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Equivariant Perturbation in Gomory and Johnson's Infinite Group Problem VII. Inverse Semigroup Theory, Closures, Decomposition of Perturbations

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Abstract

In this self-contained paper, we present a theory of the piecewise linear minimal valid functions for the 1-row Gomory–Johnson infinite group problem. The non-extreme minimal valid functions are those that admit effective perturbations. We give a precise description of the space of these perturbations as a direct sum of certain finite- and infinite-dimensional subspaces. The infinite-dimensional subspaces have partial symmetries; to describe them, we develop a theory of inverse semigroups of partial bijections, interacting with the functional equations satisfied by the perturbations. Our paper provides the foundation for grid-free algorithms for the Gomory–Johnson model, in particular for testing extremality of piecewise linear functions whose breakpoints are rational numbers with huge denominators.

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1 Introduction

1.1 Finite group relaxations $R_f(P, \mathbb{Z})$ of integer programs and hierarchies of valid inequalities

A powerful method to derive cutting planes for unstructured integer linear optimization problems is to study relaxations with more structure and convenient properties. The pioneering relaxation in this line of research on general-purpose cutting planes is Gomory's *finite group relaxation* [10], whose convex hull is known as the *corner polyhedron*.

The relaxations are structured around the simplex method, applied to the continuous relaxation, and are therefore suitable for generating cuts in a linear-programming-based cutting-plane procedure. The group relaxation is obtained by forgetting about the nonnegativity of all basic variables, retaining only their integrality. Viewed in the space of nonbasic variables, the equations of the simplex tableau are replaced by congruences modulo the abelian group ($\mathbb{Z}$-module) generated by the columns of the basis matrix. Quotienting out by this group, one obtains a "group equation," which gives the relaxation its name. Further relaxations are obtained by picking just one or a few rows of the system, or more generally by condensing the system by means of group homomorphisms; see [10] for its remarks on the use of (additive) group characters.

In the present paper, we restrict ourselves to 1-row ("cyclic") group relaxations, which after aggregation of non-basic variables with identical coefficients can be brought to the form

$$\sum_{p\in P} y(p)\,p \in f+\mathbb{Z}, \qquad y(p)\in\mathbb{Z}_+ \ \text{ for all } p\in P \tag{1}$$

where P is a finite subset of an additive group $G=\frac{1}{q}\mathbb{Z}\supset\mathbb{Z}$ and $f\in G\setminus\mathbb{Z}$, so $f+\mathbb{Z}$ is a coset of the subgroup $\mathbb{Z}$ in G. This is called Gomory's *finite (cyclic) group problem.* We denote the convex hull of y by $R_f(P,\mathbb{Z})$; it is a polyhedron of blocking type. Therefore every nontrivial valid linear inequality can be written in the form $\langle\pi,y\rangle := \sum_{p\in P}\pi(p)y(p)\geq 1$; then we call π a *valid function.* If $\pi'\leq\pi$ are two valid functions, then the valid inequality $\langle\pi,y\rangle\geq 1$ is a conic combination of $\langle\pi',y\rangle\geq 1$ and nonnegativity inequalities $y(p)\geq 0$. Thus it suffices to consider the *minimal (valid) functions* π, defined by the property

$$\text{if } \pi' \text{ is valid and } \pi'\leq\pi, \text{ then } \pi'=\pi. \tag{M}$$

A stronger notion is that of *extreme functions* π, defined by the property

$$\text{if } \pi^+,\pi^- \text{ are minimal and } \pi=\tfrac{1}{2}(\pi^++\pi^-), \text{ then } \pi=\pi^+=\pi^-. \tag{E}$$

Extreme functions correspond to facet-defining inequalities for $R_f(P,\mathbb{Z})$. Following the traditions of polyhedral combinatorics, we are interested in describing families of extreme functions and making them available for cutting-plane algorithms.

1.2 Master problems $R_f(G,\mathbb{Z})$ and the subadditive characterization of minimal functions

Gomory's approach was to consider *master problems* for this purpose. The sets of solutions y to 1-row group relaxations $R_f(P,\mathbb{Z})$ for subsets P of the same group G inject into the *master group relaxation*

$$\sum_{p\in G} y(p)\,p \in f+\mathbb{Z} \qquad y\colon G\to\mathbb{Z}_+ \text{ has finite support} \tag{2}$$

by setting $y(p)=0$ for $p\notin P$. We denote its convex hull by $R_f(G,\mathbb{Z})$. This is an infinite-dimensional set. By Gomory's master theorem [10, Theorem 13], every extreme function π' for $R_f(P,\mathbb{Z})$ is obtained from some extreme function π for a master problem $R_f(G,\mathbb{Z})$ with $P\subseteq G$ by restricting the function, $\pi'=\pi|_P$. Moreover, Gomory [10] gave a characterization of the minimal functions for the master problem $R_f(G,\mathbb{Z})$ by the following functional inequalities and equations:

$$\pi(x)\geq 0 \qquad \text{for } x\in G, \tag{3a}$$
$$\pi(x+z)=\pi(x) \qquad \text{for } x\in G,\ z\in\mathbb{Z} \qquad \text{(periodicity)} \tag{3b}$$
$$\pi(0)=0,\ \pi(f)=1, \tag{3c}$$
$$\Delta\pi(x,y)\geq 0 \qquad \text{for } x,y\in G \qquad \text{(subadditivity)}, \tag{3d}$$
$$\Delta\pi(x,f-x)=0 \qquad \text{for } x\in G \qquad \text{(symmetry condition)}, \tag{3e}$$

where $\Delta\pi(x,y)=\pi(x)+\pi(y)-\pi(x+y)$ is the *subadditivity slack* function. By quotienting out by $\mathbb{Z}$, this system describes a polyhedron in $\mathbb{R}^{G/\mathbb{Z}}$. Extreme functions are then simply the vertices of this polyhedron; thus some of the subadditivity inequalities $\Delta\pi(x,y)\geq 0$ are tight, i.e., additivity holds.

1.3 Continuous interpolations of extreme functions and the infinite group problem $R_f(\mathbb{R},\mathbb{Z})$

Gomory and Johnson, in their seminal papers [11, 12], noted that many extreme functions for finite master group problems follow simple patterns that become apparent in the piecewise linear interpolations of these functions. The simplest pattern is that of the well-known two-slope function giving the Gomory mixed integer cut (gmic ), which can be found in all finite group problems; see Figure 1 (left).[1] Gomory and Johnson initiated a program to study such functions of a real variable systematically. The technical framework is that of the *infinite group problem*, in which the group G in (2) is enlarged from $\frac{1}{q}\mathbb{Z}$ to $\mathbb{R}$. Gomory and Johnson proved that the characterization (3) of minimal functions still holds in this setting.

[1] A function name shown in sans serif font refers to the software [21], which includes the Electronic Compendium of Extreme Functions [17].

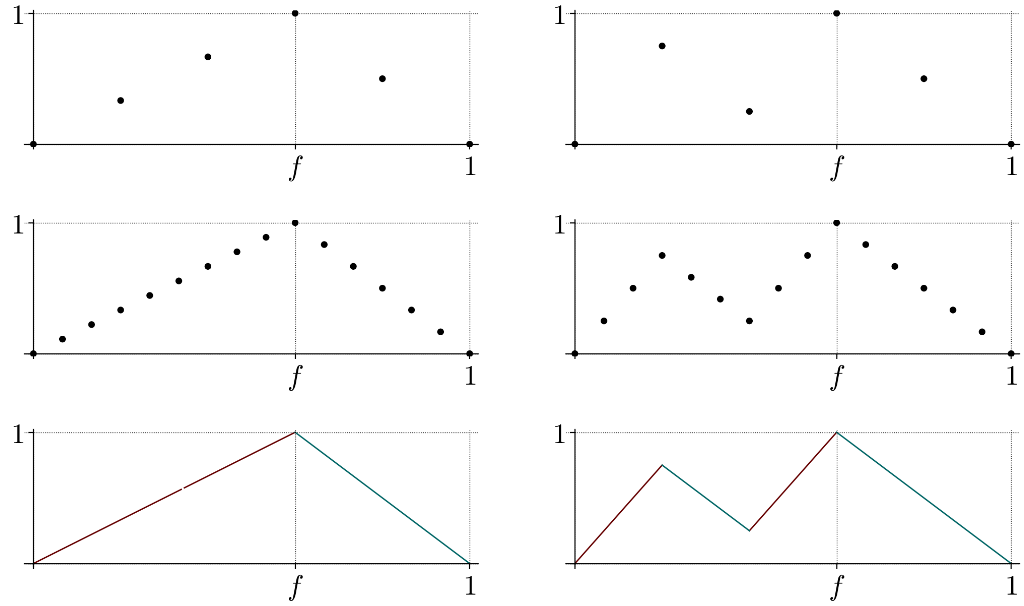
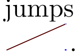


Figure 1 Extreme functions for finite master group problems following simple patterns that become apparent in the piecewise linear interpolations. *Left,* the Gomory mixed-integer cut (gmic). *Right,* another two-slope extreme function.

For an extreme function $\pi|_G$ for a finite master problem $R_f(G,\mathbb{Z})$, the piecewise linear interpolation $\pi =$ interpolate_to_infinite_group($\pi|_G$) is a minimal function, but not necessarily extreme. (A partial converse is true; the restriction of a continuous piecewise linear extreme function π for $R_f(\mathbb{R},\mathbb{Z})$ to a group G that includes all breakpoints of π is extreme for $R_f(G,\mathbb{Z})$.) There is a possible viewpoint on the extreme functions for the infinite group problems as "robust" cut-generating functions that ignore the specific number-theoretic properties of a particular group problem $R_f(\frac{1}{q}\mathbb{Z},\mathbb{Z})$. As a matter of fact, in a numerical implementation, the value q and exact rational value of f would not be readily available.

A natural algorithmic focus lies on piecewise linear valid functions, though a part of the literature [1, 2] also studies more complicated functions. (When we discuss piecewise linear functions in this paper, we include the discontinuous case with possible jumps at breakpoints, which includes important examples such as the Gomory fractional cut, gomory_fractional  ..)

For $\mathbb{Z}$-periodic piecewise linear functions, the characterization (3) of minimal functions gives a simple algorithm for testing minimality, based on enumerating the vertices of a certain polyhedral complex; see [3, Section 2.2] and [16, Section 5]. For testing the extremality of a piecewise linear minimal function, however, in contrast to the finite group case, we cannot directly use polyhedral methods any more. Since the quotient $\mathbb{R}/\mathbb{Z}$ is not finite, we have to use infinite-dimensional methods of functional equations and inequalities. The most important lemma from this theory is the Gomory–Johnson interval lemma, variants of which has been used in virtually all proofs of extremality in the literature.

1.4 The space $\tilde{\Pi}^\pi$ of effective perturbations $\tilde{\pi}$ of a minimal valid function

Recall that by definition (E), a minimal valid function π is extreme if it cannot be written as a convex combination of two other minimal valid functions π^+, π^-. A fruitful approach to extremality testing, introduced by Basu et al. in Part I of the present series of papers [3], has been to consider the difference function (*perturbation*) $\bar{\pi} = \pi^+ - \pi$, which allows us to write $\pi^+ = \pi + \bar{\pi}$ and $\pi^- = \pi - \bar{\pi}$. (Recently, Di Summa [9] obtained a breakthrough result on the question of piecewise linearity of extreme functions using this approach.) It is convenient to build a space from the notion of perturbation functions. Following Part V [16, Section 6], we define the space

$$\tilde{\Pi}^\pi = \left\{\, \tilde{\pi}\colon \mathbb{R} \to \mathbb{R} \mid \exists\, \epsilon > 0 \text{ s.t. } \pi^\pm = \pi \pm \epsilon\tilde{\pi} \text{ are minimal valid} \,\right\} \tag{4}$$

of *effective perturbation functions* for the minimal valid function π. This is a vector space (Lemma 68), a subspace of the space of bounded functions. The function π is extreme if and only if the space $\tilde{\Pi}^\pi$ is trivial.

If additivity ($\Delta\pi(x,y) = 0$) holds for some (x,y), then by convexity also $\Delta\tilde{\pi}(x,y) = 0$ holds for every effective perturbation $\tilde{\pi} \in \tilde{\Pi}^{\pi}$. This is also true for additivity in the limit [3, Lemma 2.7]; see also [16, Lemma 6.1]. Because π is assumed to be piecewise linear, the infinite system of functional equations describing additivity and limit-additivity of $\tilde{\pi}$ can be structured ("combinatorialized") according to a certain polyhedral complex [3, 16].

1.5 Finite-dimensional and equivariant perturbations

In Part I of the present series, Basu et al. [3] gave the first algorithm to decide extremality of a piecewise linear function with rational breakpoints in some "grid" (group) $G = \frac{1}{q}\mathbb{Z}$.

In a first step, one tests whether there exists a nontrivial perturbation for π in the finite-dimensional subspace of $\tilde{\Pi}^{\pi}$ that consists of the functions interpolate_to_infinite_group($\tilde{\pi}|_G$), where $\tilde{\pi}|_G$ is an effective perturbation function for the restriction $\pi|_G$ to the finite group problem $R_f(G, \mathbb{Z})$.

Otherwise, one may assume that $\tilde{\pi}|_G = 0$. Under this assumption, the interval lemma forces $\tilde{\pi}|_C = 0$ for certain *directly covered* intervals C. Basu et al.'s crucial observation was that if there are any remaining *uncovered* intervals, then one-dimensional families of additivity equations impose a type of symmetry of the perturbation function. By analyzing the required symmetry, one can construct a perturbation function and prove nonextremality of π.

Consider the additivity equations

$$\Delta\tilde{\pi}(x,t) = \tilde{\pi}(x) + \tilde{\pi}(t) - \tilde{\pi}(x+t) = 0, \quad \text{for } x \in D, \tag{5}$$

where D is a proper interval and $t \in \frac{1}{q}\mathbb{Z}$ is a grid point. Because $\tilde{\pi}(t) = 0$, this simplifies to

$$\tilde{\pi}(x) = \tilde{\pi}(x+t) \quad \text{for } x \in D. \tag{6a}$$

We then say that $\tilde{\pi}$ is *invariant* under the action of the *translation* $\tau_t\colon x \mapsto x+t$ (restricted to the proper interval D). Likewise, a second type of one-dimensional families of additivity equations simplifies to

$$\tilde{\pi}(x) = -\tilde{\pi}(r-x) \quad \text{for } x \in D. \tag{6b}$$

Here a negative sign comes in. We call $\rho_r\colon x \mapsto r-x$ a *reflection.* By assigning a *character* $\chi(\tau_t) = +1$ and $\chi(\rho_r) = -1$ to the translations and reflections, we can unify equations (6) as $\tilde{\pi}(x) = \chi(\gamma)\,\tilde{\pi}(\gamma(x))$ for $x \in D$, where γ is either a translation or a reflection. We say that $\tilde{\pi}$ is *equivariant* under the action of γ.

By analyzing the group Γ of affine transformations of $\mathbb{R}$ generated by all relevant translations and reflections, Basu et al. constructed a universal template function $\psi\colon \mathbb{R} \to \mathbb{R}$, a continuous piecewise linear function with breakpoints in $\frac{1}{4q}\mathbb{Z}$, which is equivariant under the action of the group Γ. Taking

$$\tilde{\pi}(x) = \begin{cases} \psi(x) & \text{for } x \text{ in uncovered intervals,} \\ 0 & \text{for } x \text{ in covered intervals} \end{cases} \tag{7}$$

then gives an effective perturbation function. (A revised construction in Basu et al.'s survey [6, Section 8.2] gives a continuous piecewise linear function $\tilde{\pi}$ with breakpoints in $\frac{1}{3q}\mathbb{Z}$.)

1.6 Contributions of the present paper

It has been a long-term research project to develop a complete, *grid-free* algorithmic theory and software implementation for piecewise linear minimal valid functions, extending the reach of the grid-based extremality test introduced in Part I of the series [3], which we described in (1.5) above. While Parts II–IV develop a grid-based theory for 2-row relaxations, Part V of our series [16] returned to the one-row case. It introduced our software [21] and prepared the grid-free theory with several results. Part VI of the series [20] discussed the case of piecewise linear functions that are discontinuous on both sides of the origin and have irrational breakpoints. The present paper, Part VII of the series, and a computational companion paper, Part VIII of the series, are the culmination of the project for the case of piecewise linear functions of one variable.

1.6.1 Method: Inverse semigroups as the language of partial symmetries

Group actions are the standard language to describe symmetries of mathematical objects. The use of group actions was fruitful in Part I of our series to obtain the first algorithm for testing extremality. However, group

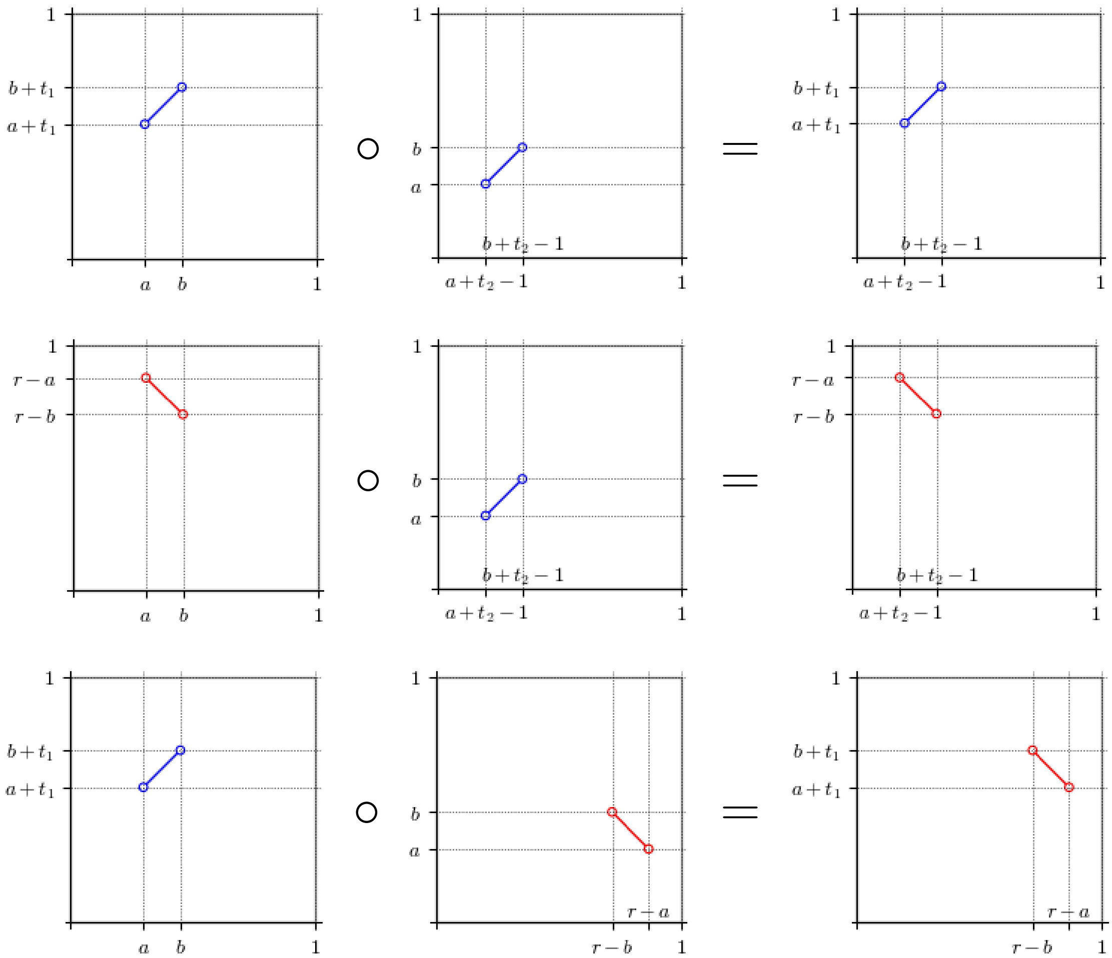
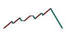


Figure 2 Operations of the inverse semigroup I: Composition

actions do not provide a complete theory of the effective perturbations. This becomes most apparent in [3, Section 5], where Basu et al. introduce a family of extreme functions with irrational breakpoints, bhk_irrational . Here the group Γ generated by the translations and reflections only gives the correct result when a certain non-group-theoretic reachability condition [3, Assumption 5.1, Lemma 5.2] is satisfied. The underlying reason is that the restriction of the translations and reflections to the interval domains D in (6) is not considered in the reflection group. Indeed, what the translations and reflections describe is not a full symmetry of the perturbation function, but only a *partial symmetry* within the uncovered intervals.

The correct language to describe partial symmetries is the less well-known theory of inverse-semigroup actions. An *inverse semigroup* $(\Gamma, \circ, \cdot^{-1})$, following [22, p. 7], is a semigroup, i.e., a set Γ closed under an associative operation $\circ$, satisfying the additional axiom that

$$\text{for every } \omega \in \Gamma, \text{ there exists a unique element } \omega^{-1} \in \Gamma \quad (\exists! \text{ inverse})$$
$$\text{such that } \omega = \omega \circ \omega^{-1} \circ \omega \text{ and } \omega^{-1} = \omega^{-1} \circ \omega \circ \omega^{-1}.$$

The equations in the axiom describe the familiar properties of a pseudoinverse, but due to the required uniqueness, we will simply refer to ω^{-1} as the *inverse* of ω. In his monograph [22], Lawson points out that *the relationship between inverse semigroups and partial symmetries is a generalization of the relation between groups and symmetries.*

Concretely, inverse semigroups arise as semigroups of partial bijections of a set, where the operation $\circ$ is the composition and $\cdot^{-1}$ is the inverse of a partial bijection. We define the restrictions of the previously defined translations and reflections to open intervals D. We denote them by $\tau_t|_D$ and $\rho_r|_D$ and consider them as partial bijections of $\mathbb{R}$ to itself, with domains $\mathrm{dom}(\tau_t|_D) = D = \mathrm{dom}(\rho_r|_D)$ and images $\mathrm{im}(\tau_t|_D) = \tau_t(D) = D + t$ and $\mathrm{im}(\rho_r|_D) = \rho_r(D) = r - D$. We refer to these partial bijections as *moves*. The composition of two moves $\gamma_1|_{D_1}$

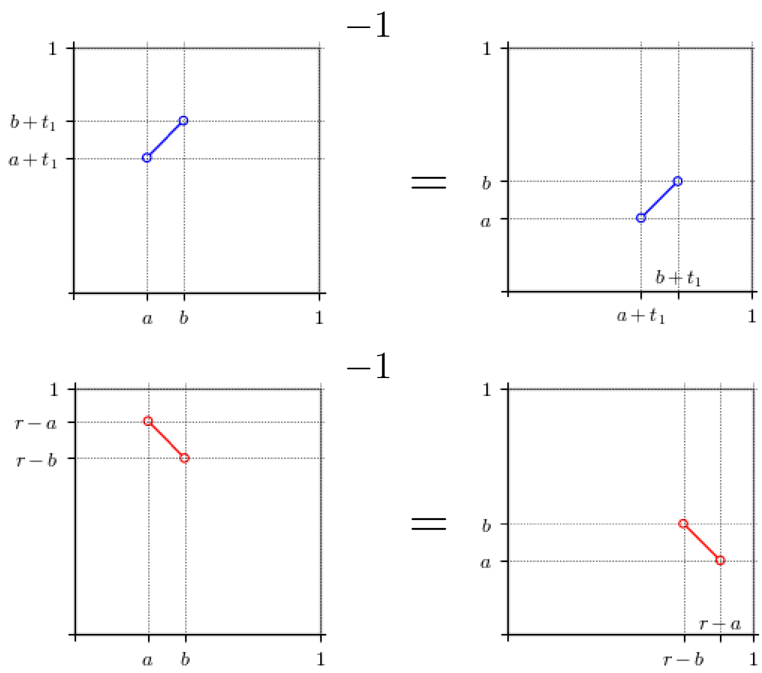


Figure 3 Operations of the inverse semigroup II: Inverse

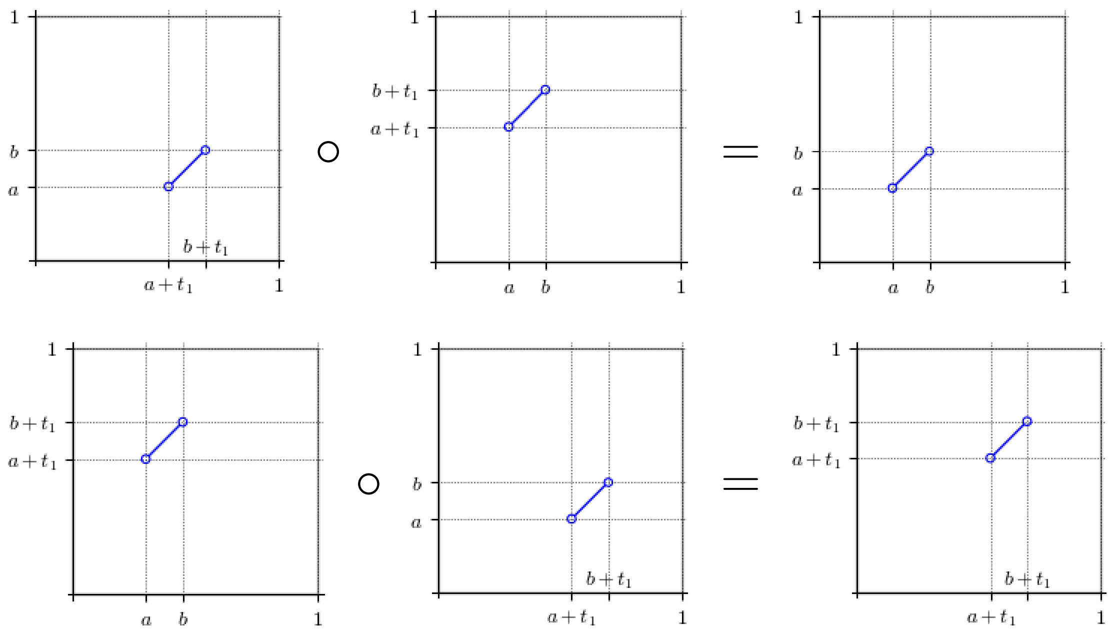


Figure 4 Operations of the inverse semigroup III: Composition with inverse. The partial identities $\tau_0|_D$ are the idempotents of the inverse semigroup.

and $\gamma_2|_{D_2}$ is defined as

$$\gamma_2|_{D_2} \circ \gamma_1|_{D_1} = \gamma_2 \circ \gamma_1|_{D_1 \cap \gamma_1^{-1}(D_2)}; \tag{8}$$

see Figure 2. The domain of the composition is an open interval (including possibly the empty set). (By definition, there are exactly two empty moves: the empty translation $\tau|_\emptyset$ and the empty reflection $\rho|_\emptyset$.) The inverse of a move $\gamma|_D$ is given by $(\gamma|_D)^{-1} = \gamma^{-1}|_{\gamma(D)}$, see Figure 3. Note that it is not an inverse in a group-theoretic sense: The compositions

$$\gamma|_D \circ (\gamma|_D)^{-1} = \tau_0|_{\gamma(D)} \quad \text{and} \quad (\gamma|_D)^{-1} \circ \gamma|_D = \tau_0|_D \tag{9}$$

are only *partial identities* (restrictions of the identity τ_0 to intervals) and therefore not neutral elements but merely idempotents (Figure 4).

We develop methods that center around inverse semigroups of moves and their generating sets. We study the set of moves that are respected by the effective perturbations of a given minimal function π. We analyze the closure properties (axioms) that it satisfies: algebraically, it is an inverse semigroup; but additional order-theoretic and analytic closure properties come in. Starting from an initial set (*move ensemble*) Ω^0, we can then form the closure with respect to the axioms. We call it the *moves closure* of Ω^0 (or *closed move semigroup* generated by Ω^0) and denote it by $\operatorname{clsemi}_A(\Omega^0)$, where A is the maximal open subset of $(0, 1)$ on which π is continuous. (The additional subscript A is important because certain properties apply where π is continuous.)

In the first part of the paper, we develop these methods in full generality, without using any specific properties of the Gomory–Johnson model. Then we turn to the study of piecewise linear functions; here we make the assumption of continuity from at least one side of the origin.

For all piecewise linear functions with rational breakpoints, we will show that $\operatorname{clsemi}_A(\Omega^0)$ has a simple structure: Its graph consists of a finite union of line segments and rectangles. (We say that it is *finitely presented.*) It will become clear that we can compute $\operatorname{clsemi}_A(\Omega^0)$ in finitely many steps using a completion-type algorithm, using only the algebraic and order-theoretic axioms, by manipulating finite presentations of generating systems. However, this algorithm is not the focus of the present paper: We defer all computational questions to the forthcoming companion paper [15].

Instead, an important point of our paper is that finitely presented closures $\operatorname{clsemi}_A(\Omega^0)$ arise in a more general context, through the interplay of the order-theoretic, algebraic, and analytic closure properties. Move ensembles whose graphs are connected open sets extend to open rectangles already in the joined semigroup (Corollary 12). Our key theorem using the analytic properties is Theorem 40: Rectangles appear in the closure whenever there is a convergent sequence of moves. (In Part I of our series, we have observed a glimpse of this phenomenon already, in a specific arithmetic context.) Empirically, for all families of piecewise linear minimal valid functions in the literature (see [17] for an electronic compendium), even if the breakpoints are irrational, the closure has a finite presentation. This includes the function bhk_irrational , which we mentioned above. Again, we defer questions regarding the computation of this closure, which then needs to use the additional axioms, to our forthcoming paper [15].

1.6.2 Result: Precise description of the space of equivariant perturbations

Under the above assumptions, the finite presentation of $\operatorname{clsemi}_A(\Omega^0)$ allows us to read off a precise description of the space of equivariant perturbations as a direct sum decomposition of vector subspaces (Theorem 102).

One component in the decomposition is a finite-dimensional space, consisting of (possibly discontinuous) piecewise linear functions. In contrast to the grid-based algorithm, the set of breakpoints of these functions is not fixed, but it is computed by our algorithm. The finite-dimensional space is then described by a system of finitely many linear equations (Lemma 97).

Then, for each of the finitely many *uncovered components* (defined in Section 10), there is a component that is an infinite-dimensional space isomorphic to the space of Lipschitz functions on a compact interval that vanish on the boundary. More specifically, our algorithm computes an open interval D, the *fundamental domain*, on which we take the space of Lipschitz functions $\tilde{\pi}$ that vanish on the boundary ∂D. Additionally there are finitely many moves $\gamma_j|_D$ with pairwise disjoint images $\gamma_j(D)$ that together extend the functions equivariantly to the whole uncovered component. Outside of the component, the functions in this space are zero. This is Theorem 100.

This description of the space strengthens previous results. The method of Part I [3], described in Subsection 1.5, guarantees to construct a piecewise linear effective perturbation if the space is nontrivial; but it does not provide a complete description of the space. A theorem regarding direct sum decomposition appeared in [5, Theorem 3.14], but it is limited to the grid case.

We remark that the precise description of the perturbation space of a minimal function π enables us to strengthen (*lift*) it by constructing a direction in the space of effective perturbations. By our theorem, the problem of finding such a direction decomposes into subproblems; one finite-dimensional, the others independent variational problems over Lipschitz functions.

1.6.3 Computational implications: Grid-free algorithms, natural proofs

We only sketch the computational implications of the present paper because we will elaborate on them in our companion paper. The inverse semigroup theory lays the foundation for grid-free algorithms for minimal valid functions, including automated extremality tests, which are detached from the finite group problem. A grid-free test is faster for functions whose breakpoints are rational numbers with huge denominators; and it enables computations for functions with irrational breakpoints. More importantly, the grid-free algorithms can give natural extremality proofs, similar to the general proof pattern of extremality proofs in the published literature. In this way, the grid-free algorithms enable automated extremality proofs for smoothly parameterized families of extreme functions, as described in [18].

Key to our grid-free algorithm is the *breakpoint stabilization theorem* (Theorem 89), which allows us to dynamically determine the set of breakpoints needed for our tests. This result could pave the way to generalizations in higher dimensions. See Remark 90 for more details.

1.7 Structure of the paper

In Sections 2–4, we introduce moves as partial bijections of $\mathbb{R}$. We study *ensembles* (sets) of such moves, which can be equipped with both an order-theoretic structure (restriction and continuation) and an algebraic structure (inverse semigroups). In Section 5 we describe how move ensembles and semigroups describe partial symmetries of a function by a system of functional equations. Move ensembles for bounded functions have additional properties, which we explore in Section 6. Then, in Section 7, we study closure properties that capture the additional properties of move ensembles for continuous functions. This development culminates in the notion of *closed move semigroups* in Subsection 7.3.

We then apply this theory to compute the effective perturbation space of a piecewise linear minimal valid function π. In Section 8, we introduce the *initial additive move ensemble* Ω^0, which describes functional equations satisfied by every effective perturbation of π. For piecewise linear functions π, it is related to the *additive faces* of a polyhedral complex (Section 9). In Section 10, working with a finite presentation of the closed move semigroup $\operatorname{clsemi}_A(\Omega^0)$ generated by Ω^0, we prove the main theorem of the paper, the decomposition theorem for the space of effective perturbations of π. Finally, in Section 11, under the same assumptions, we establish the precise relation between $\operatorname{clsemi}_A(\Omega^0)$ and semigroups of all moves respected by perturbations.

We end the paper in Section 12 with a discussion of the limitations of our approach and an outlook on the computational companion paper [15].

2 Translation and reflection moves. Their algebraic and order-theoretic structure

2.1 Group $\Gamma(\mathbb{R})$ of unrestricted translations τ_t and reflections ρ_r, character χ

▶ **Definition 1.** *For a point $r \in \mathbb{R}$, define the (unrestricted)* reflection $\rho_r \colon \mathbb{R} \to \mathbb{R}$, $x \mapsto r - x$. *For a vector $t \in \mathbb{R}$, define the (unrestricted)* translation $\tau_t \colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x + t$.

The set $\Gamma(\mathbb{R}) = \{\, \rho_r, \tau_t \mid r \in \mathbb{R},\ t \in \mathbb{R} \,\}$ of all translations and reflections, with the operations of function composition $\circ$ and inverse $\cdot^{-1}$, has the structure of a group. It is a subgroup of the group $\operatorname{Aff}(\mathbb{R})$ of regular affine transformations of $\mathbb{R}$.

To denote an element that can be either a translation or a reflection, we will usually use the letter γ. To recover whether an element γ is a translation or a reflection, we assign a *character* $\chi(\rho_r) = -1$ to every reflection and $\chi(\tau_t) = +1$ to every translation. The map $\gamma \mapsto \chi(\gamma)$ is a group character, i.e., a homomorphism, so compositions of elements follow the rule $\chi(\gamma_1 \circ \gamma_2) = \chi(\gamma_1) \cdot \chi(\gamma_2)$.

2.2 Restricted moves $\gamma|_D \in \Gamma^{\subseteq}(\mathbb{R})$ as partial bijections of $\mathbb{R}$

As we mentioned in the introduction, compared to [3], where finitely generated subgroups of $\Gamma(\mathbb{R})$ were used for the grid-based extremality test algorithm, in this paper we develop a more detailed theory using restricted moves with domains. Our terminology is based on the monograph [22] on inverse semigroups. We begin by restricting translations and reflections $\gamma \in \Gamma(\mathbb{R})$ to open interval domains $D \subseteq \mathbb{R}$.

▶ **Definition 2.** *Let $\gamma \in \Gamma(\mathbb{R})$ be a translation or reflection, and let $D \subseteq \mathbb{R}$ be an open interval. The* move $\gamma|_D$ *is the partial function with domain D and image $\gamma(D)$, defined by $\gamma|_D(x) = \gamma(x)$ for $x \in D$. The* character *of*

Table 1 Notation for move ensembles and semigroups

$\Gamma(\mathbb{R})$	Group of unrestricted translations and reflections of $\mathbb{R}$		8
τ_t, ρ_r	translation, reflection		
γ	some element		
$\Gamma^{\subseteq}(\mathbb{R})$	Inverse semigroup of translations, reflections with domains		8
$\tau_t\vert_D$	translation restricted to open interval D		
$\rho_r\vert_D$	reflection restricted to open interval D		
$\gamma\vert_D$	unrestricted move restricted to open interval D		
Ω	A move ensemble: a subset of $\Gamma^{\subseteq}(\mathbb{R})$		10
Ω^{inv}	... satisfying (inv)		12
Γ	A move semigroup: an inverse subsemigroup of $\Gamma^{\subseteq}(\mathbb{R})$		12
$\Omega^{\subseteq}$, $\Gamma^{\subseteq}$	A move ensemble, or semigroup, satisfying (restrict)		10
$\Omega^{\vee}$, $\Gamma^{\vee}$	... satisfying (restrict), (cont)		12
$\Omega^{\boxtimes}$, $\Gamma^{\boxtimes}$	... satisfying (restrict), (cont), (kaleido)		17
$\bar{\Omega}$, $\bar{\Gamma}$	... satisfying limit axiom (lim) or (arblim)		19
$\overline{\Omega}^{\vee}$, $\overline{\Gamma}^{\vee}$	... satisying (extend$_A$)		
Ω^{fin}	A finite move ensemble	finite presentation	17
$\mathcal{C}$	Connected covered components		
Ω^{red}	A reduced finite move ensemble		18
$\mathbb{L}$, $\mathbb{M}$	Families of move ensembles		24

Table 2 List of axioms for move ensembles

(comp)	*move semigroup* $\Gamma = \mathrm{isemi}(\Omega)$	*joined semigroup* $\Gamma^{\vee} = \mathrm{jsemi}(\Omega)$	*closed move semigroup* $\overline{\Gamma}^{\vee} = \mathrm{clsemi}_A(\Omega)$	12
(inv)				12
(restrict)	*joined ensemble* $\Omega^{\vee} = \mathrm{join}(\Omega)$			10
(cont)				10
(kaleido)	*kaleidoscopic ensemble* $\Omega^{\boxtimes}$			17
(lim), (arblim)	*limits-closed ensemble* $\bar{\Omega} = \mathrm{arblim}(\Omega)$			19
(extend$_A$)	*extended ensemble* $\overline{\Omega}^{\vee} = \mathrm{extend}_A(\Omega)$			22

$\gamma|_D$ is that of γ. Two moves $\gamma_1|_{D_1}, \gamma_2|_{D_2}$ with nonempty open interval domains D_1, D_2 are equal *if $\gamma_1 = \gamma_2$ and $D_1 = D_2$. A move with a nonempty open interval domain is not* equal *to a move with an empty domain. We identify all translations with empty domain and denote this object by $\tau|_\emptyset$. Likewise, we identify all reflections with empty domain and denote this object by $\rho|_\emptyset$. The empty translation and the empty reflection are not* equal*; they are distinct objects with $\chi(\tau|_\emptyset) = +1$ and $\chi(\rho|_\emptyset) = -1$. Finally, the set of all moves is denoted by $\Gamma^{\subseteq}(\mathbb{R})$.*

▶ Remark 3. Inverse semigroups of partial homeomorphisms between open subsets of a topological space are known as *pseudogroups* [22, Section 1.2]. However, our theory differs in the following ways: (1) We only allow open intervals (and the empty set) as domains of the partial functions, rather than arbitrary open subsets. The reason for our choice will become clear in Section 5, where we will use moves to describe systems of functional equations. (2) Less importantly, we have two empty moves, one for each possible character, rather than a unique empty move.

2.3 Graphs of moves

We find it convenient to describe the graphs of moves. The *graph* of $\gamma|_D$ is the set $\mathrm{Gr}(\gamma|_D) = \{\, (x, y) \in \mathbb{R} \times \mathbb{R} \mid x \in D,\ \gamma(x) = y \,\}$. Figures showing the graphs have already appeared in Figure 2 and Figure 3. To emphasize that the domains of all moves are open intervals, we decorate the endpoints of the nonempty moves by hollow circles, indicating that the endpoints are not part of the graphs.

2.4 Restriction partial order $\subseteq$ on moves

The set of all moves comes with a natural partial order. $\gamma_1|_{D_1}$ is a *restriction* of $\gamma_2|_{D_2}$, denoted $\gamma_1|_{D_1} \subseteq \gamma_2|_{D_2}$, if $D_1 \subseteq D_2$, $\chi(\gamma_1) = \chi(\gamma_2)$, and $\gamma_1(x) = \gamma_2(x)$ for $x \in D_1$. Thus, in this partial order, translations and reflections are incomparable. We have $\tau|_\emptyset \subseteq \tau_t|_D$ for all translations and likewise $\rho|_\emptyset \subseteq \rho_r|_D$.

Given $\gamma|_D$ and an open interval $D' \subseteq D$, the *restriction* of $\gamma|_D$ to D' is the move $(\gamma|_D)\big|_{D'} = \gamma|_{D'}$. Given an open interval $I' \subseteq \gamma(D)$, the *corestriction* of $\gamma|_D$ to I' is the move ${}_{I'}\big|(\gamma|_D) = \gamma|_{D \cap \gamma^{-1}(I')}$.

2.5 Inverse semigroup structure $(\Gamma^{\subseteq}(\mathbb{R}), \circ, \cdot^{-1})$

Let $\gamma_1|_{D_1}$ and $\gamma_2|_{D_2}$ be two moves. As noted in the introduction, their composition $\gamma_2|_{D_2} \circ \gamma_1|_{D_1}$ is defined as $\gamma_2 \circ \gamma_1|_{D_1 \cap \gamma_1^{-1}(D_2)}$ (Figure 2). The domain of this partial bijection is an open interval; so it is again a move. It is clear that the composition operation $\circ$ is associative. Hence the moves form a semigroup $(\Gamma^{\subseteq}(\mathbb{R}), \circ)$.

As we have noted already, a move $\gamma|_D$ also has a (unique) inverse given by $(\gamma|_D)^{-1} = \gamma^{-1}|_{\gamma(D)}$ (Figure 3) satisfying the laws (9) (Figure 4). Hence the moves form an inverse semigroup $(\Gamma^{\subseteq}(\mathbb{R}), \circ, \cdot^{-1})$. Its idempotent elements are exactly the *partial identities*, which are restrictions of the identity translation τ_0 to open intervals D. (The empty translation $\tau|_\emptyset$ is idempotent. The empty reflection is not idempotent; we have $\rho|_\emptyset \circ \rho|_\emptyset = \tau|_\emptyset$.)

The inverse semigroup structure interacts with the restriction partial order (Subsection 2.4) as follows [22, Proposition 1.1.4]. If $\gamma|_{D'} \subseteq \gamma|_D$, then $\gamma|_{D'}^{-1} \subseteq \gamma|_D^{-1}$; moreover, this restriction can be expressed as a composition with an idempotent: $\gamma|_{D'} = (\gamma|_D)\big|_{D'} = \gamma|_D \circ \tau_0|_{D'}$. Finally, if $\gamma_i|_{D_i'} \subseteq \gamma_i|_{D_i}$ for $i = 1, 2$, then $\gamma_2|_{D_2'} \circ \gamma_1|_{D_1'} \subseteq \gamma_2|_{D_2} \circ \gamma_1|_{D_1}$.

3 Ensembles Ω of moves

Now we consider *move ensembles* Ω, i.e., arbitrary subsets of the inverse semigroup $\Gamma^{\subseteq}(\mathbb{R})$. We denote elements by $\gamma|_D$, where $\gamma \in \Gamma(\mathbb{R})$ is an unrestricted move and D is the domain.

3.1 Order-theoretic structure

We shall say that a move ensemble $\Omega^{\subseteq}$ is *restriction-closed* if it satisfies the following axiom.

If $\gamma|_D \in \Omega^{\subseteq}$ and $D' \subseteq D$ is an open interval, then $\gamma|_{D'} \in \Omega^{\subseteq}$. (restrict)

(Throughout the paper, a superscript like $\subseteq$ in $\Omega^{\subseteq}$ indicates an axiom that the set $\Omega^{\subseteq}$ satisfies. See Table 1 for an overview of notation.) For a move ensemble Ω, the restriction closure $\mathrm{restrict}(\Omega)$ is the smallest restriction-closed move ensemble containing Ω. It consists of all restrictions of moves of Ω.

► **Example 4.** The inverse semigroup $\Gamma^{\subseteq}(\mathbb{R})$ of all restricted translations and reflections is a restriction-closed move ensemble.

A move ensemble $\Omega^{\vee}$ is said to be *(completely) join-closed* if it satisfies (restrict) and the following continuation condition, which connects overlapping intervals.

If there is a family $\Omega_{\mathfrak{I}} = \{\, \gamma|_I \mid I \in \mathfrak{I} \,\} \subseteq \Omega^{\vee}$ s.t. $D = \bigcup_{I \in \mathfrak{I}} I$ is an open interval, then $\gamma|_D \in \Omega^{\vee}$. (cont)

We define the *joined ensemble* $\mathrm{join}(\Omega)$ of Ω as the smallest set of moves containing Ω that satisfies (cont) and (restrict).

► **Lemma 5.** *For a move ensemble Ω, the joined ensemble $\mathrm{join}(\Omega)$ consists of the following moves.*

$$\Big\{ \gamma|_D \;\Big|\; D \subseteq \bigcup_{I \in \mathfrak{I}} I \text{ s.t. } \gamma|_I \in \Omega \text{ for } I \in \mathfrak{I},\ D \text{ open interval} \Big\}. \tag{10}$$

Proof. This set clearly satisfies (cont) and (restrict), i.e., it is join-closed. On the other hand, join(Ω) needs to contain this set. ◀

For a move ensemble Ω, let $\mathrm{Max}(\Omega)$ denote the set of maximal elements of Ω in the restriction partial order.

▶ **Lemma 6.** *A join-closed move ensemble $\Omega^\vee$ is equal to the restriction closure and to the joined ensemble of its maximal elements in the restriction partial order:*

$$\Omega^\vee = \mathrm{restrict}(\mathrm{Max}(\Omega^\vee)) = \mathrm{join}(\mathrm{Max}(\Omega^\vee)) \tag{11}$$

Proof. Let $\gamma|_D \in \Omega^\vee$. Let $\mathfrak{D} = \{\, D' \supseteq D \mid \gamma|_{D'} \in \Omega^\vee \,\}$. Let $\bar D = \bigcup \mathfrak{D}$, an open interval. Then $\gamma|_{\bar D} \in \Omega^\vee$ because $\Omega^\vee$ satisfies (cont). Moreover, $\gamma|_D \subseteq \gamma|_{\bar D} \in \mathrm{Max}(\Omega^\vee)$ and thus $\gamma|_D \in \mathrm{restrict}(\mathrm{Max}(\Omega^\vee))$. The other inclusions are trivial. ◀

3.2 Move ensembles as set-valued maps $\mathbb{R} \to 2^{\mathbb{R}}$

Let Ω be a move ensemble and R be a disjoint union of proper open intervals, $R = \bigcup_{R' \in \mathfrak{I}} R'$. The *restriction* $\Omega|_R$ is the move ensemble consisting of the restrictions $\gamma|_{D \cap R'}$ whenever $\gamma|_D \in \Omega$, $R' \in \mathfrak{I}$, and $D = \emptyset$ or $D \cap R' \neq \emptyset$. Similarly, we define the *corestriction* ${}_R|\Omega$ and the *double restriction* ${}_R|\Omega|_R$. In the restrictions, domains of moves are restricted to subintervals of R. Note that by our definition, the restrictions contain empty moves if and only if Ω contains empty moves. Therefore we have the following two convenient properties:

▶ **Lemma 7.** *For a move ensemble $\Omega^\subseteq$ that satisfies* (restrict)*, the restrictions satisfy* (restrict)*, and we have*

$$\begin{aligned} \Omega^\subseteq|_R &= \{\, \gamma|_D \in \Omega^\subseteq \mid D \subseteq R \,\}, \\ {}_R|\Omega^\subseteq &= \{\, \gamma|_D \in \Omega^\subseteq \mid \gamma(D) \subseteq R \,\}, \\ {}_R|\Omega^\subseteq|_R &= \{\, \gamma|_D \in \Omega^\subseteq \mid D, \gamma(D) \subseteq R \,\}. \end{aligned}$$

Likewise, restrictions also preserve (cont)*.*

▶ **Lemma 8.** *Let $\Omega^{\max} = \mathrm{Max}(\Omega^\vee)$, where $\Omega^\vee$ is a joined ensemble. Then each of the restrictions $\Omega^{\max}|_R$, ${}_R|\Omega^{\max}$, and ${}_R|\Omega^{\max}|_R$ consists of the maximal elements of $\Omega^\vee|_R$, ${}_R|\Omega^\vee$, and ${}_R|\Omega^\vee|_R$, respectively.*

We associate with any $x \in \mathbb{R}$ a subset $\Omega(x)$ of $\mathbb{R}$, defined as $\Omega(x) = \{\, \gamma(x) \mid \gamma|_D \in \Omega,\ x \in D \,\}$. Define the *domain* of a move ensemble Ω as $\mathrm{dom}(\Omega) = \bigcup \{\, D \mid \gamma|_D \in \Omega \text{ for some } \gamma \,\}$ and its *image* as $\mathrm{im}(\Omega) = \bigcup \{\, \gamma(D) \mid \gamma|_D \in \Omega \text{ for some } \gamma \,\}$. In these notions, a move ensemble behaves like a set-valued map $\Omega \colon \mathbb{R} \to 2^{\mathbb{R}}$. Now if $X \subseteq \mathbb{R}$ is a set, we also define the image of the set under the ensemble, $\Omega(X) = \{\, \gamma(x) \mid \gamma|_D \in \Omega,\ x \in X \cap D \,\}$.

3.3 Graphs $\mathrm{Gr}(\Omega), \mathrm{Gr}_+(\Omega), \mathrm{Gr}_-(\Omega)$ of move ensembles Ω

We introduced graphs of moves in Subsection 2.3. For a move ensemble Ω we define the *translation moves graph*

$$\mathrm{Gr}_+(\Omega) = \bigcup \{\, \mathrm{Gr}(\gamma|_D) \mid \gamma|_D \in \Omega \text{ and } \chi(\gamma) = 1 \,\},$$

consisting of line segments with slopes $+1$, and the *reflection moves graph*

$$\mathrm{Gr}_-(\Omega) = \bigcup \{\, \mathrm{Gr}(\gamma|_D) \mid \gamma|_D \in \Omega \text{ and } \chi(\gamma) = -1 \,\},$$

consisting of line segments with slopes -1. The *graph* of Ω is $\mathrm{Gr}(\Omega) = \mathrm{Gr}_+(\Omega) \cup \mathrm{Gr}_-(\Omega)$. Further, the *character conflict graph* is $\mathrm{Gr}_\pm(\Omega) = \mathrm{Gr}_+(\Omega) \cap \mathrm{Gr}_-(\Omega)$. The map $\Omega \mapsto (\mathrm{Gr}_+(\Omega), \mathrm{Gr}_-(\Omega))$ becomes an injection if restricted to the join-closed move ensembles $\Omega^\vee$. Hence these pairs of graphs faithfully represent all join-closed move ensembles. (In figures showing these graphs, we superimpose the translation graph (blue) and reflection graph (red).)

We can go back from graphs to ensembles using the following notation. Let $O \subseteq \mathbb{R}^2$. We define the (join-closed) move ensembles

$$\begin{aligned} \mathrm{moves}_+(O) &= \{\, \tau_t|_D \mid \mathrm{Gr}(\tau_t|_D) \subseteq O,\ D \text{ an open interval} \,\}, \\ \mathrm{moves}_-(O) &= \{\, \rho_r|_D \mid \mathrm{Gr}(\rho_r|_D) \subseteq O,\ D \text{ an open interval} \,\}, \\ \mathrm{moves}(O) &= \{\, \gamma|_D \mid \mathrm{Gr}(\gamma|_D) \subseteq O,\ D \text{ an open interval} \,\}. \end{aligned}$$

Thus, $\mathrm{moves}(O) = \mathrm{moves}_+(O) \cup \mathrm{moves}_-(O)$.

4 Inverse semigroups generated by move ensembles

Now we turn to the study of inverse semigroups generated by move ensembles.

4.1 Move semigroups Γ; move semigroups $\operatorname{isemi}(\Omega)$ generated by ensembles Ω

A move ensemble Γ is said to be a *move semigroup* (or, an *inverse subsemigroup* of $\Gamma^{\subseteq}(\mathbb{R})$) if it satisfies the following axioms:

$$\gamma'|_{D'} \circ \gamma|_D \in \Gamma \text{ for all } \gamma|_D, \gamma'|_{D'} \in \Gamma, \tag{comp}$$

$$(\gamma|_D)^{-1} \in \Gamma \text{ for all } \gamma|_D \in \Gamma. \tag{inv}$$

For a move ensemble Ω, the *move semigroup* $\operatorname{isemi}(\Omega)$ generated by Ω is the smallest move semigroup containing Ω. A move semigroup Γ is *finitely generated* if there exists a finite set Ω such that $\Gamma = \operatorname{isemi}(\Omega)$. Let Ω^{inv} be a move ensemble satisfying (inv). Then $\operatorname{isemi}(\Omega^{\text{inv}})$ clearly is the set of all finite compositions $\gamma^k|_{D_k} \circ \dots \circ \gamma^1|_{D_1}$ of moves $\gamma^i|_{D_i} \in \Omega^{\text{inv}}$.

▶ Remark 9. Since the domains of moves in Ω are open intervals, any move $\gamma|_D \in \operatorname{isemi}(\Omega)$ also has a domain D that is an open interval. If $\gamma|_D \in \Omega$, then the idempotent $(\gamma|_D)^{-1} \circ \gamma|_D = \tau_0|_D$ is an element of $\operatorname{isemi}(\Omega)$. The inverse semigroup generated by the empty set is the empty set.

4.2 Move semigroups and joins; joined move semigroups $\operatorname{jsemi}(\Omega)$ generated by ensembles Ω

Move semigroups generated by joined ensembles are not automatically join-closed. On the other hand, joining does preserve the semigroup properties.

▶ **Lemma 10.** *Let Γ be a move semigroup. Then the joined ensemble $\operatorname{join}(\Gamma)$ is a move semigroup. In particular, for a move ensemble Ω, we have*

$$\operatorname{join}(\operatorname{isemi}(\Omega)) = \operatorname{isemi}(\operatorname{join}(\operatorname{isemi}(\Omega))).$$

Proof. Let $\gamma|_D, \gamma'|_{D'} \in \operatorname{join}(\Gamma)$. We first show that $\operatorname{join}(\Gamma)$ satisfies the axiom (comp). By equation (10), there exist collections $\mathfrak{I}$ and $\mathfrak{I}'$ of open intervals, such that $D \subseteq \bigcup_{I \in \mathfrak{I}} I$, $D' \subseteq \bigcup_{I' \in \mathfrak{I}'} I'$, and $\gamma|_I, \gamma'|_{I'} \in \Gamma$ for all $I \in \mathfrak{I}, I' \in \mathfrak{I}'$. We know that

$$\gamma'|_{I'} \circ \gamma|_I = (\gamma' \circ \gamma)|_{\gamma^{-1}(I') \cap I} \in \Gamma, \text{ for all } I \in \mathfrak{I} \text{ and } I' \in \mathfrak{I}',$$

since Γ satisfies (comp), and that

$$\gamma^{-1}(D') \cap D \subseteq \gamma^{-1}\Big(\bigcup_{I' \in \mathfrak{I}'} I'\Big) \cap \Big(\bigcup_{I \in \mathfrak{I}} I\Big) = \bigcup_{I \in \mathfrak{I},\, I' \in \mathfrak{I}'} \big(\gamma^{-1}(I') \cap I\big).$$

Therefore, by equation (10), $\gamma'|_{D'} \circ \gamma|_D = (\gamma' \circ \gamma)|_{\gamma^{-1}(D') \cap D} \in \operatorname{join}(\Gamma)$.

We will now show that $\operatorname{join}(\Gamma)$ satisfies axiom (inv). We know that $(\gamma|_I)^{-1} = \gamma^{-1}|_{\gamma(I)} \in \Gamma$ for all $I \in \mathfrak{I}$, since Γ satisfies (inv), and that $\gamma(D) \subseteq \gamma(\bigcup_{I \in \mathfrak{I}} I) = \bigcup_{I \in \mathfrak{I}} \gamma(I)$. Therefore, $(\gamma|_D)^{-1} = \gamma^{-1}|_{\gamma(D)} \in \operatorname{join}(\Gamma)$. We conclude that $\operatorname{join}(\Gamma)$ is a move semigroup, so $\operatorname{join}(\Gamma) = \operatorname{isemi}(\operatorname{join}(\Gamma))$. ◀

Let Ω be a move ensemble. Then the *joined move semigroup* of Ω is defined as $\operatorname{jsemi}(\Omega) = \operatorname{join}(\operatorname{isemi}(\Omega))$.

4.3 Move semigroups $\operatorname{moves}(O)$, $\operatorname{moves}_+(O)$, $\operatorname{moves}_-(O)$ generated by connected open ensembles

Finitely generated inverse semigroups, as defined in Subsection 4.1, are not general enough for our purposes. As we will see later, we need to consider move ensembles Ω whose graphs are open connected sets. They generate inverse semigroups $\operatorname{isemi}(\Omega)$ that are not finitely generated. However, they have the following simple structure (see Figure 5).

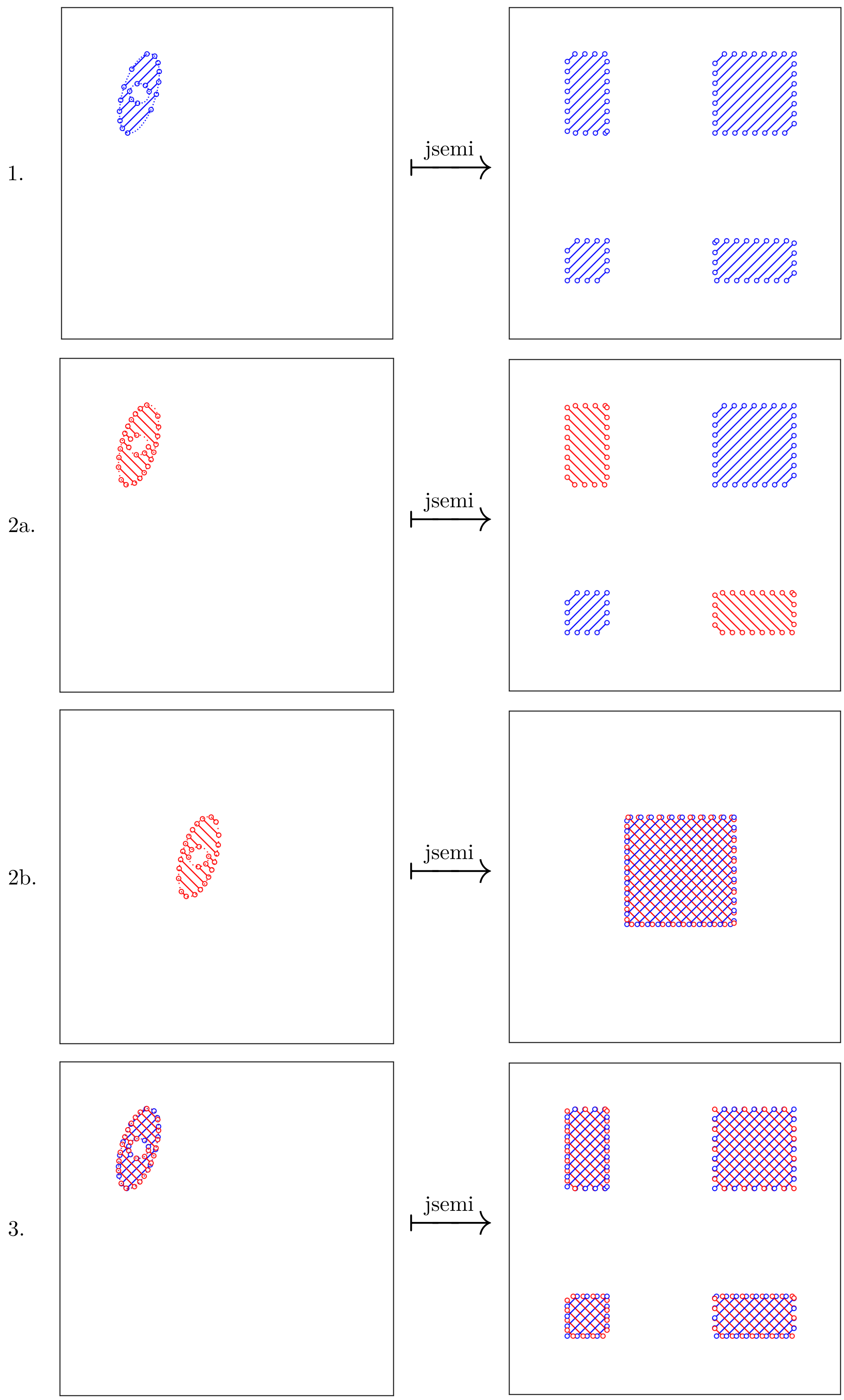


Figure 5 Illustrations for Theorem 11 and Corollary 12. Here, only a finite set of moves is considered. If, however, an infinite set is used by considering all moves in the O-shaped set in the left plots, then the entire rectangles would be filled in on the right plots. The numbering on the left corresponds to the numbering in Corollary 12.

▶ **Theorem 11.** *Let Ω be an ensemble of moves. Let $O \subseteq \mathbb{R}^2$ be a non-empty connected open set. Let $D = \operatorname{dom}(O) := \operatorname{dom}(\operatorname{moves}(O))$ and $I = \operatorname{im}(O) := \operatorname{im}(\operatorname{moves}(O))$.*

1. *If $\mathrm{Gr}_+(\Omega)$ contains O, then $\mathrm{Gr}_+(\operatorname{isemi}(\Omega))$ contains $(D \cup I) \times (D \cup I)$.*
2. *If $\mathrm{Gr}_-(\Omega)$ contains O, then $\mathrm{Gr}_-(\operatorname{isemi}(\Omega))$ contains $(D \times I) \cup (I \times D)$ and $\mathrm{Gr}_+(\Omega)$ contains $(D \times D) \cup (I \times I)$.*
3. *If $\mathrm{Gr}_\pm(\Omega)$ contains O, then $\mathrm{Gr}_\pm(\operatorname{isemi}(\Omega))$ contains $(D \cup I) \times (D \cup I)$.*

Proof. Part 2. We show that (2a) $\mathrm{Gr}_-(\operatorname{isemi}(\Omega))$ contains $D \times I$ and (2b) $\mathrm{Gr}_+(\operatorname{isemi}(\Omega))$ contains $D \times D$; the other two containments of $I \times D$ and $I \times I$ follow from the fact that $\operatorname{isemi}(\Omega)$ is closed under inverse.

Let $(x, y), (x', y') \in O$ be two arbitrary points in the connected open set O. Since there is a path between (x, y) and (x', y') contained in O, and the path is compact, it is covered by finitely many open ℓ_∞-balls $O_1, \dots, O_n \subseteq O$ with $(x_1, y_1) := (x, y) \in O_1$, $(x_2, y_2) \in O_1 \cap O_2, \dots, (x_n, y_n) \in O_{n-1} \cap O_n$ and $(x_{n+1}, y_{n+1}) := (x', y') \in O_n$. Since $(x_1, y_1), (x_2, y_1), (x_2, y_2), \dots, (x_n, y_n), (x_{n+1}, y_n), (x_{n+1}, y_{n+1}) \in O$, there exist $\rho_{r_1}|_{D_1}$, $\rho_{r'_1}|_{D'_1}$, $\rho_{r_2}|_{D_2}, \dots,$ $\rho_{r_n}|_{D_n}$, $\rho_{r'_n}|_{D'_n}$ and $\rho_{r_{n+1}}|_{D_{n+1}} \in \Omega$ such that $\rho_{r_i}|_{D_i}(x_i) = y_i$ for $i = 1, \dots, n+1$ and $\rho_{r'_i}|_{D'_i}(x_{i+1}) = y_i$ for $i = 1, \dots, n$. Notice that the inverse restricted reflections $(\rho_{r'_i}|_{D'_i})^{-1} \in \operatorname{isemi}(\Omega)$ with $(\rho_{r'_i}|_{D'_i})^{-1}(y_i) = x_{i+1}$ for $i = 1, \dots, n$. We have

$$x_1 \xmapsto{\rho_{r_1}|_{D_1}} y_1 \xmapsto{(\rho_{r'_1}|_{D'_1})^{-1}} x_2 \mapsto \cdots \mapsto y_n \xmapsto{(\rho_{r'_n}|_{D'_n})^{-1}} x_{n+1} \xmapsto{\rho_{r_{n+1}}|_{D_{n+1}}} y_{n+1}.$$

The composition of the $2n+1$ reflections

$$\rho_r|_{D_r} := \rho_{r_{n+1}}|_{D_{n+1}} \circ (\rho_{r'_n}|_{D'_n})^{-1} \circ \rho_{r_n}|_{D_n} \circ \cdots \circ (\rho_{r'_1}|_{D'_1})^{-1} \circ \rho_{r_1}|_{D_1}$$

is a restricted reflection, satisfying that $\rho_r|_{D_r} \in \operatorname{isemi}(\Omega)$ and $\rho_r|_{D_r}(x) = y'$. Therefore, (2a) holds. The composition of the $2n$ reflections

$$\tau_t|_{D_t} := (\rho_{r'_n}|_{D'_n})^{-1} \circ \rho_{r_n}|_{D_n} \circ \cdots \circ (\rho_{r'_1}|_{D'_1})^{-1} \circ \rho_{r_1}|_{D_1}$$

is a restricted translation, satisfying that $\tau_t|_{D_t} \in \operatorname{isemi}(\Omega)$ and $\tau_t|_{D_t}(x) = x'$. Therefore, (2b) holds.

Part 1. It follows exactly the same proof as part 2 using instead restricted translations $\tau_{t_1}|_{D_1}$, $\tau_{t'_1}|_{D'_1}$, $\tau_{t_2}|_{D_2}$, $\dots$, $\tau_{t_n}|_{D_n}$, $\tau_{t'_n}|_{D'_n}$, $\tau_{t_{n+1}}|_{D_{n+1}} \in \Omega$.

Part 3. Let $(x, y), (x', y') \in O$. By part 1 and 2, there exist restricted translation and reflection $\tau_t|_{D_t}, \rho_r|_{D_r} \in \operatorname{isemi}(\Omega)$ such that $x \xmapsto{\tau_t|_{D_t}} y \xmapsto{\rho_r|_{D_r}} x'$. The composition $\rho_r|_{D_r} \circ \tau_t|_{D_t}$ is a restricted reflection in $\operatorname{isemi}(\Omega)$. Therefore, $\mathrm{Gr}_-(\operatorname{isemi}(\Omega))$ contains $D \times D$. By part 1, part 2 and the fact that $\operatorname{isemi}(\Omega)$ is closed under inverse, we obtain that part 3 holds. ◀

The following corollary sharpens the result.

▶ **Corollary 12.** *Let $O \subseteq \mathbb{R}^2$ be a non-empty connected open set, with $D = \operatorname{dom}(O) = \operatorname{dom}(\operatorname{moves}(O))$ and $I = \operatorname{im}(O) = \operatorname{im}(\operatorname{moves}(O))$.*

1. $\operatorname{jsemi}(\operatorname{moves}_+(O)) = \operatorname{moves}_+((D \cup I) \times (D \cup I))$.
2a. $\operatorname{jsemi}(\operatorname{moves}_-(O)) = \operatorname{moves}_-((D \times I) \cup (I \times D)) \cup \operatorname{moves}_+((D \times D) \cup (I \times I))$, *if* $D \cap I = \emptyset$.
2b. $\operatorname{jsemi}(\operatorname{moves}_-(O)) = \operatorname{moves}((D \cup I) \times (D \cup I))$, *if* $D \cap I \neq \emptyset$.
3. $\operatorname{jsemi}(\operatorname{moves}(O)) = \operatorname{moves}((D \cup I) \times (D \cup I))$

Proof. By applying Theorem 11 (1), (2) and (3) to $\Omega = \operatorname{moves}_+(O)$, $\Omega = \operatorname{moves}_-(O)$ and $\Omega = \operatorname{moves}(O)$, we obtain that $\operatorname{jsemi}(\Omega)$ on the left-hand side of the equation in (1), (2a) and (3) contains the move ensemble on the right-hand side, respectively. In case (2b) where $D \cap I \neq \emptyset$, by applying Theorem 11 (2) to $\Omega = \operatorname{moves}_-(O)$, we have that $\operatorname{jsemi}(\Omega)$ contains $\operatorname{moves}((D \cup I) \times (D \cap I))$. It then follows from Theorem 11 (3) that $\operatorname{jsemi}(\Omega)$ contains the right-hand side of (2b), $\operatorname{moves}((D \cup I) \times (D \cup I))$. Conversely, the right-hand side of the equation in each case is a joined move semigroup that contains Ω. Hence, the equality holds. ◀

▶ Remark 13. Theorem 11 suggests to consider the following class of generating ensembles for inverse semigroups. Take a finite ensemble $\Omega^{\mathrm{fin}} = \{\gamma^1|_{D_1}, \dots, \gamma^n|_{D_n}\}$ together with a finite list of infinite ensembles of the form $\operatorname{moves}_+(D_i \times I_i)$, $i = n+1, \dots, n+m$ and $\operatorname{moves}_-(D_i \times I_i)$, $i = n+m+1, \dots, n+m+\ell$, where D_i and I_i are proper open intervals. However, we suppress the details of this. In Section 6, an additional assumption will allow us to use a more convenient class of generating ensembles.

5 Ω-equivariant functions

5.1 Spaces of Ω-equivariant functions

Move ensembles encode a system of functional equations as follows.

▶ **Definition 14.** *Let Ω be a move ensemble and let $\theta\colon \mathbb{R} \to \mathbb{R}$ be a function.*

a. *We say that θ is* affinely Ω-equivariant *(in short, θ* respects Ω*) provided that for every $\gamma|_D \in \Omega$ there exists a constant $c^\theta_{\gamma|_D}$ such that*

$$\theta(\gamma|_D(x)) = \chi(\gamma)\theta(x) + c^\theta_{\gamma|_D} \quad \text{for } x \in D, \tag{12}$$

where $\chi(\gamma) = \pm 1$ is the character of γ.

b. *If all constants $c^\theta_{\gamma|_D}$ can be chosen to be zero, then we say that θ is* Ω-equivariant *(or,* equivariant under the action of Ω*).*

Throughout the paper, we will be working with affinely Ω-equivariant functions. At the very end, in Section 10, an important space of Ω-equivariant functions will appear.

▶ Remark 15. It now becomes clear why singletons $\{x\}$ are not allowed as the domain D of a move. The functional equation (12) would degenerate to a single equation with an independent constant $c^\theta_{\gamma|_{\{x\}}}$. The equation and the constant can be eliminated from the system.

Some trivial relations between the constants $c^\theta_{\gamma|_D}$ are induced by the restriction partial order on moves (Subsection 2.4). If $\emptyset \neq D \subset D'$, thus $\gamma|_D \subseteq \gamma|_{D'}$ and $D \neq \emptyset$, then necessarily $c^\theta_{\gamma|_D} = c^\theta_{\gamma|_{D'}}$. Thus it is natural to work with restriction-closed ensembles, as defined in Section 3.

▶ **Lemma 16.** *For a set Θ of functions, we denote by Θ^Ω the set of affinely Ω-equivariant functions in Θ. If Θ is a vector space, then so is Θ^Ω.*

Proof. Let $\theta_1, \theta_2 \in \Theta$ and $a_1, a_2 \in \mathbb{R}$. Let $\theta = a_1\theta_1 + a_2\theta_2$. Then $\theta \in \Theta$. Moreover, let $c^{\theta_1}_{\gamma|_D}$ for $\gamma|_D \in \Omega$ and $c^{\theta_2}_{\gamma|_D}$ for $\gamma|_D \in \Omega$ be the families of constants that satisfy (12) for θ_1 and θ_2, respectively. Then $c^\theta_{\gamma|_D} = a_1 c^{\theta_1}_{\gamma|_D} + a_2 c^{\theta_2}_{\gamma|_D}$ for $\gamma|_D \in \Omega$ is a family of constants that satisfy (12) for θ. ◀

5.2 Join-closed semigroup Γ^{resp} of moves respected by given functions

▶ **Definition 17.** *For a function $\theta\colon \operatorname{dom}(\theta) \to \mathbb{R}$, we denote the* ensemble of moves respected by θ *as*

$$\Gamma^{\mathrm{resp}}(\theta) = \bigl\{\, \gamma|_D \in \Gamma^{\subseteq}(\mathbb{R}) \bigm| D, \gamma(D) \subseteq \operatorname{dom}(\theta),\ \exists c^\theta_{\gamma|_D} \in \mathbb{R} \text{ s.t. } (12) \text{ holds} \,\bigr\}.$$

(Clearly $\Gamma^{\mathrm{resp}}(\theta)$ is the largest move ensemble that θ respects.) For a set Θ' of functions, we denote $\Gamma^{\mathrm{resp}}(\Theta') = \bigcap_{\theta\in\Theta'} \Gamma^{\mathrm{resp}}(\theta)$.

▶ **Theorem 18.** *Let Ω be a move ensemble. If a function θ respects Ω, then θ respects the joined semigroup* $\operatorname{jsemi}(\Omega)$*.*

To prove this, we use the following lemma.

▶ **Lemma 19.** *Let $\mathfrak{I}$ be a collection of proper open intervals that cover the open interval (l, u). If a function g is constant over each interval I from the collection $\mathfrak{I}$, then g is constant over (l, u).*

Proof. Let $m = \frac{l+u}{2}$ and $a = g(m)$. Consider the interval $J = \{\, y \in (l, m) \mid g(x) = a \text{ for all } x \in [y, m] \,\}$. Since m is contained in some open interval $I \in \mathfrak{I}$ and $g(x) = a$ for $x \in I$, we know that J is non-empty. Let $l' = \inf J$. We now show that $l = l'$. Suppose that $l \neq l'$. Then there exists an open interval $I \in \mathfrak{I}$ such that $l' \in I$, and g is constant over I. Since $I \cap J \neq \emptyset$ and $g(x) = a$ for $x \in J$, we have that $g(x) = a$ for $x \in I$, a contradiction to $l' = \inf J$. Hence $g(x) = a$ for all $l < x \leq m$. Similarly, one shows that $g(x) = a$ for all $m \leq x < u$. Therefore, g is constant over (l, u). ◀

Proof of Theorem 18. Let $\gamma|_D \in \operatorname{jsemi}(\Omega)$. Thus, there exists a collection $\mathfrak{I}$ of proper open intervals, such that $D \subseteq \bigcup_{I\in\mathfrak{I}} I$ and $\gamma|_I \in \operatorname{isemi}(\Omega)$ for each $I \in \mathfrak{I}$.

Define $g(x) = \theta(\gamma(x)) - \chi(\gamma)\theta(x)$ for $x \in D$. We first show that g is constant over each interval $I \in \mathfrak{I}$. Let $I \in \mathfrak{I}$. Since $\gamma|_I \in \operatorname{isemi}(\Omega)$, we can write it in the form $\gamma|_I = \gamma_k|_{D_k} \circ \gamma_{k-1}|_{D_{k-1}} \circ \cdots \circ \gamma_1|_{D_1}$, where $\gamma_i|_{D_i} \in \Omega$ or $(\gamma_i|_{D_i})^{-1} \in \Omega$ for $i = 1, 2, \dots, k$. Since θ respects Ω, according to (12), we have that

a. $\theta(\gamma_i|_{D_i}(x)) = \chi(\gamma_i)\theta(x) + \text{constant}$ for all $x \in D_i$, when $\gamma_i|_{D_i} \in \Omega$; and
b. $\theta((\gamma_i|_{D_i})^{-1}(y)) = \chi(\gamma_i^{-1})\theta(y) + \text{constant}$ for all $y \in \gamma_i(D_i)$, when $(\gamma_i|_{D_i})^{-1} \in \Omega$.

By using $y = \gamma_i|_{D_i}(x)$ and $\chi(\gamma_i) = \chi(\gamma_i^{-1}) = \pm 1$, the equation in (b) can be rewritten as that in (a). As a result, if θ respects a move, then θ also respects its inverse. Let $x_0 \in I$ and denote $x_i = \gamma_i(x_{i-1})$ for $i = 1, 2, \dots, k$. Then, $x_i \in D_{i+1}$ for $i = 0, 1, \dots, k-1$, and $x_k = \gamma|_I(x_0) = \gamma(x_0)$. Since θ respects all the moves $\gamma_i|_{D_i}$, the equation $\theta(x_i) = \chi(\gamma_i)\theta(x_{i-1}) + c_i^\theta$ holds for every $i = 1, 2, \dots, k$, where the constants c_i^θ are independent of the choice of $x_0 \in I$. We also know that $\chi(\gamma) = \chi(\gamma_1)\chi(\gamma_2)\dots\chi(\gamma_k)$. Therefore,

$$g(x_0) = \theta(\gamma(x_0)) - \chi(\gamma)\theta(x_0) = \theta(x_k) - \chi(\gamma_k)\chi(\gamma_{k-1})\dots\chi(\gamma_1)\theta(x_0) = \sum_{j=1}^{k}\left(\prod_{i=j+1}^{k}\chi(\gamma_i)\right)c_j^\theta$$

is constant for $x_0 \in I$. Then, it follows from Lemma 19 that g is constant over D. ◀

▶ **Corollary 20.** *For a function θ, the ensemble $\Gamma^{\text{resp}}(\theta)$ defined in Definition 17 is a join-closed move semigroup. The same holds for the ensemble $\Gamma^{\text{resp}}(\Theta')$, where Θ' is a space of functions.*

6 Kaleidoscopic joined ensembles and bounded functions. Finite presentations by moves and components

6.1 Cauchy–Pexider functional equation $f(x) + g(y) = h(x + y)$

Recall from Subsection 5.1 that move ensembles encode systems of functional equations. We now bring a first result on functional equations to use. The following result on the Cauchy–Pexider functional equation on bounded domains appeared in [5, Theorem 4.3]. Here we state it for functions of a single real variable. It is a variant of the Gomory–Johnson interval lemma, which has been used throughout the extreme functions literature. Note that it requires a weak assumption regarding the function space. Boundedness is sufficient; see [5] for a more detailed discussion.

▶ **Lemma 21** (Convex additivity domain lemma)**.** *Let $f, g, h\colon \mathbb{R} \to \mathbb{R}$ be bounded functions and let $E \subseteq \mathbb{R}^2$ be open, convex, and bounded. Suppose that $f(x) + g(y) = h(x + y)$ for all $(x, y) \in E$. Define the projections $p_1(x, y) = x$, $p_2(x, y) = y$, $p_3(x, y) = x + y$ as functions from $\mathbb{R}^2$ to $\mathbb{R}$. Then f, g, h are affine with the same slopes on the domains $p_1(E), p_2(E), p_3(E)$, respectively.*

6.2 Kaleidoscopic move ensembles

When we are only interested in *bounded* functions that respect a move ensemble Ω, then it follows from Lemma 21 that we can replace Ω by a move ensemble $\Omega^{\boxtimes}$ with more convenient properties.

▶ **Lemma 22.** *Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be a bounded function. Let $D, I \subseteq \mathbb{R}$ be proper open intervals. The following are equivalent:*

1. *θ respects* $\text{moves}_+(D \times I)$,
2. *θ respects* $\text{moves}_-(D \times I)$,
3. *θ respects* $\text{moves}(D \times I)$,
4. *θ is affine on D and I with the same slope.*

Proof. We first show that (1) implies (4). By assumption, the function θ satisfies equation (12) for all $\tau_t|_{D_t}$, where $t \in \{\, y - x \mid x \in D,\ y \in I \,\}$ and $D_t = \{\, x \in D \mid x + t \in I \,\}$. Thus, there exists a function $c\colon I + (-D) \to \mathbb{R}$ such that

$$\theta(x + t) = \theta(x) + c(t) \qquad \text{for all } (x, x + t) \in D \times I.$$

The function c is bounded because θ is bounded. Then, by Lemma 21 with $f = h = \theta$ and $g = c$, we have that θ is affine on D and I with the same slope. The proofs that each of (2) and (3) implies (4) are similar; we omit them.

Now we show that (4) implies (1). Fix $t = y - x$ for some $x \in D, y \in I$. Since θ is affine on D and I with the same slope, there exist scalars a, b, b' such that $\theta(x) = a \cdot x + b$ for all $x \in D$ and $\theta(x) = a \cdot x + b'$ for all $x \in I$. But then for all $x \in D$ such that $x + t \in I$, we have that $\theta(x + t) - \theta(x) = a \cdot t + b' - b$, which is constant. Therefore, θ respects $\tau_t|_{D_t}$. Again the proofs that (4) also implies (2) and (3) are similar and we omit them. ◀

Motivated by these results, we make the following definitions.

▶ **Definition 23.** *A move ensemble* $\Omega^{\boxtimes}$ *is a* kaleidoscopic joined ensemble *if it satisfies* (restrict), (cont)*, and the following axiom:*

$$\begin{aligned}&\textit{for proper open intervals } D, I \subseteq \mathbb{R} \\ &\mathrm{moves}_+(D \times I) \subseteq \Omega^{\boxtimes} \textit{ if and only if } \mathrm{moves}_-(D \times I) \subseteq \Omega^{\boxtimes}.\end{aligned} \tag{kaleido}$$

6.3 Covered intervals, connected covered components

▶ **Definition 24.** *For a kaleidoscopic joined ensemble* $\Omega^{\boxtimes}$ *and a proper open interval* D *such that* $\mathrm{moves}(D \times D) \subseteq \Omega^{\boxtimes}$*, we say that* D *is a* covered interval *in* $\Omega^{\boxtimes}$.

Let $\Gamma^{\boxtimes}$ be a kaleidoscopic joined move semigroup. For two proper open intervals D_1, D_2, if $\mathrm{moves}\big((D_1 \cup D_2) \times (D_1 \cup D_2)\big) \subseteq \Gamma^{\boxtimes}$, then we say that both D_1 and D_2 are covered intervals in the same *connected covered component* of $\Gamma^{\boxtimes}$. (Here the word "connected" does not refer to the topology of $\mathbb{R}$, in contrast to Subsection 4.3.) It follows from Corollary 12 that this is an equivalence relation. However, we want to define the notion of a *connected covered component* also for kaleidoscopic joined ensembles $\Omega^{\boxtimes}$ that are not semigroups. In this case there is no equivalence relation (transitivity fails), but we still use the word "components" in the following definition.

▶ **Definition 25.** *Let* $\Omega^{\boxtimes}$ *be a kaleidoscopic joined ensemble. Let* C *be a non-empty open set such that* $\mathrm{moves}(C \times C) \subseteq \Omega^{\boxtimes}$*. Then* C *is called a* connected covered component *of* $\Omega^{\boxtimes}$*. Any two covered intervals* $D_1, D_2 \subseteq C$ *are said to be* connected *by the component* C*.*

The connected covered components of $\Omega^{\boxtimes}$ are partially ordered by set inclusion. The maximal elements in this partial order suffice to describe all covered intervals.

▶ **Corollary 26.** *Let* θ *be a bounded function. Suppose* θ *respects a kaleidoscopic joined ensemble* $\Omega^{\boxtimes}$*. Let* C *be a connected covered component of* $\Omega^{\boxtimes}$*. Then* θ *is affine on all open intervals in* C *with a common slope.*

Proof. Let $D, I \subseteq C$ be proper open intervals. Then $D \times I \subseteq C \times C$, and hence θ respects $\mathrm{moves}(D \times I)$. By the equivalence of conditions (3) and (4) of Lemma 22, θ is affine on D and I with the same slope. ◀

(In Section 10, we will also consider connected *uncovered* components.)

6.4 Presentations by moves Ω^{fin} and components $\mathcal{C} = \{C_1, \dots, C_k\}$

Now we are prepared to define a convenient finite presentation for a large class of kaleidoscopic joined ensembles, which we announced in Remark 13.

▶ **Definition 27.** *Take a finite list of connected covered components* $\mathcal{C} = \{C_1, \dots, C_k\}$*, where each* C_i *is a finite union of disjoint proper open intervals. Define*

$$\mathrm{moves}(\mathcal{C}) = \bigcup_{i=1}^{k} \mathrm{moves}(C_i \times C_i) = \big\{\, \gamma|_D \in \Gamma^{\subseteq}(\mathbb{R}) \;\big|\; D, \gamma(D) \subseteq C_i \textit{ for some } i = 1, \dots, k \,\big\}.$$

The graph $\mathrm{Gr}(\mathrm{moves}(\mathcal{C}))$ is a union of open rectangles. See Figure 6 for a visualization. We plot the components with different colors.

Note that any ensemble of the form $\mathrm{moves}(\mathcal{C})$ or $\mathrm{restrict}(\Omega^{\mathrm{fin}}) \cup \mathrm{moves}(\mathcal{C})$, where Ω^{fin} is a finite move ensemble, satisfies (restrict) and (kaleido), but is not necessarily join-closed. To make a kaleidoscopic joined ensemble, we use the following.

▶ **Definition 28.** *For any finite move ensemble* Ω^{fin} *and a finite list* $\mathcal{C}$ *of connected covered components, define* $\mathrm{jmoves}(\Omega^{\mathrm{fin}}, \mathcal{C}) = \mathrm{join}(\Omega^{\mathrm{fin}} \cup \mathrm{moves}(\mathcal{C}))$*. If* $\Omega^{\mathrm{fin}} = \emptyset$*, we simply write* $\mathrm{jmoves}(\mathcal{C})$.

▶ **Definition 29.** *The ordered pair* $(\Omega^{\mathrm{fin}}, \mathcal{C})$ *is said to be a* finite presentation *(by moves* Ω^{fin} *and components* $\mathcal{C}$*) of the kaleidoscopic joined ensemble* $\mathrm{jmoves}(\Omega^{\mathrm{fin}}, \mathcal{C})$.

▶ **Corollary 30.** *Let* θ *be a bounded function. Suppose* θ *respects a move ensemble* $\Omega^{\boxtimes}$ *that has the finite presentation* $(\Omega^{\mathrm{fin}}, \mathcal{C})$*. Then* θ *is affine on all intervals in* $\mathcal{C}$ *and shares a common slope on all intervals of each component* C_i *of* $\mathcal{C}$*.*

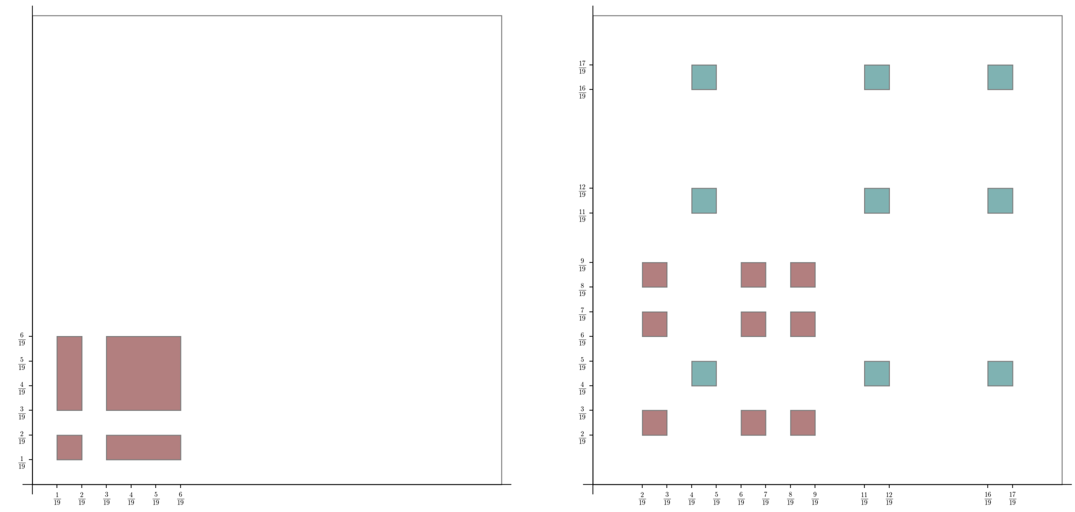


Figure 6 Move ensemble moves($\mathcal{C}$) from connected covered components $\mathcal{C}$. *Left*, $\mathcal{C} = \{C_1\}$ (one component), where $C_1 = (\frac{1}{19}, \frac{2}{19}) \cup (\frac{3}{19}, \frac{6}{19})$, shown in *red*. *Right*, $\mathcal{C} = \{C_1, C_2\}$ (two components), where $C_1 = (\frac{2}{19}, \frac{3}{19}) \cup (\frac{6}{19}, \frac{7}{19}) \cup (\frac{8}{19}, \frac{9}{19})$ is shown in *red* and $C_2 = (\frac{4}{19}, \frac{5}{19}) \cup (\frac{11}{19}, \frac{12}{19}) \cup (\frac{16}{19}, \frac{17}{19})$ is shown in *cyan*.

Proof. This is a restatement of Corollary 26. ◀

It is clear that these presentations are not unique, which motivates the next subsection.

6.5 Finite presentation in reduced form $(\Omega^{\text{red}}, \mathcal{C})$

A finite presentation $(\Omega^{\text{red}}, \mathcal{C})$ of a kaleidoscopic joined ensemble $\Omega^{\boxtimes}$ is said to be in *(long) reduced form* if the following holds:

$$\Omega^{\text{red}} \subseteq \operatorname{Max}(\Omega^{\boxtimes}) \setminus \operatorname{jmoves}(\mathcal{C}), \tag{reduce}$$

that is, each move $\gamma|_D \in \Omega^{\text{red}}$ is maximal in $\Omega^{\boxtimes}$ with respect to the restriction partial order $\sqsubseteq$, and the graph $\operatorname{Gr}(\gamma|_D)$ is not covered by the union of open rectangles $C_i \times C_i$, $C_i \in \mathcal{C}$.

▶ **Lemma 31.** *If a kaleidoscopic joined ensemble $\Omega^{\boxtimes}$ has a finite presentation $(\Omega^{\text{fin}}, \mathcal{C})$, then there is a unique finite ensemble Ω^{red} such that $(\Omega^{\text{red}}, \mathcal{C})$ is in reduced form and $\Omega^{\boxtimes} = \operatorname{jmoves}(\Omega^{\text{red}}, \mathcal{C})$.*

Figure 7 illustrates the operation of going from a finite presentation to a reduced presentation of the same ensemble.

▶ Remark 32. As the examples in Figure 7 illustrate, the domains of moves in Ω^{fin} may be extended.

6.6 Finite presentations of generating ensembles of move semigroups

Move ensembles have a crucial rôle as generating sets of move semigroups. We now describe an operation that changes the generating ensemble, but preserves the move semigroup that is generated by it.

▶ **Lemma 33** (Extend component by move). *Let $\mathcal{C}$ be a list of connected components and let Ω be a move ensemble such that* $\operatorname{moves}(\mathcal{C}) \subseteq \Omega$*. If $\gamma|_D \in \Omega$ and $D \subseteq C_i$ for some $C_i \in \mathcal{C}$, then* $\operatorname{moves}(\mathcal{C}') \subseteq \operatorname{isemi}(\Omega)$*, where $C'_i = C_i \cup \gamma(D)$ and all other components of $\mathcal{C}'$ are the same as $\mathcal{C}$.*

See Figure 8 for an illustration.

Proof. Let $x \in C_i$, $z \in \gamma(D)$, and $y = \gamma^{-1}(z) \in D$. Since $x \in C_i$, x is in the domain of moves τ_0 and ρ_0 in Ω. Thus, we can both translate and reflect x to z by

$$x \xmapsto{\tau_{y-x}} y \xmapsto{\gamma} z, \qquad \text{and} \qquad x \xmapsto{\rho_0} x \xmapsto{\tau_{y-x}} y \xmapsto{\gamma} z.$$

Note that which one above is a translation or reflection depends on the character $\chi(\gamma)$. ◀

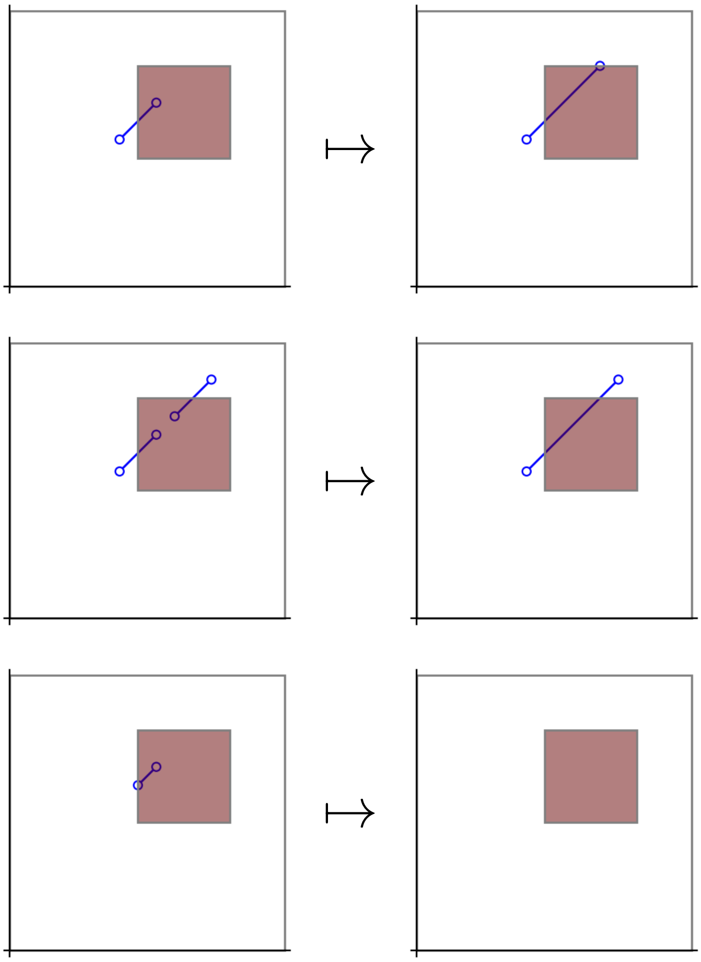


Figure 7 Finite presentation in reduced form. *Left,* finite presentations $(\Omega^{\text{fin}}, \mathcal{C})$ of kaleidoscopic joined ensembles $\Omega^{\boxtimes}$. *Right,* finite presentations $(\Omega^{\text{red}}, \mathcal{C})$ in reduced form of the same ensembles. (a) A move poking into a component is extended to become a maximal move of $\Omega^{\boxtimes}$. (b) Two restrictions of the same move are extended to become a maximal move of $\Omega^{\boxtimes}$. (c) A move that lies completely in a component is removed.

7 Limit-closed ensembles and continuous functions. Closed move semigroups

Let $A \subseteq \mathbb{R}$ be an open set. We now consider the space $\mathcal{C}_\mathrm{b}(A)$ of bounded continuous functions on A. For $\mathcal{C}_\mathrm{b}(A)$, some notions of convergence of moves are natural to study.

7.1 Limit-closed move ensembles $\bar{\Omega}$; closures $\lim(\Omega)$, $\operatorname{arblim}(\Omega)$

We consider a sequence $\{\gamma^i\}_{i\in\mathbb{N}} \subseteq \Gamma(\mathbb{R})$ of unrestricted moves to *converge* to an unrestricted translation $\tau_t \in \Gamma(\mathbb{R})$ if all but finitely many γ^i are translations τ_{t^i} and $t^i \to t$; and to an unrestricted reflection $\rho_r \in \Gamma(\mathbb{R})$ if all but finitely many γ^i are reflections ρ_{r^i} and $r^i \to r$.

We define the *limits closure* $\lim(\Omega)$ of a moves ensemble Ω to be the smallest (by set inclusion) moves ensemble $\bar{\Omega}$ containing Ω that satisfies the following axiom.

$$\text{Let } D \text{ be an open interval. If } \gamma^i \to \gamma \text{ and } \gamma^i|_D \in \bar{\Omega} \text{ for all } i, \text{ then } \gamma|_D \in \bar{\Omega}. \tag{lim}$$

We note that the domain D is fixed for all moves in the sequence. Thus, the limits closure will in general not satisfy (cont) and (inv). Instead we can consider the following axiom.

▶ **Definition 34.** *Define* $\operatorname{arblim}(\Omega)$ *to be the smallest moves ensemble* $\bar{\Omega}$ *containing* Ω *that satisfies the following axiom.*

$$\textit{If } \gamma^i \to \gamma,\ l^i \to l,\ u^i \to u \textit{ and } \gamma^i|_{(l^i,u^i)} \in \bar{\Omega} \textit{ for all } i, \textit{ then } \gamma|_{(l,u)} \in \bar{\Omega}. \tag{arblim}$$

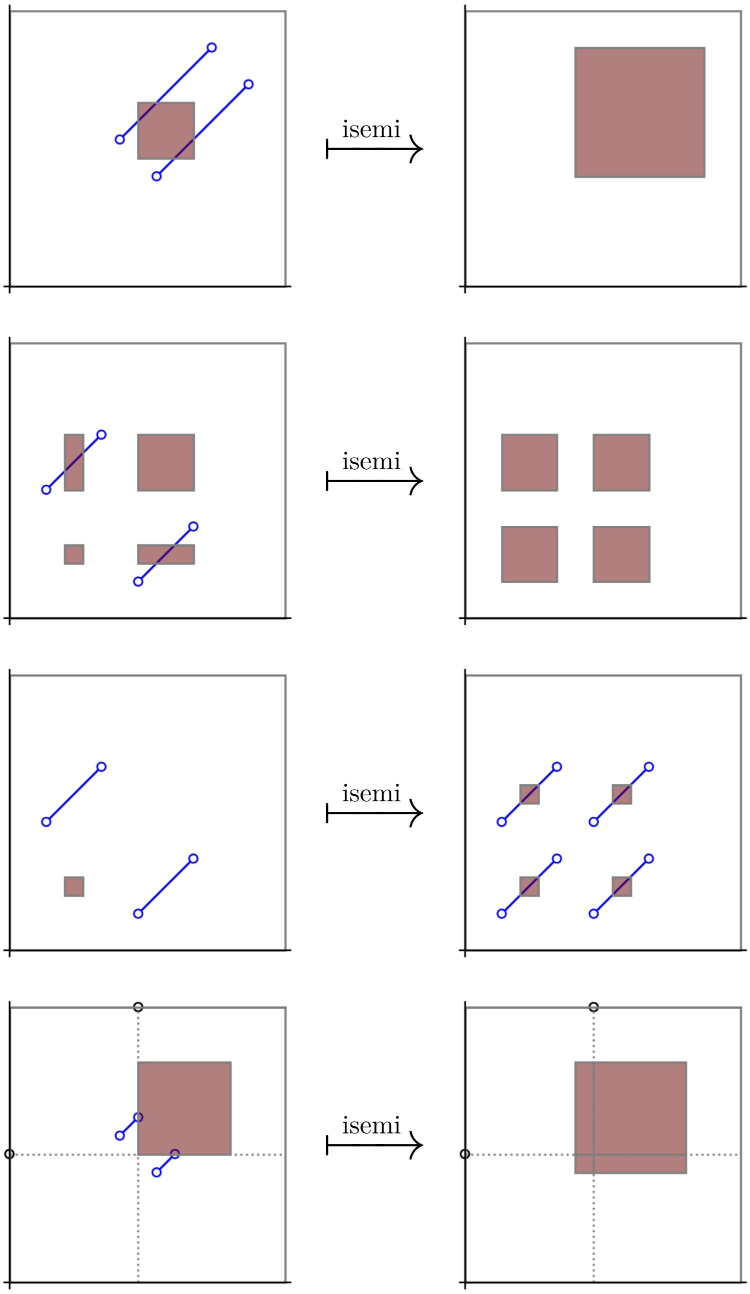


Figure 8 Extending components by moves, Lemma 33. *Left,* reduced finite presentations of a kaleidoscopic joined ensemble $\Omega^{\boxtimes}$. *Right,* reduced finite presentations of $\operatorname{isemi}(\Omega^{\boxtimes})$.

For our purposes, when considered together with (cont), the notions turn out to be equivalent.

▶ **Theorem 35.** *Let $\Omega^{\vee}$ be a join-closed move ensemble. Then*

$$\operatorname{join}(\lim(\Omega^{\vee})) = \operatorname{join}(\operatorname{arblim}(\Omega^{\vee})).$$

Proof. It is clear that $\lim(\Omega^{\vee}) \subseteq \operatorname{arblim}(\Omega^{\vee})$. Hence, it suffices to show that $\operatorname{arblim}(\Omega^{\vee}) \subseteq \operatorname{join}(\lim(\Omega^{\vee}))$. Let $\tau_t|_{(l,u)} \in \operatorname{arblim}(\Omega^{\vee})$, where $l < u$. By (arblim), there is a sequence $\{\tau_{t^i}|_{(l^i,u^i)}\}_{i\in\mathbb{N}}$ of moves in $\Omega^{\vee}$ such that $l^i \to l$, $u^i \to u$ and $t^i \to t$. For every integer $j > \frac{2}{u-l}$, there exists a large integer n_j such that for any $i \geq n_j$, we have $l_i < l + \frac{1}{j}$ and $u - \frac{1}{j} < u_i$. Since $\Omega^{\vee}$ satisfies (restrict), $\tau_{t^i}|_{D_j} \in \Omega^{\vee}$ for any $i \geq n_j$, where $D_j := (l + \frac{1}{j}, u - \frac{1}{j})$. Since $t_i \to t$, we have $\tau_t|_{D_j} \in \lim(\Omega^{\vee})$ for every j, hence $\tau_t|_{(l,u)} \in \operatorname{join}(\lim(\Omega^{\vee}))$. The proof for reflections is similar. ◀

▶ **Theorem 36.** *Let $\Omega^\vee$ be a join-closed move ensemble. The following are equivalent:*
- **i.** $\Omega^\vee$ *satisfies* (lim).
- **ii.** $\Omega^\vee$ *satisfies* (arblim).

The proof is essentially the same and we omit it.

7.1.1 Respecting limits

▶ **Lemma 37** (Limits)**.** *Let D be an open interval and let θ be continuous on D. If there exists a sequence $\gamma^i \to \gamma$ such that θ respects $\gamma^i|_D$ for all i, then θ also respects $\gamma|_D$.*

Proof. We prove the lemma for a sequence $t_i \to t$ such that θ respects the translations $\tau_{t_i}|_D$ for all i. We will show that θ also respects $\tau_t|_D$. Since θ is continuous on D, θ is also continuous on $\tau_{t_i}(D)$ for all i. Fix $\bar{x} \in D$. Since $t_i \to t$, and $\bar{x} \in \operatorname{int}(D)$ since D is open, there exists an $\hat{i}$ such that for all $i \geq \hat{i}$, we have $\bar{x} + t \in D + t_i$. Hence, for a neighborhood $N_{\bar{x}}$ of $\bar{x}$, θ is continuous in $N_{\bar{x}} + t$. Now, for all $x \in N_{\bar{x}}$,

$$\theta(x+t) - \theta(x) = \lim_{t_i \to t} \theta(x + t_i) - \theta(x) = \lim_{i\to\infty} c^\theta_{\tau_{t_i}|_D}.$$

Since the limit on the right-hand side is independent of x, we define $c^\theta_{\tau_t|_{N_{\bar{x}}}}$ to be this limit. Thus, θ respects $\tau_t|_{N_{\bar{x}}}$. Now the connected open set D is covered by the open neighborhoods $N_{\bar{x}}$ of each $\bar{x} \in D$. It follows that $c^\theta_{\tau_t|_{N_{\bar{x}}}} = c^\theta_{\tau_t|_{N_{\bar{x}'}}}$ for all $\bar{x}, \bar{x}' \in D$. Therefore, θ respects $\tau_t|_D$. Moreover, θ is continuous on $\tau_t|_D$. The proof for a sequence of reflections is the same. ◀

7.1.2 Limit-closed move semigroups

▶ **Lemma 38.** *Let Γ be a move semigroup. Then* $\operatorname{arblim}(\Gamma)$ *is also a move semigroup.*

Proof. It is clear that $\operatorname{arblim}(\Gamma)$ satisfies (inv), as Γ satisfies (inv). We now show that $\operatorname{arblim}(\Gamma)$ satisfies (comp). Let $\gamma_1|_{D_1}, \gamma_2|_{D_2} \in \operatorname{arblim}(\Gamma)$ such that $\gamma_1|_{D_1} \circ \gamma_2|_{D_2}$ is not an empty move. $\gamma_1|_{D_1}$ and $\gamma_2|_{D_2}$ are the arblim of sequences of moves $\{\gamma_1^i|_{D_1^i}\}_{i\in\mathbb{N}}$ and $\{\gamma_2^i|_{D_2^i}\}_{i\in\mathbb{N}}$ in Γ. Since Γ satisfies (comp), $\gamma^i|_{D^i} := (\gamma_1^i|_{D_1^i}) \circ (\gamma_2^i|_{D_2^i}) \in \Gamma$ for every i. The arblim of the sequence $\{\gamma^i|_{D^i}\}_{i\in\mathbb{N}}$ is $\gamma_1|_{D_1} \circ \gamma_2|_{D_2}$. Thus, we obtain that $\gamma_1|_{D_1} \circ \gamma_2|_{D_2} \in \operatorname{arblim}(\Gamma)$. This show that $\operatorname{arblim}(\Gamma)$ is a semigroup. ◀

▶ **Lemma 39.** *Let $\Gamma^\vee$ be a join-closed move semigroup. Then* $\operatorname{join}(\lim(\Gamma^\vee)) = \operatorname{join}(\operatorname{arblim}(\Gamma^\vee))$ *is a semigroup.*

Proof. It follows from Lemma 38, Lemma 10 and Theorem 35. ◀

▶ **Theorem 40** (Limits imply components)**.** *Let $\Gamma^\vee$ be a join-closed move semigroup. Assume that $\gamma|_D$ is the limit move (in the sense of* lim *or* arblim*) of a sequence $\{\gamma^i|_{D^i}\}_{i\in\mathbb{N}}$ of moves in $\Gamma^\vee$ with $\gamma^i \neq \gamma$ for every i. Let $I = \gamma(D)$. Then the following holds.*
1. *If γ is a translation, then* $\operatorname{moves}_+((D \cup I) \times (D \cup I)) \subseteq \operatorname{join}(\lim(\Gamma^\vee))$.
2. *If γ is a reflection, then* $\operatorname{moves}_-((D \times I) \cup (I \times D)), \operatorname{moves}_+((D \times D) \cup (I \times I)) \subseteq \operatorname{join}(\lim(\Gamma^\vee))$.

Proof. Let $D = (l, u)$. If a sequence $\{\gamma^i|_{D^i}\}_{i\in\mathbb{N}}$ of moves in $\Gamma^\vee$ with $\gamma^i \neq \gamma$ converges to $\gamma|_D$ in the sense of arblim, then $\gamma^i|_{D^i \cap (l+\epsilon, u-\epsilon)} \to \gamma|_{(l+\epsilon,u-\epsilon)}$ in the sense of lim for any small $\epsilon > 0$. Thus, it suffices to prove the statement for a limit move $\gamma|_D$ in the sense of lim; the statement for arblim follows from Lemma 39 and continuation.

We first show that $\operatorname{moves}_+(D \times D) \subseteq \operatorname{join}(\lim(\Gamma^\vee))$. Let $\epsilon > 0$ be an arbitrary small number. Since $\gamma|_D$ is a limit move, there exist $\gamma^i|_D, \gamma^j|_D \in \Gamma^\vee$ in the convergent sequence such that the constant-valued functions $\gamma - \gamma^i$ and $\gamma - \gamma^j$ have the same sign, and $0 < \delta < \epsilon$, where δ denotes the constant value of $\gamma^j - \gamma^i$. Let $D^1 = (l, u) \cap (l - \delta, u - \delta)$. We notice that $(\gamma^i|_D)^{-1} \circ \gamma^j|_D = \tau_\delta|_{D^1}$ when γ is a translation, and $(\gamma^j|_D)^{-1} \circ \gamma^i|_D = \tau_\delta|_{D^1}$ when γ is a reflection. Therefore, $\tau_\delta|_{D^1} \in \Gamma^\vee$. Let $D^k := (l, u) \cap (l - k\delta, u - k\delta)$ for $k \in \mathbb{Z}$. For $k \geq 1$, $\tau_{k\delta}|_{D^k}$ is the k times composition of $\tau_\delta|_{D^1}$, hence it is in $\Gamma^\vee$. For $k = -1$, $\tau_{-\delta}|_{D^{-1}} = (\tau_\delta|_{D^1})^{-1} \in \Gamma^\vee$. For $k \leq -2$, $\tau_{k\delta}|_{D^k}$ is the $-k$ times composition of $\tau_{-\delta}|_{D^{-1}}$, and hence is in $\Gamma^\vee$. Finally, for $k = 0$, we have $(\tau_\delta|_{D^1}) \circ (\tau_{-\delta}|_{D^{-1}}), (\tau_{-\delta}|_{D^{-1}}) \circ (\tau_\delta|_{D^1}) \in \Gamma^\vee$, so their join $\tau_0|_{D^0}$ is also in $\Gamma^\vee$. Therefore, for every $k \in \mathbb{Z}$ such that D^k is not empty, we have $\tau_{k\delta}|_{D^k} \in \Gamma^\vee$. By letting $\epsilon \to 0$, we obtain that $\operatorname{moves}_+(D \times D) \subseteq \operatorname{join}(\lim(\Gamma^\vee))$.

Since $\gamma|_D \in \lim(\Gamma^\vee) \subseteq \operatorname{join}(\lim(\Gamma^\vee))$ and $\operatorname{join}(\lim(\Gamma^\vee))$ is a semigroup by Lemma 39, we have that $\operatorname{moves}_+(D \times I) \subseteq \operatorname{join}(\lim(\Gamma^\vee))$ when γ is a translation, and $\operatorname{moves}_-(D \times I) \subseteq \operatorname{join}(\lim(\Gamma^\vee))$ when γ is a reflection.

The other two subsets follow from applying the above argument to $(\gamma|_D)^{-1}$ instead of $\gamma|_D$. ◀

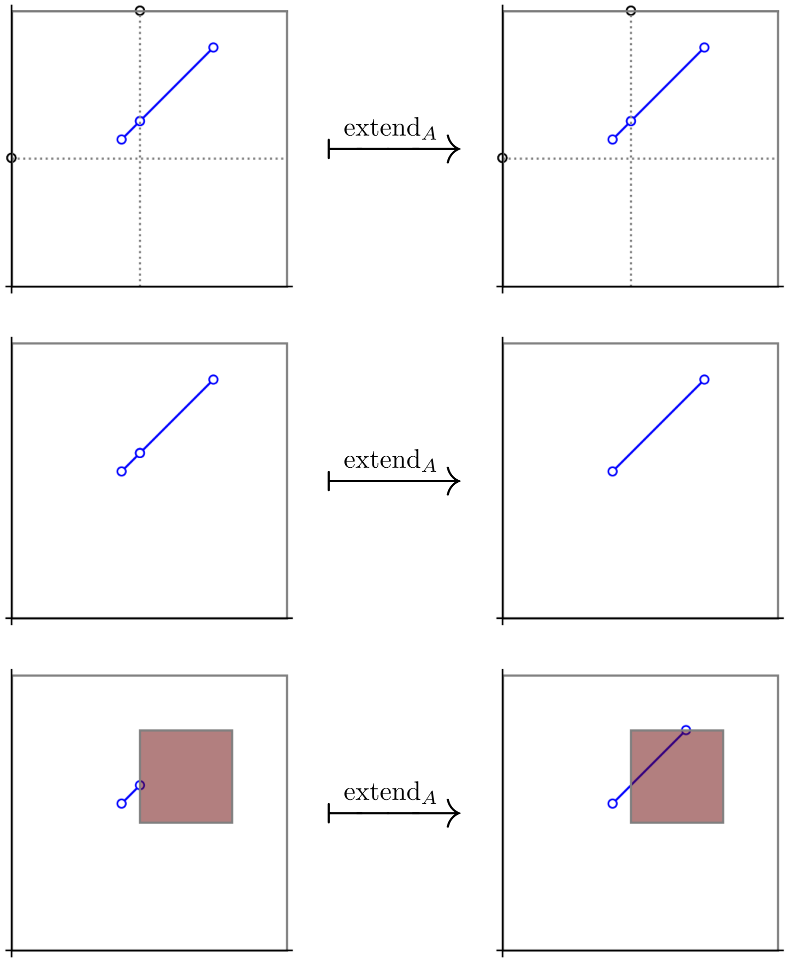


Figure 9 Extended move ensembles $\operatorname{extend}_A(\Omega)$ of ensembles Ω. Points not in the set A are indicated by *black circles at the top and left border*. The set A is $\mathbb{R} \setminus \{\text{a single point}\}$ in the top example, and $A = \mathbb{R}$ in the bottom two examples.

7.2 Continuous domain extension $\operatorname{extend}_A(\Omega)$

Next we introduce a topological version of axiom (cont). Let Ω be a move ensemble with $\operatorname{dom}(\Omega), \operatorname{im}(\Omega) \subseteq A$, where $A \subseteq \mathbb{R}$ is an open set. Then the *extended move ensemble* $\operatorname{extend}_A(\Omega)$ of Ω is defined to be the smallest set $\overline{\Omega}^{\vee}$ containing Ω that satisfies the following axiom:

$$\text{Let } \gamma \in \Gamma(\mathbb{R}) \text{ and let } D \text{ be an open interval.} \tag{extend$_A$}$$
$$\text{If there is an ensemble } \{\gamma|_{D^i}\}_{i\in\mathfrak{I}} \subseteq \overline{\Omega}^{\vee} \text{ s.t. } D \subseteq \operatorname{cl}(\textstyle\bigcup_{i\in\mathfrak{I}} D^i) \cap A \cap \gamma^{-1}(A), \text{ then } \gamma|_D \in \overline{\Omega}^{\vee}.$$

Clearly an ensemble satisfying (extend$_A$) is join-closed. The most simple application of (extend$_A$) allows us to join two adjacent moves across a point of continuity; see Figure 9.

▶ **Lemma 41.** *Let $\overline{\Omega}^{\vee}$ be a move ensemble satisfying* (extend$_A$)*. Then we have:*

$$\textit{If } \gamma|_{(l,m)}, \gamma|_{(m,u)} \in \overline{\Omega}^{\vee}, \textit{ where } l < m < u, \textit{ and } m, \gamma(m) \in A, \textit{ then } \gamma|_{(l,u)} \in \overline{\Omega}^{\vee}. \tag{2-extend$_A$}$$

The following is clear from the definition.

▶ **Lemma 42.** *Let Ω be a move ensemble with* $\operatorname{dom}(\Omega) = \operatorname{im}(\Omega) \subseteq A$*. Let* $\overline{\Omega}^{\vee} = \operatorname{extend}_A(\Omega)$*. Then* $\operatorname{dom}(\overline{\Omega}^{\vee}) = \operatorname{im}(\overline{\Omega}^{\vee}) \subseteq A$*.*

▶ Remark 43. If $\Omega^{\vee}$ is a joined ensemble with finite $\operatorname{Max}(\Omega^{\vee})$, then repeated application of (2-extend$_A$), followed by applying (cont), suffices to obtain $\operatorname{extend}_A(\Omega^{\vee})$.

However, this is not true for arbitrary joined ensembles $\Omega^{\vee}$. As an example, let $A = \mathbb{R}$ and consider $\Omega^{\vee}$ consisting of the restrictions of a move γ to all subintervals of $(-1, 0)$ and $(\frac{1}{n+1}, \frac{1}{n})$ for $n \in \mathbb{N}$. (These maximal intervals are disjoint.) Domains of moves of $\operatorname{extend}_A(\Omega^\vee)$ are all subintervals of $(-1, 1)$. The domains of moves of 2-$\operatorname{extend}_A(\Omega^\vee)$ are $(-1, 0)$ and its subintervals and the infinite chain $(\frac{1}{m}, 1)$ for $m \in \mathbb{N}$ and some of its subintervals; the supremum of the chain, $(0, 1)$ is not an element. Then the domains of maximal moves of join(2-$\operatorname{extend}_A(\Omega^\vee)$) are $(-1, 0)$ and $(0, 1)$. It takes *another round* of 2-extend_A to arrive at $\operatorname{extend}_A(\Omega^\vee)$.

We have an explicit description of the moves in the extended move ensemble $\operatorname{extend}_A(\Omega)$, similar to Lemma 5 for $\operatorname{join}(\Omega)$.

▶ Remark 44. For a move ensemble Ω with $\operatorname{dom}(\Omega), \operatorname{im}(\Omega) \subseteq A$, where $A \subseteq \mathbb{R}$ is an open set, we have

$$\operatorname{extend}_A(\Omega) = \bigl\{\, \gamma|_D \bigm| \gamma \in \Gamma(\mathbb{R}),\ D \text{ open interval},\ D \subseteq \operatorname{cl}(C_\gamma) \cap A \cap \gamma^{-1}(A) \,\bigr\}, \tag{13}$$

where $C_\gamma := \bigcup\{\, I \mid \gamma|_I \in \Omega \,\}$, which is a subset of $A \cap \gamma^{-1}(A)$.

7.2.1 Domain extension and semigroups

▶ **Lemma 45.** *Let* Γ *be a move semigroup with* $\operatorname{dom}(\Gamma), \operatorname{im}(\Gamma) \subseteq A$*, where* $A \subseteq \mathbb{R}$ *is an open set. Then* $\operatorname{extend}_A(\Gamma)$ *is a move semigroup that satisfies* $(\operatorname{extend}_A)$.

Proof. Since Γ satisfies (inv), it is clear that $\operatorname{extend}_A(\Gamma)$ satisfies (inv). We now show that $\operatorname{extend}_A(\Gamma)$ satisfies (comp), too. Let $\gamma_1|_{D_1}, \gamma_2|_{D_2} \in \operatorname{extend}_A(\Gamma)$. Let

$$C_1 = C_{\gamma_1} = \bigcup\{\, I \mid \gamma_1|_I \in \Gamma \,\} \qquad \text{and} \qquad C_2 = C_{\gamma_2} = \bigcup\{\, I \mid \gamma_2|_I \in \Gamma \,\}.$$

By equation (13), the open sets D_1 and D_2 satisfy that

$$D_1 \subseteq \operatorname{cl}(C_1) \cap A \cap \gamma_1^{-1}(A) \qquad \text{and} \qquad D_2 \subseteq \operatorname{cl}(C_2) \cap A \cap \gamma_2^{-1}(A).$$

Let $\gamma = \gamma^2 \circ \gamma^1$, $C = C_\gamma = \bigcup\{\, I \mid \gamma|_I \in \Gamma \,\}$ and let $D = \gamma_1^{-1}(D_2) \cap D_1$ be a non-empty open set. We will show that

$$D \subseteq \operatorname{cl}(C) \cap A \cap \gamma^{-1}(A). \tag{14}$$

It then follows again from (13) that $\gamma_2|_{D_2} \circ \gamma_1|_{D_1} = \gamma|_D \in \operatorname{extend}_A(\Gamma)$, and hence $\operatorname{extend}_A(\Gamma)$ is a move semigroup. It suffices to show (14) for

$$D_1 = \operatorname{int}\bigl(\operatorname{cl}(C_1) \cap A \cap \gamma_1^{-1}(A)\bigr) \ \text{ and } \ D_2 = \operatorname{int}\bigl(\operatorname{cl}(C_2) \cap A \cap \gamma_2^{-1}(A)\bigr).$$

We have on the left hand side of (14)

$$\begin{aligned} D &= \gamma_1^{-1}(D_2) \cap D_1 \\ &= \operatorname{int}\bigl(\gamma_1^{-1}\bigl(\operatorname{cl}(C_2)\bigr) \cap \gamma_1^{-1}(A) \cap \gamma^{-1}(A)\bigr) \cap \operatorname{int}\bigl(\operatorname{cl}(C_1) \cap A \cap \gamma_1^{-1}(A)\bigr) \\ &= \operatorname{int}\bigl(\operatorname{cl}(C_1)\bigr) \cap \gamma_1^{-1}\bigl(\operatorname{int}\bigl(\operatorname{cl}(C_2)\bigr)\bigr) \cap A \cap \gamma_1^{-1}(A) \cap \gamma^{-1}(A), \end{aligned}$$

and on the right hand side of (14) $\operatorname{cl}(C) \cap A \cap \gamma^{-1}(A)$. Thus, it suffices to prove that if $x \in \operatorname{int}\bigl(\operatorname{cl}(C_1)\bigr)$ such that $\gamma_1(x) \in \operatorname{int}\bigl(\operatorname{cl}(C_2)\bigr)$, then $x \in \operatorname{int}\bigl(\operatorname{cl}(C)\bigr)$. This holds since Γ satisfies (comp). ◀

7.2.2 Respecting extensions

▶ **Lemma 46** (Extend moves by continuity)**.** *Let* θ *be a function that respects a move ensemble* Ω *with* $\operatorname{dom}(\Omega), \operatorname{im}(\Omega) \subseteq A$*. Then it respects the extended move ensemble* $\operatorname{extend}_A(\Omega)$*.*

Proof. We use the characterization of $\operatorname{extend}_A(\Omega)$ from Remark 44. Let $\gamma \in \Gamma(\mathbb{R})$ and let $C_\gamma \subseteq A \cap \gamma^{-1}(A)$ be as in Remark 44. The function $x \mapsto \theta(\gamma(x)) - \chi(\gamma)\theta(x)$ is constant on the connected components of C_γ and it is continuous on $A \cap \gamma^{-1}(A)$. Then it is constant on the connected components of $\operatorname{cl}(C_\gamma) \cap A \cap \gamma^{-1}(A)$. ◀

Applied to the simple case of Lemma 41, we have the following.

▶ **Corollary 47.** *Suppose* θ *respects the moves* $\gamma|_{(l,m)}, \gamma|_{(m,u)}$ *with* $l < m < u$ *and suppose* θ *is continuous at* m*,* $\gamma(m)$*. Then* θ *respects* $\gamma|_{(l,u)}$*.*

▶ Remark 48. The assumption regarding continuity at both m and $\gamma(m)$ cannot be removed, which explains why we use $A \cap \gamma^{-1}(A)$ in $(\operatorname{extend}_A)$. We illustrate this by the following example. Let $A = (0, 2) \cup (2, 3)$. Let $\gamma = \tau_1$ and $\Omega = \{\gamma|_{(0,1)}, \gamma|_{(1,2)}\}$, so $\operatorname{dom}(\Omega) = (0, 1) \cup (1, 2) \subseteq A$ and $\operatorname{im}(\Omega) = (1, 2) \cup (2, 3) \subseteq A$. Then $1 \in A$, but $\gamma(1) = 2 \notin A$. Define $\theta = 0$ on A and $\theta(2) = 1$, so it is continuous at 1 but not at $\gamma(1) = 2$. Then θ respects Ω, but it does not respect the move $\gamma|_{(0,2)}$.

7.3 Closed move semigroups, the moves closure $\mathrm{clsemi}_A(\Omega)$

Now all axioms that we have introduced above come together.

▶ **Definition 49.** *A* closed move semigroup *is a limits-closed extension-closed kaleidoscopic joined move semigroup, i.e., a move ensemble that satisfies the axioms:* (comp)*,* (inv)*,* (cont)*,* (restrict)*,* (extend$_A$)*,* (lim)*, and* (kaleido)*.*

▶ **Definition 50.** *Let Ω be a move ensemble with $\mathrm{dom}(\Omega), \mathrm{im}(\Omega) \subseteq A$. We define the* closed move semigroup $\mathrm{clsemi}_A(\Omega)$ *generated by Ω (or just* moves closure *of Ω) to be the smallest (by set inclusion) closed move semigroup containing Ω.*

▶ **Lemma 51.** *Let $\mathbb{L}$ be the family of closed move semigroups containing Ω. Then $\mathrm{clsemi}_A(\Omega) = \bigcap \mathbb{L} = \bigcap_{\Omega' \in \mathbb{L}} \Omega'$.*

Proof. First of all, $\bigcap \mathbb{L}$ contains Ω. We show that $\bigcap \mathbb{L}$ is a closed move semigroup. Note that each axiom is a closure property of a set Ω' of the form: For all pairs of moves ensembles (Ω_1, Ω_2) obtained from the operation in the axiom (see an example below), if $\Omega_1 \subseteq \Omega'$, then $\Omega_2 \subseteq \Omega'$. Now if $\Omega_1 \subseteq \bigcap \mathbb{L}$, then $\Omega_1 \subseteq \Omega'$ for all $\Omega' \in \mathbb{L}$, and thus $\Omega_2 \subseteq \Omega'$ for all $\Omega' \in \mathbb{L}$. This implies $\Omega_2 \subseteq \bigcap \mathbb{L}$.

For example, we show that $\bigcap \mathbb{L}$ satisfies the axiom (lim) as follows. Let D be an open interval, and let $\Omega_1 \subseteq \bigcap \mathbb{L}$ be an ensemble of moves $\{\gamma^i|_D\}_{i=1,2,\dots}$ in $\bigcap \mathbb{L}$ such that $\gamma^i \to \gamma$ as $i \to \infty$. We want to show that $\gamma|_D \in \bigcap \mathbb{L}$. Let $\Omega_2 = \{\gamma|_D\}$. Since $\Omega_1 \subseteq \bigcap \mathbb{L}$, $\Omega_1 \subseteq \Omega'$ for all $\Omega' \in \mathbb{L}$. Each $\Omega' \in \mathbb{L}$ is a closed move semigroup, which satisfies the axiom (lim) in particular, so $\Omega_2 \subseteq \Omega'$. This implies $\Omega_2 \subseteq \bigcap \mathbb{L}$.

On the other hand, $\bigcap \mathbb{L}$ is contained in each of the ensembles $\Omega' \in \mathbb{L}$ and is therefore the smallest closed move semigroup containing Ω. ◀

▶ Remark 52. In contrast to Lemma 10 (regarding (cont) and (restrict) and the axioms of an inverse semigroup), we do not know whether $\mathrm{clsemi}_A(\Omega)$ can be obtained by applying a finite sequence of closures with respect to the individual axioms.

▶ **Theorem 53** (Main theorem on the moves closure)**.** *Suppose θ is bounded and continuous on A. If θ respects a move ensemble Ω with $\mathrm{dom}(\Omega), \mathrm{im}(\Omega) \subseteq A$, then θ respects the moves closure $\mathrm{clsemi}_A(\Omega)$.*

Proof. Let $\theta|_A$ denote the restriction of θ to A. We consider the ensemble $\Gamma = \Gamma^{\mathrm{resp}}(\theta|_A)$ of moves that $\theta|_A$ respects, introduced in Subsection 5.2. By definition, $\mathrm{dom}(\Gamma), \mathrm{im}(\Gamma) \subseteq A$. Since, by assumption, θ respects Ω, we have $\Gamma \supseteq \Omega$. By Theorem 18, Γ is a join-closed move semigroup. By Lemma 22, because $\theta|_A$ is bounded, Γ satisfies the axiom (kaleido). Because $\theta|_A$ is continuous, we can apply Lemma 37 to all convergent sequences $\{\gamma^i|_D\}_{i\in\mathbb{N}} \subseteq \Gamma$, and thus Γ satisfies the axiom (lim). Finally, by Lemma 46, it satisfies the axiom (extend$_A$). Hence, $\Gamma^{\mathrm{resp}}(\theta)$ is a closed move semigroup. By Lemma 51, we conclude that θ respects $\mathrm{clsemi}_A(\Omega)$. ◀

8 The initial additive move ensemble Ω^0 of a subadditive function

We will now apply the theory of the previous sections to compute the effective perturbation spaces of minimal valid functions. Let $\pi\colon \mathbb{R} \to \mathbb{R}$ be a minimal valid function. Recall from the introduction that π is nonnegative, $\mathbb{Z}$-periodic, and satisfies $\pi(0) = 0$, $\pi(f) = 1$. Its key property is subadditivity, which we express using the *subadditivity slack* function $\Delta\pi(x, y) = \pi(x) + \pi(y) - \pi(x + y)$ as $\Delta\pi(x, y) \geq 0$. Moreover, the symmetry condition $\Delta\pi(x, f - x) = 0$ holds for all x. This is the characterization that appeared in the introduction as (3).

Since π is $\mathbb{Z}$-periodic, we will work with its fundamental domain $[0, 1]$. For the rest of the paper, we will let $A = A(\pi)$ be the maximal open subset of $(0, 1)$ on which π is continuous.

8.1 The initial move ensemble Ω^0

We begin by defining an ensemble of *initial moves* $\Omega^0 = \Omega^0(\pi)$ that consists of *additive moves* and *limit additive moves*, together with their inverses and the empty moves. We define these moves $\gamma|_D$ on domains D that are open intervals such that the domain D and the image $\gamma(D)$ are subsets of A .

▶ **Definition 54.**

i. *An* additive move *is any translation $\tau_t|_D$, where $t \in (-1, 1)$ and $D \subseteq A$ is a proper open interval such that $\tau_t(D) \subseteq A$ and*

$$\Delta\pi(x, t) = \pi(x) + \pi(t) - \pi(x + t) = 0 \qquad \forall x \in D$$

or any reflection $\rho_r|_D$*, where* $r \in (0,2)$*, and* $D \subseteq A$ *is a proper open interval such that* $\rho_r(D) \subseteq A$ *such that*

$$\Delta\pi(x, r-x) = \pi(x) + \pi(r-x) - \pi(r) = 0 \qquad \forall x \in D.$$

ii. *A* limit-additive move *is any translation* $\tau_{\bar t}|_D$*, where* $\bar t \in (-1,1)$ *and* $D \subseteq A$ *is a proper open interval such that* $\tau_{\bar t}(D) \subseteq A$ *and*

$$\lim_{t\to\bar t^+} \Delta\pi(x,t) = 0 \quad \text{or} \quad \lim_{t\to\bar t^-} \Delta\pi(x,t) = 0 \qquad \forall x \in D$$

or any reflection $\rho_{\bar r}|_D$*, where* $\bar r \in (0,2)$*, and* $\rho_{\bar r}(D) \subseteq A$ *such that*

$$\lim_{r\to\bar r^+} \Delta\pi(x, r-x) = 0 \quad \text{or} \quad \lim_{r\to\bar r^-} \Delta\pi(x, r-x) = 0 \qquad \forall x \in D.$$

iii. *An* initial move *in* $\Omega^0(\pi)$ *is a move that is either additive or limit-additive, or an inverse of such a move, or an empty move.*

▶ Remark 55. The property of the moves $\gamma|_D \in \Omega^0$ that the function π is continuous on the domain D and image $\gamma(D)$ will be preserved throughout.

▶ Remark 56. The initial move ensemble Ω^0 is join-closed. Therefore, by Lemma 6, it is equal to the restriction closure of its maximal elements. Moreover, by definition, Ω^0 satisfies (inv). However, Ω^0 in general is not a semigroup.

The function π is affinely Ω^0-equivariant (Subsection 5.1), i.e., it respects all moves in Ω^0.

8.2 Moves from connected open sets of additivities

We now specialize our results from Subsection 4.3 regarding connected open ensembles to the initial moves. We have the following corollary. Recall from Subsection 6.1 the projections $p_1(x,y) = x$, $p_2(x,y) = y$, and $p_3(x,y) = x+y$ as functions from $\mathbb{R}^2$ to $\mathbb{R}$.

▶ **Corollary 57.** *Let* $E \subseteq \mathbb{R}^2$ *be a connected open set on which* π *is additive, i.e.,* $\Delta\pi(x,y) = 0$ *for* $(x,y) \in E$*. Let* $C = p_1(E) \cup p_2(E) \cup p_3(E)$ *and assume that* $C \subseteq A$*. Then* $\operatorname{moves}(C \times C) \subseteq \operatorname{jsemi}(\Omega^0)$*.*

See Figure 12 for an illustration. We remark that in [16], the intervals $p_1(E), p_2(E), p_3(E)$ are referred to as *directly covered intervals.*

Proof of Corollary 57. Denote $\Gamma^\vee = \operatorname{jsemi}(\Omega^0)$. By Lemma 10, $\Gamma^\vee = \operatorname{isemi}(\Gamma^\vee)$. We first show that $\operatorname{Gr}_\pm(\Gamma^\vee)$ contains E. Let $(x,y) \in E$. Since E is open, there exists an open interval $D \ni x$ such that the diagonal segment $\{\,(x', r-x') \mid x' \in D\,\} \subseteq E$, where $r = x+y$. By Definition 54, we have $\rho_r|_D \in \Omega^0$, with $\rho_r|_D(x) = y$. Thus, $(x,y) \in \operatorname{Gr}_-(\Gamma^\vee)$. There exist open intervals $D_y \ni y$ and $D_x \ni x$ such that the vertical segment $\{x\} \times D_y$ and the horizontal segment $D_x \times \{y\}$ are contained in E. Again by Definition 54, we have $\tau_y|_{D_x}, \tau_x|_{D_y} \in \Omega^0$. Notice that

$$x \xmapsto{\tau_y|_{D_x}} (x+y) \xmapsto{(\tau_x|_{D_y})^{-1}} y.$$

Thus, $(x,y) \in \operatorname{Gr}_+(\Gamma^\vee)$. We showed that $\operatorname{Gr}_\pm(\Gamma^\vee)$ contains E. By Theorem 11 (3), $\operatorname{moves}((p_1(E) \cup p_2(E)) \times (p_1(E) \cup p_2(E))) \subseteq \Gamma^\vee$.

For any point $x+y \in p_3(E)$, where $x \in p_1(E)$ and $y \in p_2(E)$, the above translation move $\tau_y|_{D_x}$ satisfies that $\tau_y|_{D_x} \in \Omega^0$ and $\tau_y|_{D_x}(x) = x+y$. By applying Lemma 33 to $\mathcal{C} = \{p_1(E) \cup p_2(E)\}$ and all such moves $\tau_y|_{D_x}$, we obtain that $\operatorname{moves}((p_1(E) \cup p_2(E) \cup p_3(E)) \times (p_1(E) \cup p_2(E) \cup p_3(E))) \subseteq \Gamma^\vee$. ◀

9 Piecewise linear functions, polyhedral complexes, effective perturbations

We now specialize our theory to the important case of piecewise linear functions. We begin with the basic definitions and review some tools that were developed in the previous papers of the present series.

9.1 Continuous and discontinuous piecewise linear functions π, complex $\mathcal{P}_B$

We begin by giving a definition of $\mathbb{Z}$-periodic piecewise linear functions $\pi\colon \mathbb{R} \to \mathbb{R}$ that are allowed to be discontinuous, following [20]. [16] discusses how these functions are represented in the software [21].

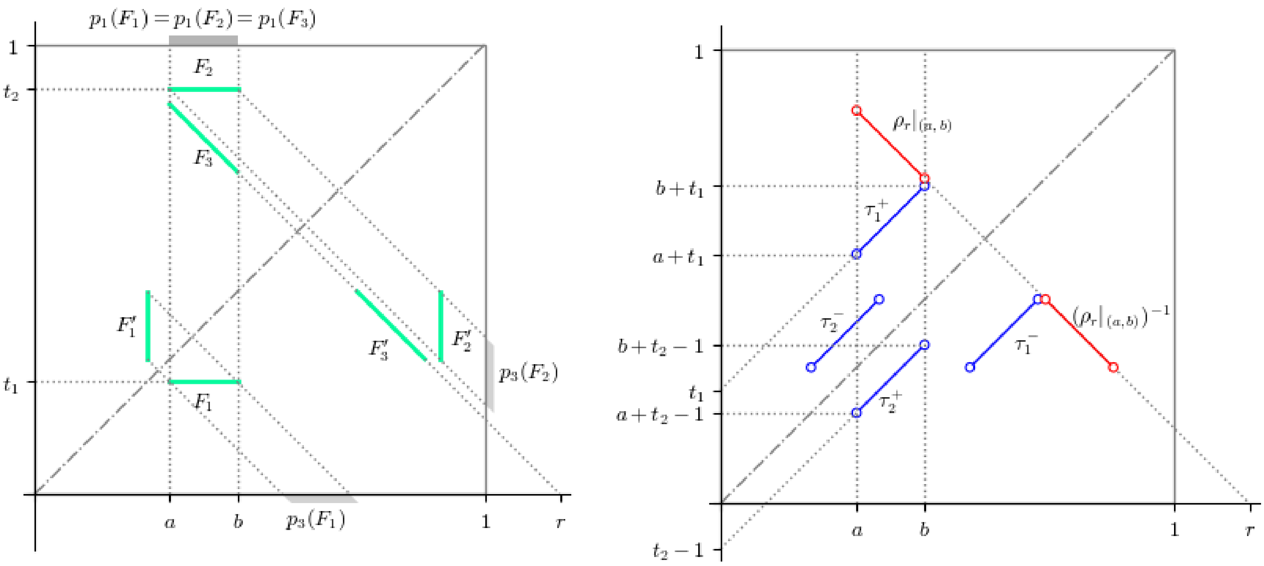


Figure 10 *Left,* additive edges (one-dimensional additive faces) in $\Delta\mathcal{P}_B$. *Right,* the corresponding initial moves (additive moves and their inverses) in Ω^0.

Let $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$. Denote by $B = \{\, x_0 + t,\ x_1 + t,\ \ldots,\ x_{n-1} + t \mid t \in \mathbb{Z} \,\}$ the set of all breakpoints. The 0-dimensional faces are defined to be the singletons, $\{x\}$, $x \in B$, and the 1-dimensional faces are the closed intervals, $[x_i + t, x_{i+1} + t]$, $i = 0, \ldots, n-1$, $t \in \mathbb{Z}$. The empty face, the 0-dimensional and the 1-dimensional faces form $\mathcal{P} = \mathcal{P}_B$, a locally finite polyhedral complex, periodic modulo $\mathbb{Z}$.

▶ **Definition 58.** *We call a function $\pi\colon \mathbb{R} \to \mathbb{R}$* piecewise linear *over $\mathcal{P}_B$ if for each face $I \in \mathcal{P}_B$, there is an affine linear function $\pi_I\colon \mathbb{R} \to \mathbb{R}$, $\pi_I(x) = c_I x + d_I$ such that $\pi(x) = \pi_I(x)$ for all $x \in \operatorname{rel\,int}(I)$.*

Under this definition, piecewise linear functions can be discontinuous. Let $I = [a, b] \in \mathcal{P}_B$ be a 1-dimensional face. The function π can be determined on $\operatorname{int}(I) = (a, b)$ by linear interpolation of the limits $\pi(a^+) = \lim_{x\to a, x>a} \pi(x) = \pi_I(a)$ and $\pi(b^-) = \lim_{x\to b, x<b} \pi(x) = \pi_I(b)$.

9.2 Two-dimensional polyhedral complex $\Delta\mathcal{P}$ and additive faces

For a piecewise linear function (see Subsection 9.1 for our notation), we now explain the structure of the initial moves. We will use the notion of the polyhedral complex $\Delta\mathcal{P}$ and its *additive faces* from [16, Section 4]. $\Delta\mathcal{P}$ is a two-dimensional polyhedral complex, which expresses the domains of linearity of the subadditivity slack $\Delta\pi(x, y)$ introduced in Subsection 1.2.

▶ **Definition 59.** *The polyhedral complex $\Delta\mathcal{P}$ of $\mathbb{R} \times \mathbb{R}$ consists of the faces*

$$F(I, J, K) = \{\, (x, y) \in \mathbb{R} \times \mathbb{R} \mid x \in I,\ y \in J,\ x + y \in K \,\},$$

where $I, J, K \in \mathcal{P}$, so each of I, J, K is either empty, a breakpoint of π, or a closed interval delimited by two consecutive breakpoints.

In the continuous case, since the function π is piecewise linear over $\mathcal{P}$, we have that $\Delta\pi$ is affine linear over each (closed) face $F \in \Delta P$. We say that a face $F \in \Delta\mathcal{P}$ is *additive* if $\Delta\pi = 0$ over all F. If π is subadditive, then the *set of additivities* $E(\pi) = \{\, (x, y) \mid \Delta\pi(x, y) = 0 \,\}$ is the union of all additive faces $F \in \Delta\mathcal{P}$; see [7, Section 3.4].

For a discontinuous function π, the subadditivity slack $\Delta\pi$ is affine linear only over the relative interior of each face F. For additivity, beside the subadditivity slack $\Delta\pi(x, y)$ at a point (x, y), we also consider its limits.

▶ **Definition 60.** *The limit value of $\Delta\pi$ at the point (x, y) approaching from the relative interior of a face $F \in \Delta\mathcal{P}$ containing (x, y) is denoted by*

$$\Delta\pi_F(x, y) = \lim_{\substack{(u,v)\to(x,y)\\ (u,v)\in \operatorname{rel\,int}(F)}} \Delta\pi(u, v).$$

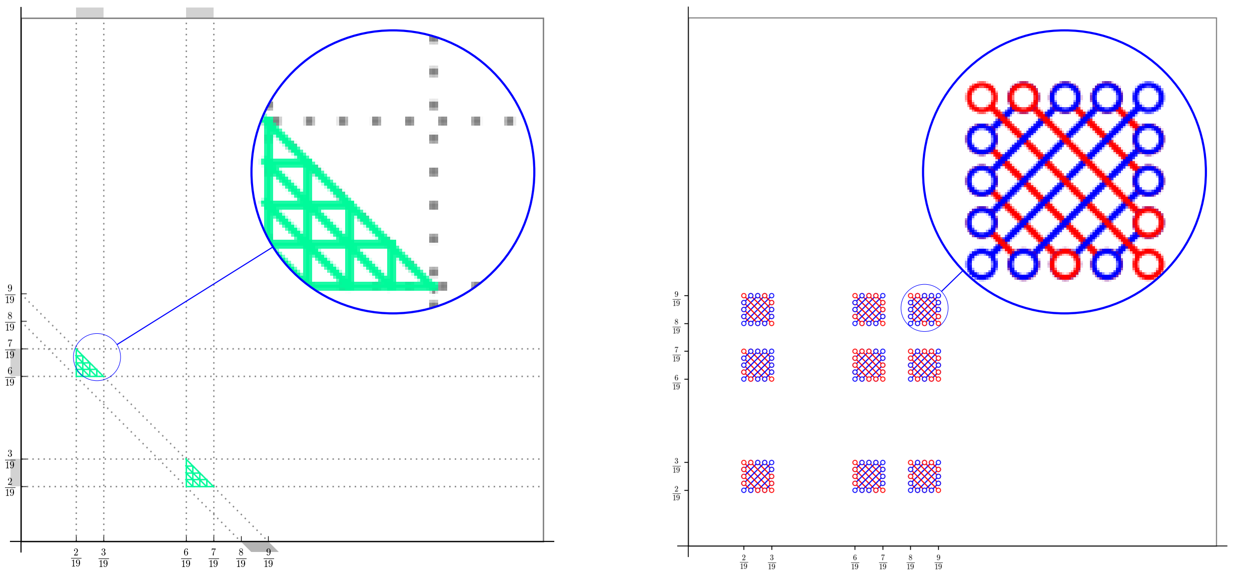


Figure 11 Additivities and initial moves. *Left,* additivities sampled from two-dimensional additive faces of $\Delta\mathcal{P}$. *Right,* the move semigroup $\operatorname{jsemi}(\Omega^0)$ generated by the initial moves. The graphs $\operatorname{Gr}_+(\Omega)$ (*blue*) and $\operatorname{Gr}_-(\Omega)$ (*red*) are plotted on top of each other. For illustration purposes, only a finite set of additive moves is considered.

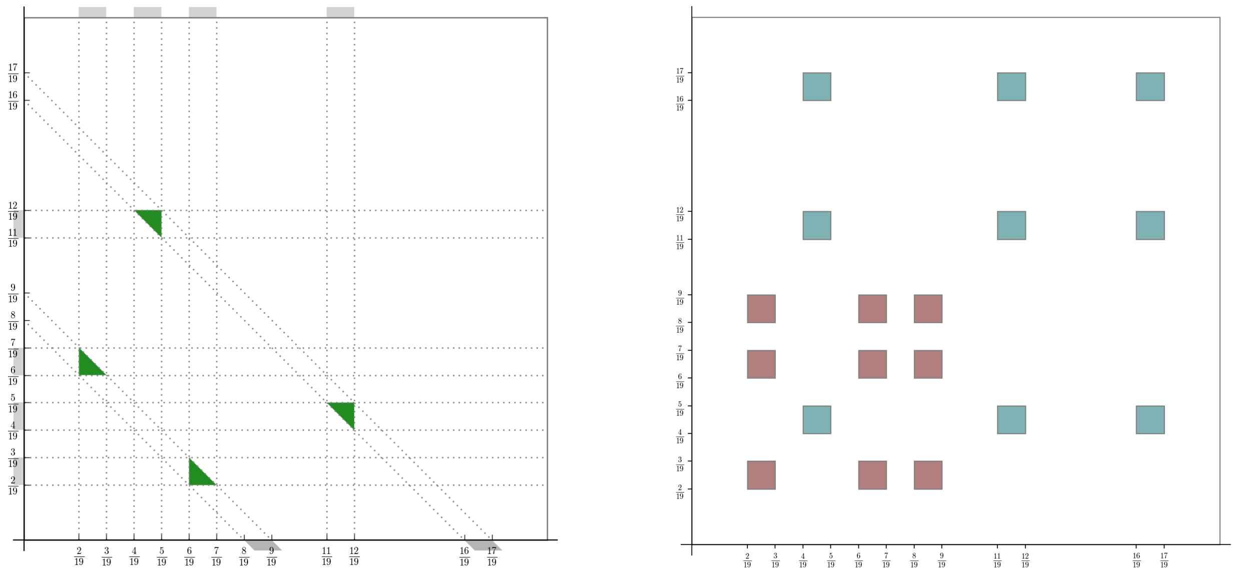


Figure 12 Additivities in $E(\pi)$ and the corresponding connected covered components of moves

▶ **Definition 61.** *Let* $F \in \Delta\mathcal{P}$*. Define the* set of additivities and limit-additivities approaching from the relative interior of F *as*

$$E_F(\pi) = \left\{\, (x, y) \in F \;\middle|\; \Delta\pi_F(x, y) \text{ exists, and } \Delta\pi_F(x, y) = 0 \,\right\}. \tag{17}$$

▶ Remark 62. The points $(x, y) \in E_F(\pi)$ that lie in $\operatorname{rel\,int}(F)$ capture all additivities of π, whereas those that lie on the relative boundary capture all limit-additivities. The set $E(\pi)$ that we introduced in the continuous case can be partitioned as $E(\pi) = \bigcup_{F\in\Delta\mathcal{P}}(E_F(\pi) \cap \operatorname{rel\,int}(F))$.

▶ **Lemma 63.** *Let* π *be a subadditive function that is piecewise linear over* $\mathcal{P}$*. Let* $F \in \Delta\mathcal{P}$*. Let* $(x_0, y_0) \in E_F(\pi) \subseteq F$ *and let* E *be the unique face of* F *containing* (x_0, y_0) *in its relative interior. Then* $E \subseteq E_F(\pi)$*.*

We make the following general definition, which is equivalent to the one found in [16, 20].

▶ **Definition 64.** *In the situation of Lemma 63, we say that the face* E *is* additive*.*

Now the following lemma is clear from the definition. [16] only states this fact for the case of continuous π.

▶ **Lemma 65.** *Let π be a subadditive function that is piecewise linear over $\mathcal{P}$. Then the set of additive faces of π is a polyhedral subcomplex of $\Delta\mathcal{P}$, i.e., it is closed under taking subfaces. In particular, each additive face is the convex hull of some additive vertices (zero-dimensional additive faces).*

For a piecewise linear function π, a finite presentation of the initial moves is easy to compute using the additive faces of the complex $\Delta\mathcal{P}$. For a detailed explanation of diagrams visualizing the additivities and limit-additivities, we refer to [16, Sections 4.2–4.3]. See Figure 10 for the moves from one-dimensional additive faces (edges) and Figure 11 and Figure 12 for the moves from two-dimensional additive faces. In the forthcoming paper [15], we will give a more detailed description how to compute the finite presentation of the initial moves.

▶ Remark 66. The zero-dimensional additive faces (i.e., additive vertices) of $\Delta\mathcal{P}_B$ do not give rise to moves (cf. Remark 15). Instead they will be considered in Section 10 to determine a refinement of $\mathcal{P}_B$ for the decomposition of perturbations.

9.3 Effective perturbations $\tilde\pi$

We recall the notion of effective perturbations from Subsection 1.4. An effective perturbation is a function $\tilde\pi\colon \mathbb{R}\to\mathbb{R}$ for which there exists an $\epsilon > 0$ such that $\pi^\pm = \pi \pm \epsilon\tilde\pi$ are minimal valid functions.

▶ Remark 67. Let π be a minimal valid function for $R_f(\mathbb{R},\mathbb{Z})$. From (3a), (3c), and (3e) it follows that $0 \le \pi \le 1$, so π is a bounded function. Now if $\tilde\pi$ is an effective perturbation, then $\pi^\pm = \pi \pm \epsilon\tilde\pi$ for some $\epsilon > 0$, where also $0 \le \pi^\pm \le 1$, and so $\tilde\pi$ is a bounded function as well.

We note that the space $\tilde\Pi^\pi$ of effective perturbations, introduced in Subsection 1.4, is a vector space.

▶ **Lemma 68.** *Let π be a minimal valid function. The space $\tilde\Pi^\pi$ of effective perturbation functions is a vector space, a subspace of the space $\mathcal{B}(\mathbb{R})$ of bounded functions.*

For the case of piecewise linear functions π that are continuous from at least one side of the origin, we have the following regularity theorem for effective perturbations.

▶ **Lemma 69** ([16, Lemma 6.4])**.** *Let π be a piecewise linear minimal valid function that is continuous from the right at 0 or continuous from the left at 1. If π is continuous on a proper interval $I \subseteq [0,1]$, then for any $\tilde\pi \in \tilde\Pi^\pi$ we have that $\tilde\pi$ is Lipschitz continuous on the interval I.*

(This is a strengthening of [8, Theorem 2].)

The purpose of the additive move ensemble is to infer properties of the effective perturbation functions. For additive moves $\gamma|_D$, it follows from convexity that every effective perturbation $\tilde\pi$ respects $\gamma|_D$. In the case of piecewise linear functions, this extends to limit-additive moves. The following lemma is shown by the proof of [16, Theorem 6.3], along with [16, Footnote 13] and also by [20, Theorem 3.3] in the case where π is two-sided discontinuous at the origin.

▶ **Lemma 70.** *Let π be a piecewise linear minimal valid function for $R_f(\mathbb{R},\mathbb{Z})$. Let $\gamma|_D \in \Omega^0$ be an initial move, where $D \subseteq (0,1)$ is an open interval. Then π respects $\gamma|_D$, and every effective perturbation function $\tilde\pi \in \tilde\Pi^\pi$ respects $\gamma|_D$.*

▶ **Corollary 71.** *Let π be a piecewise linear minimal valid function for $R_f(\mathbb{R},\mathbb{Z})$. Then π respects the moves closure* $\operatorname{clsemi}_A(\Omega^0)$*. If π is continuous from at least one side of the origin, then every effective perturbation function $\tilde\pi \in \tilde\Pi^\pi$ also respects the moves closure* $\operatorname{clsemi}_A(\Omega^0)$*.*

Proof. Let π be a function that satisfies the assumptions in the corollary. Let $\tilde\pi \in \tilde\Pi^\pi$ be an effective perturbation. By Lemma 70, π and $\tilde\pi$ both respect the initial move ensemble Ω^0. Recall that $\operatorname{dom}(\Omega^0), \operatorname{im}(\Omega^0) \subseteq A$, and π is continuous on A. By Lemma 69, $\tilde\pi$ is also continuous on A. By Remark 67, π and $\tilde\pi$ are bounded functions. Therefore, π and $\tilde\pi$ both respect the moves closure $\operatorname{clsemi}_A(\Omega^0)$ by Theorem 53. ◀

9.4 Closed move semigroup generated by Ω^0, rational case

We have the following theorem.

▶ **Theorem 72** (Finite presentation of the moves closure, rational case)**.** *Let π be a piecewise linear function whose breakpoints are rational, i.e., $B \subseteq G = \frac{1}{q}\mathbb{Z}$ for some $q \in \mathbb{N}$. Then the moves closure* $\operatorname{clsemi}_A(\Omega^0)$ *has a finite presentation $(\Omega^{\mathrm{red}}, \mathcal{C})$ in reduced form, where (i) the endpoints of all domains and the values t and r of moves $\tau_t, \rho_r|_D \in \Omega^{\mathrm{red}}$ lie in $G \cap [0,1]$, (ii) the endpoints of all maximal intervals of all $C_i \in \mathcal{C}$ lie in $G \cap [0,1]$.*

Proof sketch. We can compute $\operatorname{clsemi}_A(\Omega^0)$ in finitely many steps using a completion-type algorithm that manipulates finite presentations, maintaining properties (i) and (ii), using only the algebraic and order-theoretic axioms and (extend_A). The initialization is provided by Corollary 12, noting that vertices of additive faces of $\Delta\mathcal{P}$ lie in $G \times G$. There are only finitely many finite presentations satisfying (i) and (ii); this implies the finiteness of the algorithm. ◀

We defer all details about such an algorithm, as well as its generalization to non-rational input, to the forthcoming paper [15].

Instead, in the next section, we assume that a finite presentation $(\Omega^{\text{fin}}, \mathcal{C})$ of the moves closure $\operatorname{clsemi}_A(\Omega^0)$ is given. Using the finite presentation, we can give a description of the space of effective perturbations.

10 Perturbation space

Let $\pi\colon \mathbb{R} \to \mathbb{R}$ be a minimal valid function. In this section, we work with the following assumptions. (We will mention them explicitly only in statements of main theorems.)

10.1 Assumptions: Piecewise linear π, one-sided continuous at 0, finitely presented moves closure $\operatorname{clsemi}_A(\Omega^0)$

▶ Assumption 73. The minimal valid function π is piecewise linear (Subsection 9.1) and continuous from at least one side of the origin.

▶ Assumption 74. The set B is minimal, i.e., $\mathcal{P}_B$ is the coarsest polyhedral complex over which π is piecewise linear.

Let $\Omega^0 = \Omega^0(\pi)$ be the initial additive move ensemble (Section 8) of π. Recall that $A = A(\pi)$ is the maximal open subset of $(0, 1)$ on which π is continuous.

▶ Assumption 75. The moves closure $\operatorname{clsemi}_A(\Omega^0)$ has a finite presentation $(\Omega, \mathcal{C})$ in reduced form (Subsection 6.5). Thus Ω has finitely many moves and $\mathcal{C}$ has finitely many connected covered components $C_1, C_2, \dots, C_k$, each of which is a finite union of proper open intervals. Each $\gamma|_D \in \Omega$ is maximal in the restriction partial order of $\operatorname{clsemi}_A(\Omega^0)$ and is not contained in $\operatorname{jmoves}(\mathcal{C})$. Figures 13 (right), 15, and 16 show examples of $\operatorname{clsemi}_A(\Omega^0)$ satisfying Assumption 75.

10.2 Properties of the finitely presented moves closure

Let $C := C_1 \cup C_2 \cup \cdots \cup C_k$ denote the open set of points in $(0, 1)$ that are covered. We will refer to the open set $U := (0, 1) \setminus \operatorname{cl}(C)$ as the set of points in $(0, 1)$ that are *uncovered*. Let

$$X := \{0\} \cup \partial C \cup \{1\} = \{0\} \cup \partial U \cup \{1\} \tag{18}$$

be the set of endpoints of all covered and uncovered intervals. Thus we have the partition $[0, 1] = C \cup X \cup U$.

▶ **Example 76.** Consider the discontinuous minimal valid function for $f = \frac{1}{2}$, defined by

$$\pi(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } 0 < x < \frac{1}{2} \\ 2(1 - x) & \text{if } \frac{1}{2} \le x < 1. \end{cases}$$

It is provided by the software [21] as $\pi =$ equiv7_example_1(). Figure 13 shows the two-dimensional polyhedral complex $\Delta\mathcal{P}$ and the moves closure $\operatorname{clsemi}_A(\Omega^0)$. The interval $C = (\frac{1}{2}, 1)$ is covered, $U = (0, \frac{1}{2})$ is uncovered. We have $X = \{0, \frac{1}{2}, 1\}$.

▶ **Example 77.** Consider the continuous minimal valid function π that is provided as equiv7_example_xyz_2() by the software [21], shown in Figure 14. Figures 14 and 15 show the additive faces and the moves closure. See the caption of Figure 15 for a description of C. We have, according to (18), that $X = \{0, \frac{1}{24}, \frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{2}{3}, \frac{3}{4}, \frac{19}{24}, \frac{7}{8}, \frac{11}{12}, 1\}$.

▶ **Example 78.** Consider the minimal valid function π that is provided as equiv7_minimal_2_covered_2_uncovered() by the software [21]; see Figure 16. It has two connected covered components. The set of uncovered points is $U = (\frac{12}{49}, \frac{13}{49}) \cup (\frac{14}{49}, \frac{15}{49}) \cup \cdots \cup (\frac{20}{49}, \frac{21}{49})$. Thus we have $X = \{0, \frac{12}{49}, \frac{13}{49}, \dots, \frac{20}{49}, \frac{21}{49}, 1\}$.

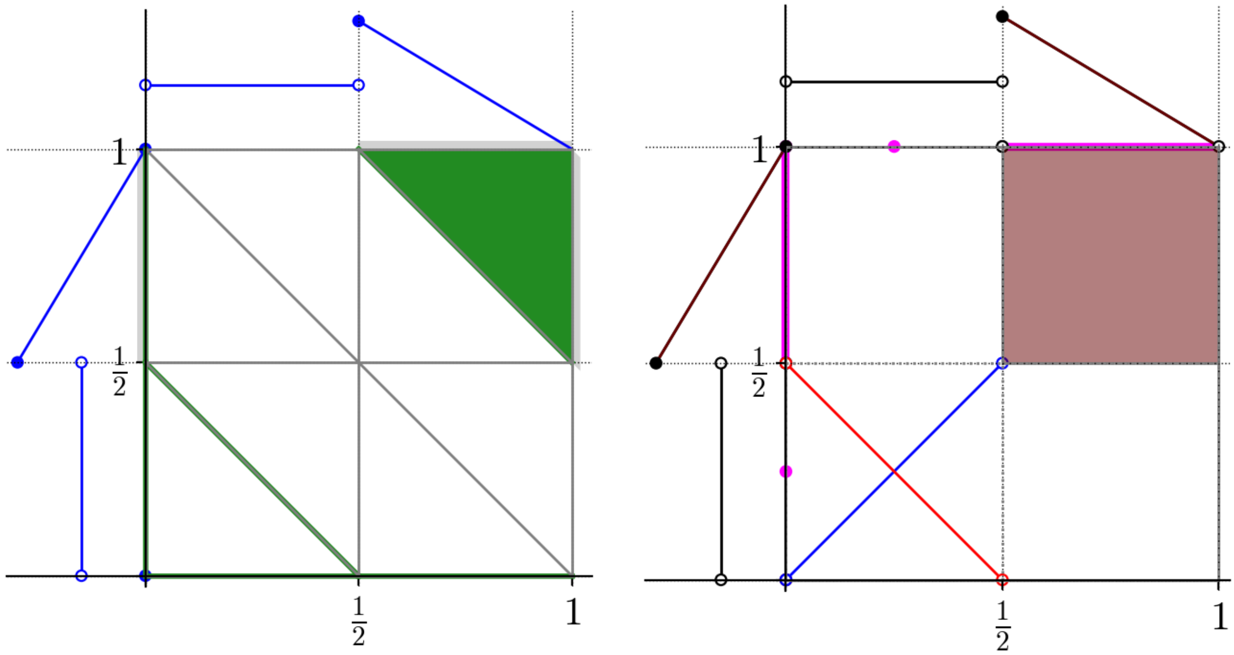


Figure 13 (*Left*) Two-dimensional polyhedral complex $\Delta\mathcal{P}$ of the one-sided discontinuous minimal valid function $\pi =$ equiv7_example_1() from Example 76 (*blue graph at the left and top borders*), where the additive faces are *colored in green.* (*Right*) The graph of the moves closure $\operatorname{clsemi}_A(\Omega^0)$ of π. It has a finite presentation by $\Omega = \{\tau_0|_{(0,1/2)}, \rho_{1/2}|_{(0,1/2)}\}$ (*blue and red line segments of slopes* ± 1) and one (maximal) connected covered component $C = (\frac{1}{2}, 1)$ (the *brown square* shows $C \times C$). The set $C \cup X \cup Y \cup Z$ of covered points and refined breakpoints is marked in *magenta on the left and top borders.*

Recall the two-dimensional polyhedral complex $\Delta\mathcal{P}_B$ and its additive faces, introduced in Subsection 9.2. Let

$$V := \left\{ p_i(x,y) \;\middle|\; (x,y) \text{ additive vertex of } \Delta\mathcal{P}_B,\ i = 1,2,3 \right\} \cap [0,1] \tag{19}$$

be the set of p_1, p_2 and p_3 projections (within the fundamental domain) of the zero-dimensional additive faces (i.e., additive vertices). By Remark 56, the initial move ensemble Ω^0 is join-closed. We consider the ensemble $\Omega^0|_U$ of moves restricted to U, as defined in Subsection 3.2. By Lemma 7 it is also join-closed and therefore, by Lemma 6, has a presentation by its maximal elements. It follows from Lemma 65 that its maximal elements have the following relation to the set V.

▶ **Lemma 79.** *If $\gamma|_{(a,b)} \in \operatorname{Max}(\Omega^0|_U)$, then the endpoints a, b lie in $V \cap U$ or ∂U.*

The following lemma shows the relation between the breakpoints B and the sets U, V. In particular, it implies that $B \cap C = \emptyset$, i.e., the connected covered components do not contain any breakpoints under Assumption 74.

▶ **Lemma 80.** *Let $b \in B \cap [0,1]$. Then, the breakpoint b lies in $V \cap U$ or ∂U.*

Proof. Since $(b, 0)$ is an additive vertex of $\Delta\mathcal{P}_B$, and $p_1(b,0) = b$, we have $b \in V$. We now show that $b \notin C$. Suppose, for the sake of contradiction, that b in contained in some connected covered component C_i . Then, Corollary 26 implies that the function π is affine on an open interval containing b, which is a contradiction to Assumption 74. ◀

Next we consider the orbit of $V \cap U$ under Ω, which is a finite set by Assumption 75,

$$\Omega(V \cap U) = \{\, \gamma|_D(x) \mid x \in V \cap U,\ x \in D \text{ and } \gamma|_D \in \Omega \,\}. \tag{20}$$

In terms of graphs of ensembles, the above set can be rewritten as $\{\, y \mid \exists x \in V \cap U \text{ such that } (x,y) \in \operatorname{Gr}(\Omega) \,\}$. We define the set Y to be $V \cap U$ union its orbit under Ω,

$$Y := (V \cap U) \cup \Omega(V \cap U). \tag{21}$$

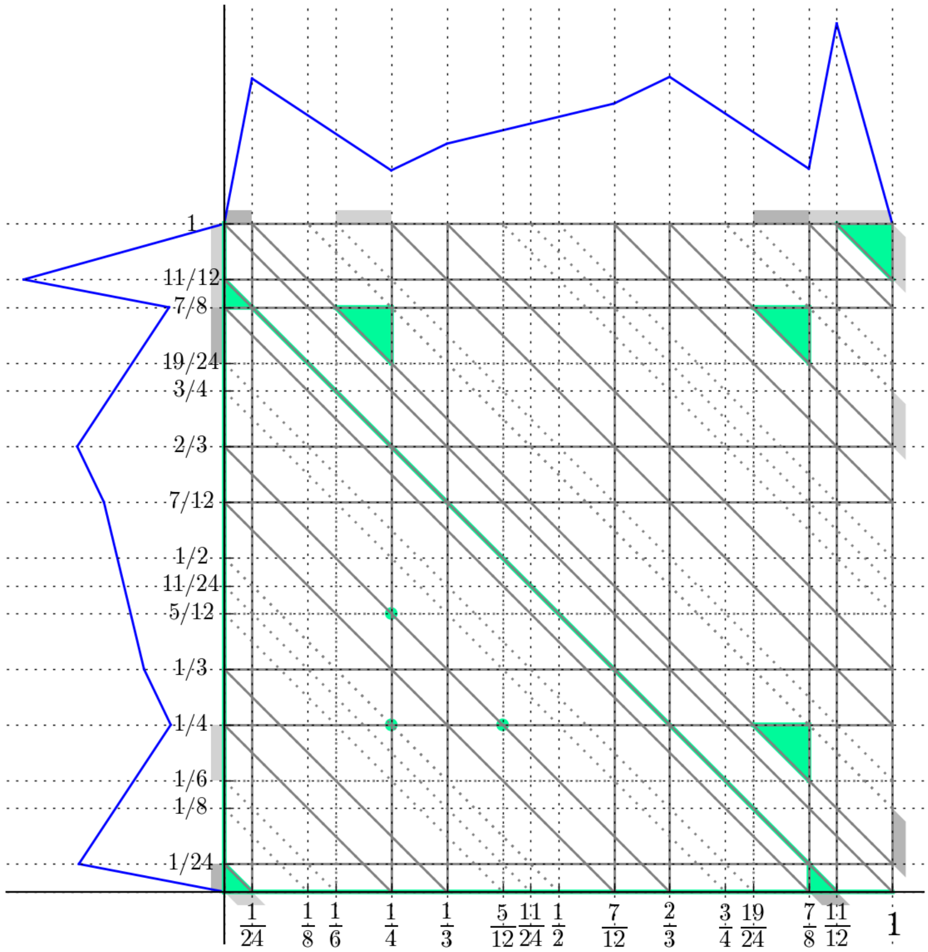


Figure 14 The function π = equiv7_example_xyz_2() from Example 77 (*blue graph at the left and top borders*) and its two-dimensional polyhedral complex $\Delta\mathcal{P}$ (*solid gray lines*), where the additive faces are *colored in green*. The refined complex $\Delta\mathcal{T}$ is shown with *dotted gray lines*.

▶ **Lemma 81.** *We have $Y \subseteq U$.*

Proof. Suppose for the sake of contradiction that there is $y \in Y$ but $y \in \operatorname{cl}(C)$. Then, $y \in \Omega(V \cap U)$. We can write y as $y = \gamma|_D(x)$ where $x \in V \cap U$, $x \in D$ and $\gamma|_D \in \Omega$. Under Assumption 75, by Lemma 33 applied to $\mathcal{C}$ and $\operatorname{clsemi}_A(\Omega^0)$, we have that C is invariant under the action of moves from $\operatorname{clsemi}_A(\Omega^0)$. Since the inverse move $(\gamma|_D)^{-1} \in \Omega \subseteq \operatorname{clsemi}_A(\Omega^0)$, we obtain that $x = (\gamma|_D)^{-1}(y) \in \operatorname{cl}(C)$. This contradicts $x \in U$. ◀

▶ **Example 82** (Example 77, continued)**.** In the example shown in Figure 15, we have $V \cap U = \{\frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \frac{7}{12}\}$. This set is already closed under the action of Ω, as $\rho_{11/12}(\frac{1}{3}) = \frac{7}{12}$ and $\rho_{11/12}(\frac{5}{12}) = \frac{1}{2}$. Thus $Y = V \cap U$ in the example.

We consider the ensembles $\Omega|_U$ and ${}_U|\Omega|_U$ of moves restricted and double-restricted to U, as defined in Subsection 3.2. We have the following results.

▶ **Lemma 83.** *The move ensemble $\Omega|_U$ satisfies:*

a. $\Omega|_U = {}_U|\Omega|_U$.
b. $\Omega|_U$ *is a finite move ensemble.*

Proof. It follows directly from Assumption 75. ◀

▶ **Lemma 84** (Filtration of $\operatorname{isemi}(\Omega^0|_U)$ by word length; maximal moves)**.** *For $k \in \mathbb{N}$, let*

$$\Omega^0|_U{}^k = \left\{\, \gamma^k|_{D^k} \circ \cdots \circ \gamma^1|_{D^1} \;\middle|\; \gamma^i|_{D^i} \in \Omega^0|_U \text{ for } 1 \le i \le k \,\right\}.$$

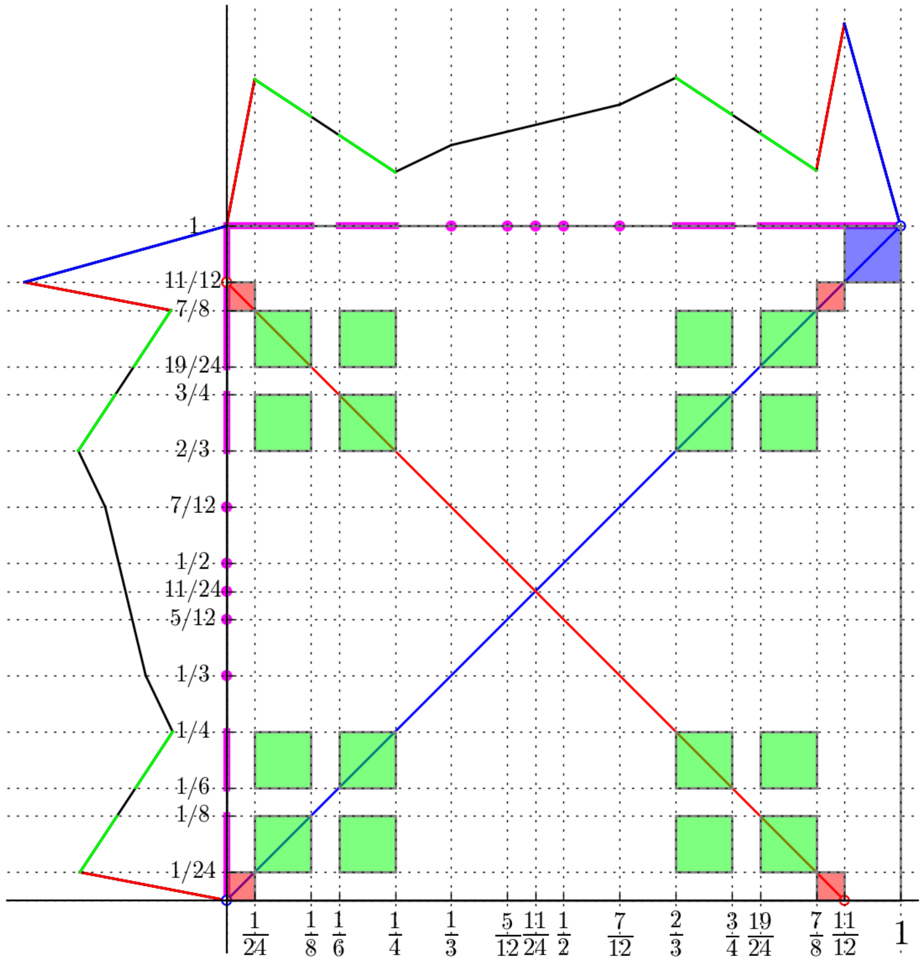


Figure 15 The function π = equiv7_example_xyz_2() from Example 77 (*colored graph at the left and top borders*) and the graph of the moves closure clsemi$_A(\Omega^0)$ of π, as computed by the command igp.equiv7_mode = True; igp.extremality_test(igp.equiv7_example_xyz_2(), True, show_all_perturbations=True). It has a finite presentation by $\Omega = \{\tau_0|_{(0,1)}, \rho_{11/12}|_{(0,11/12)}\}$ (*blue and red line segments of slopes* ± 1) and a set $\mathcal{C} = \{C_1, C_2, C_3\}$ of (maximal) connected covered components $C_1 = (\frac{11}{12}, 1)$ (the *lavender square* shows $C_1 \times C_1$), $C_2 = (0, \frac{1}{24}) \cup (\frac{7}{8}, \frac{11}{12})$ (*coral*), and $C_3 = (\frac{1}{24}, \frac{1}{8}) \cup (\frac{1}{6}, \frac{1}{4}) \cup (\frac{2}{3}, \frac{3}{4}) \cup (\frac{19}{24}, \frac{7}{8})$ (*lime*). The set $C \cup B' = C \cup X \cup Y \cup Z$ of covered points and refined breakpoints is marked in *magenta on the left and top borders.*

Then $\Omega^0|_U{}^1 \subseteq \Omega^0|_U{}^2 \subseteq \dots$ *and* $\operatorname{isemi}(\Omega^0|_U) = \bigcup_{k\in\mathbb{N}} \Omega^0|_U{}^k$*. For each* $k \in \mathbb{N}$*, the ensemble* $\Omega^0|_U{}^k$ *satisfies* (restrict) *and has a presentation by the set* $\operatorname{Max}(\Omega^0|_U{}^k)$ *of its maximal elements, which is a finite set. For* $\gamma|_{(a,b)} \in \operatorname{Max}(\Omega^0|_U{}^k)$*, we have* $a, b, \gamma(a), \gamma(b) \in X \cup Y$*.*

Proof. Because Ω^0 satisfies (inv), (cont), and (restrict) by Remark 56, so does its double restriction ${}_U|\Omega^0|_U$ to the uncovered set U. Clearly, $\Omega^0|_U$ is a subset of the moves closure with the finite presentation $(\Omega, \mathcal{C})$. Since $U \cap \operatorname{dom}(\operatorname{moves}(\mathcal{C})) = \emptyset$, we have $\Omega^0|_U \subseteq \operatorname{restrict}(\Omega|_U)$, and hence $\operatorname{im}(\Omega^0|_U) \subseteq \operatorname{im}(\Omega|_U)$. By Lemma 83 (a), $\operatorname{im}(\Omega|_U) = \operatorname{im}({}_U|\Omega|_U) \subseteq U$. Thus, $\Omega^0|_U = {}_U|\Omega^0|_U$. Recall that $\Omega^0|_U$ is join-closed and therefore has a presentation by its maximal elements. We note that the initial move ensemble Ω^0 is constructed from the additive faces of $\Delta\mathcal{P}_B$ (see Definitions 54 and 64). By Corollary 57 (see also Figures 11 and 12), a two-dimensional additive face E gives rise to moves in $\operatorname{moves}(\mathcal{C})$, whose domains are outside U. This shows that $\Omega^0|_U$ only corresponds to the additive edges of $\Delta\mathcal{P}_B$ (see Figure 10), which are finitely many. Thus, $\operatorname{Max}(\Omega^0|_U)$ is finite.

Let $\operatorname{Max}(\Omega^0|_U)^k = \{\, \gamma^k|_{\hat{D}^k} \circ \dots \circ \gamma^1|_{\hat{D}^1} \mid \gamma^i|_{\hat{D}^i} \in \operatorname{Max}(\Omega^0|_U) \,\}$, a finite set. Then $\operatorname{Max}(\Omega^0|_U{}^k) \subseteq \operatorname{Max}(\Omega^0|_U)^k$ is finite, and every element of $\Omega^0|_U{}^k$ is the restriction of an element of $\operatorname{Max}(\Omega^0|_U{}^k)$. The chain of inclusions $\Omega^0|_U{}^1 \subseteq \Omega^0|_U{}^2 \subseteq \dots$ holds because the idempotents $\tau_0|_D$ for intervals $D \subseteq U$ are elements of $\Omega^0|_U$.

Last, we prove the claim regarding the endpoints; we actually prove the slightly stronger claim $a, b, \gamma(a), \gamma(b) \in \partial U \cup Y$ by induction on word length k. Since each $\Omega^0|_U{}^k$ satisfies (inv), it suffices to prove $a, b \in \partial U \cup Y$. For

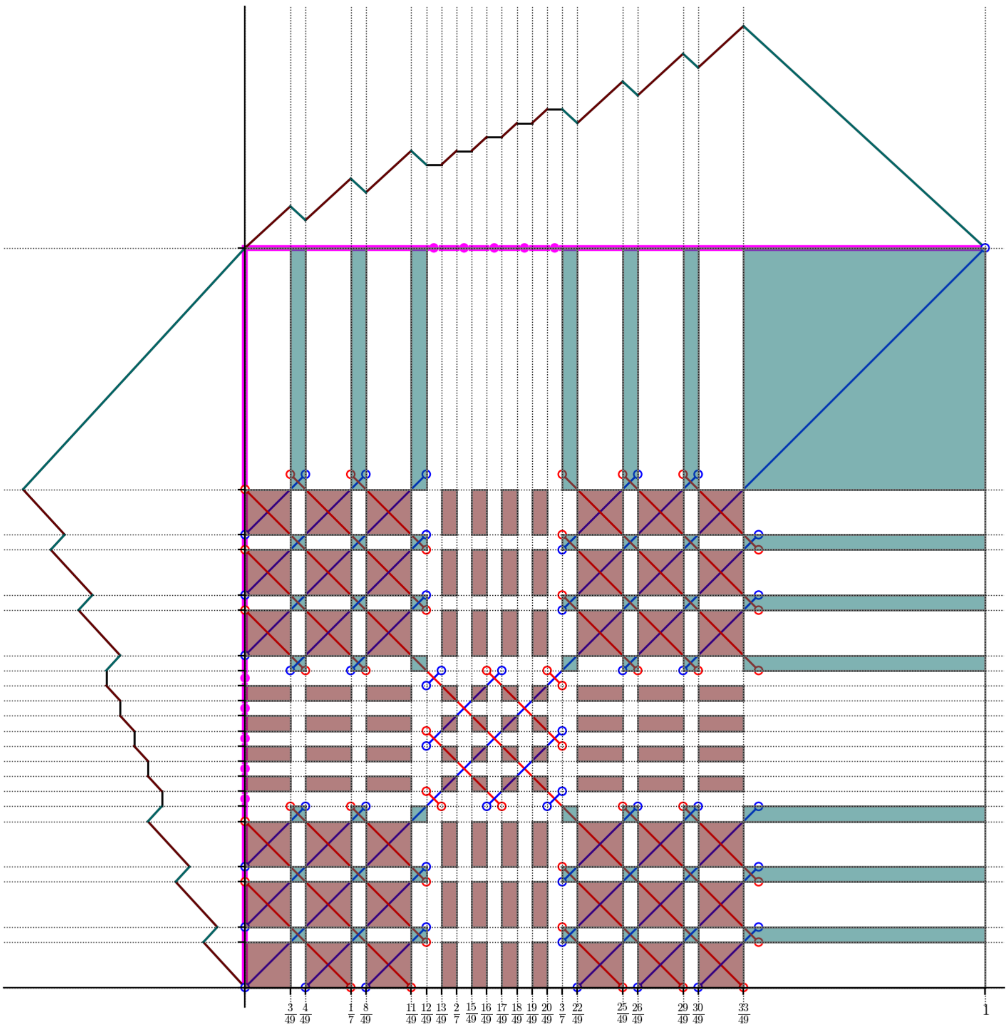


Figure 16 Moves closure $\operatorname{clsemi}_A(\Omega^0)$ for the function from Example 78, π = equiv7_minimal_2_covered_2_uncovered(). It has two connected covered components (*cyan, brown rectangles*).

$k = 1$, let $\gamma|_{(a,b)} \in \operatorname{Max}(\Omega^0|_U{}^1) = \operatorname{Max}(\Omega^0|_U)$. Then, by Lemma 79, each of the endpoints a, b lies in $V \cap U \subseteq Y$, or it lies in ∂U. Now we proceed by induction. Take $\gamma^1|_{(a,b)} \in \operatorname{Max}(\Omega^0|_U)$ and $\gamma^2|_{(c,d)} \in \operatorname{Max}(\Omega^0|_U{}^{k-1})$, so $a, b, c, d \in \partial U \cup Y$. Then, by (8), $\gamma^2|_{(c,d)} \circ \gamma^1|_{(a,b)}$ has domain $(a, b) \cap (\gamma^1)^{-1}((c, d))$. If the domain is nonempty, let x be an endpoint of it. If $x = a, b$, nothing is to show, so assume $x \in (a, b)$ and $x = (\gamma^1)^{-1}(y)$, where $y = c$ or $y = d$, so $y \in \partial U \cup Y$. But $y = \gamma^1|_{(a,b)}(x) \in U$, so $y \in Y$. Then it follows that also $x \in Y$. ◀

By Assumption 75, all elements of Ω are maximal moves of the moves closure $\operatorname{clsemi}_A(\Omega^0)$. Therefore, by Lemma 8, all elements of $\Omega|_U$ are maximal moves of $\operatorname{clsemi}_A(\Omega^0)|_U$.

After these preliminaries, we are able to state the main theorem.

▶ **Theorem 85** (Structure and generation theorem for finitely presented moves closures)**.** *Under Assumption 75, we have*

a. $\operatorname{clsemi}_A(\Omega^0) = \operatorname{extend}_A(\operatorname{clsemi}_A(\Omega^0|_U) \cup \operatorname{clsemi}_A(\Omega^0|_C))$.
b. $\Omega|_U = \operatorname{Max}(\operatorname{extend}_A(\operatorname{isemi}(\Omega^0|_U)))$.
c. $a, b, \gamma(a), \gamma(b) \in X \cup Y$ *for any* $\gamma|_{(a,b)} \in \Omega|_U$.

We emphasize that the theorem does *not* depend on an algorithm to compute the moves closure.

Proof. Part a. Let Ω' denote the right hand side of the equation in part (a). Clearly, $\Omega^0 \subseteq \Omega' \subseteq \operatorname{clsemi}_A(\Omega^0)$. We now show that Ω' is a closed move semigroup. By Lemma 83 (a), we have that

$$\operatorname{clsemi}_A(\Omega^0|_U) \subseteq \operatorname{restrict}(\Omega|_U) \subseteq \operatorname{moves}(U \times U);$$

$$\operatorname{clsemi}_A(\Omega^0|_C) = \operatorname{moves}(\mathcal{C}) \subseteq \operatorname{moves}(C \times C),$$

where the open sets U and C are disjoint. Thus, we have that $\text{clsemi}_A(\Omega^0|_U) \cup \text{clsemi}_A(\Omega^0|_C)$ is a move semigroup, under Assumption 75. It follows from Lemma 45 that Ω' is a move semigroup that satisfies (extend_A). Note that for any proper open intervals D and I such that $\text{moves}(D \times I) \subseteq \text{clsemi}_A(\Omega^0)$, we have $\text{moves}(D \times I) \subseteq \text{clsemi}_A(\Omega^0|_C)$. Therefore, Ω' also satisfies (kaleido). Moreover, (lim) holds by Theorem 40. We conclude that Ω' is a closed move semigroup. Hence, part (a) holds.

Part b. By restricting the moves ensembles on both sides of the equation in part (a) to domain U, we obtain that

$$\text{restrict}(\Omega|_U) = \text{clsemi}_A(\Omega^0)|_U = \text{clsemi}_A(\Omega^0|_U) \tag{22}$$

Next, we show that

$$\text{clsemi}_A(\Omega^0|_U) = \text{extend}_A(\text{isemi}(\Omega^0|_U)). \tag{23}$$

It follows from Lemma 45 that $\text{extend}_A(\text{isemi}(\Omega^0|_U))$ is a move semigroup that satisfies (extend_A) (and also (cont) and (restrict)). Since

$$\text{extend}_A(\text{isemi}(\Omega^0|_U)) \subseteq \text{clsemi}_A(\Omega^0|_U) = \text{restrict}(\Omega|_U), \tag{24}$$

where the equality follows from (22), and $\Omega|_U$ is a finite move ensemble by Lemma 83 (b), we obtain that the move semigroup $\text{extend}_A(\text{isemi}(\Omega^0|_U))$ also satisfies (kaleido) and (lim). Therefore, $\text{extend}_A(\text{isemi}(\Omega^0|_U))$ is a closed move semigroup which contains $\Omega^0|_U$. Since $\text{clsemi}_A(\Omega^0|_U)$ is the smallest closed move semigroup containing $\Omega^0|_U$, we have

$$\text{clsemi}_A(\Omega^0|_U) \subseteq \text{extend}_A(\text{isemi}(\Omega^0|_U)).$$

Together with (24), we conclude that (23) holds. Since Ω has only maximal moves, (22) and (23) imply the equation in part (b).

Part c. Let $\gamma|_{(a,b)} \in \Omega|_U$. By symmetry, it suffices to show that $a, b \in X \cup Y$. Consider $x = a$ or $x = b$. Part (b) implies that

$$\Omega|_U = \text{Max}(\text{extend}_A(\text{jsemi}(\Omega^0|_U))).$$

Together with (13), we know that x is the limit of a sequence $\{x^j\}_{j\in\mathbb{N}}$, where x^j is an endpoint of the domain D^j of a move $\gamma|_{D^j} \in \text{Max}(\text{jsemi}(\Omega^0|_U))$. By Lemma 84 and Lemma 5, for any $j \in \mathbb{N}$, we have that D^j is a maximal subinterval of $\bigcup\{\, D \mid \gamma|_D \in \bigcup_{k\in\mathbb{N}} \text{Max}(\Omega^0|_{U^k})\,\}$. Thus for every $j \in \mathbb{N}$, there exists a sequence $\{x^j_k\}_{k\in\mathbb{N}}$ such that each x^j_k is an endpoint of the domain of a move $\gamma|_{D^j_k} \in \text{Max}(\Omega^0|_{U^k})$, and $x^j_k \to x^j$ as $k \to \infty$. We obtain that $x^k_k \to x$ as $k \to \infty$, where each $x^k_k \in X \cup Y$ by Lemma 84. Since $X \cup Y$ is a finite discrete set under Assumption 75, we obtain that $x \in X \cup Y$. ◀

10.3 Refined breakpoints B', complex $\mathcal{T}$

In addition to the finite sets X and Y, we define

$$Z := \{\, x \mid x \in U,\ x = \rho|_D(x) \text{ for some reflection move } \rho|_D \in \Omega \,\}, \tag{25}$$

the set of *uncovered character conflicts*.

▶ Remark 86. In terms of Gr_+ and Gr_- notations, the set Z is the set of projections of the intersection of the translation and reflection moves graphs restricted to the uncovered intervals, $Z = \{\, x \mid x \in U,\ (x, x) \in \text{Gr}_\pm(\Omega) \,\}$.

▶ **Example 87** (Example 78, continued)**.** In the example shown in Figure 15, we have $Z = \{\frac{11}{24}\}$.

▶ **Theorem 88.** *Under Assumption 75, the sets X, Y, and Z are closed under the action of all moves from* $\text{clsemi}_A(\Omega^0)$.

Proof. Let $\gamma|_D$ be a move in $\text{clsemi}_A(\Omega^0)$, which has a finite presentation $(\Omega, \mathcal{C})$.

Let $x \in X$ such that $x \in D$. Since C is invariant under the action of all moves from $\text{clsemi}_A(\Omega^0)$, we have that $\gamma|_D(x) \in X$.

Let $y \in Y$ such that $y \in D$. By Lemma 81, $y \in U$, so the move $\gamma|_D$ is actually in restrict(Ω). It follows from the definition of Y in equation (21) that $y \in V \cap U$ or there exist $x \in V \cap U$ and $\gamma'|_{D'} \in \Omega$ such that $\gamma'|_{D'}(x) = y$. In the former case, $\gamma|_D(y) \in \Omega(V \cap U) \subseteq Y$. In the latter case, we have $\gamma|_D(y) = \gamma|_D \circ \gamma'|_{D'}(x)$, where $\gamma|_D \circ \gamma'|_{D'} \in$ restrict(Ω). Therefore, $\gamma|_D(y) \in Y$.

Let $z \in Z$ such that $z \in D$. By definition, $z \in U$ and $z = \rho|_{D'}(z)$ for some reflection move $\rho|_{D'} \in \Omega$. Let $z' = \gamma|_D(z)$. We have that $z' \in U$ and $z' = \gamma|_D \circ \rho|_{D'} \circ (\gamma|_D)^{-1}(z')$, where $\gamma|_D \circ \rho|_{D'} \circ (\gamma|_D)^{-1} \in$ restrict(Ω). Therefore, $z' = \gamma|_D(z) \in Z$. ◀

Under Assumption 75, the sets X, Y, Z are finite. We then define B', which is a finite set of points under Assumption 75, a *refined set of breakpoints*,

$$B' := (X \cup Y \cup Z) + \mathbb{Z}. \tag{26}$$

By Lemma 80, a breakpoint $b \in B \cap [0, 1]$ lies in $V \cap U$ or ∂U. Since $V \cap U \subseteq Y$ and $\partial U \subseteq X$, we have $B \subseteq B'$. Hence, the polyhedral complex $\mathcal{T} := \mathcal{P}_{B'}$ is a refinement of $\mathcal{P}_B$, so our function π is piecewise linear over $\mathcal{T}$. The following result shows that each of the p_1, p_2 and p_3 projections of any additive vertex of the two-dimensional polyhedral complex $\Delta\mathcal{T}$ is either in B' or covered by C.

▶ **Theorem 89** (Breakpoint stabilization theorem)**.** *Let (x, y) be an additive vertex of $\Delta\mathcal{T}$. Let $z = x + y$. Then, $x, y, z \in B' \cup (C + \mathbb{Z})$.*

Proof. Let F be the unique face of $\Delta\mathcal{P}_B$ such that $(x, y) \in \operatorname{rel\,int}(F)$. Since (x, y) is an additive vertex of $\Delta\mathcal{T}$, and $\Delta\pi$ is non-negative and affine linear over F, we have that F is an additive face of $\Delta\mathcal{P}_B$. Consider $t = x, y$ or z. By $\mathbb{Z}$-periodicity, we can assume $t \in [0, 1]$. To show that $t \in (B' \cap [0, 1]) \cup C$, we distinguish three cases, as follows. We recall that $B' \cap [0, 1] = X \cup Y \cup Z$ and $U = (0, 1) \setminus \operatorname{cl}(C)$.

Assume that F is a zero-dimensional additive face of $\Delta\mathcal{P}_B$. Then, (x, y) is an additive vertex of $\Delta\mathcal{P}_B$, and thus $t \in V$. If $t = 0$, $t = 1$, or $t \in \operatorname{cl}(C)$, then $t \in X \cup C \subseteq B' \cup C$. Otherwise, $t \in V \cap U$. Since $V \cap U \subseteq Y$ by Lemma 81, we obtain that $t \in Y \subseteq B'$.

Assume that F is a one-dimensional additive face (say, a horizontal additive edge) of $\Delta\mathcal{P}_B$. Then, $y \in B \subseteq B'$ and the move $\tau_y|_D$ with $x \in D := \operatorname{int}(p_1(F))$ is in Ω^0. Since (x, y) is a vertex of $\Delta\mathcal{T}$, at least two of x, y, z are in B', and hence at least one of x and z is in B'. Without loss of generality, we assume that $x \in B'$. By Theorem 88, $z = \tau_y|_D(x) \in B'$ as well. We showed that $x, y, z \in B'$ in this case. We omit the proof of the cases where F is a vertical or diagonal additive edge of $\Delta\mathcal{P}_B$, which are similar to the above proof.

Assume that F is a two-dimensional additive face of $\Delta\mathcal{P}_B$. Then, by Corollary 57, we have $t \in C$. ◀

▶ Remark 90. Theorem 89 is key to our grid-free theory. In the grid case of [3], where $B = \frac{1}{q}\mathbb{Z}$, the projections $p_1\colon (x, y) \mapsto x$, $p_2\colon (x, y) \mapsto y$, and $p_3\colon (x, y) \mapsto x + y$ map all vertices of $\Delta\mathcal{P}_B$ back to the set B. We have stabilization of breakpoints due to unimodularity. Going to higher dimension (minimal valid functions of several variables), the piecewise linear functions defined on a standard triangulation of $\mathbb{R}^2$ studied in [7, 4] also stabilize. However, the non-existence of triangulations with stabilization for $\mathbb{R}^k$, $k \geq 3$ [13] blocks the path for further generalizations of the approach of [3, 7, 4]. Our Theorem 89 depends on more detailed data of the function than the group G generated by B. This "dynamic" stabilization result could pave the way to generalizations to higher dimension.

10.4 Connected uncovered components U_i

Define $U' := U \setminus B'$. The interval $[0, 1]$ is partitioned into the set C of covered points, the set U' of uncovered points and the set $B' \cap [0, 1]$ of breakpoints of $\mathcal{T}$. We consider the ensemble $\Omega|_{U'}$ of maximal moves restricted to U' as defined in Subsection 3.2. Lemma 83 and Theorems 85 and 88 imply the following corollary.

▶ **Corollary 91.** *Under Assumptions 74 and 75, the move ensemble $\Omega|_{U'}$ satisfies that:*

a. $\Omega|_{U'} = {}_{U'}|\Omega|_{U'}$.
b. $\Omega|_{U'}$ *is a finite move ensemble.*
c. *For any $\gamma|_D \in \Omega|_{U'}$, $\operatorname{cl}(D)$ and $\operatorname{cl}(\gamma(D))$ are faces of $\mathcal{T}$.*

We partition the set of uncovered points U' into the (maximal) *connected uncovered components* $\{U_1, \dots, U_l\}$, as follows.[2] A connected uncovered component U_i ($1 \leq i \leq l$) is a maximal subset of U' that is the disjoint

[2] This extends the terminology of [3] where connected components are grid-based.

union of all the uncovered intervals $I_1, I_2, \ldots, I_p \subseteq U'$ such that any pair of intervals I_j and I_k $(1 \leq j, k \leq p)$ are connected by a maximal move $\gamma|_{I_k} \in \Omega|_{U'}$ with domain I_k and image $I_j = \gamma(I_k)$.

▶ Remark 92. The set $\Omega|_{U'}$ only has moves $\gamma|_D$ whose domain D and image $\gamma(D)$ are both contained in the same U_i, for $i = 1, 2, \ldots, l$.

Since the function π is piecewise linear over $\mathcal{T}$ and it respects $\Omega|_{U'}$, we have that π is affine linear with the same slope on the maximal intervals $I_1, I_2, \ldots, I_p$ of the same connected uncovered component U_i. Since an effective perturbation $\tilde{\pi} \in \tilde{\Pi}^\pi$ also respects $\Omega|_{U'}$, it takes the same shape on the uncovered intervals $I_1, I_2, \ldots, I_p \subseteq U_i$. We pick $D \in \{I_1, I_2, \ldots, I_p\}$ arbitrarily as the *fundamental domain*, and write $I_j = \gamma_j(D)$ where $\gamma_j|_D \in \Omega|_{U'}$ for $j = 1, 2, \ldots, p$. Then, the connected uncovered component $U_i \subseteq U'$ can be written as $U_i = \bigcup \gamma_j(D)$.

10.5 Finite-dimensional and equivariant perturbation subspaces

Under Assumption 73, we define the following spaces.

▶ **Definition 93.** *Define the* space of finite-dimensional perturbations *that are piecewise linear over $\mathcal{T}$:*

$$\tilde{\Pi}^\pi_{\mathcal{T}} := \bigl\{\, \tilde{\pi} \in \tilde{\Pi}^\pi \bigm| \tilde{\pi} \text{ is piecewise linear over } \mathcal{T} \,\bigr\}. \tag{27}$$

Thus, functions in $\tilde{\Pi}^\pi_{\mathcal{T}}$ are allowed to be discontinuous.

▶ **Definition 94.** *Define the* space of equivariant perturbations *that vanish on the vertices of $\mathcal{T}$:*

$$\tilde{\Pi}^\pi_{\mathrm{zero}(\mathcal{T})} := \left\{\, \tilde{\pi} \in \tilde{\Pi}^\pi \;\middle|\; \tilde{\pi}(t) = \lim_{\substack{x \to t \\ x < t}} \tilde{\pi}(t) = \lim_{\substack{x \to t \\ x > t}} \tilde{\pi}(t) = 0, \ \forall t \in \operatorname{vert}(\mathcal{T}) \,\right\}.$$

We will show in Theorem 100 that all functions in $\tilde{\Pi}^\pi_{\mathrm{zero}(\mathcal{T})}$ are Lipschitz continuous. We will also show that the space is equivariant under the action of $\mathrm{clsemi}_A(\Omega^0)$, in the sense of Subsection 5.1. This will justify the name.

▶ Remark 95. In Lemma 68 we showed that the space $\tilde{\Pi}^\pi$ of effective perturbations is a vector space. The space $\tilde{\Pi}^\pi_{\mathcal{T}}$ of finite-dimensional perturbations and the space $\tilde{\Pi}^\pi_{\mathrm{zero}(\mathcal{T})}$ of equivariant perturbations are vector subspaces of it.

▶ Remark 96. The vector spaces $\tilde{\Pi}^\pi_{\mathcal{T}}$ and $\tilde{\Pi}^\pi_{\mathrm{zero}(\mathcal{T})}$ should not be confounded with the vector spaces $\bar{\Pi}^E_{\mathcal{T}}$ and $\bar{\Pi}^E_{\mathrm{zero}(\mathcal{T})}$ with prescribed additivities $E = \{\, (x, y) \mid \Delta\pi(x, y) = 0 \,\}$, used in [5, Lemma 3.14], where the function π is assumed to be continuous piecewise linear over $\mathcal{T}$ with $\operatorname{vert}(\mathcal{T}) = \frac{1}{q}\mathbb{Z}$, $q \in \mathbb{N}$.

10.6 Finite-dimensional linear algebra for $\tilde{\Pi}^\pi_{\mathcal{T}}$

Let $\tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}^\pi_{\mathcal{T}}$ be a finite-dimensional perturbation. Note that $\tilde{\pi}_{\mathcal{T}}$ is a piecewise linear function, and it is uniquely determined by its values $\tilde{\pi}_{\mathcal{T}}(x)$ and limits $\tilde{\pi}_{\mathcal{T}}(x^-) := \lim_{t \to x, t < x} \tilde{\pi}_{\mathcal{T}}(t)$, $\tilde{\pi}_{\mathcal{T}}(x^+) := \lim_{t \to x, t > x} \tilde{\pi}_{\mathcal{T}}(t)$ at the breakpoints $x \in B' + \mathbb{Z} = \operatorname{vert}(\mathcal{T})$.

▶ **Lemma 97.** *A function $\tilde{\pi}_{\mathcal{T}} \colon \mathbb{R} \to \mathbb{R}$ is a finite-dimensional perturbation, $\tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}^\pi_{\mathcal{T}}$, if and only if $\tilde{\pi}_{\mathcal{T}}$ is piecewise linear over $\mathcal{T}$ and satisfies the following conditions.*

i. $\tilde{\pi}_{\mathcal{T}}(0) = 0$ *and* $\tilde{\pi}_{\mathcal{T}}(f) = 0$*;*
ii. $\tilde{\pi}_{\mathcal{T}}(x) = \tilde{\pi}_{\mathcal{T}}(x + t)$ *for all* $x \in \mathbb{R}$, $t \in \mathbb{Z}$*;*
iii. *For any additive vertex (x, y) of $\Delta\mathcal{T}$ and any face $F \in \Delta\mathcal{T}$ such that $(x, y) \in F$, $\Delta\pi_F(x, y) = 0$ implies $\Delta(\tilde{\pi}_{\mathcal{T}})_F(x, y) = 0$.*

Before we give the proof, we define another space $\bar{\Pi}^{E_\bullet(\pi, \mathcal{T})}$, following [19]. Recall from Subsection 9.2 the family of sets $E_F(\pi)$, indexed by faces F of a polyhedral complex, which capture the set of additivities and limit-additivities of π. Here we use this family with the refined polyhedral complex $\Delta\mathcal{T}$, considering π as a piecewise linear function on $\mathcal{T}$.

▶ **Definition 98.** *For a family $E_\bullet = \{E_F\}_{F \in \Delta\mathcal{T}}$, define the* space of perturbation functions with prescribed additivities and limit-additivities $E_\bullet$*,*

$$\bar{\Pi}^{E_\bullet} = \left\{\, \bar{\pi} \colon \mathbb{R} \to \mathbb{R} \;\middle|\; \begin{aligned} \bar{\pi}(0) = \bar{\pi}(f) &= 0 \\ \Delta\bar{\pi}_F(x, y) &= 0 && \text{for } (x, y) \in E_F,\ F \in \Delta\mathcal{T} \\ \bar{\pi}(x + t) &= \bar{\pi}(x) && \text{for } x \in \mathbb{R},\ t \in \mathbb{Z} \end{aligned} \,\right\}.$$

Proof of Lemma 97. We consider π as piecewise linear over $\mathcal{T}$, which is a refinement of $\mathcal{P}_B$. Let $\tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}^{\pi}_{\mathcal{T}}$. Then by definition, $\tilde{\pi}_{\mathcal{T}}$ is also piecewise linear over $\mathcal{T}$. Since $\tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}^{\pi}$, we have that $\tilde{\pi}_{\mathcal{T}} \in \bar{\Pi}^{E_\bullet}$, where $E_\bullet = E_\bullet(\pi, \mathcal{T})$ is the family of sets $E_F(\pi)$, indexed by $F \in \Delta\mathcal{T}$. Namely, $\tilde{\pi}_{\mathcal{T}}$ satisfies the conditions (i), (ii) and

iii′. For any face $F \in \Delta\mathcal{T}$ and any $(x, y) \in F$, if $\Delta\pi_F(x, y) = 0$ then $\Delta(\tilde{\pi}_{\mathcal{T}})_F(x, y) = 0$.

The condition (iii′) clearly implies (iii). Thus, we proved the "only if" direction. Now let $\tilde{\pi}_{\mathcal{T}}$ be a piecewise linear function over $\mathcal{T}$ that satisfies (i)–(iii). Notice that function π is subadditive and also piecewise linear over $\mathcal{T}$. Hence, the condition (iii) implies (iii′). We obtain that $\tilde{\pi}_{\mathcal{T}} \in \bar{\Pi}^{E_\bullet}$, where $E_\bullet = E_\bullet(\pi, \mathcal{T})$. It then follows from [19, Theorem 3.1] that $\tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}^{\pi}$. Therefore, $\tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}^{\pi}_{\mathcal{T}}$, we proved the "if" direction. ◀

Assume that $B' = \{x'_0 = 0, x'_1, \dots, x'_{n'-1}, x'_n = 1\}$ and we identify $\tilde{\pi}_{\mathcal{T}}(x)$ and $\tilde{\pi}_{\mathcal{T}}(x+t)$ for all $t \in \mathbb{Z}$. Lemma 97 shows that $\big(\tilde{\pi}_{\mathcal{T}}({x'_0}^-),\ \tilde{\pi}_{\mathcal{T}}(x'_0),\ \tilde{\pi}_{\mathcal{T}}({x'_0}^+),\ \tilde{\pi}_{\mathcal{T}}({x'_1}^-),\ \dots,\ \tilde{\pi}_{\mathcal{T}}({x'_{n'-1}}^-),\ \tilde{\pi}_{\mathcal{T}}(x'_{n'-1}),\ \tilde{\pi}_{\mathcal{T}}({x'_{n'-1}}^+)\big)$ is a solution to the finite-dimensional linear system defined by (i) and (iii). The interpolation of such a solution gives an effective perturbation function $\tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}^{\pi}_{\mathcal{T}}$. We know that $(0, 0, \dots, 0)$ is a trivial solution. If a nontrivial solution exists, then its interpolation $\tilde{\pi}_{\mathcal{T}} \not\equiv 0$, implying that the function π is not extreme.

▶ Remark 99. In fact, one can reduce the number of variables in the above linear system of equations to solve, by considering the connected components, as follows. Corollary 71 and (27) imply that $\tilde{\pi}_{\mathcal{T}}$ is affine linear with the same slope over all the intervals from a connected covered component C_i $(i = 1, 2, \dots, k)$ or from a connected uncovered component U_i $(i = 1, 2, \dots, l)$. Let $\tilde{s}^c_1, \dots, \tilde{s}^c_k$ and $\tilde{s}^u_1, \dots, \tilde{s}^u_l$ denote the corresponding slope variables.

In the discontinuous case, by Lemma 69, using Assumption 73, the perturbation $\tilde{\pi}_{\mathcal{T}}$ can only be discontinuous at the points where π is discontinuous. Let the variables $\tilde{d}_i$ $(i = 1, 2, \dots, m)$ denote the changes of the value of $\tilde{\pi}_{\mathcal{T}}$ at the m discontinuity points of π. In other words, the variables $\tilde{d}_i$ denote jumps $\tilde{\pi}_{\mathcal{T}}(x) - \tilde{\pi}_{\mathcal{T}}(x^-)$ when π is discontinuous at x on the left, or $\tilde{\pi}_{\mathcal{T}}(x^+) - \tilde{\pi}_{\mathcal{T}}(x)$ when π is discontinuous at x on the right.

Then, for any fixed $x \in \mathbb{R}$, the value $\tilde{\pi}_{\mathcal{T}}(x)$ is uniquely determined by the slope variables $\tilde{s}^c_i$ $(i = 1, 2, \dots, k)$, $\tilde{s}^u_i$ $(i = 1, 2, \dots, l)$ and the jump variables $\tilde{d}_i$ $(i = 1, 2, \dots, m)$. These $k + l + m \le 3n'$ variables satisfy the system of linear equations given by Lemma 97, where $(0, 0, \dots, 0)$ is a trivial solution. See [16, Example 7.2] for a concrete example.

10.7 Equivariant perturbation space $\tilde{\Pi}^{\pi}_{\mathbf{zero}(\mathcal{T})}$

Let $\tilde{\pi}_{\mathrm{zero}(\mathcal{T})} \in \tilde{\Pi}^{\pi}_{\mathrm{zero}(\mathcal{T})}$ be an equivariant perturbation of π. By Corollary 26 (or Corollary 30) and Corollary 71, $\tilde{\pi}_{\mathrm{zero}(\mathcal{T})}$ is affine linear on all covered intervals. By definition, $\tilde{\pi}_{\mathrm{zero}(\mathcal{T})}(t) = \tilde{\pi}_{\mathrm{zero}(\mathcal{T})}(t^-) = \tilde{\pi}_{\mathrm{zero}(\mathcal{T})}(t^+) = 0$ for every $t \in \mathrm{vert}(\mathcal{T})$, and $\partial C \subseteq \mathrm{vert}(\mathcal{T})$. Therefore, $\tilde{\pi}_{\mathrm{zero}(\mathcal{T})}$ is zero on $\mathrm{cl}(C)$. If the set of uncovered points $U' = \emptyset$, then $\tilde{\pi}_{\mathrm{zero}(\mathcal{T})} \equiv 0$. Otherwise, recall from Subsection 10.4 that U' is partitioned into connected uncovered components $U_1, U_2, \dots, U_l$. The following theorem gives the characterization of the projection of a perturbation $\tilde{\pi}_{\mathrm{zero}(\mathcal{T})}$ onto the space of functions with support contained in a connected uncovered component U_i.

▶ **Theorem 100** (Characterization of the equivariant perturbations supported on an uncovered component)**.** *Suppose that Assumptions 73, 74 and 75 hold. Let $U_i = \bigcup \gamma_j(D)$ be a connected uncovered component, where D is the fundamental domain for U_i and $\gamma_j|_D \in \Omega|_{U'}$ $(j = 1, \dots, p)$. Let $\tilde{\pi}_i \colon \mathbb{R} \to \mathbb{R}$ be a $\mathbb{Z}$-periodic function such that $\tilde{\pi}_i(x) = 0$ for $x \notin U_i$. Then $\tilde{\pi}_i \in \tilde{\Pi}^{\pi}_{\mathrm{zero}(\mathcal{T})}$ if and only if*

i. *$\tilde{\pi}_i$ is Lipschitz continuous on D;*
ii. *$\tilde{\pi}_i(x) = \tilde{\pi}_i(x^-) = \tilde{\pi}_i(x^+) = 0$ for $x \in \partial D$;*
iii. *$\tilde{\pi}_i(x) = \chi(\gamma_j)\tilde{\pi}_i(\gamma_j(x))$ for $x \in D$, $j = 1, \dots, p$.*

Proof. Let $\tilde{\pi}_i \in \tilde{\Pi}^{\pi}_{\mathrm{zero}(\mathcal{T})}$. Since π is continuous on D, by Lemma 69, $\tilde{\pi}_i$ is Lipschitz continuous on $\mathbb{R}$. Hence, the condition (i) holds. The condition (ii) is clearly satisfied, as $\tilde{\pi}_i(x) = \tilde{\pi}_i(x^-) = \tilde{\pi}_i(x^+) = 0$ for each $x \in \mathrm{vert}(\mathcal{T})$. Since $\tilde{\pi}_i$ respects $\Omega|_{U'}$, the condition (iii) also holds.

Conversely, let $\tilde{\pi}_i \colon \mathbb{R} \to \mathbb{R}$ be a $\mathbb{Z}$-periodic function such that $\tilde{\pi}_i(x) = 0$ for $x \notin U_i$ and the conditions (i)–(iii) hold. It follows from (ii) that $\tilde{\pi}_i(x) = \tilde{\pi}_i(x^-) = \tilde{\pi}_i(x^+) = 0$ for $x \in \partial U_i$. Since $\tilde{\pi}_i(x) = 0$ for $x \notin U_i$, we have

$$\tilde{\pi}_i(x) = \tilde{\pi}_i(x^-) = \tilde{\pi}_i(x^+) = 0 \text{ for } x \in [0, 1] \setminus U_i \supseteq B' \cup C. \tag{28}$$

We claim that $\tilde{\pi}_i$ satisfies all the additivities (including the limits) that π has. Indeed, let F be a face of $\Delta\mathcal{T}$ and let $(x, y) \in F$ such that $\Delta\pi_F(x, y) = 0$. We show that $(\Delta\tilde{\pi}_i)_F(x, y) = 0$ by distinguishing the following three cases.

a. If (x, y) is an additive vertex of $\Delta\mathcal{T}$, then by Theorem 89 and (28), we have $(\Delta\tilde{\pi}_i)_F(x, y) = 0$.

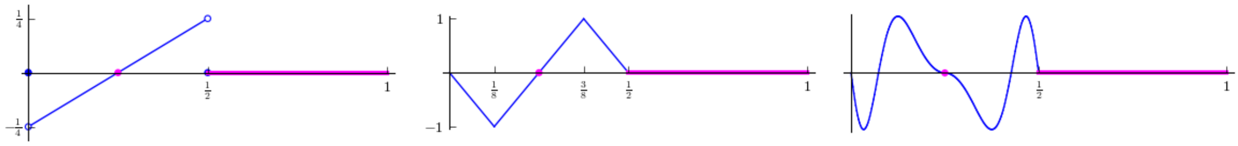


Figure 17 (*Left*) Finite-dimensional perturbation $\tilde{\pi}_{\mathcal{T}}$ of $\pi =$ equiv7_example_1() from Example 76. (*Middle–right*) Examples of equivariant perturbations $\tilde{\pi}_{\text{zero}(\mathcal{T})}$ of π.

b. If (x, y) is contained in the relative interior of an edge F' of $\Delta\mathcal{T}$, then $F' \subseteq F$ and F' is an additive face of $\Delta\mathcal{T}$. Consider the move $\gamma|_{D'}$ associated with F'. We have either D' and $\gamma(D') \subseteq (0,1) \setminus U_i$, or D' and $\gamma(D') \subseteq U_i$. In the former case, the claim holds because of (28); and in the latter case, the structure of $\Delta\mathcal{T}$ (Theorem 89) implies that $\gamma|_{D'} \in \Omega|_{U'}$, and thus $(\Delta\tilde{\pi}_i)_F(x, y) = 0$ by the condition (iii).

c. If (x, y) is contained in the relative interior of a two-dimensional face F' of $\Delta\mathcal{T}$, then $F' = F$ is a two-dimensional additive face of $\Delta\mathcal{T}$. We have $x, y, (x + y) \bmod 1 \in C$, hence the claim follows from (28).

We showed that $\tilde{\pi}_i \in \bar{\Pi}^{E_\bullet}$, where $E_\bullet = E_\bullet(\pi, \mathcal{T})$ is the family from the proof of Lemma 97. Then, [19, Theorem 3.1] implies that $\tilde{\pi}_i \in \tilde{\Pi}^{\pi}$. Therefore, $\tilde{\pi}_i \in \tilde{\Pi}^{\pi}_{\text{zero}(\mathcal{T})}$. ◀

For $i = 1, \ldots, l$, denote the space of functions $\tilde{\pi}_i$ as in the theorem by $\tilde{\Pi}^{\pi}_{U_i}$. It is independent of the choice of fundamental domain.

▶ **Theorem 101** (Direct sum decomposition of equivariant perturbations by uncovered components). *We have the direct sum decomposition $\tilde{\Pi}^{\pi}_{\text{zero}(\mathcal{T})} = \tilde{\Pi}^{\pi}_{U_1} \oplus \cdots \oplus \tilde{\Pi}^{\pi}_{U_l}$, i.e., if $\tilde{\pi} \in \tilde{\Pi}^{\pi}_{\text{zero}(\mathcal{T})}$, then it has a unique decomposition $\tilde{\pi} = \tilde{\pi}_1 + \tilde{\pi}_2 + \cdots + \tilde{\pi}_l$ such that $\tilde{\pi}_i \in \tilde{\Pi}^{\pi}_{U_i}$ for $i = 1, \ldots, l$.*

Proof. Let $\tilde{\pi} \in \tilde{\Pi}^{\pi}_{\text{zero}(\mathcal{T})}$. For $i = 1, 2, \ldots, l$, define $\tilde{\pi}_i : \mathbb{R} \to \mathbb{R}$, $\tilde{\pi}_i(x) = \tilde{\pi}(x)$ if $x \in U_i$ and $\tilde{\pi}_i(x) = 0$ otherwise. Then $\tilde{\pi} = \tilde{\pi}_1 + \tilde{\pi}_2 + \cdots + \tilde{\pi}_l$, where each $\tilde{\pi}_i$ $(i = 1, 2, \ldots, l)$ satisfies the conditions in Theorem 100. ◀

Each of the component functions $\tilde{\pi}_i$ $(i = 1, 2, \ldots, l)$ is supported on the connected uncovered component U_i and is obtained by choosing an arbitrary Lipschitz continuous template on the fundamental domain D_i, then by extending equivariantly to the other intervals through the moves in $\Omega|_{U'}$.

10.8 Decomposition theorem for effective perturbations

The following perturbation decomposition theorem, a generalization of [5, Lemma 3.14] without assuming π is continuous and $\text{vert}(\mathcal{T}) = \frac{1}{q}\mathbb{Z}$, shows that the effective perturbation space $\tilde{\Pi}^{\pi}$ is the direct sum of the finite-dimensional perturbation space $\tilde{\Pi}^{\pi}_{\mathcal{T}}$ and the equivariant perturbation space $\tilde{\Pi}^{\pi}_{\text{zero}(\mathcal{T})}$.

▶ **Theorem 102** (Perturbation decomposition theorem). *Under Assumptions 73, 74 and 75, for every effective perturbation $\tilde{\pi} \in \tilde{\Pi}^{\pi}$, there exist a unique finite-dimensional perturbation $\tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}^{\pi}_{\mathcal{T}}$ and a unique equivariant perturbation $\tilde{\pi}_{\text{zero}(\mathcal{T})} \in \tilde{\Pi}^{\pi}_{\text{zero}(\mathcal{T})}$ such that $\tilde{\pi} = \tilde{\pi}_{\mathcal{T}} + \tilde{\pi}_{\text{zero}(\mathcal{T})}$.*

Proof. Let $\tilde{\pi} \in \tilde{\Pi}^{\pi}$ be an effective perturbation. By [16, Corollary 6.5] , the limits $\tilde{\pi}(t^-)$ and $\tilde{\pi}(t^+)$ exist for every $t \in \text{vert}(\mathcal{T})$. Let $\tilde{\pi}_{\mathcal{T}}$ be the unique piecewise linear function over $\mathcal{T}$ such that $\tilde{\pi}_{\mathcal{T}}(t) = \tilde{\pi}(t)$, $\tilde{\pi}_{\mathcal{T}}(t^-) = \tilde{\pi}(t^-)$ and $\tilde{\pi}_{\mathcal{T}}(t^+) = \tilde{\pi}(t^+)$ for every $t \in \text{vert}(\mathcal{T})$. Define $\tilde{\pi}_{\text{zero}(\mathcal{T})} = \tilde{\pi} - \tilde{\pi}_{\mathcal{T}}$. Note that $\tilde{\pi}_{\mathcal{T}}$ is the unique piecewise linear function over $\mathcal{T}$ such that $\tilde{\pi}_{\text{zero}(\mathcal{T})}(t) = \tilde{\pi}_{\text{zero}(\mathcal{T})}(t^-) = \tilde{\pi}_{\text{zero}(\mathcal{T})}(t^+) = 0$ for every $t \in \text{vert}(\mathcal{T})$. It is left to show that $\tilde{\pi}_{\mathcal{T}}, \tilde{\pi}_{\text{zero}(\mathcal{T})} \in \tilde{\Pi}^{\pi}$.

We first show that $\tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}^{\pi}$, by applying Lemma 97. It suffices to show that $\tilde{\pi}_{\mathcal{T}}$ satisfies condition (iii) of Lemma 97. Let (x, y) be an additive vertex of a face $F \in \Delta\mathcal{T}$ with $\Delta\pi_F(x, y) = 0$. By [16, Lemma 6.1] , $\Delta\pi_F(x, y) = 0$ implies that $\Delta\tilde{\pi}_F(x, y) = 0$. Since (x, y) is an additive vertex of $\Delta\mathcal{T}$, Theorem 89 implies that $x, y, z \in B' \cup C$, where $z = (x + y) \bmod 1$. We have $\tilde{\pi}_{\mathcal{T}}(t) = \tilde{\pi}(t)$, $\tilde{\pi}_{\mathcal{T}}(t^-) = \tilde{\pi}(t^-)$ and $\tilde{\pi}_{\mathcal{T}}(t^+) = \tilde{\pi}(t^+)$ for $t = x, y$ or z, and hence $\Delta(\tilde{\pi}_{\mathcal{T}})_F(x, y) = \Delta\tilde{\pi}_F(x, y) = 0$. Therefore, $\tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}^{\pi}$.

Since the vector space $\tilde{\Pi}^{\pi}$ contains both $\tilde{\pi}$ and $\tilde{\pi}_{\mathcal{T}}$, we obtain that $\tilde{\pi}_{\text{zero}(\mathcal{T})} = \tilde{\pi} - \tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}^{\pi}$. ◀

▶ **Example 103** (Example 76, continued). For the function in Figure 13, $\pi =$ equiv7_example_1(), the refined polyhedral complex $\mathcal{T}$ has vertices $B' = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$. The finite-dimensional perturbation space $\tilde{\Pi}^{\pi}_{\mathcal{T}}$ has dimension

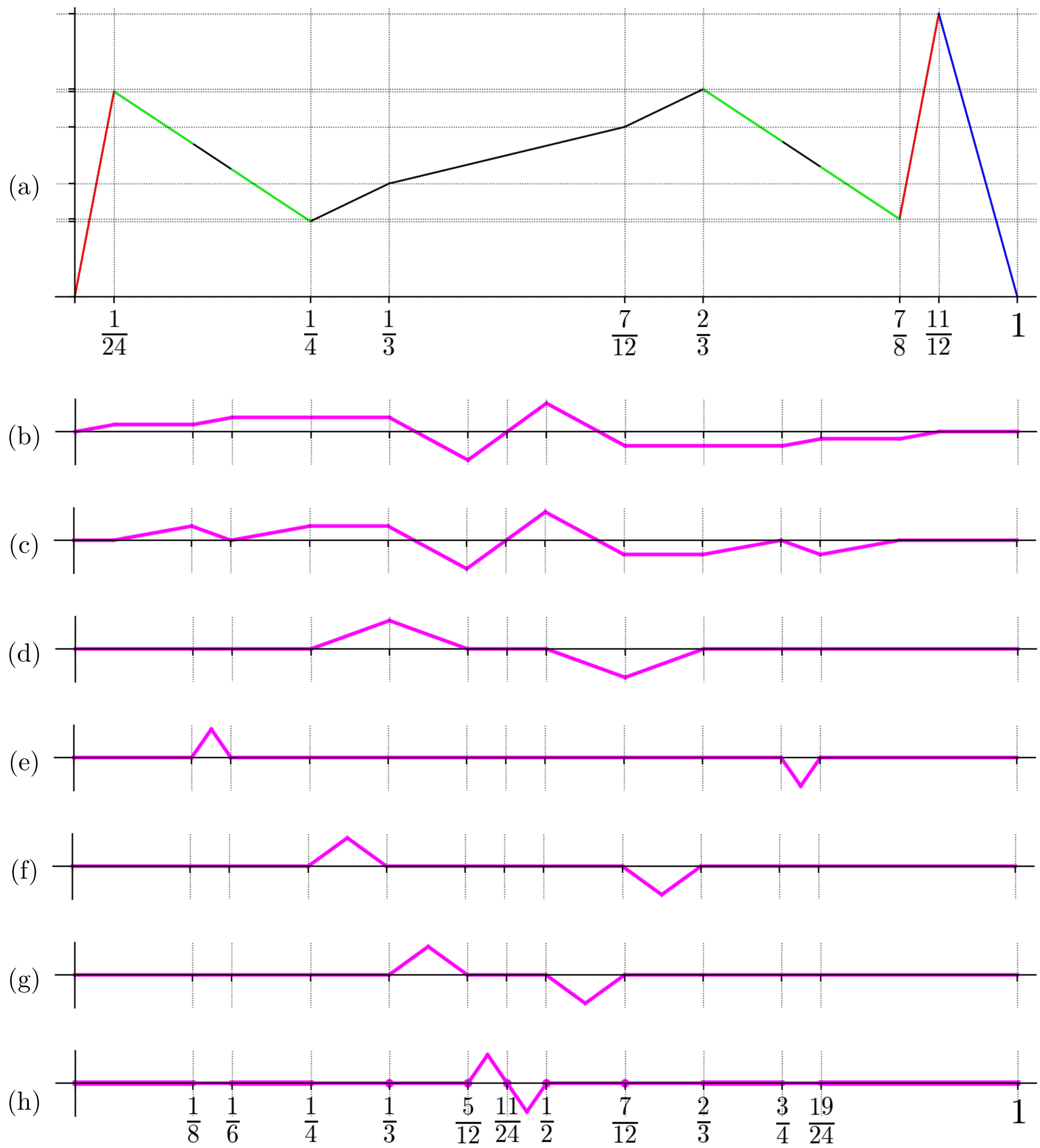


Figure 18 Decomposition of the space of effective perturbations for the function from Example 77/104, π = equiv7_example_xyz_2(). (a) The function π_2. (b–d) basis of the space $\tilde{\Pi}_{\mathcal{T}}$ of finite-dimensional perturbations. (e–h) representatives of the equivariant perturbation spaces $\tilde{\Pi}^{\pi}_{U_i}$ for the 4 connected uncovered components U_i.

1 and is spanned by the basic perturbation

$$\tilde{\pi}_{\mathcal{T}}(x) = \begin{cases} 0 & \text{if } x = 0 \\ x - \frac{1}{4} & \text{if } 0 < x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \le x < 1, \end{cases}$$

see Figure 17 (left). The two intervals $I_1 = (0, \frac{1}{4})$ and $I_2 = (\frac{1}{4}, \frac{1}{2})$ are uncovered, and they are connected through the move $\rho_f|_{(0,1/2)}$ in Ω. Because there is only one connected uncovered component, the equivariant perturbation space $\tilde{\Pi}^{\pi}_{\text{zero}(\mathcal{T})}$ consists of all Lipschitz continuous functions $\tilde{\pi}_{\text{zero}(\mathcal{T})}$ satisfying that $\tilde{\pi}_{\text{zero}(\mathcal{T})}(x) = 0$ for $x \in C \cup B'$ and that $\tilde{\pi}_{\text{zero}(\mathcal{T})}(x) = -\tilde{\pi}_{\text{zero}(\mathcal{T})}(f - x)$ for $x \in U'$. See Figure 17 (middle, right) for examples of such functions.

▶ **Example 104** (Example 77, continued)**.** Figure 18 illustrates the decomposition of the space of effective perturbations.

▶ **Example 105** (Example 78, continued)**.** Figure 19 illustrates the decomposition of the space of effective perturbations.

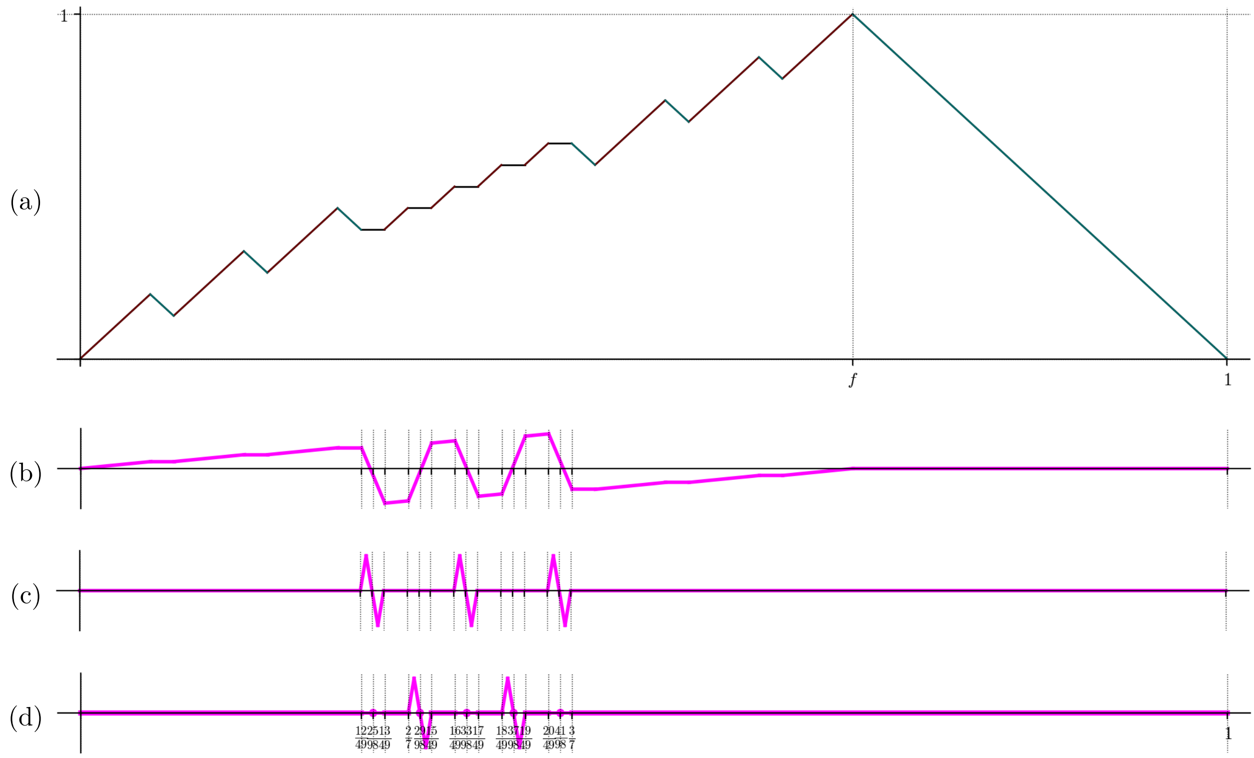


Figure 19 Decomposition of the space of effective perturbations for the function from Example 78/105, $\pi =$ equiv7_minimal_2_covered_2_uncovered(). (a) The function π. (b) finite-dimensional perturbation $\tilde{\pi}_{\mathcal{T}}$. (c), (d) examples of equivariant perturbations $\tilde{\pi}_1, \tilde{\pi}_2$ from the direct summands.

11 Relation of the moves closure to the semigroup $\Gamma^{\mathrm{resp}}(\tilde{\Pi}^\pi)$ of respected moves

In this section, still under the assumptions from Subsection 10.1, we establish the relation between $\mathrm{clsemi}_A(\Omega^0)$ and two other move semigroups:

a. the semigroup $\Gamma^{\mathrm{resp}}(\tilde{\Pi}^\pi)$ of moves respected by all effective perturbation functions $\tilde{\pi}$,
b. the semigroup $\Gamma^{\mathrm{resp}}(\{\pi\} \cup \tilde{\Pi}^\pi) = \Gamma^{\mathrm{resp}}(\pi + \tilde{\Pi}^\pi)$ of moves respected by π and its perturbations.

We already know from Corollary 71 that

$$\mathrm{clsemi}_A(\Omega^0) \subseteq \Gamma^{\mathrm{resp}}(\pi + \tilde{\Pi}^\pi) \subseteq \Gamma^{\mathrm{resp}}(\tilde{\Pi}^\pi). \tag{29}$$

In the case of an extreme function π, the space $\tilde{\Pi}^\pi$ of effective perturbations is trivial; and thus, $\Gamma^{\mathrm{resp}}(\tilde{\Pi}^\pi) = \Gamma^{\subseteq}(\mathbb{R})$.

More generally, whenever a function θ is affine on intervals $D_1, D_2, \dots, D_k$ with the same slope, then $\mathrm{moves}((D_1 \cup \dots \cup D_k) \times (D_1 \cup \dots \cup D_k)) \subseteq \Gamma^{\mathrm{resp}}(\theta)$. Thus, we have the following:

▶ **Lemma 106.** *Suppose the space $\tilde{\Pi}^\pi_{\mathcal{T}}$ of finite-dimensional perturbations is trivial.*

a. *Let C be the set of covered points. Then* $\mathrm{moves}(C \times C) \subseteq \Gamma^{\mathrm{resp}}(\tilde{\Pi}^\pi)$.
b. *Let $D_1, \dots, D_k \subseteq C$ be covered intervals on which π is affine with the same slope. Then* $\mathrm{moves}((D_1 \cup \dots \cup D_k) \times (D_1 \cup \dots \cup D_k)) \subseteq \Gamma^{\mathrm{resp}}(\pi + \tilde{\Pi}^\pi)$.

▶ **Example 107.** Consider the function $\pi =$ equiv7_example_post_3(), shown in Figure 20. It has 4 connected covered components (colored slopes in the figure) and 2 connected uncovered components $U_1 = (\frac{2}{9}, \frac{5}{18}) \cup (\frac{4}{9}, \frac{1}{2})$ and $U_2 = (\frac{5}{6}, \frac{31}{36}) \cup (\frac{31}{36}, \frac{8}{9})$. Its finite-dimensional perturbation space is trivial.

a. From Lemma 106 (a) we see that $\mathrm{moves}(C \times C) \subseteq \Gamma^{\mathrm{resp}}(\tilde{\Pi}^\pi)$.
b. For the smaller semigroup $\Gamma^{\mathrm{resp}}(\pi + \tilde{\Pi}^\pi)$, we observe that the function π is affine with slope 0 on the intervals $D_1 = (\frac{1}{18}, \frac{2}{18})$ and $D_2 = (\frac{2}{18}, \frac{3}{18})$, which belong to separate connected covered components (*cyan* and *lavender*). Because the finite-dimensional perturbation space is trivial, all functions in $\pi + \tilde{\Pi}^\pi$ take the same slope on D_1 and D_2, and hence from Lemma 106 (b) we have $\mathrm{moves}((D_1 \cup D_2) \times (D_1 \cup D_2)) \subseteq \Gamma^{\mathrm{resp}}(\pi + \tilde{\Pi}^\pi)$. By continuity, we also have $\mathrm{moves}((\frac{1}{18}, \frac{3}{18}) \times (\frac{1}{18}, \frac{3}{18})) \subseteq \Gamma^{\mathrm{resp}}(\pi + \tilde{\Pi}^\pi)$.

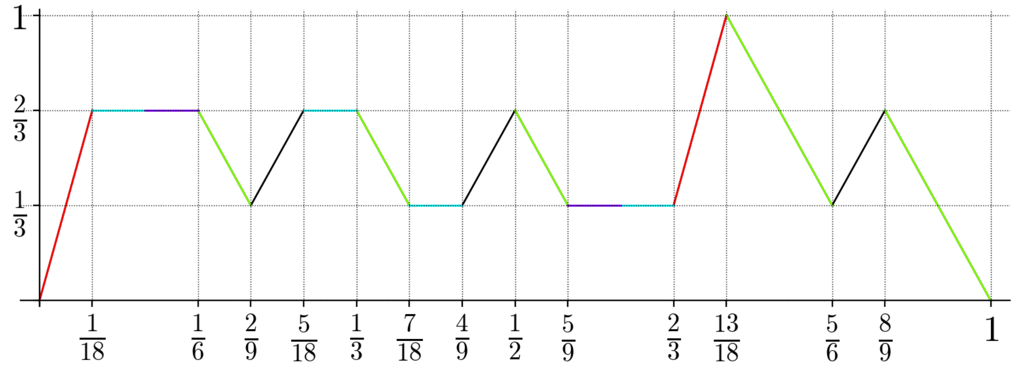


Figure 20 Function $\pi =$ equiv7_example_post_3() from Example 107.

▶ Remark 108. Suppose the finite-dimensional perturbation space has a positive dimension. Recall its description using slope variables $\tilde{s}_i^c$ (for the connected covered components C_i) and $\tilde{s}_i^u$ (for the connected uncovered components U_i) from Remark 99. Whenever for some i, j, we have that $\tilde{s}_i^c = \tilde{s}_j^c$ holds for all solutions, then $\operatorname{moves}((C_i \cup C_j) \times (C_i \cup C_j)) \subseteq \Gamma^{\mathrm{resp}}(\tilde{\Pi}^\pi)$. A similar statement holds for $\Gamma^{\mathrm{resp}}(\pi + \tilde{\Pi}^\pi)$.

Consider these move ensembles restricted to the set U of uncovered points in $(0, 1)$. We have the following theorem.

▶ **Theorem 109.** *Under Assumptions 73, 74, and 75, we have that*

$$\operatorname{clsemi}_A(\Omega^0)|_U = \Gamma^{\mathrm{resp}}(\pi + \tilde{\Pi}^\pi)|_U = \Gamma^{\mathrm{resp}}(\tilde{\Pi}^\pi)|_U. \tag{30}$$

where U is the set of uncovered points in $(0, 1)$.

Proof. We use the notations of the present section. By (29), it suffices to show that if the domain of a move $\gamma|_D \in \Gamma^{\mathrm{resp}}(\tilde{\Pi}^\pi)$ is contained in U, then $\gamma|_D \in \operatorname{clsemi}_A(\Omega^0)$.

Recall that we can write an arbitrary connected uncovered component U_i in the form of $U_i = \bigcup_{j=1}^p \gamma_j(I)$, where I is the fundamental domain for U_i, $\gamma_j|_I \in \operatorname{clsemi}_A(\Omega^0)$, and the open intervals $\gamma_j(I)$ are disjoint. As $\operatorname{clsemi}_A(\Omega^0)$ is join-closed and extension-closed, by taking sub-moves, it suffices to show that if a move $\gamma|_D$ satisfies that $D \subseteq I$ and the unrestricted move $\gamma \neq \gamma_j$ for all $j = 1, \dots, p$, then $\gamma|_D \notin \Gamma^{\mathrm{resp}}(\tilde{\Pi}^\pi)$.

Consider a move $\gamma|_D$ where $D \subseteq I$ and $\gamma \neq \gamma_j$ for all $j = 1, \dots, p$. There exists a proper open interval $D' \subseteq D$ such that $\gamma(D') \cap \gamma_j(D') = \emptyset$ for all $j = 1, \dots, p$. We can construct a perturbation $\tilde{\pi}$ such that

i. $\tilde{\pi}$ is non-zero and Lipschitz continuous on D';
ii. $\tilde{\pi}(x) = \tilde{\pi}(x^-) = \tilde{\pi}(x^+) = 0$ for $x \in \partial D'$;
iii. $\tilde{\pi}(x) = \chi(\gamma_j)\tilde{\pi}(\gamma_j(x))$ for $x \in D'$, $j = 1, \dots, p$; and
iv. $\tilde{\pi}(x) = 0$ for $x \notin \bigcup_{j=1}^p \gamma_j(D')$.

Since $\tilde{\pi}|_{D'} \not\equiv 0$ but $\tilde{\pi}|_{\gamma(D')} \equiv 0$, we have that $\gamma|_{D'} \notin \Gamma^{\mathrm{resp}}(\tilde{\pi})$, and hence $\gamma|_D \notin \Gamma^{\mathrm{resp}}(\tilde{\pi})$. By Theorem 100, $\tilde{\pi} \in \tilde{\Pi}^\pi_{\mathrm{zero}(\mathcal{T})} \subseteq \tilde{\Pi}^\pi$. Therefore, $\gamma|_D \notin \Gamma^{\mathrm{resp}}(\tilde{\Pi}^\pi)$. We conclude that (30) holds. ◀

12 Conclusion

12.1 Forthcoming computational companion paper

In the forthcoming paper [15], Part VIII of the series, we will describe a method to compute the moves closure $\operatorname{clsemi}_A(\Omega^0)$ for a large class of piecewise linear minimal valid functions, including all functions with rational breakpoints, for which the moves closure has a finite presentation. The decomposition of the perturbation space in Section 10 is already algorithmic. Thus we will obtain a grid-free extremality test.

12.2 Limits of the approach of this paper

We now discuss the limitations to the equivariant perturbation theory developed in our series of papers.

For two-sided discontinuous functions, the decomposition of the perturbation spaces breaks down. Theorems 100 and 102 do not hold when the function π is discontinuous from both sides of the origin, as the following example shows.

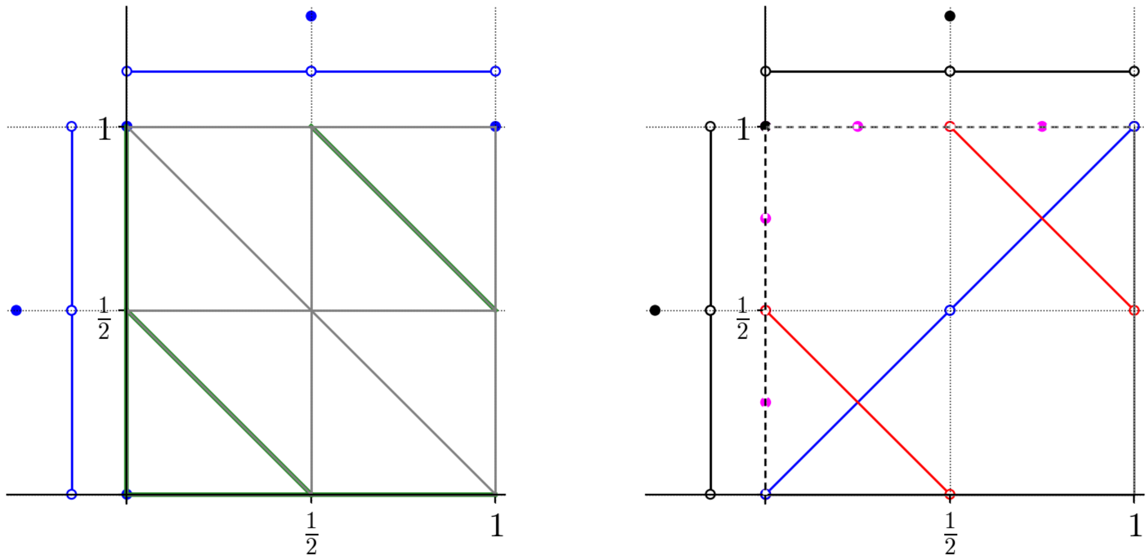


Figure 21 (*Left*) Two-dimensional polyhedral complex $\Delta\mathcal{P}$ of a two-sided discontinuous minimal valid function $\pi =$ minimal_no_covered_interval() (*blue graph at the left and top borders*) from Example 110, where the additive faces are colored in *green*. (*Right*) The graph of the move ensemble $\mathrm{clsemi}_A(\Omega^0)$ of π, where the set $C \cup B' = [0,1) \setminus U'$ of covered points and breakpoints are marked in *magenta on the left and top borders*.

▶ **Example 110.** Consider the minimal valid function $\pi =$ minimal_no_covered_interval() with $f = \frac{1}{2}$, defined by

$$\pi(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } 0 < x < \frac{1}{2} \text{ or } \frac{1}{2} < x < 1 \\ 1 & \text{if } x = \frac{1}{2}, \end{cases}$$

which is discontinuous from both sides of the origin.

Observe from Figure 21 that $C = \emptyset$, $B' = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ and the connected uncovered components are $U_1 = (0, \frac{1}{4}) \cup (\frac{1}{4}, \frac{1}{2})$ and $U_2 = (\frac{1}{2}, \frac{3}{4}) \cup (\frac{3}{4}, 1)$, where the two intervals in either U_1 or U_2 are connected through the move $\rho_f|_{(0,f)}$ or $\rho_f|_{(f,1)}$ in $\mathrm{clsemi}_A(\Omega^0)$. Any bounded $\mathbb{Z}$-periodic function $\tilde{\pi}$ satisfying that $\tilde{\pi}(x) = 0$ for $x \in B'$ and $\tilde{\pi}(x) = \tilde{\pi}(\rho_f(x))$ for $x \in [0,1)$ is an effective perturbation of π. For example, define a $\mathbb{Z}$-periodic function $\tilde{\pi} =$ equiv7_example_2_crazy_perturbation() by

$$\tilde{\pi}(x) = \begin{cases} 1 & \text{if } x \in (0, \frac{1}{4}) \text{ such that } x \in G, \text{ or} \\ & \text{if } x \in (\frac{1}{4}, \frac{1}{2}) \text{ such that } x - \frac{1}{4} \in G; \\ -1 & \text{if } x \in (0, \frac{1}{4}) \text{ such that } x - \frac{1}{4} \in G, \text{ or} \\ & \text{if } x \in (\frac{1}{4}, \frac{1}{2}) \text{ such that } x - \frac{1}{2} \in G; \\ 0 & \text{otherwise,} \end{cases}$$

where the group $G = \langle 1, \sqrt{2} \rangle_{\mathbb{Z}}$ is dense in $\mathbb{R}$. Then $\tilde{\pi}$ is an effective perturbation of π, since both $\pi \pm \epsilon\tilde{\pi}$ are minimal valid functions for $0 < \epsilon \leq \frac{1}{6}$.[3] Observe that $\tilde{\pi}$ is a highly discontinuous function, which does not have a limit at any point in $(0, \frac{1}{2})$. Thus, without Assumption 73, an equivariant perturbation is not necessarily Lipschitz continuous; and the limits of an effective perturbation at the breakpoints might not exist. For this reason, the decomposition of perturbations does not make sense when the function π is discontinuous from both sides of the origin.

Note that in [20], though an algorithm was presented that checks the effectiveness of a given perturbation function $\tilde{\pi}$, and a perturbation was constructed for an example function, it was left as an open question how to construct perturbations in general. This is still open.

[3] This positive ϵ is verified by calling find_epsilon_for_crazy_perturbation(π, $\tilde{\pi}$)

We conjecture that the equivariant perturbation theory also breaks down for the case of non–piecewise linear functions, such as the fractal functions presented in [1] and [2]. In particular we note that

1. the finite system of equations describing the space of finite-dimensional perturbations would be replaced by a system of functional equations, for which we have no lemmas available;
2. we expect that the decomposition theorem no longer holds.

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