JOURNAL
de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

Cindy (Sin Yi) TSANG

Realizable Classes and Embedding Problems

Tome 29, n° 2 (2017), p. 647-680.

<http://jtnb.cedram.org/item?id=JTNB_2017__29_2_647_0>

Journal de Théorie des Nombres
de Bordeaux **29** (2017), 647–680

Realizable Classes and Embedding Problems

par Cindy (Sin Yi) TSANG

Résumé. Soit K un corps de nombres et soit $\mathcal{O}_K$ son anneau des entiers. Soit G un groupe fini et soit K_h une K-algèbre galoisienne de groupe G. Si K_h/K est modérée, son anneau des entiers $\mathcal{O}_h$ est un $\mathcal{O}_KG$-module localement libre d'après un théorème classique d'E. Noether et définit une classe dans le groupe des classes $\mathrm{Cl}(\mathcal{O}_KG)$ des $\mathcal{O}_KG$-modules localement libres. On note $R(\mathcal{O}_KG)$ l'ensemble de toutes ces classes. En combinant les travaux de L.R. McCulloh et J. Brinkhuis, on prouve que la structure de $R(\mathcal{O}_KG)$ est liée à l'étude de problèmes de plongement lorsque G est abélien.

Abstract. Let K be a number field and denote by $\mathcal{O}_K$ its ring of integers. Let G be a finite group and let K_h be a Galois K-algebra with group G. If K_h/K is tame, then its ring of integers $\mathcal{O}_h$ is a locally free $\mathcal{O}_KG$-module by a classical theorem of E. Noether and it defines a class in the locally free class group $\mathrm{Cl}(\mathcal{O}_KG)$ of $\mathcal{O}_KG$. We denote by $R(\mathcal{O}_KG)$ the set of all such classes. By combining the work of L.R. McCulloh and J. Brinkhuis, we shall prove that the structure of $R(\mathcal{O}_KG)$ is connected to the study of embedding problems when G is abelian.

1. Introduction and Preliminaries

Let K be a number field and denote by $\mathcal{O}_K$ its ring of integers. Let K^c be a fixed algebraic closure of K and define $\Omega_K := \mathrm{Gal}(K^c/K)$. Let G be a finite group and let Ω_K act trivially on G (on the left). Then, the set of all isomorphism classes of G-Galois K-algebras (see Subsection 1.3 for a brief review of Galois algebras) is in one-to-one correspondence with the pointed Galois cohomology set $H^1(\Omega_K, G)$. For each $h \in H^1(\Omega_K, G)$, we will write K_h for a Galois algebra representative of h and $\mathcal{O}_h$ for the ring of integers in K_h.

If K_h/K is tame, then a classical theorem of E. Noether implies that $\mathcal{O}_h$ is locally free over $\mathcal{O}_KG$ and hence defines a class $\mathrm{cl}(\mathcal{O}_h)$ in the locally free

Manuscrit reçu le 7 février 2016, révisé le 12 juillet 2016, accepté le 12 septembre 2016.
Mathematics Subject Classification. 11R04, 11R32, 11R33.
Mots-clefs. Galois module, rings of integers, realizable classes, embedding problems.

class group $\mathrm{Cl}(\mathcal{O}_K G)$ of $\mathcal{O}_K G$. Such a class in $\mathrm{Cl}(\mathcal{O}_K G)$ is called *realizable*. Define

$$H_t^1(\Omega_K, G) := \{h \in H^1(\Omega_K, G) \mid K_h/K \text{ is tame}\}.$$

Then, there is a natural map

$$\mathrm{gal} : H_t^1(\Omega_K, G) \longrightarrow \mathrm{Cl}(\mathcal{O}_K G); \quad \mathrm{gal}(h) := \mathrm{cl}(\mathcal{O}_h) \tag{1.1}$$

whose image is equal to the set

$$R(\mathcal{O}_K G) := \{\mathrm{cl}(\mathcal{O}_h) : h \in H_t^1(\Omega_K, G)\}$$

of all realizable classes in $\mathrm{Cl}(\mathcal{O}_K G)$.

In [2], J. Brinkhuis related the study of these realizable classes to that of embedding problems as follows. Let K/k be a Galois subextension of K and set $\Sigma := \mathrm{Gal}(K/k)$.

Definition 1.1. Given a group extension E of Σ by G, say

$$E: \quad 1 \longrightarrow G \longrightarrow \Gamma \longrightarrow \Sigma \longrightarrow 1,$$

a *solution* to the embedding problem $(K/k, G, E)$ is a finite Galois extension N/K such that N/k is also Galois, and that there exist isomorphisms $\mathrm{Gal}(N/K) \simeq G$ and $\mathrm{Gal}(N/k) \simeq \Gamma$ making the diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Gal}(N/K) \arrow[r] \arrow[d, leftrightarrow, "\simeq"] & \mathrm{Gal}(N/k) \arrow[r] \arrow[d, leftrightarrow, "\simeq"] & \mathrm{Gal}(K/k) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & G \arrow[r] & \Gamma \arrow[r] & \Sigma \arrow[r] & 1
\end{tikzcd}$$

commute. If in addition N/K is tame, then we call N/K a *tame solution.*

Now, assume that G is abelian. The pointed set $H^1(\Omega_K, G)$ is then equal to $\mathrm{Hom}(\Omega_K, G)$ and has a group structure. Let K^t be the maximal tamely ramified extension of K lying in K^c and let $\Omega_K^t := \mathrm{Gal}(K^t/K)$ act trivially on G (on the left). Then, the subset $H_t^1(\Omega_K, G)$ may be naturally identified with $\mathrm{Hom}(\Omega_K^t, G)$ (see Remark 1.17 below) and in particular is a subgroup of $H^1(\Omega_K, G)$. The map gal, however, is not a homomorphism in general, but is only *weakly multiplicative* in the following sense. Let M_K denote the set of primes in $\mathcal{O}_K$. For $h \in \mathrm{Hom}(\Omega_K^t, G)$, define

$$d(h) := \{v \in M_K \mid K_h/K \text{ is ramified at } v\}. \tag{1.2}$$

Then, for all $h_1, h_2 \in \mathrm{Hom}(\Omega_K^t, G)$, we have

$$\mathrm{gal}(h_1 h_2) = \mathrm{gal}(h_1)\,\mathrm{gal}(h_2) \quad \text{whenever } d(h_1) \cap d(h_2) = \emptyset. \tag{1.3}$$

This weak multiplicativity of gal was first proved by Brinkhuis in [2, Proposition 3.10] and it also follows from [7, Theorem 6.7] of L.R. McCulloh.

In what follows, we will further fix a left Σ-module structure of G. In [2, Theorem 5.1], Brinkhuis constructed a commutative diagram

$$\begin{tikzcd} H^1(\mathrm{Gal}(K^t/k), G) \arrow[r, "\mathrm{res}"] & \mathrm{Hom}(\Omega_K^t, G)^\Sigma \arrow[r, "tr"] \arrow[d, "\mathrm{gal}"'] & H^2(\Sigma, G) \arrow[d, "i^*"] \\ & \mathrm{Cl}(\mathcal{O}_K G)^\Sigma \arrow[r, "\xi"] & H^2(\Sigma, (\mathcal{O}_K G)^\times) \end{tikzcd} \tag{1.4}$$

(see Section 2 below for the construction and notation), where the top row is exact and all of the maps except possibly gal are homomorphisms. This diagram (1.4) will be referred to as the *basic diagram*.

Remark 1.2. Diagram (1.4) above is a slightly modified and abridged version of the basic diagram constructed in [2, Theorem 5.1]. For example, the Picard group of $\mathcal{O}_K G$ was used in place of $\mathrm{Cl}(\mathcal{O}_K G)$, but these two groups are canonically isomorphic when G is abelian (see [4, Theorem 55.26], for example). In Theorem 2.11, we will give a proof of the facts that (1.4) commutes, that the top row is exact, and that all of the maps except possibly gal are homomorphisms.

The commutativity of (1.4) relates the study of realizable classes to that of embedding problems as follows. Let $h \in \mathrm{Hom}(\Omega_K^t, G)^\Sigma$ be given and assume that h is surjective, in which case K_h is isomorphic to $N := (K^t)^{\ker(h)}$. As we will show in Proposition 2.3 below, the field N is a tame solution to the embedding problem $(K/k, G, E_h)$, where the equivalence class of E_h is determined by $tr(h)$. Now, suppose also that i^* is injective (as is shown in [1, Theorem 4.1], this is so when K is a C.M. field and when G or Σ has odd order). If $tr(h) \neq 1$ (which corresponds to E_h being non-split), then we have $\mathrm{cl}(\mathcal{O}_h) \neq 1$ as well since (1.4) commutes and ξ is a homomorphism.

We continue to assume that G is abelian. In [7, Theorem 6.17 and Corollary 6.20], McCulloh gave a complete characterization of the set $R(\mathcal{O}_K G)$ and showed that it is in fact a subgroup of $\mathrm{Cl}(\mathcal{O}_K G)$. It is then natural to ask whether the group structure of $R(\mathcal{O}_K G)$ is also related to the study of embedding problems. More precisely, consider the subsets

$$\begin{aligned} R_\Sigma(\mathcal{O}_K G) &:= \{\mathrm{cl}(\mathcal{O}_h) : h \in \mathrm{Hom}(\Omega_K^t, G)^\Sigma\} \\ R_s(\mathcal{O}_K G) &:= \{\mathrm{cl}(\mathcal{O}_h) : h \in \mathrm{Hom}(\Omega_K^t, G)^\Sigma \text{ and } tr(h) = 1\} \end{aligned} \tag{1.5}$$

of $R(\mathcal{O}_K G)$. The classes in $R_\Sigma(\mathcal{O}_K G)$ will be called Σ-*realizable*. We want to determine whether these two subsets are also subgroups of $\mathrm{Cl}(\mathcal{O}_K G)$, and if so, whether the group structure of their quotient $R_\Sigma(\mathcal{O}_K G)/R_s(\mathcal{O}_K G)$ is related to that of $H^2(\Sigma, G)$.

In this paper, by combining the work of Brinkhuis and McCulloh, we will prove the following partial result. Given a set V of primes in $\mathcal{O}_K$, define

$$\mathrm{Hom}(\Omega_K^t, G)_V^\Sigma := \{h \in \mathrm{Hom}(\Omega_K^t, G)^\Sigma \mid K_h/K \text{ is unramified at all } v \in V\}.$$

Then, define

$$R_\Sigma(\mathcal{O}_K G)_V := \{\mathrm{cl}(\mathcal{O}_h) : h \in \mathrm{Hom}(\Omega_K^t, G)_V^\Sigma\};$$
$$R_s(\mathcal{O}_K G)_V := \{\mathrm{cl}(\mathcal{O}_h) : h \in \mathrm{Hom}(\Omega_K^t, G)_V^\Sigma \text{ with } tr(h) = 1\}.$$

We will write $\exp(G)$ for the exponent of the group G.

Theorem 1.3. *Let K/k be a Galois extension of number fields and let G be a finite abelian group. Let $\Sigma = \mathrm{Gal}(K/k)$ act trivially on G (on the left), and let $V = V_K$ denote the set of primes in $\mathcal{O}_K$ which are ramified over k. Assume also that k contains all $\exp(G)$-th roots of unity.*

(1) *Both $R_\Sigma(\mathcal{O}_K G)_V$ and $R_s(\mathcal{O}_K G)_V$ are subgroups of $\mathrm{Cl}(\mathcal{O}_K G)$. Moreover, given $h \in \mathrm{Hom}(\Omega_K^t, G)_V^\Sigma$ and any finite set T of primes in $\mathcal{O}_K$, there exists $h' \in \mathrm{Hom}(\Omega_K^t, G)_V^\Sigma$ such that*
   (a) *$K_{h'}/K$ is a field extension;*
   (b) *$K_{h'}/K$ is unramified at all $v \in T$;*
   (c) *$\mathrm{cl}(\mathcal{O}_{h'}) = \mathrm{cl}(\mathcal{O}_h)$;*
   (d) *$tr(h') = tr(h)$.*

(2) *The natural surjective map*

$$\phi : tr(\mathrm{Hom}(\Omega_K^t, G)_V^\Sigma) \longrightarrow \frac{R_\Sigma(\mathcal{O}_K G)_V}{R_s(\mathcal{O}_K G)_V}$$

defined by

$$\phi(tr(h)) := \mathrm{cl}(\mathcal{O}_h) R_s(\mathcal{O}_K G)_V \quad \text{for } h \in \mathrm{Hom}(\Omega_K^t, G)_V^\Sigma$$

is a well-defined homomorphism. Moreover, if the map i^* in the basic diagram is injective, then ϕ is an isomorphism.

Remark 1.4. Suppose now that $|G|$ is odd. Then, for each $h \in H^1(\Omega_K, G)$, there exists a fractional ideal A_h in K_h whose square is the inverse different ideal of K_h/K by Hilbert's formula (see [9, Chapter IV, Proposition 4], for example). If K_h/K is tame, by [12, Theorem 1] or [5, Theorem 1 in Section 2], we know that A_h is locally free over $\mathcal{O}_K G$ and so it defines a class $\mathrm{cl}(A_h)$ in $\mathrm{Cl}(\mathcal{O}_K G)$. For G abelian, by adapting the tools developed by McCulloh in [7], the author has shown in [10, Theorems 1.2(b) and 1.3] that the map

$$\mathrm{gal}_A : H_t^1(\Omega_K, G) \longrightarrow \mathrm{Cl}(\mathcal{O}_K G); \quad \mathrm{gal}_A(h) := \mathrm{cl}(A_h)$$

is weakly multiplicative in the sense of (1.3) and that

$$\mathcal{A}^t(\mathcal{O}_K G) := \{\mathrm{cl}(A_h) : h \in H_t^1(\Omega_K, G)\}$$

is a subgroup of $\mathrm{Cl}(\mathcal{O}_K G)$. Using these two facts, the author shows in [11, Theorems 1.4.4 and 1.4.5] that Theorems 2.11 and 1.3 still hold when gal and $\mathcal{O}_h$ are replaced by gal_A and A_h, respectively, provided that $|G|$ is odd. The proofs of the corresponding statements are essentially the same.

In the subsequent subsections, we will give a brief review of locally free class groups and Galois algebras. From Section 2 onwards, we will take G to be an abelian group. In Section 2, we construct the basic diagram and show that it commutes. In Section 3, we will first recall the necessary definitions and then state the characterization of $R(\mathcal{O}_K G)$ given in [7, Theorem 6.17]. In Section 4, we will modify this characterization and prove Theorem 1.3.

1.1. Notation and Conventions. Throughout this paper, the symbol G denotes a fixed finite group. We will also use the convention that all of the homomorphisms in the cohomology groups considered are continuous.

The symbol F will always denote either a number field or a finite extension of $\mathbb{Q}_p$ for some prime number p. Given any such F, we will define:

$$\begin{aligned}
\mathcal{O}_F &:= \text{the ring of integers in } F;\\
F^c &:= \text{a fixed algebraic closure of } F;\\
\mathcal{O}_{F^c} &:= \text{the integral closure of } \mathcal{O}_F \text{ in } F^c;\\
\Omega_F &:= \mathrm{Gal}(F^c/F);\\
F^t &:= \text{the maximal tamely ramified extension of } F \text{ in } F^c;\\
\Omega_F^t &:= \mathrm{Gal}(F^t/F);\\
M_F &:= \text{the set of all finite primes in } F.
\end{aligned}$$

We will let Ω_F and Ω_F^t act trivially on G (on the left). We will also choose a compatible set $\{\zeta_n : n \in \mathbb{Z}^+\}$ of primitive roots of unity in F^c, that is, we have $(\zeta_{mn})^m = \zeta_n$ for all $m, n \in \mathbb{Z}^+$. For G abelian, we will write $\widehat{G}$ for the group of irreducible F^c-valued characters on G.

In the case that F is a number field, for each $v \in M_F$ we will define:

$$\begin{aligned}
F_v &:= \text{the completion of } F \text{ with respect to } v;\\
i_v &:= \text{a fixed embedding } F^c \longrightarrow F_v^c \text{ extending the natural}\\
&\qquad \text{embedding } F \longrightarrow F_v;\\
\widetilde{i_v} &:= \text{the embedding } \Omega_{F_v} \longrightarrow \Omega_F \text{ induced by } i_v.
\end{aligned}$$

We will also use i_v to denote the isomorphism $F^c \longrightarrow i_v(F^c)$ induced by i_v and i_v^{-1} for the inverse of this isomorphism. The embedding $\widetilde{i_v} : \Omega_{F_v} \longrightarrow \Omega_F$ is then defined by

$$\widetilde{i_v}(\omega) := i_v^{-1} \circ \omega \circ i_v. \tag{1.6}$$

Moreover, if $\{\zeta_n : n \in \mathbb{Z}^+\}$ is the chosen compatible set of primitive roots of unity in F^c, then for each $v \in M_F$, we will choose $\{i_v(\zeta_n) : n \in \mathbb{Z}^+\}$ to be the compatible set of primitive roots of unity in F_v^c.

1.2. Locally Free Class Groups. Let F be number field. We recall the definition and an idelic description of the locally free class group $\mathrm{Cl}(\mathcal{O}_F G)$ of $\mathcal{O}_F G$ (see [4, Chapter 6] for more details).

Definition 1.5. An *$\mathcal{O}_F G$-lattice* is a (left) $\mathcal{O}_F G$-module which is finitely generated and projective as an $\mathcal{O}_F$-module. Two $\mathcal{O}_F G$-lattices X and X' are said to be *stably isomorphic* if there exists $k \in \mathbb{Z}^+$ such that

$$X \oplus (\mathcal{O}_F G)^k \simeq X' \oplus (\mathcal{O}_F G)^k.$$

The stable isomorphism class of X will be denoted by $[X]$.

Remark 1.6. If two $\mathcal{O}_F G$-lattices are isomorphic, then they are certainly stably isomorphic. The converse holds as well if G is abelian (see [4, Proposition 51.2 and Theorem 51.24], for example).

Definition 1.7. An $\mathcal{O}_F G$-lattice X is *locally free over $\mathcal{O}_F G$ (of rank one)* if $\mathcal{O}_{F_v} \otimes_{\mathcal{O}_F} X$ and $\mathcal{O}_{F_v} G$ are isomorphic as $\mathcal{O}_{F_v} G$-modules for all $v \in M_F$.

Definition 1.8. The *locally free class group of $\mathcal{O}_F G$* is the set

$$\mathrm{Cl}(\mathcal{O}_F G) := \{[X] : X \text{ is a locally free } \mathcal{O}_F G\text{-lattice}\}$$

equipped with the following group operation. Given any pair of locally free $\mathcal{O}_F G$-lattices X and X', by [3, Corollary 31.7], there exists a locally free $\mathcal{O}_F G$-lattice X'' such that $X \oplus X' \simeq \mathcal{O}_F G \oplus X''$. It is easy to see that $[X'']$ is uniquely determined by $[X]$ and $[X']$. We then define $[X][X'] := [X'']$.

The group operation of $\mathrm{Cl}(\mathcal{O}_F G)$ is often written additively. We write it multiplicatively instead since we will use an idelic description of $\mathrm{Cl}(\mathcal{O}_F G)$, which we recall below.

Definition 1.9. Let $J(FG)$ be the restricted direct product of the groups $(F_v G)^\times$ with respect to the subgroups $(\mathcal{O}_{F_v} G)^\times$ for $v \in M_F$. Let

$$\partial : (FG)^\times \longrightarrow J(FG)$$

denote the diagonal map and let

$$U(\mathcal{O}_F G) := \prod_{v \in M_F} (\mathcal{O}_{F_v} G)^\times$$

be the group of unit ideles.

For each idele $c = (c_v) \in J(FG)$, define

$$\mathcal{O}_F G \cdot c := \bigcap_{v \in M_F} (\mathcal{O}_{F_v} G \cdot c_v \cap FG). \tag{1.7}$$

Since every locally free $\mathcal{O}_F G$-lattice may be embedded into FG, the map

$$j : J(FG) \longrightarrow \mathrm{Cl}(\mathcal{O}_F G); \quad j(c) := [\mathcal{O}_F G \cdot c] \tag{1.8}$$

is surjective. It is also a homomorphism by [3, Theorem 31.19].

Theorem 1.10. *If G is abelian, then the map j induces an isomorphism*

$$\mathrm{Cl}(\mathcal{O}_F G) \simeq \frac{J(FG)}{\partial((FG)^\times)U(\mathcal{O}_F G)}.$$

Proof. See [4, Theorem 49.22 and Exercise 51.1], for example. □

1.3. Galois Algebras and Resolvends. Let F be a number field or a finite extension of $\mathbb{Q}_p$. Below, we give a brief review of Galois algebras and resolvends (see [7, Section 1] for more details).

Definition 1.11. A *G-Galois F-algebra* or *Galois F-algebra with group G* is a commutative semi-simple F-algebra N on which G acts (on the left) as a group of automorphisms such that $N^G = F$ and $[N : F] = |G|$, where $[N : F]$ denotes the dimension of N over F. Two G-Galois F-algebras are said to be *isomorphic* if there is an F-algebra isomorphism between them which preserves the action of G.

Consider the F^c-algebra $\mathrm{Map}(G, F^c)$ on which G acts (on the left) by

$$(s \cdot a)(t) := a(ts) \quad \text{for } a \in \mathrm{Map}(G, F^c) \ \text{ and } \ s, t \in G.$$

Recall that Ω_F acts trivially on G by definition. For each $h \in \mathrm{Hom}(\Omega_F, G)$, let ${}^hG$ denote the group G endowed with the twisted Ω_F-action given by

$$\omega \cdot s := h(\omega)s \quad \text{for } s \in G \ \text{ and } \ \omega \in \Omega_F.$$

Now, consider the F-subalgebra and G-submodule

$$F_h := \mathrm{Map}_{\Omega_F}({}^hG, F^c)$$

of $\mathrm{Map}(G, F^c)$ consisting of the maps ${}^hG \longrightarrow F^c$ which preserve the Ω_F-action. If $\{s_i\}$ is a set of coset representatives of $h(\Omega_F)\backslash G$ and

$$F^h := (F^c)^{\ker(h)}, \tag{1.9}$$

then evaluation at the elements s_i induces an isomorphism

$$F_h \simeq \prod_{h(\Omega_F)\backslash G} F^h$$

of F-algebras. This gives $[F_h : F] = [G : h(\Omega_F)][F^h : F] = |G|$. Identifying F with the set of constant F-valued functions in F_h, we see that $(F_h)^G = F$ as well. It follows that F_h is a G-Galois F-algebra.

It is not hard to check that every G-Galois F-algebra is isomorphic to F_h for some $h \in \mathrm{Hom}(\Omega_F, G)$, and that for any $h, h' \in \mathrm{Hom}(\Omega_F, G)$, we have

$F_h \simeq F_{h'}$ if and only if h and h' differ by an element in $\mathrm{Inn}(G)$. Hence, the map $h \mapsto F_h$ induces a bijective correspondence between the pointed set

$$H^1(\Omega_F, G) := \mathrm{Hom}(\Omega_F, G)/\,\mathrm{Inn}(G)$$

and the set of all isomorphism classes of G-Galois F-algebras. In particular, if G is abelian, then $H^1(\Omega_F, G) = \mathrm{Hom}(\Omega_F, G)$ and hence the set of all isomorphism classes of G-Galois F-algebras has a natural group structure.

Definition 1.12. Given $h \in \mathrm{Hom}(\Omega_F, G)$, let $F^h := (F^c)^{\ker h}$ be as in (1.9) and let $\mathcal{O}^h := \mathcal{O}_{F^h}$. Define the *ring of integers of F_h* by

$$\mathcal{O}_h := \mathrm{Map}_{\Omega_F}({}^hG, \mathcal{O}^h).$$

Remark 1.13. For F a number field, given $h \in \mathrm{Hom}(\Omega_F, G)$, define

$$h_v \in \mathrm{Hom}(\Omega_{F_v}, G); \quad h_v := h \circ \widetilde{i_v} \tag{1.10}$$

for each prime $v \in M_F$. It was proven in [7, (1.4)] that $(F_v)_{h_v} \simeq F_v \otimes_F F_h$, and similarly we have $\mathcal{O}_{h_v} \simeq \mathcal{O}_{F_v} \otimes_{\mathcal{O}_F} \mathcal{O}_h$.

Definition 1.14. Given $h \in \mathrm{Hom}(\Omega_F, G)$, we say that F_h/F or h is *unramified* (respectively, *tame*) if F^h/F is unramified (respectively, *tame*).

Definition 1.15. Define the *resolvend map* $\mathbf{r}_G : \mathrm{Map}(G, F^c) \longrightarrow F^cG$ by

$$\mathbf{r}_G(a) := \sum_{s \in G} a(s)s^{-1}.$$

It is clear that $\mathbf{r}_G$ is an isomorphism of F^cG-modules, but not an isomorphism of F^cG-algebras because it does not preserve multiplication.

Given $a \in \mathrm{Map}(G, F^c)$, it is easy to check that $a \in F_h$ if and only if

$$\omega \cdot \mathbf{r}_G(a) = \mathbf{r}_G(a)h(\omega) \quad \text{for all } \omega \in \Omega_F. \tag{1.11}$$

The following proposition shows that resolvends may also be used to identify elements $a \in F_h$ for which $F_h = FG \cdot a$ or $\mathcal{O}_h = \mathcal{O}_F G \cdot a$.

Proposition 1.16. *Let $a \in F_h$.*

(1) *We have $F_h = FG \cdot a$ if and only if $\mathbf{r}_G(a) \in (F^cG)^\times$.*
(2) *For G abelian, we have $\mathcal{O}_h = \mathcal{O}_F G \cdot a$ with h unramified if and only if $\mathbf{r}_G(a) \in (\mathcal{O}_{F^c}G)^\times$. Moreover, if F is a finite extension of $\mathbb{Q}_p$ and if h is unramified, then there exists $a \in \mathcal{O}_h$ such that $\mathcal{O}_h = \mathcal{O}_F G \cdot a$.*

Proof. See [7, Proposition 1.8] for (1) and [7, (2.11) and Proposition 5.5] for (2). The second statement in (2) in fact holds even when G is non-abelian by a classical theorem of Noether. □

Remark 1.17. Clearly a homomorphism $h \in \mathrm{Hom}(\Omega_F, G)$ is tame if and only if it factors through the quotient map $\Omega_F \longrightarrow \Omega_F^t$. So, the subset of $\mathrm{Hom}(\Omega_F, G)$ consisting of the tame homomorphisms may be identified with

$\mathrm{Hom}(\Omega_F^t, G)$ as follows. Any tame $\widetilde{h} \in \mathrm{Hom}(\Omega_F, G)$ may be identified with the element $h \in \mathrm{Hom}(\Omega_F^t, G)$ defined by

$$h(\omega) := \widetilde{h}(\widetilde{\omega}) \quad \text{for } \omega \in \Omega_F^t,$$

where $\widetilde{\omega} \in \Omega_F$ denotes any lift of ω. Conversely, any $h \in \mathrm{Hom}(\Omega_F^t, G)$ may be identified with the element $\widetilde{h} \in \mathrm{Hom}(\Omega_F, G)$ defined by

$$\widetilde{h}(\omega) := h(\omega|_{F^t}) \quad \text{for } \omega \in \Omega_F.$$

The above identifications will be used repeatedly in the rest of this paper. In particular, given any $h \in \mathrm{Hom}(\Omega_F^t, G)$, all of the definitions and results introduced in this subsection still apply.

2. The Basic Diagram

In this section, let K/k denote a fixed Galois extension of number fields and let $\Sigma := \mathrm{Gal}(K/k)$. We will assume that G is abelian and fix a left Σ-module structure on G. Below, we construct the basic diagram (1.4), which was first introduced by Brinkhuis in [2, Theorem 5.1] in order to relate the study of realizable classes to that of embedding problems (cf. Remark 1.2 above and the discussion following it). The map gal in (1.4) is that defined in (1.1), and the map

$$i^* : H^2(\Sigma, G) \longrightarrow H^2(\Sigma, (\mathcal{O}_K G)^\times)$$

is that induced by the natural inclusion $G \longrightarrow (\mathcal{O}_K G)^\times$. The construction of the horizontal rows will be explained in the subsequent subsections. We will also show that the top row is exact, that all of the maps except possibly gal are homomorphisms, and that (1.4) commutes.

In what follows, we will let $\mathrm{Gal}(K^t/k)$ act on G (on the left) via the natural quotient map $\mathrm{Gal}(K^t/k) \longrightarrow \Sigma$ and the given left Σ-action on G. Via the natural left $\mathrm{Gal}(K^t/k)$-action on K^t, this extends to a left $\mathrm{Gal}(K^t/k)$-action on K^tG. As mentioned in Remark 1.17, we will identify $\mathrm{Hom}(\Omega_K^t, G)$ with the subset of $\mathrm{Hom}(\Omega_K, G)$ consisting of the tame homomorphisms. We will also use the following notation.

Definition 2.1. For each $\gamma \in \Sigma$, fix a lift $\overline{\gamma} \in \mathrm{Gal}(K^t/k)$ of γ with $\overline{1} = 1$.

2.1. The Top Row: Hochschild–Serre Sequence. Recall that Ω_K^t acts trivially on G (on the left) by definition. From the Hochschild–Serre spectral sequence (see [8, Chapter II], for example) associated to the group extension

$$1 \longrightarrow \Omega_K^t \longrightarrow \mathrm{Gal}(K^t/k) \longrightarrow \Sigma \longrightarrow 1,$$

we then obtain an exact sequence

$$H^1(\mathrm{Gal}(K^t/k), G) \xrightarrow{\mathrm{res}} \mathrm{Hom}(\Omega_K^t, G)^\Sigma \xrightarrow{tr} H^2(\Sigma, G). \tag{2.1}$$

Here res is given by restriction and tr is the *transgression map*. We remark that (2.1) is also part of the five-term inflation-restriction exact sequence in group cohomology (see [8, Proposition 1.6.7], for example). We will give the definitions of the Σ-action on $\mathrm{Hom}(\Omega_K^t, G)$ and the map tr in this setting.

Definition 2.2. Given $h \in \mathrm{Hom}(\Omega_K^t, G)$ and $\gamma \in \Sigma$, define

$$(h \cdot \gamma)(\omega) := \gamma^{-1} \cdot h(\overline{\gamma}\omega\overline{\gamma}^{-1}) \quad \text{for all } \omega \in \Omega_K^t.$$

This definition is independent of the choice of the lift $\overline{\gamma}$ since G is abelian. The *transgression map* $tr : \mathrm{Hom}(\Omega_K^t, G)^\Sigma \longrightarrow H^2(\Sigma, G)$ (see [8, Proposition 1.6.6], for example) is defined by

$$tr(h) := [(\gamma, \delta) \mapsto h((\overline{\gamma})(\overline{\delta})(\overline{\gamma\delta})^{-1})],$$

where $[-]$ denotes the cohomology class. This definition is also independent of the choice of the lifts $\overline{\gamma}$ for $\gamma \in \Sigma$.

Below, we will explain how the exact sequence (2.1) is related to the study of embedding problems. To that end, observe that each group extension

$$E: \quad 1 \longrightarrow G \xrightarrow{\iota} \Gamma \longrightarrow \Sigma \longrightarrow 1$$

of Σ by G induces a canonical left Σ-module structure on G via conjugation in Γ as follows. For each $\gamma \in \Sigma$, choose a lift $\sigma(\gamma)$ of γ in Γ. Then, define

$$\gamma * s := \iota^{-1}(\sigma(\gamma)\iota(s)\sigma(\gamma)^{-1}) \quad \text{for } \gamma \in \Gamma \text{ and } s \in G. \tag{2.2}$$

This definition does not depend upon the choice of the lift $\sigma(\gamma)$ because G is abelian. In addition, define a map $c_E : \Sigma \times \Sigma \longrightarrow G$ by

$$c_E(\gamma, \delta) := \iota^{-1}(\sigma(\gamma)\sigma(\delta)\sigma(\gamma\delta)^{-1}). \tag{2.3}$$

Denote by $E(K/k, G)$ the set of all equivalence classes of the group extensions of Σ by G such that (2.2) coincides with the left Σ-action on G that we have fixed. It is well-known (see [8, Theorem 1.2.4], for example) then that the map $E \mapsto c_E$ induces a bijective correspondence between $E(K/k, G)$ and the group $H^2(\Sigma, G)$, and the map c_E represents the trivial cohomology class if and only if E splits.

Proposition 2.3. *Let $h \in \mathrm{Hom}(\Omega_K^t, G)^\Sigma$. If h is surjective, then K^h is a tame solution to the embedding problem $(K/k, G, E_h)$, where E_h is a group extension of Σ by G whose equivalence class corresponds to $tr(h)$.*

Proof. The extension K^h/k is Galois because $\mathrm{Gal}(K^t/K^h)$, which is equal to $\ker(h)$, is normal in $\mathrm{Gal}(K^t/k)$. To see why, let $\omega_k \in \mathrm{Gal}(K^t/k)$ be given and write $\omega_k = \overline{\gamma}\omega_0$ for $\gamma \in \Sigma$ and $\omega_0 \in \Omega_K^t$. For any $\omega \in \ker(h)$, we have

$$\begin{aligned} h(\omega_k\omega\omega_k^{-1}) &= h(\overline{\gamma}\omega_0\omega\omega_0^{-1}\overline{\gamma}^{-1}) \\ &= \gamma \cdot (h \cdot \gamma)(\omega_0\omega\omega_0^{-1}) \\ &= \gamma \cdot (h(\omega_0)h(\omega)h(\omega_0)^{-1}), \end{aligned}$$

where the last equality follows because h is Σ-invariant and is a homomorphism on Ω_K^t. Since $\omega \in \ker(h)$, we see that $\omega_k\omega\omega_k^{-1} \in \ker(h)$ and so $\ker(h)$ is indeed normal in $\mathrm{Gal}(K^t/k)$. We set $\Gamma_h := \mathrm{Gal}(K^h/k)$.

Observe that h induces an isomorphism $\overline{h} : \mathrm{Gal}(K^h/K) \longrightarrow G$ since h is surjective. Let $\iota : G \longrightarrow \mathrm{Gal}(K^h/K) \longrightarrow \Gamma_h$ be the homomorphism $(\overline{h})^{-1}$ followed by the natural inclusion $\mathrm{Gal}(K^h/K) \longrightarrow \mathrm{Gal}(K^h/k)$. We obtain a group extension

$$E_h : \quad 1 \longrightarrow G \xrightarrow{\ \iota\ } \Gamma_h \longrightarrow \Sigma \longrightarrow 1$$

of Σ by G for which the diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Gal}(K^h/K) \arrow[r] \arrow[d, "\overline{h}"] & \mathrm{Gal}(K^h/k) \arrow[r] \arrow[d, equal] & \mathrm{Gal}(K/k) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & G \arrow[r, "\iota"'] & \Gamma_h \arrow[r] & \Sigma \arrow[r] & 1
\end{tikzcd}$$

clearly commutes, and K^h/K is clearly tame. Hence, the extension K^h/K is a tame solution to the embedding problem $(K/k, G, E_h)$.

Finally, for each $\gamma \in \Sigma$, choose $\sigma(\gamma) := \overline{\gamma}|_{K^h}$ to be a lift of γ in Γ_h. Given any $s \in G$, there exists $\omega \in \Omega_K^t$ such that $h(\omega) = s$ because h is surjective. The left Σ-action on G defined as in (2.2) is then given by

$$\begin{aligned}
\gamma * s &= \overline{h}((\overline{\gamma}|_{K^h})(\omega|_{K^h})(\overline{\gamma}|_{K^h})^{-1}) \\
&= h(\overline{\gamma}\omega\overline{\gamma}^{-1}) \\
&= \gamma \cdot s,
\end{aligned}$$

where the last equality follows because h is Σ-invariant. This shows that the equivalence class of E_h lies in $E(K/k, G)$. Also, the map $c_{E_h} : \Sigma \times \Sigma \longrightarrow G$ in (2.3) is given by

$$\begin{aligned}
c_{E_h}(\gamma, \delta) &= \overline{h}((\overline{\gamma}|_{K^h})(\overline{\delta}|_{K^h})(\overline{\gamma\delta}|_{K^h})^{-1}) \\
&= h((\overline{\gamma})(\overline{\delta})(\overline{\gamma\delta})^{-1})
\end{aligned}$$

and so the equivalence class of E_h corresponds to $tr(h)$, as claimed. $\square$

2.2. The Bottom Row: Fröhlich-Wall Sequence. Observe that $\mathcal{O}_KG$ is equipped with a canonical left Σ-action, namely that induced by the given left Σ-action on G and $\mathcal{O}_K$. For all $\gamma \in \Sigma$ and $\beta, \beta' \in \mathcal{O}_KG$, we have

$$\gamma \cdot (\beta + \beta') = \gamma \cdot \beta + \gamma \cdot \beta' \quad \text{and} \quad \gamma \cdot (\beta\beta') = (\gamma \cdot \beta)(\gamma \cdot \beta').$$

That is, the ring $\mathcal{O}_KG$ is a Σ-*ring*. From the Fröhlich-Wall sequence associated to $\mathcal{O}_KG$ (see [2, Section 1], for example), we obtain a homomorphism

$$\xi : \mathrm{Cl}(\mathcal{O}_KG)^{\Sigma} \longrightarrow H^2(\Sigma, (\mathcal{O}_KG)^{\times}).$$

We will recall the definitions of the Σ-action on $\mathrm{Cl}(\mathcal{O}_K G)$ and the map ξ.

2.2.1. *The left* Σ*-action on* $\mathrm{Cl}(\mathcal{O}_K G)$.

Definition 2.4. Let X and X' be two (left) $\mathcal{O}_K G$-modules. A *semilinear isomorphism from* X *to* X' is a group isomorphism $\varphi : X \longrightarrow X'$ satisfying

$$\varphi(\beta \cdot x) = (\gamma \cdot \beta) \cdot \varphi(x) \quad \text{for all } \beta \in \mathcal{O}_K G \text{ and } x \in X$$

for some $\gamma \in \Sigma$. Any such $\gamma \in \Sigma$ is called a *grading of* φ.

Definition 2.5. Let $[X] \in \mathrm{Cl}(\mathcal{O}_K G)$ and $\gamma \in \Sigma$. Define $\gamma \cdot [X] := [Y]$ if there exists a semilinear isomorphism $\varphi : X \longrightarrow Y$ having γ as a grading. It is clear that the isomorphism class $[Y]$ of Y (cf. Remark 1.6) is uniquely determined by that of X. Note also that such a Y always exists, as we may take $Y := X_\gamma$ to be the abelian group X equipped with the structure

$$\beta * x := (\gamma^{-1} \cdot \beta) \cdot x \quad \text{for } \beta \in (\mathcal{O}_K G)^\times \text{ and } x \in X_\gamma \tag{2.4}$$

as an $\mathcal{O}_K G$-module and take $\varphi = \mathrm{id}_X$ to be the identity map on X.

Clearly Definition 2.5 defines a left Σ-action on the group $\mathrm{Cl}(\mathcal{O}_K G)$. The next proposition shows that $\mathrm{Cl}(\mathcal{O}_K G)$ is in fact a left Σ-module under this action and so $\mathrm{Cl}(\mathcal{O}_K G)^\Sigma$ is a subgroup of $\mathrm{Cl}(\mathcal{O}_K G)$.

Proposition 2.6. *Let* $[X], [X'] \in \mathrm{Cl}(\mathcal{O}_K G)$*. For all* $\gamma \in \Sigma$*, we have*

$$\gamma \cdot ([X][X']) = (\gamma \cdot [X])(\gamma \cdot [X']).$$

Proof. Let $[X''] \in \mathrm{Cl}(\mathcal{O}_K G)$ be such that $[X''] = [X][X']$. By Definition 1.8, this means that there exists an $\mathcal{O}_K G$-isomorphism

$$\varphi : X \oplus X' \longrightarrow \mathcal{O}_K G \oplus X''.$$

Let X_γ denote the group X equipped with the $\mathcal{O}_K G$-structure defined as in (2.4), and similarly for X'_γ and X''_γ. Let $\psi_\gamma : \mathcal{O}_K G \longrightarrow \mathcal{O}_K G$ denote the bijective map defined by $\beta \mapsto \gamma \cdot \beta$. Then, the map

$$(\psi_\gamma \oplus \mathrm{id}_{X''}) \circ \varphi : X_\gamma \oplus X'_\gamma \longrightarrow \mathcal{O}_K G \oplus X''_\gamma$$

is an isomorphism of $\mathcal{O}_K G$-modules and so $[X''_\gamma] = [X_\gamma][X'_\gamma]$, as desired. $\square$

The next proposition ensures that diagram (1.4) is well-defined.

Proposition 2.7. *Let* $h \in \mathrm{Hom}(\Omega_K^t, G)^\Sigma$*. For all* $\gamma \in \Sigma$*, the map*

$$\varphi_\gamma : \mathbf{r}_G(\mathcal{O}_h) \longrightarrow \mathbf{r}_G(\mathcal{O}_h); \quad \varphi_\gamma(\mathbf{r}_G(a)) := \overline{\gamma} \cdot \mathbf{r}_G(a) \tag{2.5}$$

is well-defined and is a semilinear isomorphism having γ *as a grading. Consequently, we have* $\gamma \cdot [\mathcal{O}_h] = [\mathcal{O}_h]$ *and so* $\mathrm{gal}(\mathrm{Hom}(\Omega_K^t, G)^\Sigma) \subset \mathrm{Cl}(\mathcal{O}_K G)^\Sigma$.

Proof. Fix $\gamma \in \Sigma$. First, we verify that $\varphi_\gamma(\mathbf{r}_G(\mathcal{O}_h)) \subset \mathbf{r}_G(\mathcal{O}_h)$ so that φ_γ is well-defined. To that end, let $a \in \mathcal{O}_h$ be given. Since $\mathbf{r}_G$ is bijective, there exists $a' \in \mathrm{Map}(G, K^c)$ such that $\mathbf{r}_G(a') = \overline{\gamma} \cdot \mathbf{r}_G(a)$. We will use (1.11) to show that $a \in K_h$. Given $\omega \in \Omega_K^t$, note that $\overline{\gamma}^{-1}\omega\overline{\gamma} \in \Omega_K^t$ and that

$$\overline{\gamma}^{-1}\omega\overline{\gamma} \cdot \mathbf{r}_G(a) = \mathbf{r}_G(a)h(\overline{\gamma}^{-1}\omega\overline{\gamma})$$

by (1.11) since $a \in K_h$. This implies that

$$\begin{aligned}\omega \cdot \mathbf{r}_G(a') &= \overline{\gamma} \cdot (\overline{\gamma}^{-1}\omega\overline{\gamma} \cdot \mathbf{r}_G(a))\\ &= \mathbf{r}_G(a')(h \cdot \gamma^{-1})(\omega)\\ &= \mathbf{r}_G(a')h(\omega),\end{aligned}$$

where the last equality holds because h is Σ-invariant. Again by (1.11), we then see that $a' \in K_h$. Since $a \in \mathcal{O}_h$, it is clear that $a' \in \mathcal{O}_h$ and so φ_γ is well-defined. It is then obvious that φ_γ is a semilinear isomorphism having γ as a grading by definition. The above shows that $\gamma \cdot [\mathbf{r}_G(\mathcal{O}_h)] = [\mathbf{r}_G(\mathcal{O}_h)]$, which in turn implies that $\gamma \cdot [\mathcal{O}_h] = [\mathcal{O}_h]$ because $\mathbf{r}_G$ restricts to an $\mathcal{O}_K G$-isomorphism $\mathcal{O}_h \simeq \mathbf{r}_G(\mathcal{O}_h)$. □

2.2.2. *The homomorphism ξ.*

Definition 2.8. Given a (left) $\mathcal{O}_K G$-module X, let $\mathrm{Sem}(X)$ denote the set of all pairs (φ, γ), where $\varphi : X \longrightarrow X$ is a semilinear isomorphism having γ as a grading, equipped with the group operation $(\varphi, \gamma)(\varphi', \gamma') := (\varphi\varphi', \gamma\gamma')$. Also, let $\mathrm{Aut}(X)$ denote the group of $\mathcal{O}_K G$-automorphisms on X. The map

$$\mathfrak{g}_X : \mathrm{Sem}(X) \longrightarrow \Sigma; \qquad \mathfrak{g}_X(\varphi, \gamma) := \gamma$$

is then a homomorphism with $\ker(\mathfrak{g}_X) = \mathrm{Aut}(X)$.

Now, consider $[X] \in \mathrm{Cl}(\mathcal{O}_K G)^\Sigma$. The map $\mathfrak{g}_X$ is surjective because $[X]$ is Σ-invariant. Also, since X is locally free over $\mathcal{O}_K G$ (of rank one), an $\mathcal{O}_K G$-automorphism on X is of the form $\psi_\beta : x \mapsto \beta \cdot x$, where $\beta \in (\mathcal{O}_K G)^\times$. So, we may identify $\mathrm{Aut}(X)$ with $(\mathcal{O}_K G)^\times$. We then obtain a group extension

$$E_X : \; 1 \longrightarrow (\mathcal{O}_K G)^\times \xrightarrow{\;\mathfrak{i}_X\;} \mathrm{Sem}(X) \xrightarrow{\;\mathfrak{g}_X\;} \Sigma \longrightarrow 1$$

of Σ by $(\mathcal{O}_K G)^\times$, where $\mathfrak{i}_X(\beta) := (\psi_\beta, 1)$.

Notice that E_X induces a left Σ-module structure on $(\mathcal{O}_K G)^\times$ via conjugation in $\mathrm{Sem}(X)$ as follows. For each $\gamma \in \Sigma$, choose a lift (φ_γ, γ) of γ in $\mathrm{Sem}(\Sigma)$. Then, define (cf. (2.2))

$$\gamma * \beta := \mathfrak{i}_X^{-1}((\varphi_\gamma \psi_\beta \varphi_\gamma^{-1}, 1)) \quad \text{for } \gamma \in \Gamma \; \text{ and } \; \beta \in (\mathcal{O}_K G)^\times. \tag{2.6}$$

But for any $x \in (\mathcal{O}_K G)^\times$, we have

$$(\varphi_\gamma \psi_\beta \varphi_\gamma^{-1})(x) = \varphi_\gamma(\beta \cdot \varphi_\gamma^{-1}(x)) = (\gamma \cdot \beta) \cdot x = \varphi_{\gamma \cdot \beta}(x).$$

This means that (2.6) coincides with the existing left Σ-action on $(\mathcal{O}_K G)^\times$. Thus, analogously to the bijective correspondence between $E(K/k, G)$ and $H^2(\Sigma, G)$ described in Subsection 2.1, the group extension E_X corresponds to a cohomology class in $H^2(\Sigma, (\mathcal{O}_K G)^\times)$. In particular, the class is represented by the 2-cocycle $d_X : \Sigma \times \Sigma \longrightarrow (\mathcal{O}_K G)^\times$ which is uniquely determined by the equations (cf. (2.3))

$$d_X(\gamma, \delta) \cdot x = (\varphi_\gamma \varphi_\delta \varphi_{\gamma\delta}^{-1})(x) \quad \text{for all } x \in X. \tag{2.7}$$

Definition 2.9. Define $\xi : \mathrm{Cl}(\mathcal{O}_K G)^\Sigma \longrightarrow H^2(\Sigma, (\mathcal{O}_K G)^\times)$ by setting

$$\xi([X]) := [d_X],$$

where $[-]$ denotes the cohomology class. It is not difficult to check that this definition depends only on the isomorphism class $[X]$ of X (cf. Remark 1.6).

Proposition 2.10. *The map ξ is a homomorphism.*

Proof. Given $[X], [X'] \in \mathrm{Cl}(\mathcal{O}_K G)^\Sigma$, define $X'' := X \otimes_{\mathcal{O}_K G} X'$. Since G is abelian, [4, Theorem 55.16] implies that X'' is a locally free $\mathcal{O}_K G$-module (of rank one), and $[X][X'] = [X'']$ (cf. the proof of [4, Theorem 55.26]).

Since $[X]$ and $[X']$ are Σ-invariant, for each $\gamma \in \Sigma$, there exist semilinear automorphisms φ_γ and φ'_γ on X and X', respectively, having γ as a grading. Then, clearly $\varphi''_\gamma := \varphi_\gamma \otimes \varphi'_\gamma$ is a semilinear automorphism on X'' having γ as a grading. Let $d_X, d_{X'}$, and $d_{X''}$ be defined as in (2.7). For all $\gamma, \delta \in \Sigma$, $x \in X$, and $x' \in X'$, we have

$$\begin{aligned} d_{X''}(\gamma, \delta) \cdot (x \otimes x') &= (\varphi''_\gamma \varphi''_\delta \varphi''^{-1}_{\gamma\delta})(x \otimes x') \\ &= (\varphi_\gamma \varphi_\delta \varphi_{\gamma\delta}^{-1})(x) \otimes (\varphi'_\gamma \varphi'_\delta \varphi'^{-1}_{\gamma\delta})(x') \\ &= (d_X(\gamma, \delta) \cdot x) \otimes (d_{X'}(\gamma, \delta) \cdot x') \\ &= (d_X(\gamma, \delta) d_{X'}(\gamma, \delta)) \cdot (x \otimes x'). \end{aligned}$$

Thus, we have $d_{X''} = d_X d_{X'}$ and so $\xi([X'']) = \xi([X])\xi([X'])$, as desired. $\square$

2.3. Commutativity.

We now show that the basic diagram commutes.

Theorem 2.11. *The basic diagram* (1.4) *commutes. Moreover, the row at the top is exact, and all of the maps except possibly gal are homomorphisms.*

Proof. Note that diagram (1.4) is well-defined by Proposition 2.7. Now, we already know that the top row is exact. The maps res, tr, and i^* are plainly homomorphisms, and ξ is a homomorphism by Proposition 2.10. Hence, it remains to verify the equality $i^* \circ tr = \xi \circ \mathrm{gal}$.

Let $h \in \mathrm{Hom}(\Omega_K^t, G)^\Sigma$ be given. By Definition 2.2, the class $(i^* \circ tr)(h)$ is represented by the 2-cocycle $d : \Sigma \times \Sigma \longrightarrow (\mathcal{O}_K G)^\times$ defined by

$$d(\gamma, \delta) := h((\overline{\gamma})(\overline{\delta})(\overline{\gamma\delta})^{-1}).$$

Next, set $X := \mathbf{r}_G(\mathcal{O}_h)$ and note that $\mathbf{r}_G$ restricts to an $\mathcal{O}_K G$-isomorphism $\mathcal{O}_h \simeq X$, so $\mathrm{gal}(h) = [X]$. For each $\gamma \in \Sigma$, let $\varphi_\gamma : X \longrightarrow X$ be as in (2.5), which is a semilinear isomorphism having γ as a grading by Proposition 2.7. Then, by Definition 2.9, the class $(\xi \circ \mathrm{gal})(h)$ is represented by the 2-cocycle $d_X : \Sigma \times \Sigma \longrightarrow (\mathcal{O}_K G)^\times$ defined by the equations

$$d_X(\gamma, \delta) \cdot x = ((\overline{\gamma})(\overline{\delta})(\overline{\gamma\delta})^{-1}) \cdot x \quad \text{for all } x \in X.$$

But $(\overline{\gamma})(\overline{\delta})(\overline{\gamma\delta})^{-1} \in \Omega_K^t$ and $x \in \mathbf{r}_G(\mathcal{O}_h)$. It then follows from (1.11) that

$$((\overline{\gamma})(\overline{\delta})(\overline{\gamma\delta})^{-1}) \cdot x = h((\overline{\gamma})(\overline{\delta})(\overline{\gamma\delta})^{-1})) \cdot x \quad \text{for all } x \in X.$$

This shows that $d_X = d$, whence $(i^* \circ tr)(h) = (\xi \circ \mathrm{gal})(h)$, as desired. $\square$

3. Characterization of Realizable Classes

In this section, we will assume that G is abelian. For the moment, let F be a number field. Recall that by a classical theorem of Noether, given any tame homomorphism $h \in \mathrm{Hom}(\Omega_F, G)$, we have that $\mathcal{O}_h$ is locally free over $\mathcal{O}_F G$ and hence defines a class $\mathrm{cl}(\mathcal{O}_h)$ in $\mathrm{Cl}(\mathcal{O}_F G)$. Below, we explain how this class $\mathrm{cl}(\mathcal{O}_h)$ may be computed using resolvends (recall Definition 1.15). It will also be helpful to recall the notation introduced in (1.7) and (1.8).

First, since $\mathcal{O}_h$ is locally free over $\mathcal{O}_F G$ (of rank one), for each $v \in M_F$ there exists $a_v \in \mathcal{O}_{h_v}$ such that

$$\mathcal{O}_{h_v} = \mathcal{O}_{F_v} G \cdot a_v. \tag{3.1}$$

Next, by the Normal Basis Theorem, there exists $b \in F_h$ such that

$$F_h = FG \cdot b. \tag{3.2}$$

Since $F_v G \cdot a_v = F_{h_v} = F_v G \cdot b$ for all $v \in M_F$ and $\mathcal{O}_{F_v} G \cdot a_v = \mathcal{O}_{F_v} G \cdot b$ for all but finitely many $v \in M_F$, there exists $c = (c_v) \in J(FG)$ such that

$$a_v = c_v \cdot b \tag{3.3}$$

for all $v \in M_F$. We see that the FG-module isomorphism $FG \longrightarrow F_h$ given by $\beta \mapsto \beta \cdot b$ restricts to an $\mathcal{O}_F G$-module isomorphism $\mathcal{O}_F G \cdot c \longrightarrow \mathcal{O}_h$ and so $\mathrm{cl}(\mathcal{O}_h) = j(c)$. Recall that the resolvend map $\mathbf{r}_G : \mathrm{Map}(G, F_v^c) \longrightarrow F_v^c G$ is an $F_v G$-module isomorphism. Equation (3.3) is then equivalent to

$$\mathbf{r}_G(a_v) = c_v \cdot \mathbf{r}_G(b). \tag{3.4}$$

Hence, in order to compute the class $\mathrm{cl}(\mathcal{O}_h)$, it is sufficient to compute the resolvends $\mathbf{r}_G(b)$ and $\mathbf{r}_G(a_v)$. Notice that the resolvend $\mathbf{r}_G(b)$ of an element $b \in F_h$ satisfying (3.2) is already characterized by Proposition 1.16(1).

The main purpose of this section is to recall the characterization of the class $\mathrm{cl}(\mathcal{O}_h)$ proved by McCulloh in [7] (see Theorem 3.10 below and the discussion following (3.22)). To avoid repetition, we will only give an overview of the ideas involved below, and then recall the necessary definitions in the subsequent subsections.

As is explained above, the characterization of the class $\mathrm{cl}(\mathcal{O}_h)$ reduces to the computation of the resolvend $\mathbf{r}_G(a_v)$ of an element $a_v \in \mathcal{O}_{h_v}$ satisfying (3.1) for each prime $v \in M_F$. To that end, recall from Remark 1.17 that h_v may be regarded as an element of $\mathrm{Hom}(\Omega_{F_v}^t, G)$. By Definition 3.8 below, the homomorphism h_v then factors into

$$h_v = h_v^{nr} h_v^{tot},$$

where $h_v^{nr}, h_v^{tot} \in \mathrm{Hom}(\Omega_{F_v}^t, G)$ are such that h_v^{nr} is unramified and $F_v^{h_v^{tot}}/F_v$ is totally ramified. As in the proof of [7, Theorem 5.6], we may decompose the resolvend $\mathbf{r}_G(a_v)$ as

$$\mathbf{r}_G(a_v) = \mathbf{r}_G(a_{v,nr})\mathbf{r}_G(a_{v,tot}), \tag{3.5}$$

where $\mathcal{O}_{h_v^{nr}} = \mathcal{O}_{F_v}G \cdot a_{v,nr}$ and $\mathcal{O}_{h_v^{tot}} = \mathcal{O}_{F_v}G \cdot a_{v,tot}$. Observe that the resolvend $\mathbf{r}_G(a_{v,nr})$ of such an element $a_{v,nr} \in \mathcal{O}_{h_v^{nr}}$ is already characterized by Proposition 1.16(2). As for the resolvend $\mathbf{r}_G(a_{v,tot})$, it may be characterized using the *Stickelberger transpose* and *local prime $\mathfrak{F}$-elements*, which we define in Subsections 3.2 and 3.3, respectively. Rather than resolvends we will in fact use *reduced resolvends*, which we define in Subsection 3.1.

3.1. Cohomology and Reduced Resolvends. Let F be a number field or a finite extension of $\mathbb{Q}_p$. Following [7, Sections 1 and 2], we use cohomology to define reduced resolvends and explain how they may be regarded as functions on characters of G.

Recall that Ω_F acts trivially on G (on the left) by definition. Define

$$\mathcal{H}(FG) := ((F^cG)^\times/G)^{\Omega_F} \quad \text{and} \quad \mathcal{H}(\mathcal{O}_FG) := ((\mathcal{O}_{F^c}G)^\times/G)^{\Omega_F}.$$

Taking Ω_F-cohomology of the short exact sequence

$$1 \longrightarrow G \longrightarrow (F^cG)^\times \longrightarrow (F^cG)^\times/G \longrightarrow 1 \tag{3.6}$$

then yields the exact sequence

$$1 \longrightarrow G \longrightarrow (FG)^\times \xrightarrow{rag} \mathcal{H}(FG) \xrightarrow{\delta} \mathrm{Hom}(\Omega_F, G) \longrightarrow 1, \tag{3.7}$$

where $H^1(\Omega_F, (F^cG)^\times) = 1$ is a consequence of Hilbert's Theorem 90. Alternatively, given $h \in \mathrm{Hom}(\Omega_F, G)$, observe that a coset $\mathbf{r}_G(a)G \in \mathcal{H}(FG)$ belongs to the preimage of h under δ if and only if

$$h(\omega) = \mathbf{r}_G(a)^{-1}(\omega \cdot \mathbf{r}_G(a)) \quad \text{for all } \omega \in \Omega_F,$$

which is equivalent to $F_h = FG \cdot a$ by (1.11) and Proposition 1.16(1). Such an element $a \in F_h$ always exists by the Normal Basis Theorem and so δ is indeed surjective.

The same argument above also shows that

$$\mathcal{H}(FG) = \{\mathbf{r}_G(a)G \mid F_h = FG \cdot a \text{ for some } h \in \mathrm{Hom}(\Omega_F, G)\}. \tag{3.8}$$

Similarly, the argument above and Proposition 1.16(2) imply that

$$\mathcal{H}(\mathcal{O}_F G) = \left\{ \mathbf{r}_G(a)G \,\middle|\, \begin{array}{c} \mathcal{O}_h = \mathcal{O}_F G \cdot a \text{ for some} \\ \text{unramified } h \in \mathrm{Hom}(\Omega_F, G) \end{array} \right\}.$$

Definition 3.1. Given $\mathbf{r}_G(a)G \in \mathcal{H}(FG)$, define $r_G(a) := \mathbf{r}_G(a)G$, called the *reduced resolvend of* a. Also, define $h_a \in \mathrm{Hom}(\Omega_F, G)$ by

$$h_a(\omega) := \mathbf{r}_G(a)^{-1}(\omega \cdot \mathbf{r}_G(a)),$$

called the *homomorphism associated to* $r_G(a)$. This definition is independent upon the choice of the representative $\mathbf{r}_G(a)$, and we have $F_h = FG \cdot a$ by (1.11) and Proposition 1.16(1).

Definition 3.2. For F a number field, let $J(\mathcal{H}(FG))$ be the restricted direct product of the groups $\mathcal{H}(F_v G)$ with respect to the subgroups $\mathcal{H}(\mathcal{O}_{F_v} G)$ for $v \in M_F$. Let

$$\eta : \mathcal{H}(FG) \longrightarrow J(\mathcal{H}(FG))$$

denote the diagonal map and let

$$U(\mathcal{H}(\mathcal{O}_F G)) := \prod_{v \in M_F} \mathcal{H}(\mathcal{O}_{F_v} G)$$

be the group of unit ideles.

Next, recall Definition 1.9 and notice that the homomorphism

$$\prod_{v \in M_F} rag_{F_v} : J(FG) \longrightarrow J(\mathcal{H}(FG)) \tag{3.9}$$

is clearly well-defined, where rag_{F_v} is the map in (3.7). The diagram

$$\begin{array}{ccc} (FG)^\times & \xrightarrow{\quad\partial\quad} & J(FG) \\ {\scriptstyle rag_F}\big\downarrow & & \big\downarrow{\scriptstyle \prod_v rag_{F_v}} \\ \mathcal{H}(FG) & \xrightarrow[\quad\eta\quad]{} & J(\mathcal{H}(FG)) \end{array}$$

clearly commutes. By abuse of notation, we will also write $rag = rag_F$ for the homomorphism in (3.9).

To interpret reduced resolvends as functions on characters of G, recall that $\widehat{G}$ denotes the group of irreducible F^c-valued characters on G. Define

$$\det : \mathbb{Z}\widehat{G} \longrightarrow \widehat{G}; \quad \det\left(\sum_\chi n_\chi \chi\right) := \prod_\chi \chi^{n_\chi}$$

and set

$$A_{\widehat{G}} := \ker(\det). \tag{3.10}$$

Applying the functor $\mathrm{Hom}(-, (F^c)^\times)$ to the short exact sequence

$$1 \longrightarrow A_{\widehat{G}} \longrightarrow \mathbb{Z}\widehat{G} \xrightarrow{\det} \widehat{G} \longrightarrow 1$$

then yields the short exact sequence

$$\begin{aligned}(3.11)\qquad 1 \longrightarrow \mathrm{Hom}(\widehat{G}, (F^c)^\times) &\longrightarrow \mathrm{Hom}(\mathbb{Z}\widehat{G}, (F^c)^\times)\\ &\longrightarrow \mathrm{Hom}(A_{\widehat{G}}, (F^c)^\times) \longrightarrow 1,\end{aligned}$$

where exactness on the right follows because $(F^c)^\times$ is divisible and therefore injective. We will identify (3.6) with (3.11) as follows.

First, we have canonical identifications

$$(F^cG)^\times = \mathrm{Map}(\widehat{G}, (F^c)^\times) = \mathrm{Hom}(\mathbb{Z}\widehat{G}, (F^c)^\times).$$

The second identification is given by extending the maps $\widehat{G} \longrightarrow (F^c)^\times$ via $\mathbb{Z}$-linearity, and the first is induced by characters as follows. Each resolvend $\mathbf{r}_G(a) \in (F^cG)^\times$ gives rise to a map $\varphi \in \mathrm{Map}(\widehat{G}, (F^c)^\times)$ defined by

$$(3.12)\qquad \varphi(\chi) := \sum_{s\in G} a(s)\chi(s)^{-1} \quad \text{for } \chi \in \widehat{G}.$$

Conversely, given $\varphi \in \mathrm{Map}(\widehat{G}, (F^c)^\times)$, one recovers $\mathbf{r}_G(a)$ by the formula

$$(3.13)\qquad a(s) := \frac{1}{|G|}\sum_{\chi} \varphi(\chi)\chi(s) \quad \text{for } s \in G.$$

Since $G = \mathrm{Hom}(\widehat{G}, (F^c)^\times)$ canonically, the third terms

$$(F^cG)^\times/G = \mathrm{Hom}(A_{\widehat{G}}, (F^c)^\times)$$

in (3.6) and (3.11), respectively, are naturally identified as well.

Taking Ω_F-invariants, we then obtain the identification

$$(3.14)\qquad \mathcal{H}(FG) = \mathrm{Hom}_{\Omega_F}(A_{\widehat{G}}, (F^c)^\times).$$

Under this identification, it is clear from (3.12) that

$$(3.15)\qquad \mathcal{H}(\mathcal{O}_FG) \subset \mathrm{Hom}_{\Omega_F}(A_{\widehat{G}}, \mathcal{O}_{F^c}^\times),$$

and from (3.13) that the above inclusion is in fact an equality when F is a finite extension of $\mathbb{Q}_p$ for a prime p not dividing $|G|$.

3.2. The Stickelberger Transpose. Let F be a number field or a finite extension of $\mathbb{Q}_p$ and let $\{\zeta_n : n \in \mathbb{Z}^+\}$ denote the chosen compatible set of primitive roots of unity in F^c. We recall the definition of the so-called Stickelberger transpose, which was first introduced by McCulloh in [7, Section 4] (see [7, Proposition 5.4] for the motivation of the definition).

Definition 3.3. For each $\chi \in \widehat{G}$ and $s \in G$, let $\upsilon(\chi, s) \in \{0, 1, \ldots, |s| - 1\}$ be the unique integer such that $\chi(s) = (\zeta_{|s|})^{\upsilon(\chi,s)}$ and define

$$\langle \chi, s \rangle := \upsilon(\chi, s)/|s|.$$

Extending this definition by $\mathbb{Q}$-linearity, we obtain a pairing

$$\langle\ ,\ \rangle : \mathbb{Q}\widehat{G} \times \mathbb{Q}G \longrightarrow \mathbb{Q},$$

called the *Stickelberger pairing.* The map

$$\Theta : \mathbb{Q}\widehat{G} \longrightarrow \mathbb{Q}G; \quad \Theta(\psi) := \sum_{s \in G} \langle \psi, s \rangle s$$

is called the *Stickelberger map.*

Proposition 3.4. *For $\psi \in \mathbb{Z}\widehat{G}$, we have $\Theta(\psi) \in \mathbb{Z}G$ if and only if $\psi \in A_{\widehat{G}}$.*

Proof. See [7, Proposition 4.3]. □

Notice that Ω_F acts on $\widehat{G}$ canonically via its action on the roots of unity in F^c. Now, up until this point, we have let Ω_F act trivially on G. Below, we introduce other Ω_F-actions on G, one of which will make the Stickelberger map preserve the Ω_F-action.

Definition 3.5. Let $m := \exp(G)$ and let μ_m be the group of m-th roots of unity in F^c. The *m-th cyclotomic character of* Ω_F is the homomorphism

$$\kappa : \Omega_F \longrightarrow (\mathbb{Z}/m\mathbb{Z})^\times$$

defined by the equations

$$\omega(\zeta) = \zeta^{\kappa(\omega)} \quad \text{for } \omega \in \Omega_F \text{ and } \zeta \in \mu_m.$$

For $n \in \mathbb{Z}$, let $G(n)$ be the group G equipped with the Ω_F-action given by

$$\omega \cdot s := s^{\kappa(\omega^n)} \quad \text{for } s \in G \text{ and } \omega \in \Omega_F.$$

We will need $G(-1)$. But of course, if F contains all $\exp(G)$-th roots of unity, then κ is trivial and Ω_F acts trivially on $G(n) = G(0)$ for all $n \in \mathbb{Z}$.

Proposition 3.6. *The map $\Theta : \mathbb{Q}\widehat{G} \longrightarrow \mathbb{Q}G(-1)$ preserves the Ω_F-action.*

Proof. See [7, Proposition 4.5]. □

From Propositions 3.4 and 3.6, we obtain an Ω_F-equivariant map

$$\Theta : A_{\widehat{G}} \longrightarrow \mathbb{Z}G(-1),$$

which in turn yields an Ω_F-equivariant homomorphism

$$\Theta^t : \mathrm{Hom}(\mathbb{Z}G(-1), (F^c)^\times) \longrightarrow \mathrm{Hom}(A_{\widehat{G}}, (F^c)^\times); \quad f \mapsto f \circ \Theta.$$

Via restriction, we then obtain a homomorphism

$$\Theta^t = \Theta^t_F : \mathrm{Hom}_{\Omega_F}(\mathbb{Z}G(-1), (F^c)^\times) \longrightarrow \mathrm{Hom}_{\Omega_F}(A_{\widehat{G}}, (F^c)^\times),$$

called the *Stickelberger transpose.*

Notice that we have a natural identification

$$\mathrm{Hom}_{\Omega_F}(\mathbb{Z}G(-1),(F^c)^\times)=\mathrm{Map}_{\Omega_F}(G(-1),(F^c)^\times).$$

To simplify notation, let

$$\begin{aligned}\Lambda(FG)&:=\mathrm{Map}_{\Omega_F}(G(-1),F^c);\\ \Lambda(\mathcal{O}_FG)&:=\mathrm{Map}_{\Omega_F}(G(-1),\mathcal{O}_{F^c}).\end{aligned}\tag{3.16}$$

Then, we may regard Θ^t as a map $\Lambda(FG)^\times\longrightarrow\mathcal{H}(FG)$ (recall (3.14)).

Definition 3.7. For F a number field, let $J(\Lambda(FG))$ be the restricted direct product of the groups $\Lambda(F_vG)^\times$ with respect to the subgroups $\Lambda(\mathcal{O}_{F_v}G)^\times$ for $v\in M_F$. Let

$$\lambda=\lambda_F:\Lambda(FG)^\times\longrightarrow J(\Lambda(FG))$$

denote the diagonal map and let

$$U(\Lambda(\mathcal{O}_FG)):=\prod_{v\in M_F}\Lambda(\mathcal{O}_{F_v}G)^\times$$

be the group of unit ideles.

Next, recall Definition 3.2 and observe that the homomorphism

$$\prod_{v\in M_F}\Theta^t_{F_v}:J(\Lambda(FG))\longrightarrow J(\mathcal{H}(FG))\tag{3.17}$$

is well-defined because (3.15) is an equality for all but finitely many $v\in M_F$. Recall from Section 1.1 that we chose $\{i_v(\zeta_n):n\in\mathbb{Z}^+\}$ to be the compatible set of primitive roots of unity in F_v^c. Hence, the diagram

$$\begin{tikzcd}\Lambda(FG)^\times \arrow[r, "\lambda"] \arrow[d, "\Theta^t_F"'] & J(\Lambda(FG)) \arrow[d, "\prod_v \Theta^t_{F_v}"] \\ \mathcal{H}(FG) \arrow[r, "\eta"'] & J(\mathcal{H}(FG))\end{tikzcd}\tag{3.18}$$

commutes. By abuse of notation, we will also use $\Theta^t=\Theta^t_F$ to denote the homomorphism in (3.17).

3.3. Local Prime $\mathfrak{F}$-Elements. Let F be a finite extension of $\mathbb{Q}_p$. Denote by $\pi=\pi_F$ a chosen uniformizer in F and write $q=q_F$ for the order of the residue field $\mathcal{O}_F/(\pi_F)$. Let $\{\zeta_n:n\in\mathbb{Z}^+\}$ be the chosen compatible set of primitive roots of unity in F^c. Also, let F^{nr} denote the maximal unramified extension of F contained in F^c and set $\Omega_F^{nr}:=\mathrm{Gal}(F^{nr}/F)$.

Recall from Remark 1.17 that a tame $h\in\mathrm{Hom}(\Omega_F,G)$ may be regarded as an element in $\mathrm{Hom}(\Omega_F^t,G)$. Any such homomorphism admits a factorization as follows. First, we will recall the structure of the group Ω_F^t (see [6, Sections 7 and 8], for example). On the one hand, the field F^{nr} is obtained by adjoining to F all roots of unity of order coprime to p. This means that Ω_F^{nr} is a procyclic group which is topologically generated by the Frobenius automorphism $\phi = \phi_F$ given by

$$\phi(\zeta_n) = \zeta_n^q \quad \text{for all } (n,p) = 1.$$

On the other hand, the field F^t is obtained by adjoining to F^{nr} all the n-th roots of π for $n \in \mathbb{N}$ coprime to p and so $\mathrm{Gal}(F^t/F^{nr})$ is a procyclic group. We will choose a coherent set of radicals

$$\{\pi^{1/n} : n \in \mathbb{Z}^+\}$$

such that $(\pi^{1/mn})^n = \pi^{1/m}$ and define $\pi^{m/n} := (\pi^{1/n})^m$ for all $m, n \in \mathbb{Z}^+$. These radicals of π, together with the chosen roots of unity, then determine a distinguished topological generator $\sigma = \sigma_F$ of $\mathrm{Gal}(F^t/F^{nr})$ given by

$$\sigma(\pi^{1/n}) = \zeta_n \pi^{1/n} \quad \text{for all } (n,p) = 1. \tag{3.19}$$

Letting ϕ also denote the lifting of ϕ in Ω_F^t that fixes the radicals $\pi^{1/n}$ for all $(n,p) = 1$, we then see that Ω_F^t is topologically generated by ϕ and σ. Notice that we have the relation $\phi\sigma\phi^{-1}\sigma^{-1} = \sigma^{q-1}$, and define

$$G_{(q-1)} := \{s \in G \mid \text{the order of } s \text{ divides } q-1\}.$$

Since G is abelian, for any $h \in \mathrm{Hom}(\Omega_F^t, G)$, we must have $h(\sigma) \in G_{(q-1)}$.

Definition 3.8. Given $h \in \mathrm{Hom}(\Omega_F^t, G)$, define

$$h^{nr} \in \mathrm{Hom}(\Omega_F^t, G); \quad h^{nr}(\phi) := h(\phi) \text{ and } h^{nr}(\sigma) := 1;$$
$$h^{tot} \in \mathrm{Hom}(\Omega_F^t, G); \quad h^{tot}(\phi) := 1 \text{ and } h^{tot}(\sigma) := h(\sigma).$$

Clearly $h = h^{nr}h^{tot}$, called the *factorization of h with respect to σ*. Notice that h^{nr} is unramified, and we know that $F^{h^{tot}} = F(\pi^{1/|s|})$ for $s := h^{tot}(\sigma)$ by [7, Proposition 5.4].

Now, let $h \in \mathrm{Hom}(\Omega_F^t, G)$ be given and suppose that $\mathcal{O}_h = \mathcal{O}_F G \cdot a$. As in the proof of [7, Theorem 5.6], we may decompose the resolvend $\mathbf{r}_G(a)$ as

$$\mathbf{r}_G(a) = \mathbf{r}_G(a_{nr})\mathbf{r}_G(a_{tot}),$$

where $\mathcal{O}_{h^{nr}} = \mathcal{O}_F G \cdot a_{nr}$ and $\mathcal{O}_{h^{tot}} = \mathcal{O}_F G \cdot a_{tot}$. This is the same decomposition mentioned in (3.5). The resolvend $\mathbf{r}_G(a_{nr})$ is already characterized by Proposition 1.16(2). In [7, Proposition 5.4], McCulloh showed that the element $a_{tot} \in \mathcal{O}_{h^{tot}}$ may be chosen such that its reduced resolvend $r_G(a_{tot})$ is equal to $\Theta^t(f_s)$ for some $f_s \in \Lambda(FG)^\times$ (recall (3.16)), where $s := h(\sigma)$.

Definition 3.9. Given $s \in G_{(q-1)}$, define $f_s = f_{F,s} \in \Lambda(FG)^\times$ by

$$f_s(t) := \begin{cases} \pi & \text{if } t = s \neq 1 \\ 1 & \text{otherwise.} \end{cases}$$

Observe that f_s indeed preserves the Ω_F-action because all $(q-1)$-st roots of unity are contained in F, so elements in $G_{(q-1)}$ are fixed by Ω_F, as is π. Such a map in $\Lambda(FG)^\times$ is called a *prime $\mathfrak{F}$-element over F*. We will write

$$\mathfrak{F}_F := \{f_s : s \in G_{(q-1)}\}$$

for the collection of all of prime $\mathfrak{F}$-elements over F.

3.4. Approximation Theorems. Let F be a number field and define

$$\mathfrak{F} = \mathfrak{F}_F := \{f \in J(\Lambda(FG)) \mid f_v \in \mathfrak{F}_{F_v} \text{ for all } v \in M_F\}. \tag{3.20}$$

We are now ready to state the characterization of the set

$$R(\mathcal{O}_F G) := \{\mathrm{cl}(\mathcal{O}_h) : \text{tame } h \in \mathrm{Hom}(\Omega_F, G)\}$$

of all realizable classes in $\mathrm{Cl}(\mathcal{O}_F G)$ that was proved by McCulloh in [7]. It will be helpful to recall (1.8) and Definition 3.2.

Theorem 3.10. *Let $h \in \mathrm{Hom}(\Omega_F, G)$, say $F_h = FG \cdot b$. Then, we have h is tame if and only if there exists $c \in J(FG)$ such that*

$$rag(c) = \eta(r_G(b))^{-1} u \Theta^t(f) \tag{3.21}$$

for some $u \in U(\mathcal{H}(\mathcal{O}_F G))$ and $f = (f_v) \in \mathfrak{F}$. Also, if (3.21) *holds, then*

1. *for all $v \in M_F$, we have $f_v = f_{F_v, s_v}$ for $s_v := h_v(\sigma_{F_v})$;*
2. *for all $v \in M_F$, we have $f_v = 1$ if and only if h_v is unramified;*
3. *$j(c) = \mathrm{cl}(\mathcal{O}_h)$.*

Proof. See [7, Theorem 6.7]. We remark that the decomposition of $rag(c)$ in (3.21) comes from equation (3.4) and the decomposition (3.5). ☐

Theorem 3.10 implies that $R(\mathcal{O}_F G)$ may be characterized as follows. For $c \in J(FG)$, we have $j(c) \in R(\mathcal{O}_F G)$ if and only if $rag(c)$ is an element of

$$\eta(\mathcal{H}(FG)) U(\mathcal{H}(\mathcal{O}_F G)) \Theta^t(\mathfrak{F}). \tag{3.22}$$

Using two approximations theorems, McCulloh showed in [7, Theorem 6.17] that in the above, the set $\mathfrak{F}$ may be replaced by the group $J(\Lambda(FG))$, which then shows that $R(\mathcal{O}_F G)$ is a subgroup of $\mathrm{Cl}(\mathcal{O}_F G)$ (see [7, Corollary 6.20]).

Below, we state the two approximation theorems, which will play a crucial role in the proof of Theorem 1.3. To that end, we need some notation. It will be helpful to recall (3.16) and Definition 3.5.

Definition 3.11. Let $\mathfrak{m}$ be an ideal in $\mathcal{O}_F$. For each $v \in M_F$, let

$$U_{\mathfrak{m}}(\mathcal{O}_{F_v^c}) := (1 + \mathfrak{m}\mathcal{O}_{F_v^c}) \cap (\mathcal{O}_{F_v^c})^\times;$$

$$U'_{\mathfrak{m}}(\Lambda(\mathcal{O}_{F_v} G)) := \{g_v \in \Lambda(\mathcal{O}_{F_v} G)^\times \mid g_v(s) \in U_{\mathfrak{m}}(\mathcal{O}_{F_v^c}) \text{ for all } s \in G \setminus \{1\}\}.$$

Define

$$U'_{\mathfrak{m}}(\Lambda(\mathcal{O}_F G)) := \left(\prod_{v \in M_F} U'_{\mathfrak{m}}(\Lambda(\mathcal{O}_{F_v} G))\right) \cap J(\Lambda(FG)).$$

Definition 3.12. For $g \in J(\Lambda(FG))$ and $s \in G$, define

$$g_s := \prod_{v \in M_F} g_v(s) \in \prod_{v \in M_F} (F_v^c)^\times.$$

Theorem 3.13. *Let* $\mathfrak{m}$ *be an ideal in* $\mathcal{O}_F$ *divisible by both* $|G|$ *and* $\exp(G)^2$. *Then, we have* $\Theta^t(U'_{\mathfrak{m}}(\Lambda(\mathcal{O}_F G))) \subset U(\mathcal{H}(\mathcal{O}_F G))$.

Proof. See [7, Proposition 6.9]. □

Theorem 3.14. *Let* $g \in J(\Lambda(FG))$ *and let* T *be any finite subset of* M_F. *Then, there exists* $f = (f_v) \in \mathfrak{F}$ *such that* $f_v = 1$ *for all* $v \in T$ *and*

$$g \equiv f \pmod{\lambda(\Lambda(FG)^\times)U'_{\mathfrak{m}}(\Lambda(\mathcal{O}_F G))}.$$

Moreover, we may choose $f \in \mathfrak{F}$ *so that for each* $s \in G(-1)$ *with* $s \neq 1$, *there exists* $\omega \in \Omega_F$ *such that* $f_{\omega \cdot s} \neq 1$.

Proof. See [7, Proposition 6.14]. □

4. Characterization of Σ-Realizable Classes

In this section, let K/k denote a fixed Galois extension of number fields and let $\Sigma := \mathrm{Gal}(K/k)$. We will assume that G is abelian and fix a left Σ-module structure on G. We will choose $K^c = \mathbb{Q}^c$ and $k^c = \mathbb{Q}^c$, where $\mathbb{Q}^c$ is a fixed algebraic closure of $\mathbb{Q}$ containing K, as well as the same compatible set $\{\zeta_n : n \in \mathbb{Z}^+\}$ of primitive roots of unity in $\mathbb{Q}^c$ for both k and K. We will identify $\mathrm{Hom}(\Omega_K^t, G)$ with the subset of $\mathrm{Hom}(\Omega_K, G)$ consisting of the tame homomorphisms as in Remark 1.17.

Definition 4.1. Let V_K denote the set of primes in M_K which are ramified over k, and let V_k denote the set of primes in M_k lying below those in V_K.

The purpose of this section is to characterize the Σ-realizable classes (recall (1.5) and see (4.8)) coming from the homomorphisms $h \in \mathrm{Hom}(\Omega_K^t, G)^\Sigma$ for which h_v is unramified at all $v \in V_K$, under the assumptions in Theorem 1.3. We will do so by refining the characterization of realizable classes sated in Theorem 3.10. The crucial step is to make suitable choices for the embeddings $i_v : \mathbb{Q}^c \longrightarrow K_v^c$ and the uniformizers π_v in K_v for $v \in M_K$. We will need the following notation.

Definition 4.2. For each prime $w \in M_k$, let $i_w : \mathbb{Q}^c \longrightarrow k_w^c$ be the chosen embedding extending the natural embedding $k \longrightarrow k_w$. This determines a distinguished prime $v_w \in M_K$ lying above w for which the v_w-adic absolute value on K is induced by i_w. For each prime $v \in M_K$ lying above w, choose an element $\gamma_v \in \Sigma$ such that $v = v_w \circ \gamma_v^{-1}$, and we choose $\gamma_{v_w} = 1$. Also, we will fix a lift $\overline{\gamma_v} \in \Omega_k$ of γ_v with $\overline{\gamma_{v_w}} = 1$.

4.1. Choices of Embeddings.

Definition 4.3. Given $v \in M_K$, let $w \in M_k$ be the prime lying below v and note that the v-adic absolute value on K is induced by $i_w \circ \overline{\gamma_v}^{-1}$. Then, by restricting $i_w \circ \overline{\gamma_v}^{-1}$, we obtain an embedding $K \longrightarrow k_w^c$ which extends to a continuous embedding $K_v \longrightarrow k_w^c$. We will then lift this to an isomorphism $\varepsilon_v^{-1} : K_v^c \longrightarrow k_w^c$ and define $i_v : \mathbb{Q}^c \longrightarrow K_v^c$ by setting $i_v := \varepsilon_v \circ i_w \circ \overline{\gamma_v}^{-1}$, which clearly extends the natural embedding $K \longrightarrow K_v$.

For all $v \in M_K$ and $w \in M_k$ with w lying below v, define $\widetilde{\gamma_v} := \varepsilon_v \circ \varepsilon_{v_w}^{-1}$. By the choices made in Definition 4.3, the diagram

$$\begin{tikzcd}
K_{v_w}^c \arrow[rr, bend left, "\widetilde{\gamma_v}"] & k_w^c \arrow[l, "\varepsilon_{v_w}"'] \arrow[r, "\varepsilon_v"] & K_v^c \\
\mathbb{Q}^c \arrow[u, "i_{v_w}"] \arrow[r, equal] & \mathbb{Q}^c \arrow[u, "i_w"] \arrow[r, "\overline{\gamma_v}"'] & \mathbb{Q}^c \arrow[u, "i_v"]
\end{tikzcd}$$

commutes. Observe also that $\widetilde{\gamma_{v_w}} = 1$ and we have the relation

$$i_v = \widetilde{\gamma_v} \circ i_{v_w} \circ \overline{\gamma_v}^{-1}. \tag{4.1}$$

Via identifying $\mathrm{Hom}(\Omega_K^t, G)$ with the subset of $\mathrm{Hom}(\Omega_K, G)$ consisting of the tame homomorphisms as in Remark 1.17, we have the following result.

Proposition 4.4. *Let $h \in \mathrm{Hom}(\Omega_K^t, G)^\Sigma$. Let $v \in M_K$ and let $w \in M_k$ be the prime lying below v. Then, we have*

$$h_v(\widetilde{\gamma_v} \circ \omega \circ \widetilde{\gamma_v}^{-1}) = \gamma_v \cdot h_{v_w}(\omega) \quad \textit{for all } \omega \in \Omega_{K_{v_w}}.$$

Proof. We have $h_v = h \circ \widetilde{i_v}$ by definition (recall (1.6) and (1.10)). Using the relation (4.1), we then see that for all $\omega \in \Omega_{K_{v_w}}$, we have

$$\begin{aligned}
h_v(\widetilde{\gamma_v} \circ \omega \circ \widetilde{\gamma_v}^{-1}) &= h(i_v^{-1} \circ \widetilde{\gamma_v} \circ \omega \circ \widetilde{\gamma_v}^{-1} \circ i_v) \\
&= h(\overline{\gamma_v} \circ i_{v_w}^{-1} \circ \omega \circ i_{v_w} \circ \overline{\gamma_v}^{-1}) \\
&= \gamma_v \cdot (h \cdot \gamma_v)(i_{v_w}^{-1} \circ \omega \circ i_{v_w}) \\
&= \gamma_v \cdot h_{v_w}(\omega),
\end{aligned}$$

where the last equality holds because h is Σ-invariant. $\square$

4.2. Choices of Uniformizers and their Radicals. For each $w \in M_k$, let π_w be a chosen uniformizer in k_w and let $\{\pi_w^{1/n} : n \in \mathbb{Z}^+\}$ be the chosen coherent set of radicals of π_w in k_w^c (recall Subsection 3.3 above).

Definition 4.5. Given $v \in M_K \setminus V_K$, let $w \in M_k$ be the prime lying below v. We choose $\pi_v := \varepsilon_v(\pi_w)$ to be the uniformizer in K_v, and $\pi_v^{1/n} := \varepsilon_v(\pi_w^{1/n})$ for $n \in \mathbb{Z}^+$ to the be coherent radicals of π_v in K_v^c. As for $v \in V_K$, we will choose the uniformizer π_v in K_v and its radicals in K_v^c arbitrarily.

Lemma 4.6. *Let $v \in M_K \setminus V_K$ and let $w \in M_k$ be the prime lying below v. Then, we have $\pi_v^{1/n} = \widetilde{\gamma_v}(\pi_{v_w}^{1/n})$ for all $n \in \mathbb{Z}^+$.*

Proof. Since K/k is Galois and $v \notin V_K$, plainly we have $v_w \notin V_K$. Using the equality $\widetilde{\gamma_v} = \varepsilon_v \circ \varepsilon_{v_w}^{-1}$, we then see that

$$\pi_v^{1/n} = \varepsilon_v(\pi_w^{1/n}) = \widetilde{\gamma_v}(\varepsilon_{v_w}(\pi_w^{1/n})) = \widetilde{\gamma_v}(\pi_{v_w}^{1/n})$$

for all $n \in \mathbb{Z}^+$ by Definition 4.5. $\square$

The choices made in Definitions 4.3 and 4.5 in turn determine a distinguished topological generator $\sigma_v = \sigma_{K_v}$ of $\mathrm{Gal}(K_v^t/K_v^{nr})$ (recall (3.19)). In particular, because we chose $\{i_v(\zeta_n) : n \in \mathbb{Z}^+\}$ to be the compatible set of primitive roots of unity in K_v^c, we have

$$\sigma_v(\pi_v^{1/n}) = i_v(\zeta_n)\pi_v^{1/n} \quad \text{for } (n, p) = 1, \tag{4.2}$$

where p is the rational prime lying below v. By abuse of notation, we will also use σ_v to denote a chosen lift of σ_v in Ω_{K_v}. By identifying $\mathrm{Hom}(\Omega_K^t, G)$ with the subset of $\mathrm{Hom}(\Omega_K, G)$ consisting of the tame homomorphisms as in Remark 1.17, we then have the following result.

Proposition 4.7. *Let $h \in \mathrm{Hom}(\Omega_K^t, G)^\Sigma$. Let $v \in M_K \setminus V_K$ and let $w \in M_k$ be the prime lying below v. Then, we have*

$$h_v(\sigma_v) = \gamma_v \cdot h_{v_w}(\sigma_{v_w})$$

provided that ζ_{e_v} is contained in k, where $e_v := |h_v(\sigma_v)|$.

Proof. We already know from Proposition 4.4 that

$$h_v(\widetilde{\gamma_v} \circ \sigma_{v_w} \circ \widetilde{\gamma_v}^{-1}) = \gamma_v \cdot h_{v_w}(\sigma_{v_w}).$$

Thus, it suffices to show that $h_v(\widetilde{\gamma_v} \circ \sigma_{v_w} \circ \widetilde{\gamma_v}^{-1}) = h_v(\sigma_v)$, or equivalently, that $\widetilde{\gamma_v} \circ \sigma_{v_w} \circ \widetilde{\gamma_v}^{-1}$ and σ_v have the same action on $L := (K_v^t)^{\ker(h_v)}$.

Let $h_v = h_v^{nr} h_v^{tot}$ be the factorization of h_v with respect to σ_v (recall Definition 3.8). Let $L^{nr} := (K_v)_{h_v^{nr}}$ and $L^{tot} := (K_v)_{h_v^{tot}}$. Then, clearly we have $L \subset L^{nr} L^{tot}$. Because L^{nr}/K_v is unramified, both $\widetilde{\gamma_v} \circ \sigma_{v_w} \circ \widetilde{\gamma_v}^{-1}$ and σ_v act as the identity on L^{nr}. We also know that $L^{tot} = K_v(\pi_v^{1/e_v})$ by [7, Proposition 5.4]. Hence, it remains to show that

$$(\widetilde{\gamma_v} \circ \sigma_{v_w} \circ \widetilde{\gamma_v}^{-1})(\pi_v^{1/e_v}) = \sigma_v(\pi_v^{1/e_v}).$$

Notice that $\pi_v^{1/e_v} = \widetilde{\gamma_v}(\pi_{v_w}^{1/e_v})$ by Lemma 4.6 because $v \notin V_K$. Using (4.2), we then obtain

$$\begin{aligned}(\widetilde{\gamma_v} \circ \sigma_{v_w} \circ \widetilde{\gamma_v}^{-1})(\pi_v^{1/e_v}) &= \widetilde{\gamma_v}(i_{v_w}(\zeta_{e_v})\pi_{v_w}^{1/e_v}) \\ &= (\widetilde{\gamma_v} \circ i_{v_w})(\zeta_{e_v})\pi_v^{1/e_v} \\ &= (\widetilde{\gamma_v} \circ i_{v_w} \circ \overline{\gamma_v}^{-1})(\zeta_{e_v})\pi_v^{1/e_v} \\ &= i_v(\zeta_{e_v})\pi_v^{1/e_v} \\ &= \sigma_v(\pi_v^{1/e_v}),\end{aligned}$$

where $\overline{\gamma_v}^{-1}(\zeta_{e_v}) = \zeta_{e_v}$ because $\zeta_{e_v} \in k$ and $i_v = \widetilde{\gamma_v} \circ i_{v_w} \circ \overline{\gamma_v}^{-1}$ by (4.1). So, indeed $\widetilde{\gamma_v} \circ \sigma_{v_w} \circ \widetilde{\gamma_v}^{-1}$ and σ_v have the same action on L, as desired. □

4.3. Embeddings of Groups of Ideles. In this subsection, assume further that k contains all $\exp(G)$-th roots of unity. Then, we have

$$\Lambda(FG) = \mathrm{Map}(G, F) \quad \text{for } F \in \{k, K, k_w, K_v\},$$

where $w \in M_k$ and $v \in M_K$ (recall (3.16) and Definition 3.5). It will also be helpful to recall Definitions 3.2 and 3.7. The isomorphisms ε_v for $v \in M_K$ in Definition 4.3 then the following embeddings of groups of ideles.

Definition 4.8. Define $\nu : J(\Lambda(kG)) \longrightarrow J(\Lambda(KG))$ by

$$\nu(g)_v := \varepsilon_v \circ g_w \quad \text{for each } v \in M_K,$$

where $w \in M_k$ is the prime lying below v.

Definition 4.9. Define $\mu : J(\mathcal{H}(kG)) \longrightarrow J(\mathcal{H}(KG))$ by

$$\mu((r_G(a))_v := r_G(\varepsilon_v \circ a_w) \quad \text{for each } v \in M_K,$$

where $w \in M_k$ is the prime lying below v, and $a_w \in \mathrm{Map}(G, k_w^c)$ is such that $r_G(a_w) = r_G(a)_w$. Notice that the definition of μ does not require that k contains all $\exp(G)$-th roots of unity.

First, we prove two basic properties of the map ν. Notice that the choices of the uniformizers π_w in k_w for $w \in M_k$ determine a subset $\mathfrak{F}_k \subset J(\Lambda(kG))$ (recall Definition 3.9 and (3.20)). Similarly, the choices of the uniformizers π_v in K_v for $v \in M_K$ in Definition 4.5 determine a subset $\mathfrak{F}_K \subset J(\Lambda(KG))$.

Proposition 4.10. *Let $f = (f_w) \in \mathfrak{F}_k$. Let $v \in M_K \setminus V_K$ and let $w \in M_k$ be the prime lying below v. If $f_w = f_{k_w,s_w}$, then $\nu(f)_v = f_{K_v,s_w}$. In particular, if $f_w = 1$ for all $w \in V_k$, then $\nu(f) \in \mathfrak{F}_K$.*

Proof. Let q_w and q_v denote the orders of the residue fields of k_w and K_v, respectively. The order of s_w divides $q_w - 1$ and hence divides $q_v - 1$. Since $v \notin V_K$, we have $\pi_v = \varepsilon_v(\pi_w)$ and it is clear that $\nu(f)_v = f_{K_v,s_w}$. Thus, we have $\nu(f)_v \in \mathfrak{F}_{K_v}$. If $f_w = 1$ for all $w \in V_k$, then $\nu(f)_v = 1$ lies in $\mathfrak{F}_{K_v}$ for all $v \in V_K$ as well. We then deduce that $\nu(f) \in \mathfrak{F}_K$ in this case. □

Proposition 4.11. *Let $f = (f_v) \in \mathfrak{F}_K$. Write $f_v = f_{K_v,s_v}$ for $v \in M_K$. If*

(1) *$s_v = 1$ for all $v \in V_K$; and*
(2) *$s_v = s_{v_w}$ for all $v \in M_K$ and $w \in M_k$ with w lying below v,*

then we have $f = \nu(g)$ for some $g \in J(\Lambda(kG))$.

Proof. For each $w \in M_k$, define $g_w \in \Lambda(k_w G)^\times = \mathrm{Map}(G, k_w^\times)$ by

$$g_w(s) := \begin{cases} \pi_w & \text{if } s = s_{v_w} \neq 1 \\ 1 & \text{otherwise.} \end{cases}$$

We have $g := (g_w) \in J(\Lambda(kG))$ because $f \in J(\Lambda(KG))$ implies that $s_v = 1$ for all but finitely many $v \in M_K$. To prove that $f = \nu(g)$, let $v \in M_K$ and let $w \in M_k$ be the prime lying below v. If $s_{v_w} \neq 1$, then $s_v \neq 1$ by (2) and so $v \notin V_K$ by (1). Since $\pi_v = \varepsilon_v(\pi_w)$ by definition and $s_v = s_{v_w}$ by (2), we have $\nu(g)_v = f_{K_v,s_v}$. If $s_{v_w} = 1$, then $s_v = 1$ by (2) and $\nu(g)_v = 1 = f_{K_v,s_v}$. This shows that $f = \nu(g)$ and so $f \in \nu(J(\Lambda(kG)))$, as claimed. □

Next, we show that certain diagrams involving ν and μ commute.

Proposition 4.12. *The diagram*

$$\begin{tikzcd} \Lambda(kG)^\times \arrow[r, "\lambda_k"] \arrow[d, "\iota_\Lambda"'] & J(\Lambda(kG)) \arrow[d, "\nu"] \\ \Lambda(KG)^\times \arrow[r, "\lambda_K"'] & J(\Lambda(KG)) \end{tikzcd}$$

commutes, where ι_Λ is the map induced by the natural inclusion $k \longrightarrow K$.

Proof. Recall that λ_k and λ_K denote the diagonal maps. Let $g \in \Lambda(kG)^\times$. Also, let $v \in M_K$ and let $w \in M_k$ be the prime lying below v. Then

$$\begin{aligned} (\nu \circ \lambda_k)(g)_v &= \varepsilon_v \circ i_w \circ g \\ &= i_v \circ \overline{\gamma_v} \circ g \\ &= i_v \circ g \\ &= (\lambda_K \circ \iota_\Lambda)(g)_v, \end{aligned}$$

where $\varepsilon_v \circ i_w = i_v \circ \overline{\gamma_v}$ by Definition 4.3 and $\overline{\gamma_v} \circ g = g$ because g takes values in k. So, we have $\nu \circ \lambda_k = \lambda_K \circ \iota_\Lambda$, as desired. □

Proposition 4.13. *The diagram*

$$\begin{tikzcd} J(\Lambda(kG)) \arrow[r, "\nu"] \arrow[d, "\Theta_k^t"'] & J(\Lambda(KG)) \arrow[d, "\Theta_K^t"] \\ J(\mathcal{H}(kG)) \arrow[r, "\mu"'] & J(\mathcal{H}(KG)) \end{tikzcd}$$

commutes.

Proof. Let $g = (g_w) \in J(\Lambda(kG))$. Also, let $v \in M_K$ and let $w \in M_k$ be the prime lying below v. On the one hand, we have

$$(\Theta_K^t \circ \nu)(g)_v = \Theta_K^t(\varepsilon_v \circ g_w). \tag{4.3}$$

On the other hand, let $r_G(a_w) \in \mathcal{H}(k_wG)$ be such that $\Theta_k^t(g_w) = r_G(a_w)$. Then, we have

$$(\mu \circ \Theta_k^t)(g)_v = r_G(\varepsilon_v \circ a_w). \tag{4.4}$$

Now, let $\widehat{G}_w$ and $\widehat{G}_v$ be the groups of irreducible k_w^c- and K_v^c-valued characters on G, respectively. Recall the notation $A_{\widehat{G}_w}$ and $A_{\widehat{G}_v}$ in (3.10). Then, we have $r_G(a_w)(\psi) = \Theta_k^t(g_w)(\psi)$ for all $\psi \in A_{\widehat{G}_w}$ via the identification

$$\mathcal{H}(k_wG) = \mathrm{Hom}_{\Omega_{k_w}}(A_{\widehat{G}_w}, (k_w^c)^\times)$$

in (3.14). We will use the analogous identification

$$\mathcal{H}(K_vG) = \mathrm{Hom}_{\Omega_{K_v}}(A_{\widehat{G}_v}, (K_v^c)^\times)$$

in (3.14) to show that the expressions in (4.3) and (4.4) are equal.

To that end, let $\psi \in A_{\widehat{G}_v}$ and write $\psi = \sum_\chi n_\chi \chi$. Define

$$\varepsilon_v^{-1} \circ \psi := \sum_\chi n_\chi(\varepsilon_v^{-1} \circ \chi),$$

which clearly lies in $A_{\widehat{G}_w}$. Since $r_G(a_w) = \Theta_k^t(g_w)$, we deduce that

$$\begin{aligned} r_G(\varepsilon_v \circ a_w)(\psi) &= \varepsilon_v(r_G(a_w)(\varepsilon_v^{-1} \circ \psi)) \\ &= \varepsilon_v\left(\prod_{s\in G} g_w(s)^{\langle \varepsilon_v^{-1}\circ\psi, s\rangle}\right) \\ &= \prod_{s\in G}(\varepsilon_v \circ g_w)(s)^{\langle \psi, s\rangle} \\ &= \Theta_K^t(\varepsilon_v \circ g_w)(\psi), \end{aligned} \tag{4.5}$$

where the third equality follows because $\langle \varepsilon_v^{-1} \circ \psi, s\rangle = \langle \psi, s\rangle$ for all $s \in G$. To see why, note that it suffices to show that $\langle \varepsilon_v^{-1} \circ \chi, s\rangle = \langle \chi, s\rangle$ holds for all $\chi \in \widehat{G}_v$ and all $s \in G$. It will be helpful to recall that we chose the same compatible set $\{\zeta_n : n \in \mathbb{Z}^+\}$ of roots of unity in $\mathbb{Q}^c$ for k and K. Also, we chose $\{i_v(\zeta_n) : n \in \mathbb{Z}^+\}$ and $\{i_w(\zeta_n) : n \in \mathbb{Z}^+\}$ to be the compatible sets of roots of unity in K_v^c and k_w^c, respectively.

Now, let $\chi \in \widehat{G}_v$ and $s \in G$ be given. Let $\upsilon = \upsilon(\chi, s)$ be defined as in Definition 3.3. Then, we have $\chi(s) = i_v(\zeta_{|s|})^\upsilon$ and $\langle \chi, s\rangle = \upsilon/|s|$. We have

$$\begin{aligned}(\varepsilon_v^{-1} \circ \chi)(s) &= (\varepsilon_v^{-1} \circ i_v)(\zeta_{|s|})^\upsilon \\ &= (i_w \circ \overline{\gamma_v}^{-1})(\zeta_{|s|})^\upsilon \\ &= i_w(\zeta_{|s|})^\upsilon,\end{aligned}$$

where $\varepsilon_v^{-1} \circ i_v = i_w \circ \overline{\gamma_v}^{-1}$ by Definition 4.3 and $\overline{\gamma_v}^{-1}(\zeta_{|s|}) = \zeta_{|s|}$ because k contains all $\exp(G)$-th roots of unity. Then, by Definition 3.3, we see that $\langle \varepsilon_v^{-1} \circ \chi, s\rangle = \upsilon/|s|$ as well. Hence, the third equality in (4.5) indeed holds. It follows that (4.3) and (4.4) are equal, and so $\Theta_K^t \circ \nu = \mu \circ \Theta_k^t$. $\square$

4.4. Preliminary Definitions. In this subsection, assume that the given left Σ-action on G is trivial. Then, the left Ω_k-action on G induced by the natural quotient map $\Omega_k \longrightarrow \Sigma$ and the left Σ-action on G is trivial, which agrees with our definition in Subsection 1.1 above. Analogous to (2.1), from the Hochschild–Serre spectral sequence associated to the group extension

$$1 \longrightarrow \Omega_K \longrightarrow \Omega_k \longrightarrow \Sigma \longrightarrow 1,$$

we obtain an exact sequence

$$\text{(4.6)} \qquad \mathrm{Hom}(\Omega_k, G) \xrightarrow{\ \mathrm{res}\ } \mathrm{Hom}(\Omega_K, G)^\Sigma \xrightarrow{\ tr\ } H^2(\Sigma, G).$$

Here res is the obvious restriction map. The Σ-action on $\mathrm{Hom}(\Omega_K, G)$ and the *transgression map tr* are defined analogously as in Definition 2.2. More precisely, for each $\gamma \in \Sigma$, fix a lift $\overline{\gamma} \in \Omega_k$ of γ with $\overline{1} = 1$.

Definition 4.14. Given $h \in \mathrm{Hom}(\Omega_K, G)$ and $\gamma \in \Sigma$, define

$$(h \cdot \gamma)(\omega) := \gamma^{-1} \cdot h(\overline{\gamma}\omega\overline{\gamma}^{-1}) \quad \text{for all } \omega \in \Omega_K.$$

The *transgression map* $tr : \mathrm{Hom}(\Omega_K, G)^\Sigma \longrightarrow H^2(\Sigma, G)$ is defined by

$$tr(h) := [(\gamma, \delta) \mapsto h((\overline{\gamma})(\overline{\delta})(\overline{\gamma\delta})^{-1})],$$

where $[-]$ denotes the cohomology class. These definitions do not depend on the choice of the lifts $\overline{\gamma}$ for $\gamma \in \Sigma$.

If we regard $\mathrm{Hom}(\Omega_K^t, G)$ as the subset of $\mathrm{Hom}(\Omega_K, G)$ consisting of the tame homomorphisms via Remark 1.17, then the Σ-action on $\mathrm{Hom}(\Omega_K^t, G)$ and the transgression map tr on $\mathrm{Hom}(\Omega_K^t, G)^\Sigma$ induced by Definition 4.14 agree with those in Definition 2.2. In particular, the identical notation does not cause any confusion.

Definition 4.15. Define (recall Definition 3.1)

$$\mathcal{H}_\Sigma(KG) := \{r_G(a) \in \mathcal{H}(KG) \mid h_a \in \mathrm{Hom}(\Omega_K, G)^\Sigma\};$$
$$\mathcal{H}_s(KG) := \{r_G(a) \in \mathcal{H}(KG) \mid h_a \in \mathrm{Hom}(\Omega_K, G)^\Sigma \text{ and } tr(h_a) = 1\}.$$

It is clear that both of the sets above are subgroups of $\mathcal{H}(KG)$.

Proposition 4.16. *If k contains all $\exp(G)$-th roots of unity, then (recall Definitions 3.2 and 3.7)*

$$(\Theta_K^t \circ \nu)(\lambda_k(\Lambda(kG)^\times)) \subset \eta(\mathcal{H}_s(KG)).$$

Proof. Suppose that k contains all $\exp(G)$-th roots of unity. The map ν is then defined and results from Subsection 4.3 apply. Now, let $g \in \Lambda(kG)^\times$. Recall that $\iota_\Lambda : \Lambda(kG)^\times \longrightarrow \lambda(KG)^\times$ denotes the map induced by the natural inclusion $k \longrightarrow K$. Then, we have

$$(\Theta_K^t \circ \nu)(\lambda_k(g)) = (\Theta_K^t \circ \lambda_K)(\iota_\Lambda(g)) = (\eta \circ \Theta_K^t)(\iota_\Lambda(g)),$$

where $\nu \circ \lambda_k = \lambda_K \circ \iota_\Lambda$ by Proposition 4.12 and $\Theta_K^t \circ \lambda_K = \eta \circ \Theta_K^t$ because diagram (3.18) commutes. It remains to show that $\Theta_K^t(\iota_\Lambda(g)) \in \mathcal{H}_s(KG)$.

Recall that $\mathcal{H}(kG) = ((\mathbb{Q}^cG)^\times/G)^{\Omega_k}$ and $\mathcal{H}(KG) = ((\mathbb{Q}^cG)^\times/G)^{\Omega_K}$ by definition. From the identification in (3.14), we get a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\Omega_k}(A_{\widehat{G}}, (\mathbb{Q}^c)^\times) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\Omega_K}(A_{\widehat{G}}, (\mathbb{Q}^c)^\times) \arrow[d, equal] \\
\mathcal{H}(kG) \arrow[r, "\iota_{\mathcal{H}}"'] & \mathcal{H}(KG)
\end{tikzcd},$$

where the two horizontal maps are the obvious inclusions induced by the inclusion $\Omega_K \subset \Omega_k$. If $\Theta_k^t(g) = r_G(a)$, then clearly $\Theta_K^t(\iota_\Lambda(g)) = \iota_{\mathcal{H}}(r_G(a))$. So, the homomorphism h associated to $\Theta_K^t(\iota_\Lambda(g))$ is equal to $res(h_a)$, where h_a is the homomorphism associated to $r_G(a)$. Since (4.6) is exact, this shows that $tr(h) = 1$ and so $\Theta_K^t(\iota_\Lambda(g)) \in \mathcal{H}_s(KG)$. This proves the claim. $\square$

4.5. Proof of Theorem 1.3(1).

Proof. Let ρ_Σ denote the composition of the homomorphism rag from Definition 3.2 followed by the natural quotient map

$$J(\mathcal{H}(KG)) \longrightarrow \frac{J(\mathcal{H}(KG))}{\eta(\mathcal{H}_\Sigma(KG))U(\mathcal{H}(\mathcal{O}_KG))(\Theta_K^t \circ \nu)(J(\Lambda(kG)))}.$$

We will show that $R_\Sigma(\mathcal{O}_KG)_V$ is a subgroup of $\mathrm{Cl}(\mathcal{O}_KG)$ by showing that

$$j^{-1}(R_\Sigma(\mathcal{O}_KG)_V) = \ker(\rho_\Sigma), \tag{4.7}$$

or equivalently, for $c \in J(KG)$, we have $j(c) \in R_\Sigma(\mathcal{O}_KG)_V$ if and only if

$$rag(c) \in \eta(\mathcal{H}_\Sigma(KG))U(\mathcal{H}(\mathcal{O}_KG))(\Theta_K^t \circ \nu)(J(\Lambda(kG))). \tag{4.8}$$

To that end, let $c \in J(KG)$ be given. First, assume that (4.8) holds, so

$$rag(c) = \eta(r_G(b))^{-1}u(\Theta^t_K \circ \nu)(g) \tag{4.9}$$

for some $r_G(b) \in \mathcal{H}_\Sigma(KG)$, $u \in U(\mathcal{H}(\mathcal{O}_K G))$, and $g \in J(\Lambda(kG))$. Let $\mathfrak{m}$ be any ideal in $\mathcal{O}_k$. Then, by Theorem 3.14, there exists $f = (f_w) \in \mathfrak{F}_k$ such that $f_w = 1$ for all primes $w \in V_k$ and

$$g \equiv f \mod \lambda_k(\Lambda(kG)^\times)U'_{\mathfrak{m}}(\Lambda(\mathcal{O}_k G)).$$

Choosing $\mathfrak{m}$ to be divisible by $|G|$ and $\exp(G)^2$, the above then yields

$$\Theta^t_k(g) \equiv \Theta^t_k(f) \mod \Theta^t_k(\lambda_k(\Lambda(kG)^\times))U(\mathcal{H}(\mathcal{O}_k G))$$

by Theorem 3.13. Since $\mu \circ \Theta^t_k = \Theta^t_K \circ \nu$ by Proposition 4.13, applying μ to the above equation and using Proposition 4.16, we then obtain

$$(\Theta^t_K \circ \nu)(g) \equiv (\Theta^t_K \circ \nu)(f) \mod \eta(\mathcal{H}_s(KG))U(\mathcal{H}(\mathcal{O}_K G)). \tag{4.10}$$

Thus, by changing b and u in (4.9) if necessary, we may assume that $g = f$. Notice that $\nu(f)_v = 1$ for all $v \in V$ and that $\nu(f) \in \mathfrak{F}_K$ by Proposition 4.10. Hence, if $h := h_b$ is the homomorphism associated to $r_G(b)$, then h is tame with h_v unramified for all $v \in V$ and $j(c) = \mathrm{cl}(\mathcal{O}_h)$ by Theorem 3.10. Since $r_G(b) \in \mathcal{H}_\Sigma(KG)$, we know that h is Σ-invariant, and so $j(c) \in R_\Sigma(\mathcal{O}_K G)_V$.

Conversely, assume that $j(c) = \mathrm{cl}(\mathcal{O}_h)$ for some $h \in \mathrm{Hom}(\Omega^t_K, G)^\Sigma_V$, with $K_h = KG \cdot b$ say. By Theorem 3.10, we know that $j(c') = \mathrm{cl}(\mathcal{O}_h)$ for some idele $c' \in J(KG)$ such that

$$rag(c') = \eta(r_G(b))^{-1}u\Theta^t_K(f'), \tag{4.11}$$

where $u \in U(\mathcal{H}(\mathcal{O}_K G))$ and $f' = (f'_v) \in \mathfrak{F}_K$. In fact, for each $v \in M_K$, we have $f'_v = f'_{K_v,s_v}$ for $s_v = h_v(\sigma_{K_v})$, and $s_v = 1$ if $v \in V$. Since the Σ-action on G is trivial, we have $s_v = s_{v_w}$ for all $v \in M_K$ and $w \in M_k$ with w lying below v from Proposition 4.7. Proposition 4.11 then implies that $f' = \nu(g)$ for some $g \in J(\Lambda(kG))$. Since $j(c) = j(c')$, by Theorem 1.10, we have

$$c \equiv c' \mod \partial((KG)^\times)U(\mathcal{O}_K G).$$

Clearly $rag((KG)^\times) \subset \mathcal{H}_s(KG)$, and so we may rewrite (4.11) as

$$rag(c) = \eta(r_G(b)r_G(b'))^{-1}uu'(\Theta^t_K \circ \nu)(g) \tag{4.12}$$

for some $r_G(b') \in \mathcal{H}_s(KG)$ and $u' \in U(\mathcal{H}(\mathcal{O}_K G))$. Note that $r_G(b) \in \mathcal{H}_\Sigma(KG)$ because h is Σ-invariant, and so (4.8) holds. This proves (4.7). It remains to show the existence of $h' \in \mathrm{Hom}(\Omega^t_K, G)^\Sigma_V$ satisfying (1a) to (1d).

Let T be a finite set of primes in $\mathcal{O}_K$. First, note that the same discussion following (4.9) shows that there exists $f = (f_w) \in \mathfrak{F}_k$ satisfying (4.10). So, by changing b' and u' in (4.12) if necessary, we may assume that $g = f$. By Theorem 3.14, we may also assume that $f_w = 1$ for all $w \in M_k$ lying below the primes in $V \cup T$, and that $f_s \neq 1$ for all $s \in G \setminus \{1\}$ (note that Ω_k acts trivially on $G(-1)$ because k contains all $\exp(G)$-th roots of unity). Then, by Proposition 4.10, we see that $\nu(f)_v = 1$ for all $v \in V \cup T$ and $\nu(f) \in \mathfrak{F}_K$.

Now, let h' be the homomorphism associated to $r_G(b)r_G(b')$. From (4.12) and Theorem 3.10, we see that h' is tame with h'_v unramified for $v \in V \cup T$ and that $j(c) = \mathrm{cl}(\mathcal{O}_h)$, so both (2) and (3) hold. Because $r_G(b') \in \mathcal{H}_s(KG)$ and $h = h_b$ is Σ-invariant, clearly $h' \in \mathrm{Hom}(\Omega_K^t, G)_V^\Sigma$ and $tr(h') = tr(h)$, which proves (4). Finally, for each $s \in G\setminus\{1\}$, we have $f_s \neq 1$ by choice and so $f_w = f_{k_w,s}$ for some $w \in M_k$. If $v \in M_K$ is the prime lying above w, then $\nu(f)_v = f_{K_v,s}$ by Proposition 4.10 and thus $h'_v(\sigma_{K_v}) = s$ by Theorem 3.10. This means that h' is surjective and so $K_{h'}$ is a field, as claimed in (1).

Since gal is weakly multiplicative in the sense of (1.3), the above implies that $R_s(\mathcal{O}_KG)_V$ is closed under multiplication. Because $\mathrm{Cl}(\mathcal{O}_KG)$ is finite, it follows that $R_s(\mathcal{O}_KG)_V$ is also a subgroup of $\mathrm{Cl}(\mathcal{O}_KG)$. This completes the proof of the theorem. □

4.6. Structure of the Quotient Group $R_\Sigma(\mathcal{O}_KG)_V/R_s(\mathcal{O}_KG)_V$. Below, we assume all of the hypotheses in Theorem 1.3. The sets $R_\Sigma(\mathcal{O}_KG)_V$ and $R_s(\mathcal{O}_KG)_V$ are subgroups of $\mathrm{Cl}(\mathcal{O}_KG)$ by Theorem 1.3(1). The next proposition is a corollary of Theorem 1.3(1), and it relates the group structure of $R_\Sigma(\mathcal{O}_KG)_V/R_s(\mathcal{O}_KG)_V$ to that of $H^2(\Sigma, G)$.

Proposition 4.17. *Let $h, h_1, h_2 \in \mathrm{Hom}(\Omega_K^t, G)_V^\Sigma$.*

(1) *There exists $h_s \in \mathrm{Hom}(\Omega_K^t, G)_V^\Sigma$ with $tr(h_s) = 1$ such that*

$$\mathrm{cl}(\mathcal{O}_{h_1})\,\mathrm{cl}(\mathcal{O}_{h_2}) = \mathrm{cl}(\mathcal{O}_{h_1h_2h_s}).$$

(2) *We have $\mathrm{cl}(\mathcal{O}_h)\,\mathrm{cl}(\mathcal{O}_{h^{-1}}) \equiv 1 \mod R_s(\mathcal{O}_KG)_V$.*

(3) *If $tr(h_1) = tr(h_2)$, then $\mathrm{cl}(\mathcal{O}_{h_1}) \equiv \mathrm{cl}(\mathcal{O}_{h_2}) \mod R_s(\mathcal{O}_KG)_V$.*

Proof. By Theorem 1.3(1), there exists $h'_2 \in \mathrm{Hom}(\Omega_K^t, G)_V^\Sigma$ such that

$$\mathrm{cl}(\mathcal{O}_{h'_2}) = \mathrm{cl}(\mathcal{O}_{h_2}), \quad tr(h'_2) = tr(h_2), \quad d(h'_2) \cap d(h_1) = \emptyset$$

(recall (1.2)). Since gal is weakly multiplicative (recall (1.3)), we obtain

$$\mathrm{cl}(\mathcal{O}_{h_1})\,\mathrm{cl}(\mathcal{O}_{h_2}) = \mathrm{cl}(\mathcal{O}_{h_1h'_2}) = \mathrm{cl}(\mathcal{O}_{h_1h_2h_s}),$$

where $h_s := h_2^{-1}h'_2$ and clearly $tr(h_s) = 1$. This proves (1), and (2) follows from applying (1) to $h_1 = h$ and $h_2 = h^{-1}$. Note that (1) and (2) together imply that there exists $h_s \in \mathrm{Hom}(\Omega_K^t, G)_V^\Sigma$ with $tr(h_s) = 1$ such that

$$\mathrm{cl}(\mathcal{O}_{h_1})\,\mathrm{cl}(\mathcal{O}_{h_2})^{-1} \equiv \mathrm{cl}(\mathcal{O}_{h_1})\,\mathrm{cl}(\mathcal{O}_{h_2^{-1}}) \equiv \mathrm{cl}(\mathcal{O}_{h_1h_2^{-1}h_s}) \mod R_s(\mathcal{O}_KG)_V.$$

Hence, if $tr(h_1) = tr(h_2)$, then $tr(h_1h_2^{-1}h_s) = 1$ and thus $\mathrm{cl}(\mathcal{O}_{h_1}) \equiv \mathrm{cl}(\mathcal{O}_{h_2})$ mod $R_s(\mathcal{O}_KG)_V$, as desired. □

4.7. Proof of Theorem 1.3(2).

Proof. The map ϕ is well-defined by Proposition 4.17(3). Next, by Proposition 4.17(1), given any $h_1, h_2 \in \mathrm{Hom}(\Omega_K^t, G)_V^{\Sigma}$, we have

$$\mathrm{cl}(\mathcal{O}_{h_1})\,\mathrm{cl}(\mathcal{O}_{h_2}) = \mathrm{cl}(\mathcal{O}_{h_1 h_2 h_s})$$

for some $h_s \in \mathrm{Hom}(\Omega_K^t, G)_V^{\Sigma}$ with $tr(h_s) = 1$. This implies that

$$\phi(tr(h_1))\phi(tr(h_2)) = \phi(tr(h_1 h_2 h_s)) = \phi(tr(h_1 h_2))$$

and so ϕ is indeed a homomorphism. This proves the first claim.

Next, let $h \in \mathrm{Hom}(\Omega_K^t, G)_V^{\Sigma}$ be such that $tr(h) \in \ker(\phi)$. In other words, we have $\mathrm{cl}(\mathcal{O}_h) \in R_s(\mathcal{O}_K G)_V$. By Theorem 1.3(1), we know that

$$\mathrm{cl}(\mathcal{O}_h)^{-1} = \mathrm{cl}(\mathcal{O}_{h_s})$$

for some $h_s \in \mathrm{Hom}(\Omega_K^t, G)_V^{\Sigma}$ with $tr(h_s) = 1$. In particular, we may assume that $d(h_s) \cap d(h) = \emptyset$ (recall (1.2)). Since gal is weakly multiplicative (recall (1.3)), we then deduce that

$$1 = \mathrm{cl}(\mathcal{O}_h)\,\mathrm{cl}(\mathcal{O}_{h_s}) = \mathrm{cl}(\mathcal{O}_{h h_s}).$$

Now, recall Theorem 2.11. Because ξ is a homomorphism, the above implies that $(\xi \circ \mathrm{gal})(h h_s) = 1$. Since the basic diagram (1.4) is commutative, this yields $(i^* \circ tr)(h h_s) = 1$. Hence, if i^* is injective, then $tr(h h_s) = 1$ and so $tr(h) = 1$. In this case, the map ϕ is injective and so is an isomorphism. ☐

5. Acknowledgments

I would like to thank my advisor Professor Adebisi Agboola for bringing this problem to my attention and for reading a draft of this paper. I would also like to thank the referee for reading the manuscript very carefully and for providing many detailed comments which helped improve the quality of the paper significantly.

References

[1] J. Brinkhuis, "Normal integral bases and embedding problems", *Math. Ann.* **264** (1983), p. 537-543.

[2] ———, "Galois modules and embedding problems", *J. Reine Angew. Math.* **346** (1984), p. 141-165.

[3] C. W. Curtis & I. Reiner, *Methods of representation theory with applications to finite groups and orders. Vol. I*, Pure and Applied Mathematics, John Wiley & Sons, 1981, xxi+819 pages.

[4] ———, *Methods of representation theory with applications to finite groups and orders. Vol. II*, Pure and Applied Mathematics, John Wiley & Sons, 1987, xv+951 pages.

[5] B. Erez, "The Galois structure of the square root of the inverse different", *Math. Z.* **208** (1991), no. 2, p. 239-255.

[6] A. Fröhlich, "Local fields", in *Algebraic number theory*, Academic Press, 1967.

[7] L. R. McCulloh, "Galois module structure of abelian extensions", *J. Reine Angew. Math.* **375/376** (1987), p. 259-306.

[8] J. Neukirch, A. Schmidt & K. Wingberg, *Cohomology of Number Fields*, 2nd ed., Grundlehren der Mathematischen Wissenschaften, vol. 323, Springer, 2008, xv+825 pages.

[9] J.-P. SERRE, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer, 1979, viii+241 pages.
[10] C. TSANG, "On the Galois module structure of the square root of the inverse different in abelian extensions", *J. Number Theory* **160** (2016), p. 759-804.
[11] ———, "On the Galois module structure of the square root of the inverse different in abelian extensions", PhD Thesis, University of California (USA), 2016.
[12] S. V. ULLOM, "Integral normal bases in Galois extensions of local fields", *Nagoya Math. J.* **39** (1970), p. 141-148.

Cindy (Sin Yi) TSANG
Yau Mathematical Sciences Center
Tsinghua University
Haidian District
Beijing 100084, P.R. China
E-mail: sinyitsang@math.tsinghua.edu.cn
URL: http://sites.google.com/site/cindysinyitsang/