phi"'] &
\end{tikzcd}$$

being commutative, we have $\phi(\theta_\phi) = \phi$.
Conversely, let θ be an element in $\operatorname{End}(\omega_R)$. Since θ is an endomorphism of functors, by Remark 3.6, the square

$$\begin{tikzcd}
V \otimes_\Lambda R \arrow[r, "\theta(V,ca)"] \arrow[d, "ca\otimes\mathrm{Id}"] & V \otimes_\Lambda R \arrow[d, "ca\otimes\mathrm{Id}"] \\
V \otimes_\Lambda C \otimes_\Lambda R \arrow[r, "\theta(V\otimes_\Lambda C,\mathrm{Id}\otimes cm)"] & V \otimes_\Lambda C \otimes_\Lambda R
\end{tikzcd}$$

is commutative. Let d be the dimension of V over Λ. The right C-comodule $(V \otimes_\Lambda C, \mathrm{Id} \otimes cm)$ being isomorphic to d copies of (C, cm), one sees that $\theta(V \otimes C, \mathrm{Id} \otimes cm) = \mathrm{Id} \otimes \theta(C, cm)$. This implies that the diagram

$$\begin{tikzcd}
V \arrow[r, "\mathrm{Id}\otimes u"] \arrow[d, "ca"] & V \otimes_\Lambda R \arrow[r, "\theta(V,ca)"] \arrow[d, "ca\otimes\mathrm{Id}"] & V \otimes_\Lambda R \arrow[d, "ca\otimes\mathrm{Id}"] \arrow[dr, bend left, equal] & \\
V \otimes_\Lambda C \arrow[r, "\mathrm{Id}\otimes\mathrm{Id}\otimes u"] \arrow[rrr, bend right, "\mathrm{Id}\otimes\phi(\theta)"'] & V \otimes_\Lambda C \otimes_\Lambda R \arrow[r, "\mathrm{Id}\otimes\theta(C,cm)"] & V \otimes_\Lambda C \otimes_\Lambda R \arrow[r, "\mathrm{Id}\otimes cu\otimes\mathrm{Id}"] & V \otimes_\Lambda R
\end{tikzcd}$$

is commutative and shows that the endomorphisms of functors θ and $\theta_{\phi(\theta)}$ coincide.
It remains to show that the Λ-linear morphism (14) is a morphism of Λ-algebras. Recall that the unit of the algebra $\operatorname{Hom}_\Lambda(C, R)$ is the morphism $C \xrightarrow{cu} \Lambda \xrightarrow{u} R$, so, using that all our comodules are counitary, it is easy to see that it maps under (14) to the idendity endomorphism of ω_R.
Let ϕ, ψ be elements in $\operatorname{Hom}_\Lambda(C, R)$. Their product $\phi \bullet \psi$ in $\operatorname{Hom}_\Lambda(C, R)$ is the morphism

$$C \xrightarrow{cm} C \otimes_\Lambda C \xrightarrow{\phi\otimes\psi} R \otimes_\Lambda R \xrightarrow{m} R.$$

Let (V, ca) be a counitary right C-comodule. The commutativity of the diagram

$$\begin{tikzcd}
V \otimes_\Lambda R \arrow[rrr, "\theta_{\phi\bullet\psi}(V,ca)"] \arrow[ddr, bend right, "\theta_\psi(V,ca)"'] & & & V \otimes_\Lambda R \\
V \arrow[u, "\mathrm{Id}\otimes u"'] \arrow[r, "ca"] \arrow[dr, "ca"'] & V \otimes_\Lambda C \arrow[r, "\mathrm{Id}\otimes cm"] & V \otimes_\Lambda C \otimes_\Lambda C \arrow[r, "\mathrm{Id}\otimes\phi\otimes\psi"] \arrow[dd, "\mathrm{Id}\otimes\mathrm{Id}\otimes\psi"] & V \otimes_\Lambda R \otimes_\Lambda R \arrow[u, "\mathrm{Id}\otimes m"'] \\
 & V \otimes_\Lambda C \arrow[ur, "ca\otimes\mathrm{Id}"'] \arrow[d, "\mathrm{Id}\otimes\psi"'] & & \\
 & V \otimes_\Lambda R \arrow[r, "ca\otimes\mathrm{Id}"] \arrow[uuurr, bend right=60, "\theta_\phi(V,ca)"'] & V \otimes_\Lambda C \otimes_\Lambda R \arrow[uur, "\mathrm{Id}\otimes\phi\otimes\mathrm{Id}"'] &
\end{tikzcd}$$

shows that $\theta_{\phi\bullet\psi} = \theta_\phi \circ \theta_\psi$ as desired. ☐

Theorem 3.7. — *Let C be a counitary Λ-coalgebra. Then C is canonically isomorphic as a counitary coalgebra to the colimit (taken over the poset of isomorphism classes of counitary right C-comodules of finite dimension V) of the coalgebras*

$$\mathrm{End}(\omega|_{\langle V\rangle})^\vee.$$

Proof. — Assume that C is a counitary Λ-coalgebra. Proposition 3.5 provides a canonical isomorphism of unitary Λ-algebras

$$C^\vee := \mathrm{Hom}_\Lambda(C, \Lambda) \xrightarrow{\sim} \mathrm{End}(\omega).$$

If C has finite dimension over Λ, then the above isomorphism can be dualized to give an isomorphism of counitary Λ-coalgebras

$$C \xrightarrow{\sim} \mathrm{End}(\omega)^\vee.$$

If C is not finite dimensional over Λ, then, using (11), we obtain a canonical isomorphism of counitary Λ-coalgebras

$$C = \mathrm{colim}_{V\in\mathrm{ob}(\mathbf{comod}(C))}\, C_V \xrightarrow{\sim} \mathrm{colim}_{V\in\mathrm{ob}(\mathbf{comod}(C))}\, \mathrm{End}(\omega|_{\langle V\rangle})^\vee. \qquad ☐$$

Note that Theorem 3.7 holds true whether C underlies a biunitary bialgebra structure or not. If C is a (commutative) biunitary bialgebra, then the category $\mathbf{comod}(C)$ is a (symmetric) monoidal category (see Proposition 2.19) and its tensor product induces a biunitary bialgebra structure on

$$\mathrm{colim}_{V\in\mathrm{ob}(\mathbf{comod}(C))}\, \mathrm{End}(\omega|_{\langle V\rangle})^\vee.$$

for which the isomorphism in Theorem 3.7 becomes an isomorphism of biunitary bialgebras. In particular, Theorem 3.7 provides an answer to the reconstruction problem which turns out to be essential also in the understanding of the recognition problem.
Let us get back to the proof of Theorem 3.1 by refining Proposition 3.5 in the case where the coalgebra underlies a biunitary bialgebra structure.

Proposition 3.8. — *Let H be a biunitary bialgebra and R be a unitary algebra. Then the isomorphism* (14) *induces a bijection*

$$\mathrm{Hom}_{\mathbf{Alg}_\Lambda}(H, R) \xrightarrow{\sim} \mathrm{End}^\otimes(\omega_R).$$

Proof. — Let θ be an endomorphism of the functor ω_R and ϕ be the corresponding Λ-linear morphism $\phi : H \to R$. Recall that ϕ is given by the composition

$$\phi\colon H \xrightarrow{\mathrm{Id}\otimes u} H \otimes_\Lambda R \xrightarrow{\theta(H,cm)} H \otimes_\Lambda R \xrightarrow{cu\otimes \mathrm{Id}} R.$$

Let $\mathbb{1}$ be the trivial comodule, that is $\mathbb{1} : \Lambda \xrightarrow{\mathrm{Id}\otimes u} \Lambda \otimes_\Lambda H$. Note that $\theta(\mathbb{1})$ is equal to the R-linear extension of

$$\Lambda \xrightarrow{\mathrm{Id}\otimes u} \Lambda \otimes_\Lambda H \xrightarrow{\mathrm{Id}\otimes \phi} \Lambda \otimes_\Lambda R = R$$

hence, $\theta(\mathbb{1})$ is the identity if and only if ϕ is unitary. Assume that θ is a tensor endomorphism and consider the diagram

$$\begin{tikzcd}
H \otimes_\Lambda H \arrow[r, equal] \arrow[d, "\mathrm{Id}\otimes u\otimes \mathrm{Id}\otimes u"] \arrow[ddd, bend right=60, "\phi\otimes\phi"'] & H \otimes_\Lambda H \arrow[r, "m"] \arrow[d, "\mathrm{Id}\otimes \mathrm{Id}\otimes u\otimes u"] & H \arrow[d, "\mathrm{Id}\otimes u"] \arrow[ddd, bend left=60, "\phi"] \\
H \otimes_\Lambda R \otimes_\Lambda H \otimes_\Lambda R \arrow[r, "\mathrm{Id}\otimes\tau\otimes \mathrm{Id}"] \arrow[d, "\theta(H,cm)\otimes\theta(H,cm)"] & H \otimes_\Lambda H \otimes_\Lambda R \otimes_\Lambda R \arrow[r, "m\otimes m"] & H \otimes_\Lambda R \arrow[d, "\theta(H,cm)"] \\
H \otimes_\Lambda R \otimes_\Lambda H \otimes_\Lambda R \arrow[r, "\mathrm{Id}\otimes\tau\otimes \mathrm{Id}"] \arrow[d, "cu\otimes \mathrm{Id}\otimes cu\otimes \mathrm{Id}"] & H \otimes_\Lambda H \otimes_\Lambda R \otimes_\Lambda R \arrow[r, "m\otimes m"] \arrow[d, "cu\otimes cu\otimes \mathrm{Id}\otimes \mathrm{Id}"] & H \otimes_\Lambda R \arrow[d, "cu\otimes \mathrm{Id}"] \\
R \otimes_\Lambda R \arrow[r, "\mathrm{Id}"] & R \otimes_\Lambda R \arrow[r, "m"] & R.
\end{tikzcd}$$

To see that ϕ is a morphism of unitary algebras, we have to check that the above diagram is commutative. This amounts to showing that the central subdiagram is commutative (all other subdiagrams are commutative). Since θ is a tensor endomorphism, the diagram

$$\begin{tikzcd}
H \otimes_\Lambda R \otimes_\Lambda H \otimes_\Lambda R \arrow[r, "\mathrm{Id}\otimes\tau\otimes \mathrm{Id}"] \arrow[d, "\theta(H,cm)\otimes\theta(H,cm)"] & H \otimes_\Lambda H \otimes_\Lambda R \otimes_\Lambda R \arrow[r, "\mathrm{Id}\otimes \mathrm{Id}\otimes m"] & H \otimes_\Lambda H \otimes_\Lambda R \arrow[d, "\theta((H,cm)\otimes(H,cm))"'] \\
H \otimes_\Lambda R \otimes_\Lambda H \otimes_\Lambda R \arrow[r, "\mathrm{Id}\otimes\tau\otimes \mathrm{Id}"] & H \otimes_\Lambda H \otimes_\Lambda R \otimes_\Lambda R \arrow[r, "\mathrm{Id}\otimes \mathrm{Id}\otimes m"] & H \otimes_\Lambda H \otimes_\Lambda R
\end{tikzcd} \tag{15}$$

is commutative. On the other hand, the multiplication $m\colon H \otimes H \to H$ is a morphism of comodules when $H \otimes H$ is endowed with the coaction of the tensor product $(H, cm) \otimes (H, cm)$ and H with the comultiplication cm. Hence, the square

$$\begin{tikzcd}
H \otimes_\Lambda H \otimes_\Lambda R \arrow[r, "m\otimes \mathrm{Id}"] \arrow[d, "\theta((H,cm)\otimes(H,cm))"'] & H \otimes_\Lambda R \arrow[d, "\theta(H,cm)"] \\
H \otimes_\Lambda H \otimes_\Lambda R \arrow[r, "m\otimes \mathrm{Id}"] & H \otimes_\Lambda R
\end{tikzcd}$$

is commutative too. This shows that ϕ is a morphism of algebras as soon as θ is a tensor endomorphism.

Conversely assume that ϕ is a morphism of unitary algebras. By Remark 3.6, we know that every comodule in $\mathbf{comod}(H)$ is a subcomodule of a finite direct sum of copies of (H, cm). Hence to show that θ is a tensor endomorphism, it is enough to show that the diagram (15) is commutative. Using the description of the endomorphism $\theta(V, ca)$ as the R-linear extension of the morphism (12), this amounts to showing that the diagram

$$\begin{tikzcd}
H \otimes_\Lambda H \arrow[d, "cm\otimes cm"] \arrow[rrd, "\text{coaction of } (H,cm)\otimes(H,cm)"] & & \\
H \otimes_\Lambda H \otimes_\Lambda H \otimes_\Lambda H \arrow[r, "\mathrm{Id}\otimes\tau\otimes\mathrm{Id}"] \arrow[d, "\mathrm{Id}\otimes\phi\otimes\mathrm{Id}\otimes\phi"] & H \otimes_\Lambda H \otimes_\Lambda H \otimes_\Lambda H \arrow[r, "\mathrm{Id}\otimes\mathrm{Id}\otimes m"] \arrow[d, "\mathrm{Id}\otimes\mathrm{Id}\otimes\phi\otimes\phi"] & H \otimes_\Lambda H \otimes_\Lambda H \arrow[d, "\mathrm{Id}\otimes\mathrm{Id}\otimes\phi"] \\
H \otimes_\Lambda R \otimes_\Lambda H \otimes_\Lambda R \arrow[r, "\mathrm{Id}\otimes\tau\otimes\mathrm{Id}"] & H \otimes_\Lambda H \otimes_\Lambda R \otimes_\Lambda R \arrow[r, "\mathrm{Id}\otimes\mathrm{Id}\otimes m"] & H \otimes_\Lambda H \otimes_\Lambda R
\end{tikzcd}$$

is commutative. That readily follows from the fact that ϕ is a morphism of algebras. □

Proof of Theorem 3.1. — Let $H = \mathscr{O}(G)$ be the commutative Λ-Hopf algebra associated with G. Then, for every commutative unitary Λ-algebra R, the group of R-points of G is given by

$$G(R) = \mathrm{Hom}_{\mathbf{Alg}_\Lambda}(H, R).$$

Now $\mathrm{Hom}_{\mathbf{Alg}_\Lambda}(H, R)$ is a submonoid of $\mathrm{Hom}_\Lambda(H, R)$ (see Remark 2.13) whose image by the isomorphism of monoids (14) coincides with the submonoid $\mathrm{End}^\otimes(\omega_R)$ of $\mathrm{End}(\omega_R)$ formed by the tensor endomorphisms of the functor ω_R. Hence, (14) induces an isomorphism of monoids

$$G(R) = \mathrm{Hom}_{\mathbf{Alg}_\Lambda}(H, R) \xrightarrow{\sim} \mathrm{End}^\otimes(\omega_R).$$

Since $G(R)$ is a group, so is $\mathrm{End}^\otimes(\omega_R)$ and therefore we have the equality $\mathrm{End}^\otimes(\omega_R) = \mathrm{Aut}^\otimes(\omega_R)$. This concludes the proof. □

3.2. The categorical recognition problem. — The problem now is to be able to determine when a given Abelian category is the category of finite dimensional representations of an affine group scheme. We first answer the weaker problem of recognizing among Abelian categories those that are equivalent to a category of finite dimensional counitary right comodules over some counitary coalgebra.
The main theorem of this section is the following.

Theorem 3.9. — *Let $\mathcal{A}$ be a Λ-linear Abelian category. The following conditions are equivalent:*

1. *the category $\mathcal{A}$ is Noetherian and Artinian and for every objects A, B in $\mathcal{A}$ the Λ-vector space $\mathcal{A}(A, B)$ is finite dimensional;*

2. *there exist a Λ-coalgebra C and a Λ-linear exact equivalence of categories between $\mathcal{A}$ and the category $\mathbf{comod}(C)$;*

3. *there exists a Λ-linear exact and faithful functor $\omega : \mathcal{A} \to \mathbf{vec}(\Lambda)$.*

The equivalence of 1. and 2. is a finite dimensional variant of a result of Takeuchi (see [51]) that we will not develop here. Instead, we will focus on the equivalence between 3. and 2. If $\mathcal{A}$ is equivalent to $\mathbf{comod}(C)$ for some coalgebra, then the composition of the equivalence with the forgetful functor defines a Λ-linear exact faithful functor $\omega : \mathcal{A} \to \mathbf{vec}(\Lambda)$. Hence 3. is a necessary condition for 2. Therefore, the main problem is to see, how a Λ-linear faithful exact functor $\omega : \mathcal{A} \to \mathbf{vec}(\Lambda)$ defines a counitary coalgebra $C(\omega)$ and enriches into an equivalence of categories between $\mathcal{A}$ and the category of counitary right $C(\omega)$-comodule of finite dimension. Theorem 3.13 is the central result for which our main reference is [50, §2.5] (see also [23, 44])

Let $\mathcal{A}$ be a Λ-linear Abelian category and $\omega : \mathcal{A} \to \mathbf{vec}(\Lambda)$ be a Λ-linear faithful exact functor. Recall that we denote by $\langle X \rangle$ the Abelian subcategory generated by X in $\mathcal{A}$. It is the strictly full subcategory of $\mathcal{A}$ whose objects are the subquotients of finite direct sums of copies of X. Let X be an object in $\mathcal{A}$, and let S be a subset of $\omega(X)$. The intersection of the subobjects Y of X such that $S \subseteq \omega(Y)$ is the smallest subobject with this property, we call it the subobject of X generated by S. An object will be said to be monogenic if it can be generated by one element. Let X be an object of $\mathcal{A}$. An element $p \in \omega(X)$ will be said to be contained in a subobjet Y of X if $\omega(Y)$ contains p.

Lemma 3.10. — *Let $\mathcal{A}$ be a Λ-linear Abelian category such that $\mathcal{A} = \langle X \rangle$ for some object X in $\mathcal{A}$. Let Y be a monogenic object in $\mathcal{A}$. Then,*

$$\dim_\Lambda \omega(Y) \leqslant (\dim_\Lambda \omega(X))^2.$$

Proof. — Let $y \in \omega(Y)$ be a generator of Y. Since $\mathcal{A} = \langle X \rangle$, the object Y is a quotient Y'/Y'' where Y' is isomorphic to a subobject of X^n for some integer $n \geqslant 1$.

Since ω is exact, the map $\omega(Y') \to \omega(Y)$ is surjective and we may lift y to an element y' in $\omega(Y')$. Let Z be the subobject of Y' generated by y'. Then, its image in Y still contains y so it is equal to Y. Hence, we get an epimorphism $Z \twoheadrightarrow Y$ where Z is a monogenic subobject of some X^n. This implies $\dim_\Lambda \omega(Y) \leqslant \dim_\Lambda \omega(Z)$ and we may assume that $Y = Z$.

The exactness of ω implies that

$$\dim_\Lambda \omega(Y) \leqslant \dim_\Lambda \omega(X^n) = n \dim_\Lambda \omega(X).$$

Therefore, it is enough to show that one may choose the integer n so that $n \leqslant \dim_\Lambda \omega(X)$. Note that $\omega(Y)$ is a linear subspace of $\omega(X)^n$ and the generator y of Y can be written $y = (y_1, \dots, y_n)$ with $y_1, \dots, y_n$ in $\omega(X)$.

If $n > \dim_\Lambda \omega(X)$, then $y_1, \dots, y_n$ are not linear independant, and there exists $(\lambda_1, \dots, \lambda_n) \in \Lambda^n \setminus \{0\}$ such that $\sum_{i=1}^n \lambda_i y_i = 0$. Let $A \in \mathscr{M}_{n-1,n}(\Lambda)$ be a matrix such that the square matrix

$$\begin{pmatrix} \lambda_1 & \cdots & \lambda_n \\ & & \\ & A & \\ & & \end{pmatrix}$$

is invertible.

Consider the morphism $X^n \to X$ defined by the matrix $(\lambda_1, \dots, \lambda_n)$ and let K be its kernel. By assumption K contains y, and since Y is generated by y, it must be contained in K. The composition

$$K \to X^n \xrightarrow{A} X^{n-1}$$

is an isomorphism in $\mathcal{A}$ since it is after applying the faithful functor ω. Therefore K is isomorphic to X^{n-1} and by induction we may assume that $n \leqslant \dim_\Lambda \omega(X)$. This concludes the proof. □

Lemma 3.11. — *Let $\mathcal{A}$ be a Λ-linear Abelian category and $\omega : \mathcal{A} \to \mathbf{vec}(\Lambda)$ be a Λ-linear faithful exact functor. If there exists an object $X \in \mathcal{A}$ such that $\mathcal{A} = \langle X \rangle$, then the functor $\omega : \mathcal{A} \to \mathbf{vec}(\Lambda)$ is representable.*

Proof. — By Lemma 3.10, the dimension of $\omega(Y)$ can take only finitely many values when Y is monogenic. Hence, among all monogenic object in $\mathcal{A}$ we may choose one whose image by ω has the greatest dimension possible. Let P be such an object and $p \in \omega(P)$ be a generator of P.
Let us show that the pair (P, p) represents the functor ω. Let A an object in $\mathcal{A}$ and $a \in \omega(A)$. We have to show that there exists one and only one morphism $f : P \to A$ in $\mathcal{A}$ such that $\omega(f)$ maps p to a. The uniqueness follows immediately from the fact that p generates P. Indeed, let f, g be two such morphisms. Then $\mathrm{Ker}(f - g)$ is a subobject of P that contains p. Since P is generated by p, we must have $\mathrm{Ker}(f - g) = P$ that is $f = g$.
Let us show the existence. Let B be the subobject of $P \oplus A$ generated by (p, a) and $\pi_1 : B \to P$, $\pi_2 : B \to A$ be the two projections. Since P is generated by P, the projection $\pi_1 : B \to P$ is an epimorphism in $\mathcal{A}$. In particular, we have

$$\dim_\Lambda \omega(B) \geqslant \dim_\Lambda \omega(P).$$

Since B is monogenic and the dimension of the image of P by ω has maximal dimension among all monogenic objects, we must have $\dim_\Lambda \omega(B) = \dim_\Lambda \omega(P)$. Hence the projection map $\pi_1 : B \to P$ is an isomorphism after applying ω and therefore is an isomorphism. The composition of the

$$P \xrightarrow{\pi_1^{-1}} B \xrightarrow{\pi_2} A$$

is a morphism in $\mathscr{A}$ whose image by ω maps p to a. ☐

Proposition 3.12. — *Let $\mathcal{A}$ be a Λ-linear Abelian category and $\omega : \mathcal{A} \to \mathbf{vec}(\Lambda)$ be a Λ-linear faithful exact functor. If there exists an object $X \in \mathcal{A}$ such that $\mathcal{A} = \langle X \rangle$, then the functor ω induces an equivalence*

$$\mathcal{A} \xrightarrow{\omega} \mathbf{mod}(\mathrm{End}(\omega)) \to \mathbf{comod}(\mathrm{End}(\omega)^\vee).$$

Proof. — By Lemma 3.11, the functor ω is representable by a pair (P, p) where P is in $\mathcal{A}$ and $p \in \omega(P)$. In other words, for every object A in $\mathcal{A}$, $\omega(A)$ is canonically isomorphic to $\mathrm{Hom}_\mathcal{A}(P, A)$ and $\mathrm{End}(\omega)$ to $\mathrm{End}_\mathcal{A}(P)$. Let $\mathbf{mod}(\mathrm{End}_\mathcal{A}(P))$ be the category of left $\mathrm{End}_\mathcal{A}(P)$-modules that are finite dimensional over Λ.
We have to prove that the Λ-linear faithful exact functor

$$h : \mathcal{A} \to \mathbf{mod}(\mathrm{End}_\mathcal{A}(P)) \qquad (16)$$
$$A \mapsto \mathrm{Hom}_\mathcal{A}(P, A)$$

is an equivalence of categories (which means that we have to show that it is full and essentially surjective). Let $R = \mathrm{End}_\mathcal{A}(P)$. Every object in $\mathbf{mod}(R)$ is of finite presentation over R, thus one can find an exact sequence

$$R^m \xrightarrow{\phi} R^n \to M \to 0.$$

The matrix of ϕ in the canonical basis is an element in $\mathscr{M}_{n,m}(R)$ that defines an element ϕ_P in $\mathrm{Hom}_\mathcal{A}(P^m, P^n)$ whose image by $\mathrm{Hom}_\mathcal{A}(P, -)$ coincides with ϕ. The image under (16) of the cokernel of ϕ_P is therefore isomorphic to M. This shows that (16) is essentially surjective and that moreover every object in $\mathcal{A}$ fits into an exact sequence

$$P^m \to P^n \to Y \to 0. \qquad (17)$$

Let Z be another object in $\mathcal{A}$. The short exact sequence (17) induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(Y,Z) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(P^n,Z) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(P^m,Z) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_R(h(Y),h(Z)) \arrow[r] & \mathrm{Hom}_R(R^n,h(Z)) \arrow[r] & \mathrm{Hom}_R(R^m,h(Z))
\end{tikzcd}$$

in which all rows are exact sequences. To conclude the proof, it is enough to remark that, for every integer $n \geqslant 1$, the map induced by (16)

$$\mathrm{Hom}_{\mathcal{A}}(P^n,Z) \to \mathrm{Hom}_R(R^n,h(Z))$$

is an isomorphism. □

Let $C(\omega)$ be the counitary Λ-coalgebra defined by

$$C(\omega) = \mathrm{colim}_{X\in \mathrm{ob}(\mathcal{A})} \mathrm{End}(\omega|_{\langle X\rangle})^{\vee}$$

where the colimit is taken over the directed poset of isomorphism classes of objects in $\mathcal{A}$ (see Theorem 3.7). For such an isomorphism class, we set

$$C_X(\omega) = \mathrm{End}(\omega|_{\langle X\rangle})^{\vee}$$

(see Lemma 3.4).

Theorem 3.13. — *Let $\mathcal{A}$ be a Λ-linear Abelian category and $\omega : \mathcal{A} \to \mathbf{vec}(\Lambda)$ be a Λ-linear faithful exact functor. Then, the functor ω induces an equivalence*

$$\mathcal{A} \xrightarrow{\omega} \mathbf{comod}(C(\omega)). \tag{18}$$

Proof. — Let X, Y be objects in $\mathcal{A}$ such that $X \in \langle Y\rangle$. Then, one has a canonical morphism

$$\mathrm{End}(\omega|_{\langle Y\rangle}) \to \mathrm{End}(\omega|_{\langle X\rangle}). \tag{19}$$

By Lemma 3.11, the functor $\omega|_{\langle X\rangle}$ is representable by a pair (P,p) where $P \in \langle X\rangle$ is monogenic and $p \in \omega(P)$ generates P. Similarly the functor $\omega|_{\langle Y\rangle}$ is representable by a pair (Q,q) where $Q \in \langle Y\rangle$ is monogenic and $q \in \omega(Q)$ generates Q.
In particular, there exists one and only morphism $\pi : Q \to P$ in $\mathcal{A}$ whose image by ω maps q to p. The induced morphism

$$\omega(\pi) : \omega(Q) \to \omega(P)$$

coincides with (19). Moreover the image of π is a subobject of P that contains p and therefore is equal to P. This means that π is an epimorphism and implies that (19) is surjective. Hence, the canonical morphism

$$C_X(\omega) = \mathrm{End}(\omega|_{\langle X\rangle})^{\vee} \to C_Y(\omega) = \mathrm{End}(\omega|_{\langle Y\rangle})^{\vee}$$

is injective. This implies that the morphism $C_X(\omega) \to C(\omega)$ is injective also.
If X is an object in $\mathcal{A}$, then the functor ω induces a commutative square

$$\begin{tikzcd}
\langle X\rangle \arrow[r] \arrow[d, "\omega|_{\langle X\rangle}"] & \mathcal{A} \arrow[d, "\omega"] \\
\mathbf{comod}(C_X(\omega)) \arrow[r] & \mathbf{comod}(C(\omega))
\end{tikzcd}$$

and we know by Proposition 3.12 that $\omega|_{\langle X\rangle}$ is an equivalence. Note that the inclusion of $\langle X\rangle$ in $\mathcal{A}$ is obviously fully faithful and that the functor

$$\mathbf{comod}(C_X(\omega)) \to \mathbf{comod}(C(\omega))$$

is also fully faithful since the canonical morphism of coalgebra $C_X(\omega) \to C(\omega)$ is injective. From this, it follows that (18) is fully faithful. It remains to show that it is essentially surjective. Let V be a right $C(\omega)$-comodule such that V is finite dimensional over Λ. The coaction is a morphism of Λ-vector spaces

$$ca\colon V \to V \otimes_\Lambda C(\omega) = \operatorname{colim}_{X\in \mathrm{ob}(\mathcal{A})} V \otimes_\Lambda C_X(\omega).$$

Since V has finite dimension over Λ, there exists an object X in $\mathcal{A}$ such that the coaction factorizes in a morphism

$$ca\colon V \to V \otimes_\Lambda C_X(\omega).$$

It is easy to see, using the injectivity of $C_X(\omega) \to C(\omega)$, that this morphism defines a structure of right $C_X(\omega)$-comodule over V. Now we can apply Proposition 3.12 to conclude that (V, ca) is in the essential image of the functor (18). □

It is possible to deduce Theorem 1.4 from Theorem 3.13. Let us briefly sketch the idea. Let $(\mathcal{A}, \omega)$ be a neutralized Tannakian category. By Theorem 3.13, the functor ω induces an equivalence

$$\mathcal{A} \xrightarrow{\omega} \mathbf{comod}(C(\omega)) \tag{20}$$

where $C(\omega)$ be the Λ-coalgebra defined by

$$C(\omega) = \operatorname{colim}_{X\in \mathrm{ob}(\mathcal{A})} \operatorname{End}(\omega|_{\langle X\rangle})^\vee.$$

The symmetric monoidal structure on $\mathcal{A}$ induces a commutative biunitary bialgebra stucture on $C(\omega)$ for which (20) becomes symmetric monoidal when $\mathbf{comod}(C(\omega))$ is endowed with the symmetric monoidal structure defined by the bialgebra structure (see [25, II, Proposition 2.16]). The rigidity of $\mathcal{A}$ implies finally that $C(\omega)$ is a Hopf algebra.
The core result behind Theorem 1.4 is thus Theorem 3.13. The refinement that Theorem 1.4 represents lies in the possibility to translate the additional requirements on $\mathcal{A}$ and ω into properties of the coalgebra $C(\omega)$.

4. Motives

For our discussion, it will be enough to assume that k is field of characteristic zero with a fixed embedding $\sigma : k \hookrightarrow \mathbb{C}$. This assumption simplifies the exposition while still retaining the main problems.
For example, under this condition, Betti cohomology provides a Weil cohomology with rational coefficients. As explained by Serre (see [28, §1.7]), such a Weil cohomology cannot exist over the finite field $\mathbf{F}_{p^2}$.

4.1. The problem of its application to (pure) motives. — Grothendieck introduced the notion of pure motive (see [22, p. 173]) as a way to encapsulate in one object the common properties of the various cohomology theories that he and his school had defined for smooth projective varieties (the so called classical Weil cohomology among which ℓ-adic cohomology, algebraic de Rham cohomology and Betti cohomology).

From the geometric point of view, the category of Chow motives introduced by Grothendieck is very natural (see Section 4.1 for its construction). Its morphisms are given by the Chow groups of algebraic cycles modulo rational equivalence and therefore the category retains all the information contained in the Chow groups.

As part of Grothendieck's vision, categories of (pure or mixed) motives were meant to provide a motivic Galois theory via the Tannakian formalism developed in the thesis of Saavedra under his direction [48] (see [23, 25] also).

However, as we shall see in this section, the natural strategy, based on using Betti cohomology, to build a Tannakian category from the category of Chow motives leads to two standard conjectures: (a) the algebraicity of Künneth projectors (b) the coincidence of homological equivalence with numerical equivalence (with rational coefficients).

In the next sections, we will explain how two different extensions of the classical Tannaka duality, either the weak Tannakian formalism of Ayoub [11, 12] or the approach of Nori using quiver representations [26], allow to construct unconditionally motivic Galois groups that encompass pure motives as well as mixed motives.

Note that, to avoid having to rely on conjectures, other approaches have been developed long before. The first approach, initiated by Deligne (see [25]), consists roughly in adding to algebraic cycles enough of the cohomology classes expected to be algebraic to avoid assuming the conjectures. It has been refined by André with his unconditional theory of motives [1]. André and Kahn have developed a different strategy in [4, 5] to circumvent the standard conjecture on numerical and homological equivalences (modulo rational coefficients).

Let us recall briefly the construction of the category of (rational) Chow motives. For a more detailed account of the definition and properties of pure motives we refer to [2, 39] or to Scholl's survey article [49].

Let X and Y be smooth projective k-varieties. If X is connected of dimension d, an algebraic correspondence of degree r (with rational coefficients) from X to Y is an element of the $\mathbb{Q}$-vector space

$$\mathrm{CH}^{r+d}(X \times_k Y, \mathbb{Q}).$$

When X is no longer connected, one sets

$$\mathrm{Corr}^r(X, Y) = \bigoplus_{i=1}^{n} \mathrm{CH}^{r+d_i}(X_i \times_k Y, \mathbb{Q})$$

where $X_1, \ldots, X_n$ are the connected components of X and $d_1, \ldots, d_n$ their dimensions. Algebraic correspondences can be composed using intersection theory via the formula

$$\begin{gathered} \mathrm{Corr}^r(Y, Z) \otimes_{\mathbb{Q}} \mathrm{Corr}^s(X, Y) \to \mathrm{Corr}^{r+s}(X, Z) \\ (\beta, \alpha) \mapsto \beta \circ \alpha = p_{13*}(p_{23}^*\beta \frown p_{12}^*\alpha) \end{gathered} \tag{21}$$

Objects in the category $\mathbf{Mrat}(k, \mathbb{Q})$ are triples (X, p, a) where X is a smooth projective k-variety, $p \in \mathrm{Corr}^0(X, X)$ is an idempotent correspondence and $a \in \mathbb{Z}$ is an integer. Morphisms are given by

$$\mathrm{Hom}_{\mathbf{Mrat}(k,\mathbb{Q})}((X,p,a),(Y,q,b)) = q\,\mathrm{Corr}^{b-a}(X,Y)p,$$

the composition being induced by (21). The category $\mathbf{Mrat}(k,\mathbb{Q})$ is a $\mathbb{Q}$-linear pseudo-abelian category (i.e. projectors split) such that $\mathrm{End}(\mathbb{1}) = \mathbb{Q}$ and we have a functor

$$h : \mathbf{SmPr}_k^{\mathrm{op}} \to \mathbf{Mrat}(k,\mathbb{Q})$$

sending a variety X to $(X, \mathrm{Id}_X, 0)$ and a morphism f to the correspondence ${}^t[\Gamma_f]$ where Γ_f denotes the graph (recall that if α is an algebraic cycle on a product $X \times_k Y$, then ${}^t\alpha$ is the algebraic cycle on $Y \times_k X$ obtained by permutation of the factors). The n-th Tate twist $M(n)$ of a motive $M = (X,p,a)$ is defined to be the motive $M(n) = (X,p,a+n)$. With this in mind, it is very convenient to denote the motive defined by a triple (X,p,a) by $ph(X)(a)$. It is the direct summand of the motive $h(X)(a)$ cut-out by p.
The unit motive is $\mathbb{1} = (\mathrm{Spec}(k), \mathrm{Id}, 0)$ and it is easy to see that there is a canonical decomposition of the motive of $\mathbb{P}^1_k$ given by $h(\mathbb{P}^1_k) = \mathbb{1} \oplus \mathbb{1}(-1)$.
The category $\mathbf{Mrat}(k,\mathbb{Q})$ is a rigid symmetric monoidal category (see [2, §4.1.4]). If X is a smooth projective k-variety of pure dimension d, p is an idempotent algebraic correspondence and a is an integer, the dual of the motive $ph(X)(a)$ is given by

$$ph(X)(a)^\vee = {}^t ph(X)(d-a)$$

(the duality is given by the transposition on morphisms).
Betti cohomology of smooth projective k-varieties (defined using the embedding $\sigma : k \hookrightarrow \mathbb{C}$) provides a Weil cohomology theory with rational coefficients (see [2, 3.3.1.1 Definition] for the definition) and therefore a $\mathbb{Q}$-linear symmetric monoidal functor

$$H^* : \mathbf{Mrat}(k,\mathbb{Q}) \to \mathbf{vec}(\mathbb{Q})^{\mathbb{Z}} \tag{22}$$

where $\mathbf{vec}(\mathbb{Q})^{\mathbb{Z}}$ denotes the category of finite dimensional $\mathbb{Z}$-graded $\mathbb{Q}$-vector spaces (see [2, Proposition 4.2.5.1]). The cycle class map in Betti cohomology is simply the map induced by the realization functor

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{Mrat}(k,\mathbb{Q})}(\mathbb{1}, h(X)(n)) \arrow[r, "H^*"] \arrow[d, equal] & \mathrm{Hom}_{\mathbf{vec}(\mathbb{Q})^{\mathbb{Z}}}(\mathbb{Q}, H^*(X)(n)) \arrow[d, equal] \\ \mathrm{CH}^n(X) \arrow[r, "\mathrm{cl}"] & H^{2n}(X)(n). \end{tikzcd}$$

Let X be a smooth projective k-variety of pure dimension d. Recall that a cycle $\alpha \in \mathrm{CH}^n(X,\mathbb{Q})$ is said to be numerically equivalent to zero if for every cycle $\beta \in \mathrm{CH}^{d-n}(X,\mathbb{Q})$ the degree of the zero cycle $\alpha \frown \beta$ obtained by intersecting α and β is equal to zero. Since

$$\deg(\alpha \frown \beta) = \mathrm{Tr}(\mathrm{cl}(\alpha) \smile \mathrm{cl}(\beta))$$

that is the degree of the intersection of the cycles α and β is equal to the image by the trace map $\mathrm{Tr} : H^{2d}(X)(d) \to \mathbb{Q}$ of the cup product of their cohomology classes, a cycle which is homologically trivial is also numerically trivial. This defines the following quotients of the Chow group

$$\mathrm{CH}^n(X,\mathbb{Q}) \twoheadrightarrow A^n_{\mathrm{hom}}(X,\mathbb{Q}) \twoheadrightarrow A^n_{\mathrm{num}}(X,\mathbb{Q}) \tag{23}$$

We can mimic the construction of the category of Chow motives by replacing, in the definition of algebraic correspondences, the Chow groups by the quotients in (23) to obtain the category $\mathbf{Mhom}(k,\mathbb{Q})$ of homological motives (aka. Grothendieck's motives) and the category $\mathbf{Mnum}(k,\mathbb{Q})$ of numerical motives. The hierarchy between the three equivalence relations implies that there are canonical $\mathbb{Q}$-linear functors

$$\mathbf{Mrat}(k,\mathbb{Q}) \to \mathbf{Mhom}(k,\mathbb{Q}) \to \mathbf{Mnum}(k,\mathbb{Q}).$$

which are symmetric monoidal. Note that the category $\mathbf{Mrat}(k,\mathbb{Q})$ is not Abelian (see [49, Corollary 3.5]) and that $\mathbf{Mnum}(k,\mathbb{Q})$ on the contrary is Abelian semi-simple by Jannsen's theorem [37].

The functor (22) is not faithful since Chow groups may be infinite dimensional while on the contrary Betti cohomology groups are finite dimensional. To obtain a faithful $\mathbb{Q}$-linear functor from (22), it is necessary (and sufficient) to kill all the cycles α that are homogically trivial. In other words, (22) induces a faithful symmetric monoidal $\mathbb{Q}$-linear functor

$$\mathbf{Mhom}(k,\mathbb{Q}) \to \mathbf{vec}(\mathbb{Q})^{\mathbb{Z}}. \tag{24}$$

Now we would like to have a fiber functor that takes its values in the category of finite dimensional vector spaces and not in the category of finite dimensional graded vector spaces. To do so, we would like to simply consider the functor

$$\begin{aligned} \mathbf{Mhom}(k,\mathbb{Q}) &\to \mathbf{vec}(\mathbb{Q}) \\ M &\mapsto \bigoplus_{i\in\mathbb{Z}} H^i(M) \end{aligned} \tag{25}$$

that is the composition of (24) and the $\mathbb{Q}$-linear functor

$$\begin{aligned} \mathbf{vec}(\mathbb{Q})^{\mathbb{Z}} &\to \mathbf{vec}(\mathbb{Q}) \\ V^* &\mapsto \bigoplus_{i\in\mathbb{Z}} V^i. \end{aligned} \tag{26}$$

The problem is that (26) is a monoidal functor but is not a *symmetric* monoidal functor. The symmetry isomorphism $V^*\otimes W^* \xrightarrow{\sim} W^*\otimes V^*$ in $\mathbf{vec}(\mathbb{Q})^{\mathbb{Z}}$ is given by $v\otimes w \mapsto (-1)^{pq} w\otimes v$ for v of degree p and w of degree q, whereas in $\mathbf{vec}(\mathbb{Q})$ the symmetry isomorphism is simply given by $v\otimes w\mapsto w\otimes v$.

As a consequence the functor (25) is not a symmetric monoidal functor. A way to remedy this problem is to redefine the commutativity constraint in $\mathbf{Mhom}(k,\mathbb{Q})$ so that (25) becomes a symmetric monoidal functor. This is where we enter the realm of conjectures: to do so we have to know that the grading on $H^*(M)$ comes from a grading on the homological motive M. It is sufficient to know it for homological motives of the form $M = h(X)$ where X is a smooth projective k-variety which is precisely the content of the standard conjecture on the algebraicity of the Künneth projectors.

***Conjecture 4.1* (standard conjecture $\mathsf{C}(X)$).** — *Let X be a smooth projective variety of dimension d over k. The Künneth projectors*

$$\pi^i \in \mathrm{End}(H^*) : H^*(X) \twoheadrightarrow H^i(X) \hookrightarrow H^*(X)$$

are algebraic i.e. belong to

$$\mathbf{Mhom}(k;\mathbb{Q})(h(X),h(X)) \hookrightarrow \mathrm{End}(H^*(X)).$$

In particular $\mathsf{C}(X)$ provides a canonical weight decomposition of the homological motive of X

$$h(X) = h^0(X) \oplus h^1(X) \oplus \cdots \oplus h^{2d-1}(X) \oplus h^{2d}(X)$$

where h^i is the direct summand of the homological motive of X cut-off by the Künneth projector π^i. Assuming standard conjecture C, we can redefine the commutativity constraint in $\mathbf{Mhom}(k, \mathbb{Q})$ using the Koszul rule of signs so that (25) becomes a $\mathbb{Q}$-linear faithful symmetric monoidal functor. As customary, we denote by $\mathbf{M\dot{h}om}(k, \mathbb{Q})$ the category of homological motives with the new commutativity constraint.
To be able to see the tensor functor

$$\begin{aligned} \mathbf{M\dot{h}om}(k, \mathbb{Q}) &\to \mathbf{vec}(\mathbb{Q}) \\ M &\mapsto \bigoplus_{i \in \mathbb{Z}} H^i(M) \end{aligned}$$

has a fiber functor of a (neutral) Tannakian category, it remains to know that the category $\mathbf{M\dot{h}om}(k, \mathbb{Q})$ is Abelian. As shown by André in [1, Appendice], the category of Grothendieck's motives (aka. homological motives) is Abelian if and only if: (a) homogical equivalence coincides with numerical equivalence (standard conjecture D) ; (b) the Künneth projectors are algebraic (standard conjecture C) . This result is a refinement of Jannsen's theorem [37] which shows that the category of pure motives for a given adequate equivalence relation is a semi-simple Abelian category if and only of the equivalence relation is numerical equivalence.

4.2. Mixed motives. — As we have seen in the previous section, pure motives are limited to the description of the cohomology of smooth projective (or proper) varieties. As soon as a variety is not proper, its cohomology is usually not pure anymore but can still be described in terms of pure pieces using resolution of singularities and a smooth proper compactification with a strict normal crossing divisor as boundary. The idea of mixed motives and the related notion of weights were introduced by Grothendieck to understand precisely this phenomenon and explain how the cohomology of (smooth non proper) varieties could be described in terms of pure motives (with applications towards the Weil conjectures in mind).
In [27, p. 193, p. 237], Grothendieck considers the formalism of the six operations and the behaviour of weights under these operations as intimately related to its theory of motives which he sees as the ultimate goal of his cohomological program started with ℓ-adic cohomology.
In [29], he envisions the existence, for every separated k-scheme of finite type X, of a $\mathbb{Q}$-linear Abelian category $\mathcal{MM}(X)$ of mixed motives (a.k.a. mixed motivic sheaves) that realizes to constructible ℓ-adic sheaves (the realization functor being exact, faithful and symmetric monoidal) and such that their derived category are endowed with a formalism of the six operations compatible with the one developed by him and his school in ℓ-adic cohomology.
This vision has been developed and made more clear by Beĭlinson in [16, 5.10]. Note that in *loc. cit.*, the idea of a perverse variant of the constructible mixed motivic sheaves is added to the picture. Categories of perverse sheaves have better (algebraic) properties than the categories of constructibles sheaves. They are Noetherian and Artinian (hence every object has finite length), which make them more natural from the viewpoint of weight filtrations (see [17, Théorème 5.3.5] and the observation of Deligne in [24, 2.1]).
We refer to [38, Conjecture 4.8] for a detailed exposition of the properties that the conjectural category of mixed motives (denoted by $\mathscr{DM}(X)$ in [38]) is expected to have. Briefly, one hopes for:

1. a formalism of the six operations and a theory of nearby cycles;
2. realization functors such as Betti realization, Hodge realization, ℓ-adic realization and so on;
3. a close relation to Chow groups that should appear as Hom groups in the triangulated category;
4. the existence of a motivic t-structure (actually one hopes for two t-structures: the constructible one and its perverse variant).

After pioneering works by Levine, Hanamura and other mathematicians, motivic stable homotopy theory developed by Morel and Voevodsky (see [40, 45, 53]) has led to the best known candidate for the triangulated categories of mixed motives. These are the categories of étale motives with rational coefficients further studied by Ayoub in [6, 7, 10] (see [9] for an overview).

Let X be a quasi-projective k-scheme. The triangulated category of étale motives with rational coefficients $\mathbf{DA}^{\text{ét}}(X,\mathbb{Q})$ has been introduced in [6, 7], where it is a particular case of the category $\mathbf{SH}_{\mathfrak{M}}(X)$ obtained by choosing the topology to be the étale topology and the model category $\mathfrak{M}$ of coefficients to be the model category $\mathsf{Ch}(\mathbb{Q})$ of chain complexes of $\mathbb{Q}$-vector spaces (with the projective model structure). It is part of a stable homotopic 2-functor $\mathbf{DA}^{\text{ét}}(-,\mathbb{Q})$ on the category of quasi-projective k-schemes as defined in [7, Définition 2.4.13]. As shown by Ayoub in [6, 7], the categories $\mathbf{DA}^{\text{ét}}(X,\mathbb{Q})$ are endowed with a formalism of the six operations as envisioned by Grothendieck. More precisely, each of these categories is equipped with a tensor product $\otimes$ and an internal Hom and every morphism of quasi-projective k-schemes $f : X \to Y$ induces four triangulated functors

$$\begin{tikzcd} \mathbf{DA}^{\text{ét}}(X,\mathbb{Q}) \arrow[r, shift right, "f_*"'] & \mathbf{DA}^{\text{ét}}(Y,\mathbb{Q}) \arrow[l, shift right, "f^*"'] \arrow[r, shift left, "f_!"] & \mathbf{DA}^{\text{ét}}(X,\mathbb{Q}). \arrow[l, shift left, "f^!"] \end{tikzcd}$$

These six functors satisfy the usual compatibilities and adjunctions. The theory developed in [6] provides also nearby cycles functors.

The category of constructible motives $\mathbf{DA}^{\text{ét}}_{\mathsf{ct}}(X,\mathbb{Q})$ is defined as the smallest strictly full triangulated subcategory of $\mathbf{DA}^{\text{ét}}(X,\mathbb{Q})$ stable by direct factors, Tate twists, and containing the homological motives of smooth quasi-projective X-schemes. As shown in [7, Scholie 2.2.34] these categories of constructible motives are stable under the six operations (and the nearby cycles functors).

With a scheme $Y \in \mathrm{Sm}/X$ is associated a homological motive $\mathrm{M}_X(Y)$ given by the T_X-spectrum $\mathrm{Sus}^0_{T_X}(X \otimes \mathbb{Q})$. In terms of the six operations, it is given by

$$\mathrm{M}_X(Y) = a_! a^! \mathbb{1}_X$$

where $a : Y \to X$ is the structural morphism. Its dual, given by

$$\mathrm{M}^\vee_X(Y) = a_* a^* \mathbb{1}_X,$$

is called the cohomological motive of Y.

The category $\mathbf{DA}^{\text{ét}}_{\mathsf{ct}}(k,\mathbb{Q})$ is equivalent to the triangulated category of geometric motives $\mathrm{DM}_{gm}(k,\mathbb{Q})$ considered by Voevodsky in [54] and built out of finite correspondences. This result is a particular case of [21, Corollary 16.2.22] (see also [10, Théorème B.1]). As a consequence, the work of Voevodsky, Friedlander and Suslin (see [55]), provides a canonical isomorphism

$$\mathrm{CH}^i(X,\mathbb{Q}) \simeq \mathrm{Hom}_{\mathbf{DA}^{\text{ét}}(k,\mathbb{Q})}(\mathbb{1}_k, \mathrm{M}_k^\vee(X)(i)[2i])$$

for every smooth quasi-projective k-scheme and every integer $i \in \mathbb{Z}$. These isomorphisms induce a $\mathbb{Q}$-linear fully faithful symmetric monoidal functor

$$\begin{aligned}\mathbf{Mrat}(k,\mathbb{Q}) &\to \mathbf{DA}^{\text{ét}}_{\mathsf{ct}}(k,\mathbb{Q})\\ h(X) &\mapsto \mathrm{M}_k^\vee(X).\end{aligned}$$

Various realization functors have been constructed [8, 10, 32, 34, 35]. The Betti realization has been constructed in [8]. It takes the form of an adjunction

$$\mathsf{Bti}_X^* : \mathbf{DA}^{\text{ét}}(X,\mathbb{Q}) \rightleftarrows \mathrm{D}(X^{\mathrm{an}},\mathbb{Q}) : \mathsf{Bti}_{X*}$$

the right hand side being the unbounded derived category of the Abelian category of sheaves of $\mathbb{Q}$-vector spaces over the analytic space X^{an}.
Moreover the functor Bti_X^* is a symmetric monoidal functor and induces a functor

$$\mathsf{Bti}_X^* : \mathbf{DA}^{\text{ét}}_{\mathsf{ct}}(X,\mathbb{Q}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X^{\mathrm{an}},\mathbb{Q})$$

which is compatible with the six operations. Here, the right-hand side $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X^{\mathrm{an}},\mathbb{Q})$ denotes the triangulated subcategory of $\mathrm{D}(X^{\mathrm{an}},\mathbb{Q})$ formed by the bounded complexes of sheaves with algebraically constructible cohomology sheaves.

Remark 4.2. — Note that to get a Betti realization in the form of an adjunction, it is necessary to use the big categories of motives and not only the smaller category of constructible motives (geometric motives). This point turns out to be very important in the weak Tannakian formalism developed by Ayoub.

For $X = \mathrm{Spec}(k)$, the Betti realization provides an adjunction

$$\mathsf{Bti}^* : \mathbf{DA}^{\text{ét}}(k,\mathbb{Q}) \rightleftarrows \mathrm{D}(\mathbb{Q}) : \mathsf{Bti}_*$$

where the right handside is the derived category of $\mathbb{Q}$-vector spaces and induces a symmetric monoidal functor

$$\mathsf{Bti}^* : \mathbf{DA}^{\text{ét}}_{\mathsf{ct}}(k,\mathbb{Q}) \to \mathrm{D}^{\mathrm{b}}\mathbf{vec}(\mathbb{Q}) \tag{27}$$

Remark 4.3. — Note that the derived category $\mathrm{D}^{\mathrm{b}}\mathbf{vec}(\mathbb{Q})$ is canonically equivalent to $\mathbf{vec}(\mathbb{Q})^{\mathbb{Z}}$ and that Bti^* is an extension of the Betti realization for Chow motives (22).

Hence, the categories of étale motives with rational coefficients satisfy the expected properties except for the existence of a motivic t-structure which remains one of the main conjectures of the theory:

Conjecture 4.4. — *There exists on the triangulated category $\mathbf{DA}^{ét}_{\mathsf{ct}}(k,\mathbb{Q})$ a non-degenerate t-structure such that the tensor product and the realization functor*

$$\mathsf{Bti}^* : \mathbf{DA}^{ét}_{\mathsf{ct}}(k,\mathbb{Q}) \to \mathrm{D}^{\mathrm{b}}\mathbf{vec}(\mathbb{Q})$$

are t-exact.

As shown by Beĭlinson in [15], the previous conjecture implies the standard conjectures. Let $\mathcal{MM}(k)$ be the heart of such a t-structure. The conjecture also implies that $\mathcal{MM}(k)$ is a neutral Tannakian category with the induced functor

$$\mathsf{Bti}^* : \mathcal{MM}(k) \to \mathbf{vec}(\mathbb{Q})$$

as fiber functor (in particular it is faithful). In particular, one could then apply the classical Tannakian duality in the form described above to this functor and develop a motivic Galois theory for smooth k-varieties.
The conjecture also implies the so-called conservativity conjecture:

Conjecture 4.5. — *The realization functor*

$$\mathsf{Bti}^* : \mathbf{DA}^{ét}_{\mathsf{ct}}(k, \mathbb{Q}) \to \mathrm{D}^{\mathrm{b}}\mathbf{vec}(\mathbb{Q})$$

is conservative, that is reflects isomorphisms.

The conservativity of the Betti realization is among the deepest conjectures in the theory of motives and implies many results in particular some deep existence statement in the theory of algebraic cycles.
The expected relation between the motivic t-structure of Conjecture 4.4 and the category of Grothendieck's motives (aka. homological motives) can be mode more clear as follows. Here we assume standard conjectures C and D to be true. One expects the existence of a $\mathbb{Q}$-linear fully faithful symmetric monoidal functor

$$\dot{\mathbf{Mhom}}(k, \mathbb{Q}) \to \mathcal{MM}(k) \tag{28}$$

such that the squares are commutative (up to isomorphisms of functors)

$$\begin{tikzcd} \mathbf{Mrat}(k,\mathbb{Q}) \arrow[r] \arrow[d] & \mathbf{DA}^{ét}_{\mathsf{ct}}(k,\mathbb{Q}) \arrow[d, "\oplus_{i\in\mathbb{Z}} H^i"] \\ \dot{\mathbf{Mhom}}(k,\mathbb{Q}) \arrow[r] & \mathcal{MM}(k) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Mrat}(k,\mathbb{Q}) \arrow[r] \arrow[d, "h^i"] & \mathbf{DA}^{ét}_{\mathsf{ct}}(k,\mathbb{Q}) \arrow[d, "H^i"] \\ \dot{\mathbf{Mhom}}(k,\mathbb{Q}) \arrow[r] & \mathcal{MM}(k). \end{tikzcd}$$

Note that, in the square on the left, both vertical functors are monoidal but not symmetric monoidal. Moreover, the essential image of (28) should be the strictly full subcategory of $\mathcal{MM}(k)$ formed by the pure (semi-simple) motives.

4.3. Étale motives: sketch of the construction. — We end this section, by sketching the construction of the category of étale motives over a quasi-projective k-scheme X. For model categories introduced by Quillen in [46] we refer e.g. to [30, 31].
Let X be a quasi-projective k-scheme. Let Sm/X be the category of smooth quasi-projective X-schemes. The starting point of the construction is the category of presheaves of $\mathbb{Q}$-vector spaces $\mathbf{PSh}(\mathrm{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ endowed with its projective model structure: the fibrations (resp. equivalences) are the maps of presheaves of complexes $\mathscr{X} \to \mathscr{Y}$ such that $\mathscr{X}(Y) \to \mathscr{Y}(Y)$ is a epimorphism (resp. a quasi-isomorphism) in $\mathsf{Ch}(\mathbb{Q})$ for every $Y \in \mathrm{Sm}/X$.
By left Bousfield localization one obtains the ét-local model structure. For the ét-local structure, the weak equivalences are the morphisms of complexes of presheaves that induce isomorphisms on the étale sheafification of the homology presheaves.
The ét-local model structure is then further localized with respect to the class of maps

$$\mathbb{A}^1_Y \otimes \mathbb{Q} \to Y \otimes \mathbb{Q}$$

where $Y \in \mathrm{Sm}/X$. The left Bousfield localization of the ét-local model structure with respect to the above maps is called the $(\mathbb{A}^1, \text{ét})$-local projective model structure. Its homotopy category

$$\mathbf{DA}^{\text{eff},\text{ét}}(X, \mathbb{Q}) := \mathsf{Ho}_{\mathbb{A}^1,\text{ét}}(\mathbf{PSh}(\mathrm{Sm}/X, \mathsf{Ch}(\mathbb{Q})))$$

is called the category of effective étale motives (with rational coefficients).
The last step of the construction is the stabilization. Let T_X be the presheaf

$$T_X := \frac{\mathbf{G}_{m,X} \otimes \mathbb{Q}}{1_X \otimes \mathbb{Q}}.$$

Consider the category $\mathbf{Spt}_{T_X}(\mathbf{PSh}(\mathrm{Sm}/X, \mathsf{Ch}(\mathbb{Q})))$ of T_X-spectra of presheaves of complexes of $\mathbb{Q}$-vector spaces (see [6, Définition 4.3.6]). The $(\mathbb{A}^1, \text{ét})$-local projective model structure induces on it a model structure (see [6, Définition 4.3.29]): the so-called $(\mathbb{A}^1, \text{ét})$-local stable projective model structure. Its homotopy category

$$\mathbf{DA}^{\text{ét}}(X, \mathbb{Q}) := \mathsf{Ho}_{(\mathbb{A}^1,\text{ét})-\mathrm{st}}(\mathbf{Spt}_{T_X}(\mathbf{PSh}(\mathrm{Sm}/X, \mathsf{Ch}(\mathbb{Q}))))$$

is the triangulated category of étale motives with rational coefficients.

5. Quiver representations and Tannaka duality

5.1. Coalgebra associated with a quiver representation. — Let Λ be a field. The theory extends with very little changes to Dedekind rings. Note that over more general Noetherian rings, a part of it still holds but one looses the relation with coalgebras (see [33]).
Let $\mathcal{Q}$ be a quiver and

$$T : \mathcal{Q} \to \mathbf{vec}(\Lambda)$$

be a representation of $\mathcal{Q}$. Recall that a quiver is simply a directed graph or equivalently a collection of vertices and for every vertices $p, q \in \mathcal{Q}$ a collection $\mathcal{Q}(p, q)$ of edges (Nori uses the terminology diagram). With such a representation, Nori associates a counitary Λ-coalgebra $H(T)$.
Let us explain its construction. It is done in two steps: one first look at the case where the quiver $\mathcal{Q}$ has only finitely many objects and then look at the general case by a colimit argument. The starting point is the ring of endomorphisms of the representation.

Definition 5.1. — The ring $\mathrm{End}_\Lambda(T)$ of endomorphisms of T is the subring of

$$\prod_{q \in \mathcal{Q}} \mathrm{End}_\Lambda(T(q)) \tag{29}$$

formed by the elements $e = (e_q)_{q \in \mathcal{Q}}$ such that for every objects $p, q \in \mathcal{Q}$ and every morphism $m \in \mathcal{Q}(p, q)$ the square

$$\begin{tikzcd} T(p) \arrow[r, "T(m)"] \arrow[d, "e_p"] & T(q) \arrow[d, "e_q"] \\ T(p) \arrow[r, "T(m)"] & T(q) \end{tikzcd}$$

is commutative.

Assume that $\mathcal{Q}$ has finitely many objects. By assumption, the product in (29) is finite and thus $\mathrm{End}_\Lambda(T)$ is a finite Λ-algebra. Its Λ-linear dual

$$H(T) := \mathrm{End}_\Lambda(T)^\vee$$

is therefore a counitary Λ-coalgebra. For every object $q \in \mathcal{Q}$, the finite dimensional Λ-vector space $T(q)$ has a natural structure of left $\mathrm{End}_\Lambda(T)$-module via the projection

$$\mathrm{End}_\Lambda(T) \to \mathrm{End}_\Lambda(T(q))$$

and thus a structure of counitary right $H(T)$-comodule. This shows that the representation T may be lifted to a representation

$$\overline{T} : \mathcal{Q} \to \mathbf{comod}(H(T))$$

by simply viewing $T(q)$ as a counitary right $H(T)$-comodule.
Let now $\mathcal{Q}$ be a quiver which may have infinitely many objects. Consider for every finite full sub-quiver $\mathcal{Q}' \subseteq \mathcal{Q}$, the induced representation

$$T|_{\mathcal{Q}'} : \mathcal{Q}' \to \mathbf{vec}(\Lambda)$$

and the associated coalgebra $H(T|_{\mathcal{Q}'})$. If $\mathcal{Q}''$ is a finite full subquiver of $\mathcal{Q}$ that contains $\mathcal{Q}'$, the inclusion $\mathcal{Q}' \subseteq \mathcal{Q}''$ induces by projection a morphism of Λ-algebras

$$\prod_{q \in \mathcal{Q}''} \mathrm{End}_\Lambda(T(q)) \to \prod_{q \in \mathcal{Q}'} \mathrm{End}_\Lambda(T(q))$$

which induces a morphism of unitary Λ-algebras $\mathrm{End}_\Lambda(T|_{\mathcal{Q}''}) \to \mathrm{End}_\Lambda(T_{\mathcal{Q}'})$. This provides a morphism of counitary Λ-coalgebras $H(T|_{\mathcal{Q}'}) \to H(T|_{\mathcal{Q}''})$. The counitary Λ-coalgebra associated by Nori with the representation $T : \mathcal{Q} \to \mathbf{mod}(\Lambda)$ is obtained by taking the colimit over all finite full sub-quivers of $\mathcal{Q}$:

$$H(T) := \mathrm{colim}_{\mathcal{Q}' \subseteq \mathcal{Q}} H(T|_{\mathcal{Q}'}).$$

For every object $q \in \mathcal{Q}$, the Λ-vector space $T(q)$ inherits a structure of counitary right $H(T)$-comodule and the representation T factors as a representation

$$\overline{T} : \mathcal{Q} \to \mathbf{comod}(H(T))$$

via the forgetful functor $\mathbf{comod}(H(T)) \to \mathbf{vec}(\Lambda)$ which is Λ-linear exact and faithful.
Now let us consider the functoriality of this construction. Let $T_1 : \mathcal{Q}_1 \to \mathbf{vec}(\Lambda)$ and $T_2 : \mathcal{Q}_2 \to \mathbf{vec}(\Lambda)$ be two representations of quivers. Let $Q : \mathcal{Q}_1 \to \mathcal{Q}_2$ be a morphism of quivers, and $\alpha : T_2 \circ Q \to T_1$ be an invertible 2-morphism. Assume first that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have finitely many objects. Consider the morphism of rings:

$$\prod_{q_2 \in \mathcal{Q}_2} \mathrm{End}_\Lambda(T_2(q_2)) \xrightarrow{\Pi_\alpha} \prod_{q_1 \in \mathcal{Q}_1} \mathrm{End}_\Lambda(T_1(q_1)) \tag{30}$$

where the map Π_α is defined for every $e = (e_{q_2})_{q_2 \in \mathcal{Q}_2}$ by

$$\Pi_\alpha(e) = (\alpha \cdot e_{Q(q_1)} \cdot \alpha^{-1})_{q_1 \in \mathcal{Q}_1}.$$

If for every objects $p_2, q_2 \in \mathcal{Q}_2$ and every morphism $m_2 \in \mathcal{Q}_2(p_2, q_2)$ the square

$$\begin{tikzcd} T_2(p_2) \arrow[r, "T_2(m_2)"] \arrow[d, "e_{p_2}"] & T_2(q_2) \arrow[d, "e_{q_2}"] \\ T_2(p_2) \arrow[r, "T_2(m_2)"] & T_2(q_2) \end{tikzcd}$$

is commutative, then in particular for every objects $p_1, q_1 \in \mathcal{Q}_1$ and every morphism $m_1 \in \mathcal{Q}_1(p_1, q_1)$ the square

$$\begin{tikzcd} T_1(p_1) \arrow[r, "T_1(m_1)"] \arrow[d, "\Pi_\alpha(e)_{p_1}"] & T_1(q_1) \arrow[d, "\Pi_\alpha(e)_{q_1}"] \\ T_1(p_1) \arrow[r, "T_1(m_1)"] & T_1(q_1) \end{tikzcd}$$

is commutative. This shows that the morphism of rings (30) induces a morphism of Λ-algebras $\mathrm{End}_\Lambda(T_2) \to \mathrm{End}_\Lambda(T_1)$ and thus a morphism of counitary Λ-coalgebras $H(T_1) \to H(T_2)$. If $\mathcal{Q}_1$ and $\mathcal{Q}_2$ do not have finitely many objects, then for any finite sub-quivers $\mathcal{Q}_1' \subseteq \mathcal{Q}_1$ and $\mathcal{Q}_2' \subseteq \mathcal{Q}_2$ such that $Q(\mathcal{Q}_1') \subseteq \mathcal{Q}_2'$ one has a morphism of counitary Λ-coalgebras

$$H(T_1|_{\mathcal{Q}_1'}) \to H(T_2|_{\mathcal{Q}_2'}).$$

By taking the colimit one obtains a morphism of counitary Λ-coalgebras $H(T_1) \to H(T_2)$. This morphism induces by corestriction a Λ-linear exact functor

$$\mathbf{comod}(H(T_1)) \to \mathbf{comod}(H(T_2)). \tag{31}$$

The invertible 2-morphism α lifts and provides an invertible natural transformation $\overline{\alpha} : \overline{T}_2 \circ Q \Rightarrow (31) \circ \overline{T}_1$. The diagram

$$\begin{tikzcd} \mathcal{Q}_1 \arrow[r, "Q"] \arrow[d, "\overline{T}_1"'] & \mathcal{Q}_2 \arrow[d, "\overline{T}_2"] \\ \mathbf{comod}(H(T_1)) \arrow[r, "(31)"] & \mathbf{comod}(H(T_2)) \end{tikzcd}$$

is commutative up to the invertible natural transformation $\overline{\alpha}$.

5.2. Relation with classical Tannaka duality. — Let $\mathcal{A}$ be a Λ-linear Abelian category and $\omega : \mathcal{A} \to \mathbf{vec}(\Lambda)$ be a Λ-linear faithful exact functor. With ω we can associate two different counitary Λ-algebras. The first one is obtained via the classical Tannaka duality and is defined by

$$C(\omega) = \mathrm{colim}_{X \in \mathrm{ob}(\mathcal{A})} \mathrm{End}(\omega|_{\langle X \rangle})^\vee$$

where the colimit is taken over the directed poset of isomorphism classes of objects in $\mathcal{A}$. The other one $H(\omega)$ is obtained via Nori's construction by viewing $\mathcal{A}$ simply as a quiver (that is we forget about the composition in $\mathcal{A}$).

We now compare these two coalgebras (see Proposition 5.2). Pick a finite full subquiver $\mathcal{Q}$ of $\mathcal{A}$. Let $X_1, \dots, X_n$ be the vertices in $\mathcal{Q}$. Let $\langle X_1, \dots, X_n \rangle$ be the strictly full Abelian subcategory of $\mathcal{A}$ generated by $X_1, \dots, X_n$. One has the inclusions

$$\mathcal{Q} \subseteq \langle X_1, \dots, X_n \rangle \subseteq \langle X_1 \oplus \cdots \oplus X_n \rangle.$$

They induce, by restriction, a morphism of finite Λ-algebras

$$\mathrm{End}(\omega|_{\langle X_1\oplus\cdots\oplus X_n\rangle}) \to \mathrm{End}(\omega|_{\mathcal{Q}}) \tag{32}$$

and therefore a morphism of counitary Λ-coalgebras

$$\mathrm{End}(\omega|_{\mathcal{Q}})^\vee \to \mathrm{End}(\omega|_{\langle X_1\oplus\cdots\oplus X_n\rangle})^\vee \to \mathrm{colim}_{X\in\mathrm{ob}(\mathcal{A})}\,\mathrm{End}(\omega|_{\langle X\rangle})^\vee =: C(\omega).$$

By enlarging the finite subquiver $\mathcal{Q}$, one obtains eventually a canonical morphism of Λ-coalgebras:

$$H(\omega) := \mathrm{colim}_{\mathcal{Q}\subseteq\mathcal{A}}\,\mathrm{End}(\omega|_{\mathcal{A}})^\vee \to \mathrm{colim}_{X\in\mathrm{ob}(\mathcal{A})}\,\mathrm{End}(\omega|_{\langle X\rangle})^\vee =: C(\omega). \tag{33}$$

Proposition 5.2. — *The canonical morphism* (33) *is an isomorphism of counitary Λ-coalgebras.*

Note that there is no reason for the morphism (32) to be already an isomorphism in general. However, with the help of Lemma 3.11, the proposition is not difficult to obtain. Indeed, given an object X in $\mathcal{A}$, it ensures that the functor $\omega|_{\langle X\rangle}$ is representable by a pair (P,p) where P is an object in $\mathcal{A}$ and p is an element in the vector space $\omega(P)$. Consequently, the morphism

$$\mathrm{End}(\omega|_{\langle X\rangle}) \to \mathrm{End}(\omega|_{\mathcal{Q}}),$$

induced by the inclusion of the quiver $\mathcal{Q} = \{P\}$ in the category $\langle X\rangle$ generated by X, is an isomorphism. This implies that (33) is an isomorphism.

5.3. Universal property. — The following theorem is the heart of Nori's approach to Tannaka duality (see [26, Proposition 1.10] and [33, Theorem 6.1.19]).

Theorem 5.3. — *Let $\mathcal{A}$ be a Λ-linear Abelian category and $\omega : \mathcal{A} \to \mathbf{vec}(\Lambda)$ be a Λ-linear exact faithful functor. Then, the representation*

$$\overline{\omega} : \mathcal{A} \to \mathbf{comod}(H(\omega))$$

is a Λ-linear functor and an equivalence of categories.

Proof. — Under our assumptions, Theorem 5.3 can be deduced from Theorem 3.13. Indeed the corestriction functor associated with the morphism of counitary Λ-coalgebras (33) fits into a commutative diagram

$$\begin{tikzcd}
 & \mathbf{comod}(H(\omega)) \arrow[d] \\
\mathcal{A} \arrow[ur, "\overline{\omega}"] \arrow[dr, "(18)"'] & \\
 & \mathbf{comod}(C(\omega)).
\end{tikzcd}$$

Therefore, it is enough to know that (18) is an equivalence (Theorem 3.13) and that the same is true for the corestriction (Proposition 5.2). □

Let us come back to the more general setting where $\mathcal{Q}$ is a quiver and $T : \mathcal{Q} \to \mathbf{vec}(\Lambda)$ is a representation. As seen before, the representation T can be factorized

$$\begin{tikzcd} \mathcal{Q} \arrow[r, "\overline{T}"] \arrow[dr, bend right, "T"'] & \mathbf{comod}(H(T)) \arrow[d, "\text{forgetful}"] \\ & \mathbf{vec}(\Lambda). \end{tikzcd}$$

Then, Theorem 5.3 implies that the category $\mathbf{comod}(H(T))$ satisfies a universal property. We refer to [36] for a precise formulation of this universal property (see also [26, Theorem 1.6] and [33, Theorem 6.1.13]).

Theorem 5.4. — *The factorization*

$$\begin{tikzcd} \mathcal{Q} \arrow[r, "\overline{T}"] \arrow[dr, bend right, "T"'] & \mathbf{comod}(H(T)) \arrow[d, "\textit{forgetful}"] \\ & \mathbf{vec}(\Lambda) \end{tikzcd}$$

is universal among all factorizations

$$\begin{tikzcd} \mathcal{Q} \arrow[r, "R"] \arrow[dr, bend right, "T"'] & \mathcal{A} \arrow[d, "F"] \\ & \mathbf{vec}(\Lambda) \end{tikzcd}$$

where $\mathcal{A}$ is a Λ-linear Abelian category, R is a representation and F a Λ-linear faithful exact functor.

Proof. — Instead of giving a detailed and complete proof of the theorem, we only sketch the construction of a functor $G : \mathbf{comod}(H(T)) \to \mathcal{A}$. Forget about the composition in the category $\mathcal{A}$, and consider it simply as a quiver. Using the functoriality of the construction, we get a canonical morphism of counitary Λ-coalgebras

$$H(T) \to H(F).$$

(With the notation used before, we set $\mathcal{Q}_1 = \mathcal{Q}$, $\mathcal{Q}_2 = \mathcal{A}$, $Q = R$, $T_1 = T$ and $T_2 = F$). By corestriction, we get a functor

$$\mathbf{comod}(H(T)) \to \mathbf{comod}(H(F))$$

but the canonical functor

$$\mathcal{A} \to \mathbf{comod}(H(F))$$

is an equivalence by Theorem 5.3. The choice of a quasi-inverse defines G. □

Note that the Abelian category $\mathbf{comod}(H(T))$ is determined up to an equivalence (unique up to a unique isomorphism of functors) by the universal property stated in Theorem 5.4.

5.4. Application to the theory of motives. — The starting point of the construction of the Abelian category of Nori's mixed motives is the quiver $\mathcal{D}$ defined as follows. A vertex in $\mathcal{D}$ is a triple (Y, Z, i) where Y is a quasi-projective k-scheme, Z is a closed subsecheme of X and $i \in \mathbb{Z}$ is an integer. Vertices are linked by two types of edges: edges of functoriality type and edges of boundary type.

– If (Y_1, Z_1, i) and (Y_2, Z_2, i) are vertices, then every morphism of k-schemes $f : Y_2 \to Y_1$ such that$f(Z_2) \subseteq Z_1$ defines an edge

$$(Y_2, Z_2, i) \to (Y_1, Z_1, i). \tag{34}$$

– If (Y, Z, i) is a vertex, then every closed subscheme $W \subseteq Z$ defines an edge

$$(Y, Z, i) \to (Z, W, i-1). \tag{35}$$

With a vertex (Y, Z, i), we can associate the relative Betti homology group $H_i(Y, Z, \mathbb{Q})$ which is a finite dimensional $\mathbb{Q}$-vector space. We may send the edge (34) to the functoriality morphism

$$H_i(Y_2, Z_2, \mathbb{Q}) \to H_i(Y_1, Z_1, \mathbb{Q}) \tag{36}$$

and the edge (35) to the boundary morphism

$$H_i(Y, Z, \mathbb{Q}) \to H_{i-1}(Z, W, \mathbb{Q}). \tag{37}$$

This provides a representation

$$T : \mathcal{D} \to \mathbf{vec}(\mathbb{Q})$$

of the quiver $\mathcal{D}$ and Nori's Abelian category of effective homological motives $\mathrm{EHM}(k, \mathbb{Q})$ is the category of finite dimensional counitary right $H(T)$-comodules. By construction every vertex (Y, Z, i) defines a motive $\overline{H}_i(Y, Z)$ in $\mathrm{EHM}(k, \mathbb{Q})$. Its underlying $\mathbb{Q}$-vector space is $H_i(Y, Z, \mathbb{Q})$ and the morphisms (36), (37) come from morphisms in $\mathrm{EHM}(k, \mathbb{Q})$.
The existence of a symmetric monoidal structure (essentially obtained from the Künneth formula) on the category of effective Nori mixed motives implies the following proposition (see [33, Theorem 9.1.5] or [20, p. 478]).

Proposition 5.5. — *The coalgebra $\mathbf{H}^{\mathrm{eff}}_{\mathrm{Nori}}(k, \sigma, \mathbb{Q}) := H(T)$ is a commutative biunitary bialgebra.*

By inverting the Tate twist, one get the Abelian category of mixed motives $\mathrm{NMM}(k, \mathbb{Q})$. One can show that this is the category of finite dimensional counitary comodules over a Hopf algebra $\mathbf{H}_{\mathrm{Nori}}(k, \sigma, \mathbb{Q})$. It defines an affine group scheme over $\mathbb{Q}$

$$\mathbf{G}_{\mathrm{Nori}}(k, \sigma, \mathbb{Q})$$

which is the motivic Galois group of Nori. Mixed Hodge theory ensures that the representation T has a factorization via the category $\mathrm{MHS}^p_{\mathbb{Q}}$ of polarizable mixed Hodge structure. The universal property of Theorem 1.5 show that there exists a canonical $\mathbb{Q}$-linear exact faithful functor

$$\mathrm{EHM}(k, \mathbb{Q}) \to \mathrm{MHS}^p_{\mathbb{Q}}.$$

This functor extends to a $\mathbb{Q}$-linear exact faithful functor

$$\mathrm{NMM}(k, \mathbb{Q}) \to \mathrm{MHS}^p_{\mathbb{Q}}.$$

Remark 5.6. — Let S be a quasi-projective k-scheme. As shown in [36], using a finite dimensional variant of a result of Takeuchi (see [51]), it is possible to develop a relative version over S of Nori's category of mixed motives modeled after perverse sheaves.

6. The weak Tannakian formalism

We give the main results of the weak Tannakian formalism introduced by Ayoub in [11] which plays a central role in Ayoub's definition of the motivic Galois group and in his work [12, 14]. For the construction of étale motives (see [6, 10] or the brief recollection in Section 4.2). The Betti realization has been constructed in [8].

6.1. Bialgebra associated with a monoidal adjunction with section. — Let us first make clear the assumptions needed to develop the weak Tannakian formalism of [11]. The starting points are the data of two symmetric monoidal categories $(\mathcal{M}, \otimes, \mathbb{1})$, $(\mathcal{E}, \otimes, \mathbb{1})$ and a symmetric monoidal functor

$$f : \mathcal{M} \to \mathcal{E}$$

that admits a right adjoint $g : \mathcal{E} \to \mathcal{M}$. Let η and δ be respectively the unit and the counit of the adjunction.
Given $A \in \mathcal{E}$ and $M \in \mathcal{M}$, we may consider the morphism of coprojection

$$c_d : g(A) \otimes M \to g(A \otimes f(M))$$

given by the composition

$$c_d : g(A) \otimes M \xrightarrow{\mathrm{Id} \otimes \eta} g(A) \otimes gf(M) \to g(A \otimes f(M))$$

where the second morphism is the one in Lemma 2.4.

Assumption 6.1. —

1. *There exists a* monoidal *functor $e : \mathcal{E} \to \mathcal{M}$ and an isomorphism of monoidal functors $fe \simeq \mathrm{Id}_{\mathcal{E}}$.*

2. *The morphism of coprojection*
$$c_d : g(A) \otimes e(B) \to g(A \otimes fe(B))$$
is an isomorphism for every objects $A, B \in \mathcal{E}$.

Note that the unit object $\mathbb{1}$ of $\mathcal{E}$ is a commutative unitary algebra, applying Lemma 2.8 to the pseudo-monoidal functor g provides $g(\mathbb{1})$ with a structure of commutative unitary algebra in $\mathcal{M}$. The multiplication is the morphism

$$g(\mathbb{1}) \otimes g(\mathbb{1}) \to g(\mathbb{1} \otimes \mathbb{1}) \xrightarrow{g(m)} g(\mathbb{1})$$

and the unit is the morphism $\mathbb{1} \to g(\mathbb{1})$. Again by Lemma 2.8, the object $\mathrm{H} := fg(\mathbb{1})$ in $\mathcal{E}$ has a natural structure of commutative unitary algebra with multiplication given by the morphism

$$\mathrm{H} \otimes \mathrm{H} = fg(\mathbb{1}) \otimes fg(\mathbb{1}) \to f(g(\mathbb{1}) \otimes g(\mathbb{1})) \to fg(\mathbb{1} \otimes \mathbb{1}) \to fg(\mathbb{1}) = \mathrm{H}$$

and unit given by the morphism $\mathbb{1} \to f(\mathbb{1}) \to fg(\mathbb{1}) = \mathrm{H}$.

The main point is to use Assumption 6.1 to define a comultiplication on H compatible with the previous algebra structure.
Let $A, B \in \mathcal{E}$. Under the assumption, we can define an isomorphism in $\mathcal{M}$

$$\theta_{A,B} : g(A \otimes B) \to g(A) \otimes e(B)$$

by identifying B with $fe(B)$ and taking the inverse of the coprojection morphism.
Let us now precise part of the statement of Theorem 1.6. Remark that e being a section of the monoidal functor f, we have, for every objects $M \in \mathcal{M}$ and $A \in \mathcal{E}$, a canonical isomorphism

$$\vartheta_{M,A} : f(M \otimes e(A)) \to f(M) \otimes fe(A) \simeq f(M) \otimes A.$$

Theorem 6.2. — *Assume that Assumption 6.1 is satisfied. Then, the morphism*

$$cm : \mathrm{H} \xrightarrow{\eta} fg(\mathrm{H}) \xrightarrow{\theta_{\mathbb{1},\mathrm{H}}} f(g(\mathbb{1}) \otimes e(\mathrm{H})) \xrightarrow{\vartheta_{g(\mathbb{1}),\mathrm{H}}} \mathrm{H} \otimes \mathrm{H}$$

defines a structure of commutative biunitary bialgebra over the unitary algebra $\mathrm{H} = fg(\mathbb{1})$ *in* $\mathcal{E}$*. The counit of this coalgebra is the counit of the adjunction* $\delta : \mathrm{H} := fg(\mathbb{1}) \to \mathbb{1}$*.*

The proof of Theorem 6.2 relies on Proposition 6.3. To state this proposition, let us first remark that, given an object A in $\mathcal{E}$, there is an isomorphim in $\mathcal{E}$

$$fg(A) \xrightarrow{\simeq} \mathrm{H} \otimes A$$

obtained via the isomorphisms

$$\mathrm{H} \otimes A \xleftarrow{\vartheta_{g(\mathbb{1}),A}} f(g(\mathbb{1}) \otimes e(A)) \xrightarrow{c_d} fg(\mathbb{1} \otimes fe(A)) \simeq fg(\mathbb{1} \otimes A) \simeq fg(A).$$

If B is an object in $\mathcal{E}$, the adjunction isomorphism

$$\mathrm{Hom}_{\mathcal{M}}(g(A), g(B)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{E}}(fg(A), B)$$

yields therefore an isomorphism

$$\mathrm{Hom}_{\mathcal{M}}(g(A), g(B)) \simeq \mathrm{Hom}_{\mathcal{E}}(\mathrm{H} \otimes A, B). \tag{38}$$

Proposition 6.3. — *Assume that Assumption 6.1 is satisfied.*

1. *Let* A, B, C *be objects in* $\mathcal{E}$*, and* $a \in \mathrm{Hom}_{\mathcal{M}}(g(A), g(B))$ *and* $b \in \mathrm{Hom}_{\mathcal{M}}(g(B), g(C))$ *be morphisms. The composition* $b \circ a$ *corresponds via the isomorphism* (38) *to the composition*

$$\mathrm{H} \otimes A \xrightarrow{cm\otimes\mathrm{Id}} (\mathrm{H} \otimes \mathrm{H}) \otimes A \xrightarrow{\mathrm{Id}\otimes\overline{a}} \mathrm{H} \otimes B \xrightarrow{\overline{b}} C$$

where $\overline{a}$ *and* $\overline{b}$ *are the morphisms corresponding to* a *and* b *via* (38)*.*

2. *Let* A *be an object in* $\mathcal{E}$*. Then the identity of* $g(A)$ *corresponds via the isomorphism* (38) *to the composition*

$$\mathrm{H} \otimes A \xrightarrow{\delta\otimes\mathrm{Id}} \mathbb{1} \otimes A \xrightarrow{\simeq} A$$

We refer to [11, Proposition 1.22] for the proof of the proposition.

Sketch of proof of Theorem 6.2. — Let us briefly explain why the morphism cm given in Theorem 6.2 defines on H a structure of counitary coalgebra. Let A, B, C, D be objects in $\mathcal{E}$. Let $\overline{a} : \mathrm{H} \otimes A \to B$, $\overline{b} : \mathrm{H} \otimes B \to C$ and $\overline{c} : \mathrm{H} \otimes C \to D$ be morphisms in $\mathcal{E}$. By Proposition 6.3, the morphism

$$\begin{tikzcd} \mathrm{H} \otimes A \arrow[r, "cm\otimes\mathrm{Id}"] & \mathrm{H} \otimes \mathrm{H} \otimes A \arrow[r, "\mathrm{Id}\otimes cm\otimes\mathrm{Id}"] & \mathrm{H} \otimes \mathrm{H} \otimes \mathrm{H} \otimes A \arrow[d, "\mathrm{Id}\otimes\mathrm{Id}\otimes\overline{a}"] \\ D & \mathrm{H} \otimes C \arrow[l, "\overline{c}"'] & \mathrm{H} \otimes \mathrm{H} \otimes B \arrow[l, "\mathrm{Id}\otimes\overline{b}"'] \end{tikzcd} \tag{39}$$

corresponds via the isomorphism (38) to the composition $c \circ (b \circ a)$ of the morphisms a, b, c associated with $\overline{a}$, $\overline{b}$, $\overline{c}$ via (38).
Similary the morphism

$$\begin{tikzcd} \mathrm{H} \otimes A \arrow[r, "cm\otimes\mathrm{Id}"] & \mathrm{H} \otimes \mathrm{H} \otimes A \arrow[r, "cm\otimes\mathrm{Id}\otimes\mathrm{Id}"] & \mathrm{H} \otimes \mathrm{H} \otimes \mathrm{H} \otimes A \arrow[d, "\mathrm{Id}\otimes\mathrm{Id}\otimes\overline{a}"] \\ D & \mathrm{H} \otimes C \arrow[l, "\overline{c}"'] & \mathrm{H} \otimes \mathrm{H} \otimes B \arrow[l, "\mathrm{Id}\otimes\overline{b}"'] \end{tikzcd} \tag{40}$$

corresponds to the composition $(c \circ b) \circ a$. Therefore, the two morphisms (39) and (40) are equal since the composition of morphisms is associative in $\mathcal{E}$.
The coassociativity of the comultiplication cm follows from this observation by taking $A = \mathbb{1}$, $B = \mathrm{H} \otimes \mathbb{1}$ and $C = \mathrm{H} \otimes \mathrm{H} \otimes \mathbb{1}$ and all the morphisms $\overline{a}, \overline{b}$ and $\overline{c}$ to be the identities.
It remains to check that H is counitary. Let $\overline{a} : \mathrm{H} \otimes A \to B$ be a morphism. By Proposition 6.3, the two morphisms

$$\mathrm{H} \otimes A \xrightarrow{cm\otimes\mathrm{Id}} \mathrm{H} \otimes \mathrm{H} \otimes A \xrightarrow{\mathrm{Id}\otimes\overline{a}} \mathrm{H} \otimes B \xrightarrow{\delta\otimes\mathrm{Id}} B$$

and

$$\mathrm{H} \otimes A \xrightarrow{cm\otimes\mathrm{Id}} \mathrm{H} \otimes \mathrm{H} \otimes A \xrightarrow{\mathrm{Id}\otimes\delta\otimes\mathrm{Id}} \mathrm{H} \otimes A \xrightarrow{\overline{a}} B$$

are equal to $\overline{a}$. The result follows by taking $A = \mathbb{1}$, $B = \mathrm{H} \otimes A$ and $\overline{a}$ the identity. □

Let H be a commutative biunitary bialgebra in $(\mathcal{E}, \otimes, \mathbb{1})$. Theorem 6.2 can be applied to reconstruct H from the category $\mathbf{coMod}(H)$ of counitary left comodules. Indeed, $\mathbf{coMod}(H)$ inherits a symmetric monoidal structure (see Section 2.3) and the forgetful functor

$$f : \mathbf{coMod}(H) \to \mathcal{E}$$

is a symmetric monoidal functor. The conditions of application of Ayoub's construction are met (see Assumption 6.1).
Let us describe the right adjoint g and the section e. For $A \in \mathcal{E}$, $g(A)$ is the object $H \otimes A$ with the coaction

$$H \otimes A \xrightarrow{cm\otimes\mathrm{Id}} H \otimes H \otimes A.$$

If (B, ca) is a counitary left H-comodule, the unit of the adjunction is given by the morphism $ca : B \to H \otimes B$ while, for an object $A \in \mathcal{E}$, the counit is given by the morphism $cu \otimes \mathrm{Id} : H \otimes A \to A$. Let A be an object in $\mathcal{E}$. Then $e(A)$ is the object A with its trivial H-comodule structure (that is with the coaction $u \otimes \mathrm{Id} : A \to H \otimes A$).
By applying the weak tannakian formalism to the forgetful functor, the biunitary bialgebra obtained from Theorem 6.2 is canonically isomorphic to H:

Proposition 6.4. — *Let H be a commutative biunitary bialgebra in $\mathcal{C}$. Then canonical isomorphism*

$$\mathrm{H} := fg(\mathbb{1}) = H \otimes \mathbb{1} \to H$$

is an isomorphism of biunitary bialgebras.

For a proof see [11, Lemme 1.54].

6.2. Universal factorization. — As shown by Ayoub (see [11, Proposition 1.28]), the functor $f : \mathcal{M} \to \mathcal{E}$ admits a canonical factorisation via the category of counitary left H-comodules

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\overline{f}"] \arrow[dr, "f"'] & \mathbf{coMod}(H) \arrow[d, "\text{forgetful}"] \\ & \mathcal{E}. \end{tikzcd}$$

More precisely, one has the following proposition (see [11, Proposition 1.28]).

Proposition 6.5. — *Assume that Assumption 6.1 is satisfied.*

1. *Let M be an object of $\mathcal{M}$. Then, the morphism*

   $$ca : f(M) \xrightarrow{\eta} fgf(M) \xrightarrow{\theta_{\mathbb{1},f(M)}} f(g(\mathbb{1}) \otimes ef(M)) \xrightarrow{\vartheta_{g(\mathbb{1}),f(M)}} \mathrm{H} \otimes f(M)$$

   defines a structure of counitary left H-comodule over $f(M)$. Moreover for every morphism $m : M \to N$ in $\mathcal{M}$, the morphism $f(m) : f(M) \to f(N)$ in $\mathcal{E}$ is a morphism of counitary left H*-comodules.*

2. *Let M, N be objects in $\mathcal{M}$. Then, the isomorphism $f(M) \otimes f(N) \xrightarrow{\simeq} f(M \otimes N)$ is an isomorphism of counitary* H*-comodules.*

3. *Let A be an object in $\mathcal{E}$. Then, the isomorphism $fe(A) \xrightarrow{\simeq} A$ is an isomorphism of counitary* H*-comodules where A is seen as a trivial* H*-comodule.*

In the canonical factorization, given an object M in $\mathcal{M}$, $\overline{f}(M)$ is the object $f(M)$ with the structure of counitary left H-comodule given in Proposition 6.5. Note that the proposition ensures also that $\overline{f} : \mathcal{M} \to \mathbf{coMod}(\mathrm{H})$ is a symmetric monoidal functor.
The proof of the universal property stated in Theorem 1.6 is very similar to the proof of the universal propery of Nori's construction (see Theorem 5.3). It relies on the study of the functoriality of Ayoub's contruction and Proposition 6.4 (see [11, Proposition 1.55]).

6.3. Hopf algebra structure. — Under more restrictive assumptions, it is possible to show that the biunitary bialgebra H associated with the monoidal adjunction is in fact a Hopf algebra. Since H is commutative, the antipode will be a morphism of unitary algebras $\iota : \mathrm{H} \to \mathrm{H}$. Let E be a unitary algebra in $\mathcal{E}$. Let us first describe (still under Assumption 6.1) the morphisms $\mathrm{H} \to E$ in $\mathcal{E}$ and among them those that are morphisms of unitary algebras in terms of the functor f.

Definition 6.6. — Let E be an object in $\mathcal{E}$. A natural transformation
$$t : f(-) \to E \otimes f(-)$$
is called an operation if for every object $M \in \mathcal{M}$ and every object $A \in \mathcal{E}$ the square
$$\begin{tikzcd}
f(M \otimes e(A)) \arrow[r, "t_{M\otimes e(A)}"] \arrow[d, "\vartheta_{M,A}"] & E \otimes f(M \otimes e(A)) \arrow[d, "\mathrm{Id}\otimes\vartheta_{M,A}"] \\
f(M) \otimes A \arrow[r, "t_M \otimes \mathrm{Id}"] & E \otimes f(M) \otimes A
\end{tikzcd}$$
is commutative.

We denote by $\mathrm{Oper}_E(f)$ the set of operations. Let us detail a consequence of the above definition.

Lemma 6.7. — *Let M is an object in $\mathcal{M}$ and t be an element in $\mathrm{Oper}_E(f)$. Then, the square*

(41)
$$\begin{tikzcd}
f(M) \arrow[r, "ca"] \arrow[d, "t_M"] & \mathrm{H} \otimes f(M) \arrow[d, "t_{g(\mathbb{1})}\otimes\mathrm{Id}"] \\
E \otimes f(M) \arrow[r, "\mathrm{Id}\otimes ca"] & E \otimes \mathrm{H} \otimes f(M)
\end{tikzcd}$$

is commutative.

Proof. — Recall that the coaction on $f(M)$ is given by the composition of the image by f of the morphism
$$M \xrightarrow{\eta} gf(M) \xrightarrow{\theta_{\mathbb{1},f(M)}} g(\mathbb{1}) \otimes ef(M)$$
and the morphism
$$\vartheta_{g(\mathbb{1}),f(M)} : f(g(\mathbb{1}) \otimes ef(M)) \to \mathrm{H} \otimes f(M)$$
(see Proposition 6.3). Hence the square (41) can be decomposed in two squares
$$\begin{tikzcd}
f(M) \arrow[r] \arrow[rr, bend left, "ca"] \arrow[d, "t_M"] & f(g(\mathbb{1}) \otimes ef(M)) \arrow[r, "\vartheta_{g(\mathbb{1}),f(M)}"] \arrow[d, "t_{g(\mathbb{1})\otimes ef(M)}"] & \mathrm{H} \otimes f(M) \arrow[d, "t_{g(\mathbb{1})}\otimes\mathrm{Id}"] \\
E \otimes f(M) \arrow[r] \arrow[rr, bend right, "\mathrm{Id}\otimes ca"'] & E \otimes f(g(\mathbb{1}) \otimes ef(M)) \arrow[r, "\vartheta_{g(\mathbb{1}),f(M)}"] & E \otimes \mathrm{H} \otimes f(M).
\end{tikzcd}$$
The first square is commutative since t is natural transformation and the second one is commutative by definition of an operation. □

Remark 6.8. — Assume that E is a unitary algebra in $\mathcal{E}$. Then $\mathrm{Oper}_E(f)$ has a natural structure of monoid. Let t, t' be two operations. Then their product $t' \bullet t$ is the natural transformation
$$f(-) \xrightarrow{t} E \otimes f(-) \xrightarrow{\mathrm{Id}\otimes t'} E \otimes E \otimes f(-) \xrightarrow{m\otimes\mathrm{Id}} E \otimes f(-).$$

The unit of this monoid is the operation

$$f(-) \xrightarrow{u\otimes \mathrm{Id}} E\otimes f(-).$$

For every $M\in\mathcal{M}$, the object $f(M)$ of $\mathcal{E}$ has a canonical structure of counitary left H-comodule (see Proposition 6.5). This coaction define a natural transformation

$$f(-) \xrightarrow{ca} \mathrm{H}\otimes f(-).$$

Therefore, every morphism $a:\mathrm{H}\to E$ in $\mathcal{E}$ defines a natural transformation

$$t_a : f(-) \xrightarrow{ca} \mathrm{H}\otimes f(-) \xrightarrow{a\otimes \mathrm{Id}} E\otimes f(-)$$

and it is easy to see that it is an operation in the sense of Definition 6.6.

Proposition 6.9. — *Let E be an object in $\mathcal{E}$. The mapping*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{E}}(\mathrm{H},E) &\to \mathrm{Oper}_E(f) \\ a &\mapsto t_a \end{aligned} \tag{42}$$

is a bijection. If E is a unitary algebra in $\mathcal{E}$, then it is an isomorphism of monoids.

Proof. — Let us denote by α the map in (42). By construction $\mathrm{H}=fg(\mathbb{1})$. Hence if t be an operation, it is possible to define a morphism $\beta(t)$ as the composition

$$\mathrm{H} \xrightarrow{t_{g(\mathbb{1})}} E\otimes \mathrm{H} \xrightarrow{\mathrm{Id}\otimes cu} E.$$

This defines a morphism $\beta:\mathrm{Oper}_E(f)\to \mathrm{Hom}_{\mathcal{E}}(\mathrm{H},E)$. Let us show that α and β are inverse to each other. If $a:\mathrm{H}\to E$ is a morphism in $\mathcal{E}$, then $\beta\circ\alpha(a)$ is the composition

$$\mathrm{H} \xrightarrow{cm} \mathrm{H}\otimes\mathrm{H} \xrightarrow{a\otimes\mathrm{Id}} E\otimes\mathrm{H} \xrightarrow{\mathrm{Id}\otimes cu} E$$

which is equal to a since H is counitary and cu is its counit.
Conversely, let t be an operation. By definition $\alpha\circ\beta(t)$ is the operation given, for every $M\in\mathcal{M}$, by the composition

$$f(M) \xrightarrow{ca} \mathrm{H}\otimes f(M) \xrightarrow{t_{g(\mathbb{1})}\otimes\mathrm{Id}} E\otimes\mathrm{H}\otimes f(M) \xrightarrow{\mathrm{Id}\otimes cu\otimes\mathrm{Id}} E\otimes f(M).$$

By Lemma 6.7 and the fact that $f(M)$ is a counitary left H-comodule, the diagram

$$\begin{tikzcd} f(M) \arrow[r, "ca"] \arrow[d, "t_M"] & \mathrm{H}\otimes f(M) \arrow[d, "t_{g(\mathbb{1})}\otimes\mathrm{Id}"] & \\ E\otimes f(M) \arrow[r, "\mathrm{Id}\otimes ca"] \arrow[rr, bend right, equal] & E\otimes\mathrm{H}\otimes f(M) \arrow[r, "\mathrm{Id}\otimes cu\otimes\mathrm{Id}"] & E\otimes f(M) \end{tikzcd}$$

is commutative. This shows that $\alpha\circ\beta(t)=t$.
Assume that E is a unitary algebra in the category $\mathcal{E}$. We have to show that $\alpha(a\bullet b)=\alpha(a)\bullet\alpha(b)$ for every morphism $a:\mathrm{H}\to E$ and $b:\mathrm{H}\to E$ in $\mathcal{E}$. More precisely, we have to show that the operation $t_{a\bullet b}$ associated with the morphism

$$a\bullet b : \mathrm{H} \xrightarrow{cm} \mathrm{H}\otimes\mathrm{H} \xrightarrow{a\otimes b} E\otimes E \xrightarrow{m} E$$

is equal to the operation $t_a\bullet t_b$ given by

$$t_a\bullet t_b : f(-) \xrightarrow{t_a} E\otimes f(-) \xrightarrow{\mathrm{Id}\otimes t_b} E\otimes E\otimes f(-) \xrightarrow{m\otimes\mathrm{Id}} E\otimes f(-).$$

Using the definition of t_a and t_b, we see that this operation is given by the commutative diagram

$$\begin{tikzcd}
f(-) \arrow[r, "t_a"] \arrow[rrr, bend left, "t_a \bullet t_b"] \arrow[dr, "ca"'] & E \otimes f(-) \arrow[r, "\mathrm{Id} \otimes t_b"] & E \otimes E \otimes f(-) \arrow[r, "m \otimes \mathrm{Id}"] & E \otimes f(-) \\
& \mathrm{H} \otimes f(-) \arrow[r, "\mathrm{Id} \otimes ca"'] & \mathrm{H} \otimes \mathrm{H} \otimes f(-). \arrow[u, "a \otimes b \otimes \mathrm{Id}"'] &
\end{tikzcd}$$

Hence, it is equal to $t_{a \bullet b}$ since $f(M)$ is a comodule over H. ☐

Remark 6.10. — It follows from the proof of Proposition 6.9 that the morphism $\mathrm{H} \to E$ associated with the operation $t \in \mathrm{Oper}_E(f)$ is the composition

$$\mathrm{H} \xrightarrow{t_{g(\mathbb{1})}} E \otimes \mathrm{H} \xrightarrow{\mathrm{Id} \otimes cu} E.$$

Definition 6.11. — Let E be a unitary algebra in $\mathcal{E}$. A natural transformation $f \to E \otimes f$ is said to be multiplicative if the diagram

$$\begin{tikzcd}
f(M) \otimes f(N) \arrow[r, "t_M \otimes t_N"] \arrow[d] & (E \otimes f(M)) \otimes (E \otimes f(N)) \arrow[r, "\mathrm{Id} \otimes \tau \otimes \mathrm{Id}"] & E \otimes E \otimes f(M) \otimes f(N) \arrow[d, "m \otimes \mathrm{Id} \otimes \mathrm{Id}"] \\
f(M \otimes N) \arrow[r, "t_{M \otimes N}"] & E \otimes f(M \otimes N) & E \otimes f(M) \otimes f(N) \arrow[l]
\end{tikzcd}$$

is commutative for every objects $M, N \in \mathcal{M}$. If moreover

$$\begin{tikzcd}
\mathbb{1} \arrow[r] \arrow[dr, bend right, "u"'] & f(\mathbb{1}) \arrow[r, "t_{\mathbb{1}}"] & E \otimes f(\mathbb{1}) \arrow[d] \\
& E & E \otimes \mathbb{1} \arrow[l]
\end{tikzcd}$$

is commutative, then it is said to be unitary.

Proposition 6.12. — *Let E be a unitary algebra in $\mathcal{E}$. Then, the bijection of Proposition 6.9 induces a bijection between the subset of $\mathrm{Hom}_{\mathcal{E}}(\mathrm{H}, E)$ formed by the morphisms of unitary algebras and the subset of $\mathrm{Oper}_E(f)$ formed by the multiplicative and unitary operations.*

Proof. — If t is a multiplicative and unitary operation, then

$$\mathrm{H} \xrightarrow{t_{g(\mathbb{1})}} E \otimes \mathrm{H} \xrightarrow{\mathrm{Id} \otimes cu} E$$

is a morphism of unitary algebras. Conversely, if $a : \mathrm{H} \to E$ is a morphism of unitary algebras, then t_a is a multiplicative and unitary operation since the coaction

$$ca : f(-) \to \mathrm{H} \otimes f(-)$$

is a natural transformation which is multiplicative and unitary. ☐

Let us come back to the issue of the antipode. Consider the following strengthening of Assumption 6.1.

Assumption 6.13. —

1. *There exists a* monoidal *functor $e : \mathcal{E} \to \mathcal{M}$ and an isomorphism of monoidal functors $fe \xrightarrow{\sim} \mathrm{Id}_{\mathcal{E}}$. Moreover the functor e admits a right adjoint $u : \mathcal{M} \to \mathcal{E}$.*

2. *The morphism of coprojection*
$$c_d : g(A) \otimes M \to g(A \otimes f(M))$$
is an isomorphism for every $A \in \mathcal{E}$ and $M \in \mathcal{M}$.

These new assumptions are clearly stronger than Assumption 6.1: the coprojection is now assumed to be an isomorphism for every object $M \in \mathcal{M}$ while previously it was only assumed to be the case for objects in the essential image of e. In particular, if A is an object in $\mathcal{E}$ and M in $\mathcal{M}$, then the inverse of the coprojection defines a morphism
$$p_d : g(A \otimes f(M)) \to g(A) \otimes M.$$
Moreover the fact that e is a section of f forces g to be a section of u and since u is right adjoint to e, for every M in $\mathcal{M}$ we have a canonical morphism $eu(M) \to M$.
Let $M \in \mathcal{M}$ and consider the morphism cd_M defined as the composition
$$\mathrm{cd}_M : f(M) \to fgf(M) \xrightarrow{p_d} f(g(\mathbb{1}) \otimes M) \to \mathrm{H} \otimes f(M)$$
where the first morphism is obtained by identifying $f(M)$ with $feugf(M)$ and composing with the counit of the adjunction between u and e.

Proposition 6.14. — *The natural transformation*
$$\mathrm{cd} : f(-) \to \mathrm{H} \otimes f(-)$$
is a multiplicative and unitary operation in the sense of Definition 6.6.

For a proof, see [11, Proposition 1.42]. In particular, by Proposition 6.12, the operation $\mathrm{cd} : f(-) \to \mathrm{H} \otimes f(-)$ defines a morphism of unitary algebras $\iota : \mathrm{H} \to \mathrm{H}$ defined by the composition

$$\mathrm{H} \xrightarrow{\mathrm{cd}_{g(\mathbb{1})}} \mathrm{H} \otimes \mathrm{H} \xrightarrow{\mathrm{Id} \otimes cu} \mathrm{H} \tag{43}$$

(see Remark 6.10). Finally, one has the following theorem (for a proof see [11, Théorème 1.45]).

Theorem 6.15. — *Under Assumption 6.13, the commutative biunitary bialgebra* H *is a Hopf algebra with antipode the map $\iota : \mathrm{H} \to \mathrm{H}$ constructed in* (43)*.*

6.4. Application to motivic Galois theory. — Recall that we have an adjunction
$$\mathsf{Bti}^* : \mathbf{DA}^{\text{ét}}(k, \mathbb{Q}) \rightleftarrows \mathrm{D}(\mathbb{Q}) : \mathsf{Bti}_*$$
where the right handside is the derived category of $\mathbb{Q}$-vector spaces.
As shown by Ayoub (see [11, Proposition 2.7]), the weak Tannakian formalism can be applied to the above adjunction. In that case, the functor $(-)_{\mathrm{cst}}$ which maps a $\mathbb{Q}$-vector space V to the constant presheaf with values V prodives a section of Bti^*.
In particular, the object of $\mathrm{D}(\mathbb{Q})$.
$$\mathscr{H}_{\mathrm{mot}}(k, \sigma, \mathbb{Q}) := \mathsf{Bti}^*\mathsf{Bti}_*\mathbb{Q}$$
has a canonical structure of commutative Hopf algebra.

Note that $\mathscr{H}_{\mathrm{mot}}(k,\sigma,\mathbb{Q})$ is a Hopf algebra in the derived category $\mathrm{D}(\mathbb{Q})$. In particular, it does not yield immediately a $\mathbb{Q}$-Hopf algebra and therefore an affine group scheme over $\mathbb{Q}$. However, in [11, Corollaire 2.105], Ayoub proves the following theorem.

Theorem 6.16. — *$\mathscr{H}_{\mathrm{mot}}(k,\sigma,\mathbb{Q})$ has no homology in negative degree that is*

$$\mathrm{H}_n(\mathscr{H}_{\mathrm{mot}}(k,\sigma,\mathbb{Q}))=0$$

for every integer $n<0$.

This theorem ensures that $\mathbf{H}_{\mathrm{mot}}(k,\sigma,\mathbb{Q}):=\mathrm{H}_0(\mathscr{H}_{\mathrm{mot}}(k,\sigma,\mathbb{Q}))$ inherits a structure of $\mathbb{Q}$-Hopf algebra. The associated affine group scheme over $\mathrm{Spec}(\mathbb{Q})$

$$\mathbf{G}_{\mathrm{mot}}(k,\sigma,\mathbb{Q})$$

is called the motivic Galois group of k.
The two approaches, that of Nori and that of Ayoub, though very different yield the same motivic Galois group. A comparison theorem has been proven by Gallauer Alves de Souza and Choudhury [20]:

Theorem 6.17. — *The Hopf algebra $\mathbf{H}_{\mathrm{mot}}(k,\sigma,\mathbb{Q})$ and $\mathbf{H}_{\mathrm{Nori}}(k,\sigma,\mathbb{Q})$ are isomorphic. In particular, the motivic Galois groups $\mathbf{G}_{\mathrm{mot}}(k,\sigma,\mathbb{Q})$ and $\mathbf{G}_{\mathrm{Nori}}(k,\sigma,\mathbb{Q})$ are isomorphic.*

The weak Tannakian formalism can be applying in other contexts. For example, Ayoub uses it in [12] to define a motivic avatar of the fundamental group.

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