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Whiskered KAM tori of conformally symplectic systems

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(Recommended by Leonid Polterovich)

Abstract. We investigate the existence of whiskered tori in some dissipative systems, called *conformally symplectic* systems, having the property that they transform the symplectic form into a multiple of itself. We consider a family f_μ of conformally symplectic maps which depends on a drift parameter μ.

We fix a Diophantine frequency of the torus and we assume to have a drift μ_0 and an embedding of the torus K_0, which satisfy approximately the invariance equation $f_{\mu_0} \circ K_0 = K_0 \circ T_\omega$ (where T_ω denotes the shift by ω). We also assume to have a splitting of the tangent space at the range of K_0 into three bundles. We assume that the bundles are approximately invariant under Df_{μ_0} and the derivative satisfies some *rate conditions*.

Under suitable nondegeneracy conditions, we prove that there exist μ_∞, K_∞ invariant under f_{μ_∞}, close to the original ones, and a splitting which is invariant under Df_{μ_∞}. The proof provides an efficient algorithm to construct whiskered tori. Full details of the statements and proofs are given in [10].

1. Introduction

A whiskered torus for a dynamical system is an invariant torus such that the motion on the torus is conjugated to a rotation and has hyperbolic directions, exponentially contracting in the future or in the past under the linearized evolution [2, 1]. Whiskered tori and their invariant manifolds are the key ingredients proposed in [2] of the so-called *Arnold diffusion* in which solutions of a nearly integrable system may drift far from their initial values.

Whiskered tori have been widely studied mainly for symplectic systems (see, e.g., [22, 17, 18]); in this paper we go over the results of [10] and we consider their existence for *conformally symplectic systems* [3, 8, 16, 28], which are characterized by the fact that the symplectic structure is transformed into a multiple of itself. Conformally symplectic systems are a very special case of dissipative systems and occur in several physical

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examples, e.g. the spin-orbit problem in Celestial Mechanics, Gaussian thermostats, Euler–Lagrange equations of exponentially discounted systems [12, 28, 14, 15, 27].

The existence of invariant tori in conformally symplectic systems needs an adjustment of parameters. This leads to considering a family f_μ of conformally symplectic maps depending parametrically on μ. Our main result (Theorem 4.1) establishes the existence of whiskered tori with frequency ω for f_μ for some μ; Theorem 4.1 is based on the formulation of an invariance equation for the parameterization of the torus, say $K = K(\theta)$, for the parameter μ and for the splitting of the space. The invariance equation expresses that the parameterization is invariant under the map f_μ and the splitting is invariant under Df_μ. The main assumption of Theorem 4.1 is that we are given a sufficiently approximate solution of (2.2) with an approximately invariant splitting. We also need to assume that the frequency ω is Diophantine and some nondegeneracy conditions are met. We note that the nondegeneracy conditions we need to assume are algebraic expressions depending only on the approximate solution and its derivatives. We do not need to assume any global properties (such as twist) for the whole system. We also note that Theorem 4.1 does not assume the system to be close to an integrable system. Theorems where the main hypothesis is that there is an approximate solution with some condition numbers are called “a-posteriori” theorems in the numerical analysis literature.

The proof of Theorem 4.1 is based on showing that a Newton-like method started on the approximate solution converges. At each step of Newton's method, the linearized equation is projected on the hyperbolic and center subspaces. The equations on the hyperbolic subspaces are solved using a contraction method (see, e.g., [7]). The invariance equation projected on the center subspace is solved using the so-called *automatic reducibility:* taking advantage from the geometry of a conformally symplectic system, one can introduce a change of coordinates in which the linearized equation along the center directions can be solved by Fourier methods.

Remarkably, we show the center bundles of whiskered tori are trivial in the sense of bundle theory, i.e., they are homeomorphic to product bundles. On the other hand, we allow the stable and unstable bundles to be nontrivial in the sense of bundle theory, and there are examples of this situation. Note that nontrivial bundles do not seem to be incorporated in some of the proofs based on normal form theory.

We remark that we do not use transformation theory unlike the pioneering works [24, 4, 5]; we do not perform subsequent changes of variables that transform the system into a form which admits an invariant torus.

Whiskered tori were studied with a similar approach in [17, 18]; the results in an a-posteriori format were proved in [17] for the case of finite-dimensional Hamiltonian systems, while generalizations to Hamiltonian lattice systems are presented in [18] and to PDEs in [22].

The method introduced in [20] (see also [21, 8, 11] for an application to quasi-periodic normally hyperbolic invariant tori) has several advantages: it leads to efficient algorithms, it does not need to work in action-angle variables, and it does not assume that the system is close to integrable. Hence, the approach is suitable to study systems close to breakdown and in the limit of small dissipation. This allows us to study the analyticity domain of K and μ as a function of a parameter ε, such that the limit of ε tending to zero represents the symplectic case. Note that the limit of dissipation going to zero is a singular limit. Full dimensional KAM tori in conformally symplectic systems have also been considered in [26, 23]. The first paper is based on transformation theory and the second also includes numerical implementations comparing the methods based on transformation theory and those based on studying (2.2).

Our second main result, Theorem 7.1, shows that if we introduce an extra perturbative parameter ε such that f_ε is a symplectic map with a solution K_0, μ_0 as in (2.2), then this map can be continued to K_ε, μ_ε which are analytic in a domain obtained by removing a sequence of smaller balls from a ball centered at the origin. The centers of the removed balls lie on a union of curves and their radii decrease quickly as they approach the origin (see also [9, 6]). The proof is based on the construction of Lindstedt series, whose finite order truncation provides an approximate solution of the a-posteriori theorem. We conjecture that this domain is essentially optimal.

The rest of this paper is organized as follows. In Section 2 we provide some preliminary notions; Section 3 presents some properties of cocycles and invariant bundles; the main result, Theorem 4.1, is stated in Section 4; a sketch of the proof of Theorem 4.1 is given in Section 5; an algorithm allowing to construct the new approximation is given in Section 6; the analyticity domains of whiskered tori are presented in Section 7.

2. Preliminary notions

This section is devoted to introducing the notion of conformally symplectic systems, the definition of Diophantine vectors, invariant rotational tori, and the introduction of function spaces.

We denote by $\mathscr{M} = \mathbb{T}^n \times B$ a symplectic manifold of dimension $2n$, where $B \subseteq \mathbb{R}^n$ is an open, simply connected domain with smooth boundary. We endow $\mathscr{M}$ with the standard scalar product and a symplectic form Ω, which does not necessarily have the standard form. For systems in the limit of small dissipation (see Section 7), we assume that the form Ω is exact.

Definition 2.1. A diffeomorphism $f : \mathscr{M} \to \mathscr{M}$ is conformally symplectic if there exists a function λ such that

$$f^*\Omega = \lambda\Omega\,. \tag{2.1}$$

We will assume λ to be constant, which is always the case when $n \geq 2$ (see [3]), since whiskered tori exist only for $n \geq 2$.

Denoting the inner product on $\mathbb{R}^{2n}$ by $\langle\cdot,\cdot\rangle$, let J_x be the matrix representing Ω at x

$$\Omega_x(u,v) = \langle u, J_x v\rangle$$

with $J_x^T = -J_x$.

Frequency vectors of whiskered tori are assumed to be Diophantine.

Definition 2.2. For $\lambda \in \mathbb{C}$, let $\nu(\lambda;\omega,\tau)$ be defined as

$$\nu(\lambda;\omega,\tau) \equiv \sup_{k\in\mathbb{Z}^d\setminus\{0\}} \left(|e^{2\pi i k\cdot\omega} - \lambda|^{-1}\,|k|^{-\tau}\right).$$

We say that λ is ω-Diophantine of class τ and constant $\nu(\lambda;\omega,\tau)$ if

$$\nu(\lambda;\omega,\tau) < \infty\,.$$

A particular case of the definition above is when $\lambda = 1$ which corresponds to the classical definition of ω. In our theorems, we will assume that ω is Diophantine and we will consider λ's which are Diophantine with respect to it.

We remark that in Theorem 4.1 we will take only $\lambda \in \mathbb{R}$, while in Theorem 7.1 we will take $\lambda \in \mathbb{C}$.

To find an invariant torus in a conformally symplectic system, we need to adjust some parameters (see [8]); hence, we consider a family f_μ of conformally symplectic maps depending on a *drift* parameter μ.

Definition 2.3. Let $f_\mu : \mathcal{M} \to \mathcal{M}$ be a family of differentiable diffeomorphisms and let $K : \mathbb{T}^d \to \mathcal{M}$ be a differentiable embedding. Denoting by T_ω the shift by $\omega \in \mathbb{R}^d$, we say that K parameterizes an invariant torus for the parameter μ if the following invariance equation is satisfied:

$$f_\mu \circ K = K \circ T_\omega \, . \tag{2.2}$$

Equation (2.2), which will be the centerpiece of our study, contains K and μ as unknowns; its linearization will be analyzed using a quasi-Newton method that takes advantage of the geometric properties of conformally symplectic systems. We remark that if (K,μ) is a solution, then $(K \circ T_\alpha, \mu)$ is also a solution. We also show that local uniqueness is obtained by choosing a suitable normalization that fixes α.

To estimate the quantities involved in the proof, we introduce an analytic function space and a norm.

Definition 2.4. Let $\rho > 0$ and let $\mathbb{T}^d_\rho$ be the set

$$\mathbb{T}^d_\rho = \{z \in \mathbb{C}^d/\mathbb{Z}^d : \operatorname{Re}(z_j) \in \mathbb{T} \, , \quad |\operatorname{Im}(z_j)| \le \rho \, , \quad j = 1, ..., d\} \, .$$

Given a Banach space X, let $\mathcal{A}_\rho(X)$ be the set of functions from $\mathbb{T}^d_\rho$ to X, analytic in $\operatorname{Int}(\mathbb{T}^d_\rho)$ and extending continuously to the boundary of $\mathbb{T}^d_\rho$. We endow $\mathcal{A}_\rho$ with the following norm, which makes $\mathcal{A}_\rho$ a Banach space

$$\|f\|_{\mathcal{A}_\rho} = \sup_{z \in \mathbb{T}^d_\rho} |f(z)| \, .$$

The norm of a vector valued function $g = (g_1, \ldots, g_n)$ is defined as

$$\|g\|_{\mathcal{A}_\rho} = \sqrt{\|g_1\|^2_{\mathcal{A}_\rho} + \ldots + \|g_n\|^2_{\mathcal{A}_\rho}},$$

while the norm of an $n_1 \times n_2$ matrix valued function G is defined as

$$\|G\|_{\mathcal{A}_\rho} = \sup_{\chi \in \mathbb{R}^{n_2}_+, |\chi| = 1} \sqrt{\sum_{i=1}^{n_1} \left(\sum_{j=1}^{n_2} \|G_{ij}\|_{\mathcal{A}_\rho} \chi_j \right)^2} \, .$$

3. Cocycles and invariant bundles

Given an approximate solution of (2.2), we will construct a new approximation with a smaller error. In order to do it, we will need to study products

$$\Gamma^j \equiv Df_\mu \circ K \circ T_{(j-1)\omega} \times \cdots \times Df_\mu \circ K \, , \tag{3.1}$$

which are quasi-periodic cocycles of the form

$$\Gamma^j = \gamma_\theta \circ T_{(j-1)\omega} \times \cdots \times \gamma_\theta \tag{3.2}$$

with $\gamma_\theta = Df_\mu \circ K(\theta)$. The cocycle (3.2) satisfies the property $\Gamma^{j+m} = \Gamma^j \circ T_{m\omega} \, \Gamma^m$. The study of the invariance equation strongly depends on the asymptotic growth of the cocycle (3.1), which leads to the definition below (see [25, 13]). We denote the number of degrees of freedom by n, the dimension of the phase space by $\mathfrak{n}$, and the dimension of the invariant tori by d.

Definition 3.1. The cocycle (3.1) admits an exponential trichotomy if there exists a decomposition

$$\mathbb{R}^{\mathfrak{n}} = E^s_\theta \oplus E^c_\theta \oplus E^u_\theta \, , \qquad \theta \in \mathbb{T}^d \, , \tag{3.3}$$

rates of decay $\lambda_- < \lambda_c^- \le \lambda \le \lambda_c^+ < \lambda_+$, $\lambda_- < 1 < \lambda_+$ and a constant $C_0 > 0$, such that

$$\begin{aligned} v \in E_\theta^s &\iff |\Gamma^j(\theta)\, v| \le C_0 \lambda_-^j |v|, \quad j \ge 0 \\ v \in E_\theta^u &\iff |\Gamma^j(\theta)\, v| \le C_0 \lambda_+^j |v|, \quad j \le 0 \\ v \in E_\theta^c &\iff \begin{array}{l} |\Gamma^j(\theta)\, v| \le C_0 (\lambda_c^+)^j |v|, \quad j \ge 0 \\ |\Gamma^j(\theta)\, v| \le C_0 (\lambda_c^-)^j |v|, \quad j \le 0\,. \end{array} \end{aligned} \tag{3.4}$$

Splitting (3.3) determines projections on E_θ^s, E_θ^c, E_θ^u which we denote by $\Pi_\theta^s, \Pi_\theta^c, \Pi_\theta^u$. Let us now consider two nearby splittings E, $\tilde{E}$; then, for each space in $\tilde{E}$, we can find a linear function $A_\theta^\sigma : E_\theta^\sigma \to E_\theta^{\hat\sigma}$ (where $E_\theta^{\hat\sigma}$ is the sum of the spaces in the splitting not indexed by σ), such that

$$\tilde{E}_\theta^\sigma = \{v \in \mathbb{R}^n,\, v = x + A_\theta^\sigma x \,|\, x \in E_\theta^\sigma\}\,. \tag{3.5}$$

We denote the orthogonal projections by $P_{E_\theta}^\perp$ and define the distance between E and $\tilde{E}$ as

$$\mathrm{dist}_\rho(E, \tilde{E}) = \| P_{E_\theta}^\perp - P_{\tilde{E}_\theta}^\perp \|_{\mathscr{A}_\rho}\,.$$

From (3.4) it is possible to show (see [19]) that the splitting depends continuously (Hölder) on θ; furthermore, by bootstrapping the regularity, the splitting is proven to be analytic. Therefore, the projections Π^σ, $\sigma = s, u, c$, are uniformly bounded (see [25]). We also notice (see [10]) that the bundles characterized by (3.4) are invariant:

$$\gamma_\theta E_\theta^\sigma = E_{\theta+\omega}^\sigma.$$

3.1. Approximately invariant splittings.

For a splitting $E_\theta^s \oplus E_\theta^u \oplus E_\theta^c$ and a cocycle γ_θ, let $\gamma_\theta^{\sigma,\sigma'}$ be

$$\gamma_\theta^{\sigma,\sigma'} = \Pi_{\theta+\omega}^\sigma \gamma_\theta \Pi_\theta^{\sigma'}\,; \tag{3.6}$$

hence, the splitting is invariant under the cocycle if and only if

$$\gamma_\theta^{\sigma,\sigma'} \equiv 0\,, \qquad \sigma \ne \sigma'\,.$$

The lack of invariance of the splitting under the cocyle γ is measured by the quantity

$$\mathscr{I}_\rho(\gamma, E) \equiv \max_{\substack{\sigma,\sigma' \in \{s,c,u\} \\ \sigma \ne \sigma'}} \|\gamma_\theta^{\sigma,\sigma'}\|_{\mathscr{A}_\rho}\,.$$

Now, we introduce a notion of hyperbolicity for approximately invariant splittings.

Definition 3.2. Let γ be a cocycle and E an approximately invariant splitting. Then, γ is approximately hyperbolic w.r.t. E if the cocycle

$$\tilde\gamma_\theta = \begin{pmatrix} \gamma_\theta^{s,s} & 0 & 0 \\ 0 & \gamma_\theta^{c,c} & 0 \\ 0 & 0 & \gamma_\theta^{u,u} \end{pmatrix}$$

satisfies (3.4) with $\gamma^{\sigma,\sigma}$ as in (3.6).

The following Lemma 3.1 shows that if we have an approximately invariant splitting for an approximately hyperbolic cocycle, then there exists a true invariant splitting.

Lemma 3.1. *Fix an analytic reference splitting on $\mathbb{T}_\rho^d$ and let $\mathscr{U}$ be a sufficiently small neighborhood of this splitting, such that all splittings from this neighborhood can be parameterized as graphs of linear maps A_θ^σ as in* (3.5) *with $\|A_\theta^\sigma\|_{\mathscr{A}_\rho} < M_1$ for some $M_1 > 0$.*

Let E be an analytic splitting in the neighborhood $\mathscr{U}$.

Let γ be an analytic cocycle over a rotation defined on $\mathbb{T}_\rho^d$ with $\|\gamma\|_{\mathscr{A}_\rho} < M_2$ for some $M_2 > 0$.

Assume that E is approximately invariant under γ

$$\mathscr{I}_\rho(\gamma, E) \leq \eta$$

and that γ is approximately hyperbolic for the reference splitting as in Definition 3.2.

Then, there is a locally unique splitting $\tilde{E}$ close to E, invariant under γ, which satisfies the trichotomy of Definition 3.1, and such that

$$\mathrm{dist}_\rho(E, \tilde{E}) \leq C\eta \ ,$$

where C, η can be chosen uniformly and depending only on M_1, M_2.

We refer to [10] for the proof of the closing Lemma 3.1, which is based on the standard method of writing the new spaces as the graphs of linear maps $A_x^\sigma : E^\sigma \to E_x^{\hat{\sigma}}$. The fact that these spaces are invariant is equivalent to some fixed point equations that can be solved by the contraction mapping principle. We refer to [10] for details.

4. Existence of whiskered tori

Whiskered tori are defined as follows.

Definition 4.1. Let $f_\mu : \mathscr{M} \to \mathscr{M}$ be a family of conformally symplectic maps with the conformal factor λ. We say that $K : \mathbb{T}^d \to \mathscr{M}$ represents a whiskered torus when for some $\omega \in \mathbb{R}^d$:

(1) K is the embedding of a rotational torus: $f_\mu \circ K = K \circ T_\omega$.
(2) The cocycle $Df_\mu \circ K$ over the rotation T_ω admits a trichotomy as in (3.4) with rates $\lambda_-, \lambda_c^-, \lambda_c^+, \lambda_+$.
(3) The spaces E_θ^c in (3.3) have dimension $2d$.

Theorem 4.1 below states the existence of whiskered tori by solving the invariance equation (2.2).

Let K, μ be an approximate solution of (2.2) with a small error term e: $f_\mu \circ K - K \circ T_\omega = e$. Let Δ, β be some corrections, such that $K' = K + \Delta$, $\mu' = \mu + \beta$ satisfy the invariance equation with an error quadratically smaller. This is obtained if Δ, β satisfy

$$(Df_\mu \circ K)\ \Delta - \Delta \circ T_\omega + (D_\mu f_\mu) \circ K\beta = -e \ .$$

Theorem 4.1. *Let $\omega \in \mathscr{D}_d(\nu, \tau)$, $d \leq n$, be as in* (2.2)*, let $f_\mu : \mathscr{M} \to \mathscr{M}$, $\mu \in \mathbb{R}^d$, be a family of real analytic, conformally symplectic mappings as in* (2.1) *with* $0 < \lambda < 1$. *We make the following assumptions.*

$(H1)$ *Approximate solution:*

Let (K_0, μ_0) with $K_0 : \mathbb{T}^d \to \mathscr{M}$, $K_0 \in \mathscr{A}_\rho$, and $\mu_0 \in \mathbb{R}^d$ represent an approximate whiskered torus for f_{μ_0} with frequency ω:

$$\| f_{\mu_0} \circ K_0 - K_0 \circ T_\omega \|_{\mathscr{A}_\rho} \leq \mathscr{E} \ , \qquad \mathscr{E} > 0 \ .$$

To ensure the composition $f_\mu \circ K$ can be defined, we assume that there exists a domain $\mathscr{U} \subset \mathbb{C}^n / \mathbb{Z}^n \times \mathbb{C}^n$ such that for all μ with $|\mu - \mu_0| \leq \eta$, f_μ is defined on $\mathscr{U}$ and

$$\mathrm{dist}(K_0(\mathbb{T}_\rho^d), \mathbb{C}^n / \mathbb{Z}^n \times \mathbb{C}^n \setminus \mathscr{U}) \geq \eta. \tag{4.1}$$

$(H2)$ *Approximate splitting:*

For all $\theta \in \mathbb{T}_\rho^d$, there exists a splitting of the tangent space of the phase space, depending analytically on the angle $\theta \in \mathbb{T}_\rho^d$; the bundles are approximately invariant under the cocycle $\gamma_\theta = Df_{\mu_0} \circ K_0(\theta)$, i.e., $\mathscr{I}_\rho(\gamma, E) \leq \mathscr{E}_h$, $\mathscr{E}_h > 0$.

$(H3)$ *Spectral condition for the bundles (exponential trichotomy):*

For all $\theta \in \mathbb{T}_\rho^d$ the spaces in $(H2)$ *are approximately hyperbolic for the cocycle γ_θ.*

$(H3')$ *Since we are dealing with conformally symplectic systems, we assume:*

$$\lambda_- < \lambda\lambda_+ < \lambda_c^- , \qquad \lambda_-\lambda_c^+ < \lambda , \qquad \lambda_c^- \le \lambda \le \lambda_c^+ .$$

$(H4)$ *The dimension of the center subspace is* $2d$.

$(H5)$ *Nondegeneracy:*

We assume that some $d \times d$ *matrix* $\mathscr{S}$ *is invertible. This matrix is obtained by solving cohomology equations using algebraic operations that involve the derivatives of the approximate solution. The explicit form of this matrix is given below by the matrix appearing at the l.h.s. of* (5.16).

Let $\alpha = \alpha(\tau)$ *be a sufficiently small number, whose value can be computed explicitly and assume for some* $0 < \delta < \rho$*, we have* $\mathscr{E} \le \delta^{2\alpha}\mathscr{E}^*$, $\mathscr{E}_h \le \mathscr{E}_h^*$*, where* $\mathscr{E}$, $\mathscr{E}^*$ *depend on* ν, τ, C_0, λ_+, λ_-, λ_c^+, λ_c^-, $\|\Pi_\theta^{s/u/c}\|_{\mathscr{A}_\rho}$, $\|DK_0\|_{\mathscr{A}_\rho}$, $\|(DK_0^T DK_0)^{-1}\|_{\mathscr{A}_\rho}$.

Then, there exists an exact solution (K_e, μ_e)*, which satisfies*

$$f_{\mu_e} \circ K_e - K_e \circ T_\omega = 0$$

with

$$\|K_e - K_0\|_{\mathscr{A}_{\rho-2\delta}} \le C\mathscr{E}\delta^{-\tau} , \qquad |\mu_e - \mu_0| \le C\mathscr{E} .$$

Moreover, the invariant torus K_e *is hyperbolic, namely there exists an invariant splitting*

$$\mathscr{T}_{K_e(\theta)}\mathscr{M} = E_\theta^s \oplus E_\theta^c \oplus E_\theta^u ,$$

satisfying Definition 3.1 and such that the splitting Π_f *of the invariant torus is close to the original one* Π_0*:*

$$\|\Pi_0^{s/u/c} - \Pi_f^{s/u/c}\|_{\mathscr{A}_{\rho-2\delta}} \le C(\mathscr{E}\delta^{-\tau} + \mathscr{E}_h) .$$

Finally, the hyperbolicity constants associated to the invariant splitting of the invariant torus (denoted by a tilde) are close to those of the approximately invariant splitting of the approximately invariant torus (see (H1), (H2)):

$$|\lambda_\pm - \widetilde{\lambda}_\pm| \le C(\mathscr{E}\delta^{-\tau} + \mathscr{E}_h) , \quad |\lambda_c^\pm - \widetilde{\lambda}_c^\pm| \le C(\mathscr{E}\delta^{-\tau} + \mathscr{E}_h) .$$

Remark 4.1. By using properties of conformally symplectic geometry, we can show the following is true (see [10]):

• The stable/unstable exponential rates given by the set of Lyapunov multipliers $\{\lambda_i\}_{i=1}^{2d}$ satisfy the pairing rule

$$\lambda_i \ \lambda_{i+d} = \lambda .$$

• Invariant tori satisfy the isotropic property: the symplectic form restricted to the invariant torus is zero.

• Because of the conformally symplectic structure, the symplectic form is nondegenerate when restricted to the center bundle E_θ^c.

An important result is that the bundle E_θ^c near a rotational invariant torus satisfying our hypotheses (notably that the dimension of the fibers of the bundle is $2d$) is trivial, that is, the bundle is isomorphic to a product bundle. Precisely, we can show that if K is an approximate solution of (2.2), we can find a linear operator $\mathscr{B}_\theta : \mathrm{Range}(DK(\theta)) \to E_\theta^c$, such that E_θ^c is the image under $\mathrm{Id} + \mathscr{B}_\theta$ of the tangent bundle of the torus.

5. A sketch of the proof of Theorem 4.1

We now proceed to sketch the proof of Theorem 4.1 (see Section 5.2), which uses the so-called automatic reducibility presented in Section 5.1. The proof leads to the algorithm described in Section 6. We refer to [10] for full details.

5.1. Automatic reducibility. We assume that there exists an invariant splitting of the tangent space of $\mathscr{M}$ at $K(\theta)$, $\mathscr{T}_{K(\theta)}\mathscr{M}$ with $\theta \in \mathbb{T}^d$:

$$\mathscr{T}_{K(\theta)}\mathscr{M} = E^s_\theta \oplus E^c_\theta \oplus E^u_\theta \ .$$

Let $N(\theta) = (DK(\theta)^T DK(\theta))^{-1}$, $P(\theta) = DK(\theta)N(\theta)$, $\chi(\theta) = DK(\theta)^T (J^c)^{-1} \circ K(\theta) DK(\theta)$, where J^c is the $2n \times 2n$ matrix of the embeddings of the center space into the ambient space, and let

$$S(\theta) = P(\theta+\omega)^T Df_\mu \circ K(\theta)(J^c)^{-1} \circ K(\theta)P(\theta) - N(\theta+\omega)^T \chi(\theta+\omega)N(\theta+\omega)\ \lambda\, \mathrm{Id}_d \ . \tag{5.1}$$

Let M be defined as

$$M(\theta) = [DK(\theta) \mid (J^c)^{-1} \circ K(\theta)\ DK(\theta)N(\theta)] \ . \tag{5.2}$$

Taking the derivative of (2.2) we get

$$Df_\mu \circ K(\theta)\, DK(\theta) - DK \circ T_\omega(\theta) = 0 \ , \tag{5.3}$$

which shows that $\mathrm{Range}(DK(\theta)) \subset E^c_\theta$ and hence:

$$DK^T(\theta) J^c \circ K(\theta) DK(\theta) = 0 \ . \tag{5.4}$$

Due to (5.4), the dimension of the range of M in (5.2) is $2d$, and from $(H4)$ we have:

$$\mathrm{Range}(M(\theta)) = E^c_\theta \ . \tag{5.5}$$

Hence, there exists a matrix $\mathscr{B}(\theta)$ such that

$$Df_\mu \circ K(\theta) M(\theta) = M(\theta+\omega)\ \mathscr{B}(\theta) \ , \tag{5.6}$$

where $\mathscr{B}(\theta)$ is upper triangular with constant matrices on the diagonal. From (5.3), the first columm of $\mathscr{B}$ is $\begin{pmatrix} \mathrm{Id}_d \\ 0 \end{pmatrix}$. From (5.5), by setting $v(\theta) = (J^c)^{-1} \circ K(\theta)\ DK(\theta)N(\theta)$, we have

$$Df_\mu \circ K(\theta)\ v(\theta) = DK(\theta+\omega)S(\theta) + v(\theta+\omega)U(\theta) \ , \tag{5.7}$$

where $U = U(\theta)$ is obtained by multiplying (5.7) on the right by $DK^T(\theta+\omega) J^c \circ K(\theta+\omega)$ and using (5.4):

$$U(\theta) = DK^T(\theta+\omega) J^c \circ K(\theta+\omega) Df_\mu \circ K(\theta)\ v(\theta) \ . \tag{5.8}$$

The center foliation is conformally symplectic and invariant. From these properties we obtain

$$Df_\mu^T(x) J^c_{f(x)} Df_\mu(x) = \lambda J^c_{f(x)} \ ,$$

from which $J^c_{f(x)} Df_\mu(x) (J^c_{f(x)})^{-1} = \lambda Df_\mu^{-T}(x)$.

Hence, we see that the left hand side of (5.8) is equal to λ, thus showing

$$U(\theta) = \lambda\ \mathrm{Id}_d \ .$$

Defining S as in (5.1), we can write (5.6) as

$$Df_\mu \circ K(\theta) M(\theta) = M(\theta+\omega) \begin{pmatrix} \mathrm{Id}_d & S(\theta) \\ 0 & \lambda\, \mathrm{Id}_d \end{pmatrix} \ . \tag{5.9}$$

5.2. Sketch of the proof. Once the automatic reducibility leading to (5.9) is established, we can proceed to sketch the proof of Theorem 4.1.

We start with an approximate solution of the invariance equation which is approximately hyperbolic and look for a correction to K and μ, such that the error of the invariance of the new embedding and the new parameter is roughly the square of the original error in a smaller domain; this is the content of the following Proposition.

Proposition 5.1. *Let $f_\mu : \mathscr{M} \to \mathscr{M}$, $\mu \in \mathbb{R}^d$, $d \le n$ be a family of real-analytic, conformally symplectic maps as in Theorem 4.1 with $0 < \lambda < 1$. Let $\omega \in D_d(\nu, \tau)$.*

Let (K, μ), $K : \mathbb{T}^d \to \mathscr{M}$, $K \in \mathscr{A}_\rho$, be an approximate solution, such that

$$f_\mu \circ K(\theta) - K \circ T_\omega(\theta) = e(\theta) \tag{5.10}$$

and let $\mathscr{E} = \|e\|_{\mathscr{A}_\rho}$.

Let $E_\theta^{s/c/u}$ be an approximately invariant hyperbolic splitting based on K, such that $\mathscr{I}_\rho(\gamma, E_\theta^{s/c/u}) < \mathscr{E}_h$ with $\mathscr{E}_h$ as in $(H2)$ of Theorem 4.1. Assume that (K, μ) satisfies assumptions $(H2)$-$(H3)$-$(H3')$-$(H4)$-$(H5)$ of Theorem 4.1 and that $\mathscr{E}$, $\mathscr{E}_h$ are sufficiently small.

Then, there exists an exact invariant splitting $\widetilde{E}_\theta^{s/c/u}$ with associated cocycle $\widetilde{\gamma}_\theta^{\sigma,\sigma}$, such that

$$dist_\rho(E_\theta^{s/c/u}, \widetilde{E}_\theta^{s/c/u}) \le C\mathscr{E}_h\ , \qquad \|\gamma_\theta^{\sigma,\sigma} - \widetilde{\gamma}_\theta^{\sigma,\sigma}\|_{\mathscr{A}_\rho} \le C\mathscr{E}_h\ .$$

Furthermore, we can find corrections Δ, β, such that $K' = K + \Delta$, $\mu' = \mu + \beta$ satisfy

$$f_{\mu'} \circ K'(\theta) - K' \circ T_\omega(\theta) = e'(\theta)$$

with

$$\|e'\|_{\mathscr{A}_{\rho-\delta}} \le C\,\delta^{-2\tau}\,\mathscr{E}^2\ , \qquad \|\Delta\|_{\mathscr{A}_{\rho-\delta}} \le C\,\delta^{-\tau}\,\mathscr{E}\ , \qquad |\beta| \le C\mathscr{E}\ .$$

Moreover, the splitting $\widetilde{E}_\theta^{s/c/u}$ is approximately invariant for $Df_{\mu'} \circ K'$.

The proof of Proposition 5.1 is based on the following ideas. By expanding the invariance equation for K', μ' in Taylor series, we have:

$$\begin{aligned} f_{\mu'} \circ K'(\theta) - K'(\theta+\omega) &= f_\mu \circ K(\theta) + Df_\mu \circ K(\theta)\,\Delta(\theta) + D_\mu f_\mu \circ K(\theta)\,\beta \\ &\quad - K(\theta+\omega) - \Delta(\theta+\omega) + O(\|\Delta\|^2) + O(|\beta|^2)\ . \end{aligned}$$

Using (5.10), the new error is quadratically smaller if the corrections Δ, β satisfy

$$Df_\mu \circ K(\theta)\ \Delta(\theta) + D_\mu f_\mu \circ K(\theta)\ \beta - \Delta(\theta+\omega) = -e(\theta)\ . \tag{5.11}$$

The solution of (5.11) is obtained by projecting it on the hyperbolic and center spaces and by using the invariant splitting (3.3). Let K_e be the exact solution of (5.10); we assume that the cocycle $Df_\mu \circ K_e$ admits an invariant splitting as in (3.3). For the initial step, this follows from $(H2)$ and the closing Lemma 3.1, while in the subsequent steps, we observe that the exactly invariant splitting for one step will be approximately invariant for the corrected one, so we can apply again Lemma 3.1 to restore the invariance.

Denoting by $\Delta^\xi(\theta) \equiv \Pi^\xi_{\theta+\omega}\Delta(\theta)$, $e^\xi(\theta) \equiv \Pi^\xi_{\theta+\omega} e(\theta)$ with $\xi = s, c, u$, we have

$$Df_\mu \circ K(\theta)\ \Delta^\xi(\theta) + \Pi^\xi_{\theta+\omega} D_\mu f_\mu \circ K(\theta)\beta - \Delta^\xi(\theta+\omega) = -e^\xi(\theta)\ , \tag{5.12}$$

which contains Δ^s, Δ^c, Δ^u, β as unknowns. The equation for Δ^c allows to determine Δ^c and β. In fact, from

$$\Delta^c = M\,W^c\ ,$$

and by recalling (5.6), it follows that the approximate solution satisfies (5.9) up to an error term, say $R = R(\theta)$:

$$Df_\mu \circ K(\theta)\ M(\theta) = M(\theta+\omega)\mathscr{B}(\theta) + R(\theta) \tag{5.13}$$

with

$$\|R\|_{\mathscr{A}_{\rho-\delta}} \le C\delta^{-1}\, \|e\|_{\mathscr{A}_\rho} \ .$$

Using (5.12) and (5.13), one obtains

$$\begin{pmatrix} \mathrm{Id}_d & S(\theta) \\ 0 & \lambda\,\mathrm{Id}_d \end{pmatrix} W^c(\theta) - W^c \circ T_\omega(\theta) = -\widetilde{e}^c(\theta) - \widetilde{A}^c(\theta)\beta\,, \tag{5.14}$$

where $\widetilde{e}^c(\theta) \equiv M^{-1} \circ T_\omega(\theta) e^c(\theta)$, $\widetilde{A}^c(\theta) \equiv M^{-1} \circ T_\omega(\theta)\, \Pi^c_{\theta+\omega} D_\mu f_\mu \circ K(\theta)$. Next, we define $\widetilde{A}^c \equiv [\widetilde{A}^c_1|\widetilde{A}^c_2]$, $\overline{W^c}$ as the average of W^c, $(W^c)^0 \equiv W^c - \overline{W^c}$, and $(W^c_2)^0$ as an affine function of β, where $(W^c_2)^0 = (W^c_a)^0 + \beta(W^c_b)^0$ for some functions W^c_a, W^c_b. With this setting, (5.14) becomes

$$\begin{aligned} &(W^c_1)^0(\theta) - (W^c_1)^0 \circ T_\omega(\theta) = -(SW^c_2)^0(\theta) - (\widetilde{e}^c_1)^0(\theta) - (\widetilde{A}^c_1)^0(\theta)\beta \\ &\lambda(W^c_a)^0(\theta) - (W^c_a)^0 \circ T_\omega(\theta) = -(\widetilde{e}^c_2)^0(\theta) \\ &\lambda(W^c_b)^0(\theta) - (W^c_b)^0 \circ T_\omega(\theta) = -(\widetilde{A}^c_2)^0(\theta)\,, \end{aligned} \tag{5.15}$$

whose solution for $(W^c_1)^0$, $(W^c_a)^0$, $(W^c_b)^0$ is found by using standard results (see, e.g., [8]).

Taking the average of (5.15) and combining the last two equations in a single equation for $(W^c_2)^0$ leads to the following system of equations:

$$\begin{pmatrix} \overline{S} & \overline{S(W^c_b)^0} + \overline{\widetilde{A}^c_1} \\ (\lambda-1)\,\mathrm{Id}_d & \overline{\widetilde{A}^c_2} \end{pmatrix} \begin{pmatrix} \overline{W^c_2} \\ \beta \end{pmatrix} = \begin{pmatrix} -\overline{S(W^c_a)^0} - \overline{\widetilde{e}^c_1} \\ -\overline{\widetilde{e}^c_2} \end{pmatrix}. \tag{5.16}$$

Using the nondegeneracy condition (H5) allows to find a solution of (5.16) and, hence, to determine $\overline{W^c_2}$, β.

Next, we solve (5.12) for the stable subspace. After denoting

$$\theta' = T_\omega(\theta), \qquad \widetilde{e}^s(\theta') \equiv \Pi^s_{\theta'} e \circ T_{-\omega}(\theta'),$$

equation (5.12) becomes

$$Df_\mu(K\circ T_{-\omega}(\theta'))\, \Delta^s(T_{-\omega}(\theta')) + \Pi^s_{\theta+\omega} D_\mu f_\mu(K\circ T_{-\omega}(\theta'))\beta - \Delta^s(\theta') = -\widetilde{e}^s(\theta')\,,$$

which can be solved for Δ^s in the form

$$\begin{aligned} \Delta^s(\theta') = \widetilde{e}^s(\theta') &+ \sum_{k=1}^\infty \Big(Df_\mu(K\circ T_{-\omega}(\theta')) \times \dots \times Df_\mu(K\circ T_{-k\omega}(\theta'))\Big)\, \widetilde{e}^s(T_{-k\omega}(\theta')) \\ &+ \Pi^s_{\theta+\omega} D_\mu f_\mu(K\circ T_{-\omega}(\theta'))\beta \\ &+ \sum_{k=1}^\infty \Big(Df_\mu(K\circ T_{-\omega}(\theta')) \times \ldots \times Df_\mu(K\circ T_{-k\omega}(\theta'))\, \Pi^s_{\theta+\omega} D_\mu f_\mu(K\circ T_{-(k+1)\omega}(\theta')\Big)\, \beta, \end{aligned}$$

where the series in the last term converges in $\mathscr{A}_\rho$, due to the growth rates (3.4).

In a similar way, one can solve the equation for the unstable subspace, thus obtaining

$$\begin{aligned} \Delta^u(\theta) = &-\sum_{k=0}^\infty \Big((Df_\mu)^{-1}(K(\theta)) \times \ldots \times (Df_\mu)^{-1}(K\circ T_{k\omega}(\theta))\Big)\, e^u(T_{k\omega}(\theta)) \\ &- \sum_{k=0}^\infty \Big((Df_\mu)^{-1}(K(\theta)) \times \ldots \times (Df_\mu)^{-1}(K\circ T_{k\omega}(\theta))\, \Pi^u_{\theta+\omega} D_\mu f_\mu(K\circ T_{k\omega}(\theta)\Big)\beta. \end{aligned}$$

Simple estimates show that the norm of $\mathscr{S}$ (defined in (H5)), the norm of the change of the projections, the change in the rates, and the constant C_0 in (3.4) slightly change after one iterative step; denoting by $\widehat{\gamma}^{\sigma,\sigma}_\theta$ the cocycle associated to K', μ', one has:

$$\begin{aligned} &\|\mathscr{S}'\|_{\mathscr{A}_{\rho-\delta}} \le \|\mathscr{S}\|_{\mathscr{A}_\rho} + C\delta^{-\tau}\|e\|_{\mathscr{A}_\rho} \\ &\|\widetilde{\Pi}^{s/c/u}_\theta - \Pi^{s/c/u}_\theta\|_{\mathscr{A}_\rho} \le C\|K' - K\|_{\mathscr{A}_\rho} \le C\delta^{-\tau}\|e\|_{\mathscr{A}_\rho} \\ &\|\widehat{\gamma}^{\sigma,\sigma}_\theta - \widetilde{\gamma}^{\sigma,\sigma}_\theta\|_{\mathscr{A}_{\rho-\delta}} \le C(\delta^{-\tau}\mathscr{E} + \mathscr{E}_h)\,. \end{aligned}$$

The last issue with proving Theorem 4.1 is showing that the inductive step can be iterated infinitely many times and that it converges to the true solution, provided the initial error is sufficiently small. This is a standard KAM argument, which is proved by introducing a sequence $\{K_h, \mu_h\}$ of approximate solutions on shrinking domains and imposing a smallness condition on the size of the initial error $\|e\|_{\mathscr{A}_\rho}$.

6. The algorithm for the new approximation

The proof of Theorem 4.1 leads to the following algorithm, which allows to construct the improved approximation, given f_μ, ω, K_0, μ_0. We fix an integer L_0, which denotes the maximum number of terms which are computed in the infinite series defining Δ^s and Δ^u. Each step is denoted as $a \leftarrow b$, meaning that the quantity a is determined from b. Note that the number of steps is less than 40 and that all the steps involve just calling a standard function, so the coding is reasonably straightforward.

Algorithm 6.1. *Let f_μ, ω, K_0, μ_0 be as in the previous sections and let $L_0 \in \mathbb{Z}$:*

- $\chi_1 \leftarrow f_{\mu_0} \circ K_0$
- $\chi_2 \leftarrow K_0 \circ T_\omega$
- $e \leftarrow \chi_1 - \chi_2$
- $e^{s/c/u} \leftarrow \Pi^{s/c/u}_{\theta+\omega} e$
- $\gamma \leftarrow Df_{\mu_0} \circ K_0$
- $\widetilde{\gamma} \leftarrow D_\mu f_{\mu_0} \circ K_0$
- $\alpha \leftarrow DK_0$
- $N \leftarrow [\alpha^T \alpha]^{-1}$
- $\widetilde{J} \leftarrow (J^c)^{-1} \circ K$
- $M \leftarrow [\alpha | \widetilde{J}\, \alpha\, N]$
- $\widetilde{M} \leftarrow M^{-1} \circ T_\omega$
- $\widetilde{e}^c \leftarrow \widetilde{M}\, e^c$
- $P \leftarrow \alpha N$
- $\chi \leftarrow \alpha^T\, \widetilde{J}\, \alpha$
- $\Lambda \leftarrow \lambda\ \mathrm{Id}_d$
- $S \leftarrow (P \circ T_\omega)^T \gamma \widetilde{J}\, P - (N \circ T_\omega)^T (\chi \circ T_\omega)^T\ N \circ T_\omega\ \Lambda$
- $\widetilde{A}^c \leftarrow \widetilde{M}\, \Pi^c_{\theta+\omega}\, \widetilde{\gamma}$
- $(W^c_a)^o$ *solves* $\lambda (W^c_a)^o - (W^c_a)^o \circ T_\omega = -(\widetilde{e}^c_2)^o$
- $(W^c_b)^o$ *solves* $\lambda (W^c_b)^o - (W^c_b)^o \circ T_\omega = -(\widetilde{A}^c_2)^o$
- *Find* $\overline{W^c_2}$, β *by solving*

$$
\overline{S}\ \overline{W^c_2} + (\overline{S (W^c_b)^o} + \overline{\widetilde{A}^c_1})\beta = -\overline{S (W^c_a)^o} - \overline{\widetilde{e}^c_1}
$$
$$
(\lambda - 1)\overline{W^c_2} + \overline{\widetilde{A}^c_2}\beta = -\overline{\widetilde{e}^c_2}
$$

- $(W^c_2)^o \leftarrow (W^c_a)^o + \beta (W^c_b)^o$
- $W^c_2 \leftarrow (W^c_2)^o + \overline{W^c_2}$
- $(W^c_1)^o$ *solves* $(W^c_1)^o - (W^c_1)^o \circ T_\omega = -(S\, W^c_2)^o - (\widetilde{e}^c_1)^o - (\widetilde{A}^c_1)^o \beta$
- $\Delta^c \leftarrow M^c\, W^c$
- $\mu_1 \leftarrow \mu_0 + \beta$
- *Compute* $\widetilde{\Gamma}_k = \gamma^{-1} \times \ldots \times \gamma^{-1} \circ T_{k\omega}$ *for* $k = 0, \ldots, L_0$
- *Compute* $e^u_k = e^u \circ T_{k\omega}$ *for* $k = 0, \ldots, L_0$
- *Compute* $\widetilde{\gamma}_k = \Pi^u_{\theta+\omega} \widetilde{\gamma} \circ T_{k\omega}$ *for* $k = 0, \ldots, L_0$
- $\Delta^u \leftarrow -\sum_{k=0}^{L_0} (\widetilde{\Gamma}_k e^u_k + \widetilde{\Gamma}_k \widetilde{\gamma}_k \beta)$
- $\theta' \leftarrow T_\omega(\theta)$
- *Compute* $\Gamma_k = \gamma \circ T_{-\omega}(\theta') \times \ldots \times \gamma \circ T_{-k\omega}(\theta')$ *for* $k = 1, \ldots, L_0$

- *Compute* $\widetilde{e}^s_k = \widetilde{e}^s \circ T_{-k\omega}(\theta')$ *for* $k = 1, ..., L_0$
- *Compute* $\widetilde{\gamma}' = \Pi^s_{\theta'} \widetilde{\gamma} \circ T_{-\omega}(\theta')$ *for* $k = 1, ..., L_0$
- *Compute* $\widetilde{\gamma}'_k = \Pi^s_{\theta'} \widetilde{\gamma} \circ T_{-(k+1)\omega}(\theta')$ *for* $k = 1, ..., L_0$
- $\Delta^{s'} \leftarrow \widetilde{e}^s(\theta') + \sum_{k=1}^{L_0} \Gamma_k \widetilde{e}^s_k + \widetilde{\gamma}'\beta + \sum_{k=1}^{L_0} \Gamma_k \widetilde{\gamma}'_k \beta$
- $\Delta^s \leftarrow \Delta^{s'} \circ T_{-\omega}$
- $K'_1 \leftarrow K_0 + \Delta^c + \Delta^u + \Delta^s$.

7. Domains of analyticity and Lindstedt expansions of whiskered tori

The study of domains of analyticity of whiskered tori of conformally symplectic systems in the limit of small dissipation is similar to that developed in [9] with added hyperbolicity. The main idea is to compute an asymptotic expansion (Lindstedt series), which can be used as starting point for the application of Theorem 4.1.

The Lindstedt series expansions to order N of K, μ, A^σ satisfy the invariance equation up to an error bounded by $C_N|\varepsilon|^{N+1}$. Then, we apply Theorem 4.1 for ε belonging to a domain with good Diophantine properties of λ. Hence, we are able to prove that there exists a true solution K, μ and

$$\|K^{[\le N]} - K\|, |\mu^{[\le N]} - \mu| \le \tilde{C}_N |\varepsilon|^{N+1}$$

in the domain, thus showing the Lindstedt series are asymptotic expansions of the true solution. The quantities $K^{[\le N]}$, $\mu^{[\le N]}$ denote the truncations to order N in ε (see (7.1) below) of the Lindstedt series expansions.

Let $f_{\mu_\varepsilon,\varepsilon} : \mathscr{M} \to \mathscr{M}$ be a family of maps such that

$$f^*_{\mu_\varepsilon,\varepsilon} \Omega = \lambda(\varepsilon)\Omega \,,$$

where the conformal factor λ is taken as

$$\lambda(\varepsilon) = 1 + \alpha\varepsilon^a + O(|\varepsilon|^{a+1}) \tag{7.1}$$

for some $a > 0$ integer and $\alpha \in \mathbb{C} \setminus \{0\}$.

Recalling Definition 2.2, we introduce the following sets, where the Diophantine constants allow to start an iterative convergent procedure (see [9]).

Definition 7.1. For $A > 0$, $N \in \mathbb{Z}_+$, $\omega \in \mathbb{R}^d$, let the set $\mathscr{G} = \mathscr{G}(A; \omega, \tau, N)$ be defined as

$$\mathscr{G}(A; \omega, \tau, N) \equiv \{\varepsilon \in \mathbb{C} : \ \nu(\lambda(\varepsilon); \omega, \tau)\ |\lambda(\varepsilon) - 1|^{N+1} \le A\} \,.$$

For $r_0 \in \mathbb{R}$, let

$$\mathscr{G}_{r_0}(A; \omega, \tau, N) = \mathscr{G} \cap \{\varepsilon \in \mathbb{C} : \ |\varepsilon| \le r_0\} \,. \tag{7.2}$$

We prove that K and μ are analytic in a domain $\mathscr{G}_{r_0}$ as in (7.2) for a sufficiently small r_0. This domain is obtained by removing a sequence of smaller balls whose centers lie along smooth lines going through the origin from a ball centered at zero (see Figure 1). The removed balls have radii decreasing faster than any power of the distance of their center from the origin. Like in [9], we conjecture that this domain is essentially optimal.

Theorem 7.1. *Let* $f_{\mu,\varepsilon} : \mathscr{M} \to \mathscr{M}$ *with* $\mu \in \Gamma$ *with* $\Gamma \subseteq \mathbb{C}^d$ *open,* $d \le n$*,* $\varepsilon \in \mathbb{C}$*, be a family of conformally symplectic maps with conformal factor satisfying* (7.1) *with* $\alpha \in \mathbb{R}$*,* $\alpha \ne 0$*,* $a \in \mathbb{N}$*. Let* $\omega \in \mathscr{D}_d(\nu, \tau)$*.*

(A1) Assume that for $\mu = \mu_0$*,* $\varepsilon = 0$ *the map* $f_{\mu_0,0}$ *admits a whiskered invariant torus, namely*

(A1.1) there exists an embedding $K_0 : \mathbb{T}^d \to \mathscr{M}$*,* $K_0 \in \mathscr{A}_\rho$ *for some* $\rho > 0$*, such that*

$$f_{\mu_0,0} \circ K_0 = K_0 \circ T_\omega \,;$$

(A1.2) there exists a splitting $\mathcal{T}_K(\theta)\mathcal{M} = E^s_\theta \oplus E^c_\theta \oplus E^u_\theta$, which is invariant under the cocycle $\gamma^0_\theta = Df_{\mu_0,0} \circ K_0(\theta)$ and satisfies Definition 3.1. The rates of the splitting satisfy the assumptions (H3), (H3') and (H4) of Theorem 4.1.

(A2) The function $f_{\mu,\varepsilon}(x)$ is analytic in all of its arguments and the analyticity domains are large enough, namely:

(A2.1) both $K_0(\theta)$ and the splitting $E^{s,c,u}_\theta$ considered as a function of θ are in $\mathcal{A}_{\rho_0}$ for some $\rho_0 > 0$;

(A2.2) there is a domain $\mathcal{U} \subset \mathbb{C}^n/\mathbb{Z}^n \times \mathbb{C}^n$ such that for $|\varepsilon| \le \varepsilon^*$ and all μ with $|\mu - \mu_0| \le \mu^*$ the function $f_{\mu,\varepsilon}$ is defined in $\mathcal{U}$ and the condition (4.1) *holds.*

(A3) The nondegeneracy condition (H5) of Theorem 4.1 is satisfied by the invariant torus.
Then:

(B.1) We can compute formal power series expansions

$$K_\varepsilon^{[\infty]} = \sum_{j=0}^{\infty} \varepsilon^j K_j \qquad \mu_\varepsilon^\infty = \sum_{j=0}^{\infty} \varepsilon^j \mu_j$$

satisfying (2.2) *and such that for any $0 < \rho' \le \rho$ and $N \in \mathbb{N}$, we have*

$$||f_{\mu_\varepsilon^{[\le N]},\varepsilon} \circ K_\varepsilon^{[\le N]} - K_\varepsilon^{[\le N]} \circ T_\omega||_{\mathcal{A}_{\rho'}} \le C_N |\varepsilon|^{N+1} .$$

(B.2) We can compute the formal power series expansions

$$A_\varepsilon^{\sigma,\infty} = \sum_{j=0}^{\infty} \varepsilon^j A_j^\sigma , \qquad A_j^\sigma(\theta) : E_0^\sigma(\theta) \to E_0^{\hat\sigma}(\theta) , \qquad \sigma = s, \hat s, u, \hat u$$

with $A_j^\sigma \in \mathcal{A}_\rho$ satisfying the equations for invariant dichotomies in the sense of power series.

(B.3) For the set $\mathcal{G}_{r_0}$ as in (7.2) *with r_0 sufficiently small and for* $0 < \rho' < \rho$, *there exists $K_\varepsilon : \mathcal{G}_{r_0} \to \mathcal{A}_{\rho'}$, $\mu_\varepsilon : \mathcal{G}_{r_0} \to \mathbb{C}^d$, analytic in the interior of $\mathcal{G}_{r_0}$ taking values in $\mathcal{A}_{\rho'}$, extending continuously to the boundary of $\mathcal{G}_{r_0}$ and such that for $\varepsilon \in \mathcal{G}_{r_0}$ the invariance equation is satisfied exactly:*

$$f_{\mu_\varepsilon,\varepsilon} \circ K_\varepsilon - K_\varepsilon \circ T_\omega = 0 .$$

Furthermore, the formal series in (B.1) give an asymptotic expansion of the exact solution, namely for $0 < \rho' < \rho$, $N \in \mathbb{N}$, one has:

$$||K_\varepsilon^{[\le N]} - K_\varepsilon||_{\mathcal{A}_{\rho'}} \le C_N |\varepsilon|^{N+1} , \qquad |\mu_\varepsilon^{[\le N]} - \mu_\varepsilon| \le C_N |\varepsilon|^{N+1} .$$

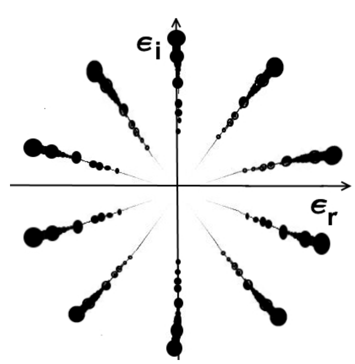

FIGURE 1. A representation of the set $\mathcal{G}$ given by the complement of the black circles, whose radii have been rescaled for graphical reasons. We took $d = 1$, $\tau = 1$, $a = 5$.

We refer to [10] for the proof of Theorem 7.1. Figure 1 is a graphical description of the set $\mathscr{G}$, which is the complement of the black circles with centers along smooth lines going through the origin and with radii decreasing quickly as the centers go to zero.

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