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Ranks For Two Partition Quadruple Functions

par Chris JENNINGS-SHAFFER

Résumé. L'auteur a récemment introduit deux fonctions de partitions entières qui satisfont des congruences du type Ramanujan modulo 3, 5, 7, et 13. On definit une statistique du type rang et obtient une amélioration de l'interprêtation combinatoire des congruences modulo 3, 5 et 7.

Abstract. Recently the author introduced two new integer partition quadruple functions, which satisfy Ramanujan-type congruences modulo 3, 5, 7, and 13. Here we reprove the congruences modulo 3, 5, and 7 by defining a rank-type statistic that gives a combinatorial refinement of the congruences.

1. Introduction and Statement of Results

In [8] the author introduced the two partition quadruple functions $u(n)$ and $v(n)$. We recall a partition of a positive integer n is a non-increasing sequence of positive integers that sum to n. For example the partitions of 5 are exactly 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, and 1+1+1+1+1. We say a quadruple $(\pi_1, \pi_2, \pi_3, \pi_4)$ of partitions is a partition quadruple of n if altogether the parts of π_1, π_2, π_3, and π_4 sum to n. For a partition π, we let $s(\pi)$ denote the smallest part of π and $\ell(\pi)$ denote the largest part of π. We use the conventions that the empty partition has smallest part ∞ and largest part 0. We let $u(n)$ denote the number of partition quadruples $(\pi_1, \pi_2, \pi_3, \pi_4)$ of n such that π_1 is non-empty, $s(\pi_1) \leq s(\pi_2)$, $s(\pi_1) \leq s(\pi_3)$, $s(\pi_1) \leq s(\pi_4)$, and $\ell(\pi_4) \leq 2s(\pi_1)$. We let $v(n)$ denote the number of partition quadruples $(\pi_1, \pi_2, \pi_3, \pi_4)$ counted by $u(n)$, where additionally the smallest part of π_1 appears at least twice. As example, $u(3) = 15$ as the relevant partition quadruples are $(3, -, -, -)$, $(2+1, -, -, -)$, $(1+1+1, -, -, -)$, $(1+1, 1, -, -)$, $(1+1, -, 1, -)$, $(1+1, -, -, 1)$, $(1, 2, -, -)$, $(1, -, 2, -)$, $(1, -, -, 2)$, $(1, 1+1, -, -)$, $(1, -, 1+1, -)$, $(1, -, -, 1+1)$, $(1, 1, 1, -)$, $(1, 1, -, 1)$, and $(1, -, 1, 1)$. Also from this list we see that $v(3) = 4$. In [8] we proved the following congruences for $u(n)$ and $v(n)$.

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Theorem 1.1.

$$\begin{aligned}
u(3n) &\equiv 0 \pmod{3},\\
u(5n) &\equiv 0 \pmod{5},\\
u(5n+3) &\equiv 0 \pmod{5},\\
u(7n) &\equiv 0 \pmod{7},\\
u(7n+5) &\equiv 0 \pmod{7},\\
u(13n) &\equiv 0 \pmod{13},\\
v(3n+1) &\equiv 0 \pmod{3},\\
v(5n+1) &\equiv 0 \pmod{5},\\
v(5n+4) &\equiv 0 \pmod{5},\\
v(13n+10) &\equiv 0 \pmod{13}.
\end{aligned}$$

The proof was to use various identities between products and generalized Lambert series to determine formulas modulo ℓ for the ℓ-dissections of generating functions for $u(n)$ and $v(n)$, with the appropriate terms of the dissections being zero.

We use the standard product notation,

$$\begin{aligned}
(z;q)_n &= \prod_{j=0}^{n-1}(1-zq^j),\\
(z;q)_\infty &= \prod_{j=0}^{\infty}(1-zq^j),\\
(z_1,\dots,z_k;q)_n &= (z_1;q)_n \dots (z_k;q)_n\,,\\
(z_1,\dots,z_k;q)_\infty &= (z_1;q)_\infty \dots (z_k;q)_\infty\,,\\
[z;q]_\infty &= (z,q/z;q)_\infty\,, \qquad [z_1,\dots,z_k;q]_\infty = [z_1;q]_\infty \dots [z_k;q]_\infty\,.
\end{aligned}$$

By summing according to n being the smallest part of the partition π_1, we find that generating functions for $u(n)$ and $v(n)$ are given by

$$\begin{aligned}
U(q) &= \sum_{n=0}^{\infty} u(n)q^n = \sum_{n=1}^{\infty} \frac{q^n}{(q^n;q)_\infty\,(q^n;q)_\infty\,(q^n;q)_\infty\,(q^n;q)_{n+1}}\\
&= q + 5q^2 + 15q^3 + 44q^4 + 105q^5 + 252q^6 + 539q^7 + 1135q^8 + \dots,\\
V(q) &= \sum_{n=0}^{\infty} v(n)q^n = \sum_{n=1}^{\infty} \frac{q^{2n}}{(q^n;q)_\infty\,(q^n;q)_\infty\,(q^n;q)_\infty\,(q^n;q)_{n+1}}\\
&= q^2 + 4q^3 + 15q^4 + 39q^5 + 105q^6 + 237q^7 + 530q^8 + 1100q^9 + \dots.
\end{aligned}$$

Recently in private correspondence Garvan conjectured that one could use the series

$$\sum_{n=1}^{\infty}\frac{q^n\left(q^n,q^n,q^{2n+1};q\right)_\infty}{\left(q^{3n};q^3\right)_\infty^2},\qquad \sum_{n=1}^{\infty}\frac{q^n\left(q^n,q^{2n+1};q\right)_\infty}{\left(q^{5n};q^5\right)_\infty},$$
$$\sum_{n=1}^{\infty}\frac{q^n\left(q^n,\zeta_7^3q^n,\zeta_7^4q^n,q^{2n+1};q\right)_\infty}{\left(q^{7n};q^7\right)_\infty},$$

where ζ_ℓ is a primitive ℓ^{th} root of unity, as rank functions to prove the congruences $u(3n)\equiv 0 \pmod 3$, $u(5n)\equiv u(5n+3)\equiv 0 \pmod 5$, and $u(7n)\equiv u(7n+5)\equiv 0 \pmod 7$ respectively. These functions correspond to the $z=\zeta_3$, ζ_5, and ζ_7 cases of $F(z^2,z^{-2},z;q)$, where $F(\rho_1,\rho_2,z;q)$ is a function the author studied in [9] defined by

$$F(\rho_1,\rho_2,z;q)=\frac{(q;q)_\infty}{\left(z,z^{-1},\rho_1,\rho_2;q\right)_\infty}\sum_{n=1}^{\infty}\frac{\left(z,z^{-1},\rho_1,\rho_2;q\right)_n\left(\frac{q}{\rho_2\rho_2}\right)^n}{(q;q)_{2n}}.$$

In this article we give the proof of this as well as give the corresponding rank function for $V(q)$. We let $RU(z,q)=F(z^2,z^{-2},z;q)$ and $RV(z,q)=G(z^2,z^{-2},z;q)$, where

$$G(\rho_1,\rho_2,z;q)=\frac{(q;q)_\infty}{\left(z,z^{-1},\rho_1,\rho_2;q\right)_\infty}\sum_{n=1}^{\infty}\frac{\left(z,z^{-1},\rho_1,\rho_2;q\right)_n\left(\frac{q^2}{\rho_2\rho_2}\right)^n}{(q;q)_{2n}}.$$

We prove the following identities for these functions.

Theorem 1.2. *Let ζ_ℓ denote a primitive ℓ^{th} root of unity. Then*

$$RU(\zeta_3,q)=\frac{q^7}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{\frac{9n^2+27n}{2}}}{(1-q^{9n+6})} - \frac{q^5}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{\frac{9n^2+21n}{2}}}{(1-q^{9n+6})}, \tag{1.1}$$

$$RV(\zeta_3,q)=-\frac{q^3}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{\frac{9n^2+15n}{2}}}{(1-q^{9n+6})} + \frac{q^5}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{\frac{9n^2+21n}{2}}}{(1-q^{9n+6})}, \tag{1.2}$$

$$RU(\zeta_5,q)=\frac{q\left(q^{25};q^{25}\right)_\infty}{\left[q^5;q^{25}\right]_\infty^2} - \frac{q^7}{\left(q^{25};q^{25}\right)_\infty\left[q^{10};q^{25}\right]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{\frac{25n^2+45n}{2}}}{1-q^{25n+10}} - \frac{q^4}{\left(q^{25};q^{25}\right)_\infty\left[q^5;q^{25}\right]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{\frac{25n^2+35n}{2}}}{1-q^{25n+10}}, \tag{1.3}$$

$$RV(\zeta_5,q)=-\frac{q^5}{\left(q^{25};q^{25}\right)_\infty\left[q^5;q^{25}\right]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{\frac{25n^2+35n}{2}}}{1-q^{25n+15}} + \frac{q^{12}}{\left(q^{25};q^{25}\right)_\infty\left[q^{10};q^{25}\right]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{\frac{25n^2+55n}{2}}}{1-q^{25n+15}} + \frac{q^2\left(q^{25};q^{25}\right)_\infty}{\left[q^5,q^{10};q^{25}\right]_\infty} - \frac{q^3\left(q^{25};q^{25}\right)_\infty}{\left[q^{10};q^{25}\right]_\infty^2}, \tag{1.4}$$

$$RU(\zeta_7,q)=\frac{q\left(q^{49};q^{49}\right)_\infty\left[q^{21};q^{49}\right]_\infty}{\left[q^7;q^{49}\right]_\infty\left[q^{14};q^{49}\right]_\infty^2} - \frac{(\zeta_7^2+\zeta_7^5)q^{15}}{\left(q^{49};q^{49}\right)_\infty\left[q^{21};q^{49}\right]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{\frac{49n^2+91n}{2}}}{1-q^{49n+21}} - \frac{(\zeta_7^3+\zeta_7^4)q^2\left(q^{49};q^{49}\right)_\infty}{\left[q^7,q^{14};q^{49}\right]_\infty} + \frac{q^3\left(q^{49};q^{49}\right)_\infty}{\left[q^7,q^{21};q^{49}\right]_\infty} + \frac{(\zeta_7+\zeta_7^6)q^4\left(q^{49};q^{49}\right)_\infty}{\left[q^{14};q^{49}\right]_\infty^2} + \frac{(\zeta_7+\zeta_7^6)q^{11}}{\left(q^{49};q^{49}\right)_\infty\left[q^{14};q^{49}\right]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{\frac{49n^2+77n}{2}}}{1-q^{49n+21}} + \frac{(1+\zeta_7^3+\zeta_7^4)q^6\left(q^{49};q^{49}\right)_\infty}{\left[q^{21};q^{49}\right]_\infty^2} - \frac{(1+\zeta_7^3+\zeta_7^4)q^6}{\left(q^{49};q^{49}\right)_\infty\left[q^7;q^{49}\right]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{\frac{49n^2+63n}{2}}}{1-q^{49n+21}}. \tag{1.5}$$

The congruences modulo 3, 5, and 7 of Theorem 1.1 are a corollary to Theorem 1.2 by the standard argument of ranks and cranks. We let

$$RU(z,q)=\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}ru(m,n)z^mq^n, \quad RV(z,q)=\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}rv(m,n)z^mq^n,$$

$$ru(k,\ell,n) = \sum_{m\equiv k \pmod{\ell}} ru(m,n), \quad rv(k,\ell,n) = \sum_{m\equiv k \pmod{\ell}} rv(m,n).$$

Since $U(q) = RU(1,q)$ and $V(q) = RV(1,q)$ we have that

$$u(n) = \sum_{k=0}^{\ell-1} ru(k,\ell,n), \qquad v(n) = \sum_{k=0}^{\ell-1} rv(k,\ell,n).$$

Also we see that

$$RU(\zeta_\ell, q) = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\ell-1} ru(k,\ell,n)\zeta_\ell^k \right) q^n,$$
$$RV(\zeta_\ell, q) = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\ell-1} rv(k,\ell,n)\zeta_\ell^k \right) q^n.$$

If ℓ is prime and the coefficient of q^N in $RU(\zeta_\ell, q)$ is zero, then

$$ru(0,\ell,N) + ru(1,\ell,N)\zeta_\ell + \cdots + ru(\ell-1,\ell,N)\zeta_\ell^{\ell-1} = 0,$$

and so

$$ru(0,\ell,N) = ru(1,\ell,N) = \cdots = ru(\ell-1,\ell,N),$$

as the minimal polynomial for ζ_ℓ is $1 + x + \cdots + x^{\ell-1}$. Thus $ru(N) = \ell \cdot ru(k,\ell,N) \equiv 0 \pmod{\ell}$. By Theorem 1.2 the coefficients of q^{3n} in $RU(\zeta_3, q)$; q^{5n} and q^{5n+3} in $RU(\zeta_5, q)$; and q^{7n} and q^{7n+5} in $RU(\zeta_7, q)$ are all zero which yields the congruences for $u(n)$. The explanation for $rv(n)$ is similar.

It is worth noting that the $ru(k,\ell,N)$ being equal and the $rv(k,\ell,N)$ being equal is a stronger result than the congruences alone. Also the $ru(m,n)$ and $rv(m,n)$ give statistics on the partition quadruples counted by $u(n)$ and $v(n)$ that yield a combinatorial refinement of the congruences. For this we recall that $s(\pi)$ is the smallest part of π and let $\#(\pi)$ denote the number of a parts of π. For a quadruple $(\pi_1, \pi_2, \pi_3, \pi_4)$ we let $\omega(\pi_1, \pi_2, \pi_3, \pi_4)$ denote the number of parts of π_1 that are either $s(\pi_1)$ or are larger than $s(\pi_1) + \#(\pi_4)$. We note that if $\#(\pi_4) = 0$, then $\omega(\pi) = \#(\pi_1)$. We define the u-rank and v-rank of $(\pi_1, \pi_2, \pi_3, \pi_4)$ by

$$u\text{-rank}(\pi_1,\pi_2,\pi_3,\pi_4) = \omega(\pi_1,\pi_2,\pi_3,\pi_4) - 1 + 2\#(\pi_2) - 2\#(\pi_3) - \#(\pi_4),$$
$$v\text{-rank}(\pi_1,\pi_2,\pi_3,\pi_4) = \omega(\pi_1,\pi_2,\pi_3,\pi_4) - 2 + 2\#(\pi_2) - 2\#(\pi_3) - \#(\pi_4).$$

We will prove the following.

Theorem 1.3. *Let U denote the set of partitions quadruples counted by $u(n)$, that is to say,*

$$U = \{(\pi_1, \pi_2, \pi_3, \pi_4) : \ell(\pi_4) \leq 2s(\pi_1) < \infty \quad \text{and } s(\pi_1) \leq s(\pi_i) \text{ for } i = 2, 3, 4\}.$$

In the same fashion let V denote the set of partitions quadruples counted by $v(n)$. Then $ru(m, n)$ is the number of partition quadruples from U of n with u-rank equal to m and $rv(m, n)$ is the number of partition quadruples from V of n with v-rank equal to m. Furthermore,

(i) *the residue of the u-rank mod 3 divides the partition quadruples from U of $3n$ into 3 equal classes,*
(ii) *the residue of the v-rank mod 3 divides the partition quadruples from V of $3n + 1$ into 3 equal classes,*
(iii) *the residue of the u-rank mod 5 divides the partition quadruples from U of $5n$ and of $5n + 3$ into 5 equal classes,*
(iv) *the residue of the v-rank mod 5 divides the partition quadruples from V of $5n + 1$ and of $5n + 4$ into 5 equal classes, and*
(v) *the residue of the u-rank mod 7 divides the partition quadruples from U of $7n$ and of $7n + 5$ into 7 equal classes.*

As example of Theorem 1.3, we have $u(3) \equiv 0 \pmod 3$ and $u(3) \equiv 0 \pmod 5$ by considering the following table of values:

π_1	π_2	π_3	π_4	ω	u-rank	(mod 3)	(mod 5)
3	–	–	–	1	0	0	0
$2+1$	–	–	–	2	1	1	1
$1+1+1$	–	–	–	3	2	2	2
$1+1$	1	–	–	2	3	0	3
$1+1$	–	1	–	2	-1	2	4
$1+1$	–	–	1	2	0	0	0
1	2	–	–	1	2	2	2
1	–	2	–	1	-2	1	3
1	–	–	2	1	-1	2	4
1	$1+1$	–	–	1	4	1	4
1	–	$1+1$	–	1	-4	2	1
1	–	–	$1+1$	1	-2	1	3
1	1	1	–	1	0	0	0
1	1	–	1	1	1	1	1
1	–	1	1	1	-3	0	2

Two remarks are in order. The first is that Theorem 1.3 shows that we are doing something more than reproving some of the congruences for $u(n)$ and $v(n)$. Previously we simply knew the congruences held, whereas now we have a much stronger refinement. This refinement fits into the rich framework of partition rank and cranks. To review the history of this subject, one should consult the works in [2, 3, 4, 6, 7]. The second is that the proof of Theorem 1.2 is considerably easier than the original proofs of the congruences. One possible explanation for this is that it is not entirely clear what kind of functions are $U(q)$ and $V(q)$; are they modular, mock modular, or quasi-mock modular? However given a form of $RU(z,q)$ and $RV(z,q)$ in terms of generalized Lambert series, it clear by the works of Zwegers [10] that for z a root of unity, other than ±1, they are mock modular forms.

In Section 2 we give a few preliminary identities necessary for the proof of Theorem 1.2, in Section 3 we prove Theorem 1.2, in Section 4 we prove Theorem 1.3, and in Section 5 we give a few concluding remarks.

2. Preliminaries

We begin with expressing the two rank functions as generalized Lambert series.

Proposition 2.1. *We have*

$$RU(z,q) = \frac{1}{(1+z)\,(q,z,z^{-1};q)_\infty}\sum_{j=-\infty}^{\infty}\frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^j q^{\frac{j(j+3)}{2}}}{(1-z^2q^j)(1-z^{-2}q^j)}, \tag{2.1}$$

$$RV(z,q) = \frac{1}{(1+z)\,(q,z,z^{-1};q)_\infty}\sum_{j=-\infty}^{\infty}\frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^j q^{\frac{j(j+1)}{2}}}{(1-z^2q^j)(1-z^{-2}q^j)}. \tag{2.2}$$

Proof. By [9, Theorem 2.3] we have that

$$\begin{aligned}
&(1+z)\left(z,z^{-1};q\right)_\infty F(\rho_1,\rho_2,z;q)\\
&= \frac{1}{\left(\rho_1,\rho_2,\frac{q}{\rho_1\rho_2};q\right)_\infty}\sum_{j=1}^{\infty}(1-z^j)(1-z^{j-1})z^{1-j}\rho_1^{1-j}\rho_2^{1-j}(-1)^{j+1}q^{\frac{j(j-1)}{2}}\\
&\quad\times(\rho_1,\rho_2;q)_{j-1}\left(\rho_1^{-1}q^{j+1},\rho_2^{-1}q^{j+1};q\right)_\infty\\
&\quad\times(1-\rho_1^{-1}q^j-\rho_2^{-1}q^j+\rho_1^{-1}q^{3j-1}+\rho_2^{-1}q^{3j-1}-q^{4j-2}),\\
&(1+z)\left(z,z^{-1};q\right)_\infty G(\rho_1,\rho_2,z;q)\\
&= \frac{1}{\left(\rho_1,\rho_2,\frac{q^2}{\rho_1\rho_2};q\right)_\infty}\sum_{j=1}^{\infty}(1-z^j)(1-z^{j-1})z^{1-j}\rho_1^{1-j}\rho_2^{1-j}(-1)^{j+1}q^{\frac{j(j+1)}{2}-1}\\
&\quad\times(1-q^{2j-1})\,(\rho_1,\rho_2;q)_{j-1}\left(\rho_1^{-1}q^{j+1},\rho_2^{-1}q^{j+1};q\right)_\infty.
\end{aligned}$$

With $\rho_2 = \rho_1^{-1}$ and $\rho_1 = z^2$ we find the above reduces to

$$\begin{aligned}
&(1+z)\left(z,z^{-1};q\right)_\infty RU(z,q)\\
&\quad= \frac{1}{(q;q)_\infty}\sum_{j=1}^{\infty}\frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^{j+1}q^{\frac{j(j-1)}{2}}}{(1-z^2q^{j-1})(1-z^{-2}q^{j-1})(1-z^2q^j)(1-z^{-2}q^j)}\\
&\qquad\quad\times(1-z^2q^j-z^{-2}q^j+z^2q^{3j-1}+z^{-2}q^{3j-1}-q^{4j-2}),\\
&(1+z)\left(z,z^{-1};q\right)_\infty RV(z,q)\\
&\quad= \frac{1}{(q;q)_\infty}\sum_{j=1}^{\infty}\frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^{j+1}q^{\frac{j(j+1)}{2}-1}(1-q)(1-q^{2j-1})}{(1-z^2q^{j-1})(1-z^{-2}q^{j-1})(1-z^2q^j)(1-z^{-2}q^j)}.
\end{aligned}$$

We note that

$$\begin{aligned}
&\frac{1-z^2q^j-z^{-2}q^j+z^2q^{3j-1}+z^{-2}q^{3j-1}-q^{4j-2}}{(1-z^2q^{j-1})(1-z^{-2}q^{j-1})(1-z^2q^j)(1-z^{-2}q^j)}\\
&\qquad= \frac{1}{(1-z^2q^{j-1})(1-z^{-2}q^{j-1})}-\frac{q^{2j}}{(1-z^2q^j)(1-z^{-2}q^j)},\\
&\frac{(1-q)(1-q^{2j-1})}{(1-z^2q^{j-1})(1-z^{-2}q^{j-1})(1-z^2q^j)(1-z^{-2}q^j)}\\
&\qquad= \frac{1}{(1-z^2q^{j-1})(1-z^{-2}q^{j-1})}-\frac{q}{(1-z^2q^j)(1-z^{-2}q^j)},
\end{aligned}$$

so that

$$\begin{aligned}
&(1+z)\left(z,z^{-1};q\right)_\infty RU(z,q)\\
&\qquad= \frac{1}{(q;q)_\infty}\sum_{j=1}^{\infty}\frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^{j+1}q^{\frac{j(j-1)}{2}}}{(1-z^2q^{j-1})(1-z^{-2}q^{j-1})}\\
&\qquad\quad - \frac{1}{(q;q)_\infty}\sum_{j=1}^{\infty}\frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^{j+1}q^{\frac{j(j+3)}{2}}}{(1-z^2q^j)(1-z^{-2}q^j)}\\
&\qquad= \frac{1}{(q;q)_\infty}\sum_{j=-\infty}^{\infty}\frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^{j}q^{\frac{j(j+3)}{2}}}{(1-z^2q^j)(1-z^{-2}q^j)},\\
&(1+z)\left(z,z^{-1};q\right)_\infty RV(z,q)\\
&\qquad= \frac{1}{(q;q)_\infty}\sum_{j=1}^{\infty}\frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^{j+1}q^{\frac{j(j+1)}{2}-1}}{(1-z^2q^{j-1})(1-z^{-2}q^{j-1})}\\
&\qquad\quad - \frac{1}{(q;q)_\infty}\sum_{j=1}^{\infty}\frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^{j+1}q^{\frac{j(j+1)}{2}}}{(1-z^2q^j)(1-z^{-2}q^j)}
\end{aligned}$$

$$= \frac{1}{(q;q)_\infty} \sum_{j=-\infty}^{\infty} \frac{(1-z^j)(1-z^{j-1})z^{1-j}(-1)^j q^{\frac{j(j+1)}{2}}}{(1-z^2q^j)(1-z^{-2}q^j)}. \qquad \square$$

The proof of Theorem 1.2 will be to replace n by $\ell n + k$, with $k = 0, 1, \ldots, \ell - 1$ in the series in (2.1) and (2.1) and find cancellations between the resulting terms. For this, we let

$$T(a,b,\ell) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{\ell^2 n(n+1)}{2} + \ell b n}}{(1-q^{\ell^2 n + \ell a})}.$$

By letting $n \mapsto -n$ we have the useful fact that $T(-a,b,\ell) = -q^{\ell a} T(a,-b,\ell)$. Here and in the proof of Theorem 1.2 we use the notation

$$E(a) = (q^a;q^a)_\infty, \qquad\qquad P(a) = \left[q^{\ell a}; q^{\ell^2}\right]_\infty,$$

where ℓ will always be clear from the context. We note that $P(\ell - a) = P(a)$ and $P(\ell + a) = P(-a) = -q^{-\ell a} P(a)$. The $r = 1$ and $s = 2$ case of [5, Theorem 2.1] with $q \mapsto q^{\ell^2}$, $a_1 \mapsto q^{\ell a}$, $b_1 \mapsto q^{\ell b_1}$, and $b_1 \mapsto q^{\ell b_2}$ gives

$$\frac{P(a)E(\ell^2)^2}{P(b_1)P(b_2)} = \frac{P(a-b_1)}{P(b_2-b_1)} T(b_1, a-b_2, \ell) + \frac{P(a-b_2)}{P(b_1-b_2)} T(b_2, a-b_1, \ell),$$

so that

$$(2.3)\quad T(b_2, a-b_1, \ell) = q^{\ell(b_1-b_2)} \frac{P(a-b_1)}{P(a-b_2)} T(b_1, a-b_2, \ell) - q^{\ell(b_1-b_2)} \frac{P(a)P(b_2-b_1)E(\ell^2)^2}{P(b_1)P(b_2)P(a-b_2)}.$$

By setting $b_1 = -b$, $b_2 = b$, and using that $T(-b, a-b, \ell) = -q^{\ell b} T(b, b-a, \ell)$, we also deduce that

$$(2.4)\quad T(b, a+b, \ell) = -q^{-\ell b} \frac{P(a+b)}{P(a-b)} T(b, b-a, \ell) + q^{-\ell b} \frac{P(a)P(2b)E(\ell^2)^2}{P(b)^2 P(a-b)}.$$

Additionally we need an identity for $\left(q, \zeta_\ell, \zeta_\ell^{-1}; q\right)_\infty$. Using the Jacobi triple product identity,

$$\left(zq, z^{-1}, q; q\right)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{\frac{n(n+1)}{2}},$$

we easily deduce that for odd ℓ we have

$$(2.5)\quad \left(q, \zeta_\ell, \zeta_\ell^{-1}; q\right)_\infty = (1-\zeta_\ell) E(\ell^2) \sum_{k=0}^{\frac{\ell-3}{2}} (-1)^k (\zeta_\ell^k - \zeta_\ell^{-k-1}) q^{\frac{k(k+1)}{2}} P(\tfrac{\ell-1}{2} - k).$$

3. Proof of Theorem 1.2

Proof of (1.1). By (2.1) we have

$$
\begin{aligned}
&RU(\zeta_3, q)\\
&\quad= \frac{1}{(1+\zeta_3)\left(q,\zeta_3,\zeta_3^{-1};q\right)_\infty} \sum_{j=-\infty}^{\infty} \frac{(1-\zeta_3^j)(1-\zeta_3^{j-1})\zeta_3^{1-j}(-1)^j q^{\frac{j(j+3)}{2}}}{(1-\zeta_3^2 q^j)(1-\zeta_3^{-2}q^j)}\\
&\quad= \frac{1}{(1+\zeta_3)(1-\zeta_3)(1-\zeta_3^{-1})\,(q^3;q^3)_\infty}\\
&\qquad\times \sum_{j=-\infty}^{\infty} \frac{(1-\zeta_3^2)(1-\zeta_3)\zeta_3^{-1}(-1)^j q^{\frac{9j^2+21j}{2}+5}}{(1-\zeta_3^2 q^{3j+2})(1-\zeta_3^{-2}q^{3j+2})}\\
&\quad= \frac{-1}{(q^3;q^3)_\infty} \sum_{j=-\infty}^{\infty} \frac{(-1)^j q^{\frac{9j^2+21j)}{2}+5}(1-q^{3j+2})}{(1-q^{9j+6})}\\
&\quad= \frac{q^7}{(q^3;q^3)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^j q^{\frac{9j^2+27j}{2}}}{(1-q^{9j+6})} - \frac{q^5}{(q^3;q^3)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^j q^{\frac{9j^2+21j}{2}}}{(1-q^{9j+6})},
\end{aligned}
$$

which is (1.1). □

Proof of (1.2). By (2.2) we have

$$
\begin{aligned}
&RV(\zeta_3, q)\\
&\quad= \frac{1}{(1+\zeta_3)\left(q,\zeta_3,\zeta_3^{-1};q\right)_\infty} \sum_{j=-\infty}^{\infty} \frac{(1-\zeta_3^j)(1-\zeta_3^{j-1})\zeta_3^{1-j}(-1)^j q^{\frac{j(j+1)}{2}}}{(1-\zeta_3^2 q^j)(1-\zeta_3^{-2}q^j)}\\
&\quad= \frac{1}{(1+\zeta_3)(1-\zeta_3)(1-\zeta_3^{-1})\,(q^3;q^3)_\infty}\\
&\qquad\times \sum_{j=-\infty}^{\infty} \frac{(1-\zeta_3^2)(1-\zeta_3)\zeta_3^{-1}(-1)^j q^{\frac{9j^2+15j}{2}+3}}{(1-\zeta_3^2 q^{3j+2})(1-\zeta_3^{-2}q^{3j+2})}\\
&\quad= \frac{-1}{(q^3;q^3)_\infty} \sum_{j=-\infty}^{\infty} \frac{(-1)^j q^{\frac{9j^2+15j}{2}+3}(1-q^{3j+2})}{(1-q^{9j+6})}\\
&\quad= -\frac{q^3}{(q^3;q^3)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^j q^{\frac{9j^2+15j}{2}}}{(1-q^{9j+6})} + \frac{q^5}{(q^3;q^3)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^j q^{\frac{9j^2+21j}{2}}}{(1-q^{9j+6})},
\end{aligned}
$$

which is (1.2). □

Proof of (1.3). By (2.1) we have

$$
\begin{aligned}
&RU(\zeta_5,q)\\
&\quad= \frac{1}{(1+\zeta_5)\left(q,\zeta_5,\zeta_5^{-1};q\right)_\infty}\sum_{j=-\infty}^{\infty}\frac{(1-\zeta_5^j)(1-\zeta_5^{j-1})\zeta_5^{1-j}(-1)^j q^{\frac{j(j+3)}{2}}}{(1-\zeta_5^2q^j)(1-\zeta_5^{-2}q^j)}\\
&\quad= \frac{1}{(1+\zeta_5)\left(q,\zeta_5,\zeta_5^{-1};q\right)_\infty}\sum_{k=2}^{4}(-1)^k(1-\zeta_5^k)(1-\zeta_5^{k-1})\zeta_5^{1-k}q^{\frac{k(k+3)}{2}}\\
&\qquad\times\sum_{j=-\infty}^{\infty}\frac{(-1)^j q^{\frac{25j^2+15j}{2}+5jk}}{(1-\zeta_5^2q^{5j+k})(1-\zeta_5^{-2}q^{5j+k})}\\
&\quad= \frac{1}{(1+\zeta_5)\left(q,\zeta_5,\zeta_5^{-1};q\right)_\infty}\sum_{k=2}^{4}(-1)^k(1-\zeta_5^k)(1-\zeta_5^{k-1})\zeta_5^{1-k}q^{\frac{k(k+3)}{2}}\\
&\qquad\times\sum_{j=-\infty}^{\infty}\frac{(-1)^j q^{\frac{25j(j+1)}{2}-5j+5jk}(1-q^{5j+k})(1-\zeta_5q^{5j+k})(1-\zeta_5^{-1}q^{5j+k})}{(1-q^{25j+5k})}\\
&\quad= \frac{1}{(1+\zeta_5)\left(q,\zeta_5,\zeta_5^{-1};q\right)_\infty}\sum_{k=2}^{4}(-1)^k(1-\zeta_5^k)(1-\zeta_5^{k-1})\zeta_5^{1-k}q^{\frac{k(k+3)}{2}}\\
&\qquad\times\Big(T(k,k-1,5)-(1+\zeta_5+\zeta_5^4)q^kT(k,k,5)\\
&\qquad\qquad+(1+\zeta_5+\zeta_5^4)q^{2k}T(k,k+1,5)-q^{3k}T(k,k+2,5)\Big).
\end{aligned}
$$

In (2.3) we set $\ell=5$, $a=2+k+c$, $b_1=2$, and $b_2=k$ to get

$$
\begin{aligned}
T(k,k+c,5) = q^{10-5k}\frac{P(k+c)}{P(2+c)}T(2,2+c,5)\\
- q^{10-5k}\frac{P(2+k+c)P(k-2)E(25)^2}{P(2)P(k)P(2+c)},
\end{aligned}
$$

for $k=3,4$ and $c=-1,0,1,2$. We set $\ell=5$, $a=1$, and $b=2$ in (2.4) and simplify the products to get

$$
T(2,3,5) = q^{-5}\frac{P(2)}{P(1)}T(2,1,5) - q^{-5}\frac{P(1)E(25)^2}{P(2)^2}.
$$

With these identities we write each of the $T(a,b,5)$ in terms of $T(2,1,5)$ and $T(2,2,5)$ and carefully simplify to find that

$$\begin{aligned}(3.1)\qquad &(1+\zeta_5)\left(q,\zeta_5,\zeta_5^{-1};q\right)_\infty RU(\zeta_5,q)\\ &= T(2,1,5)\left(-(2+2\zeta_5+\zeta_5^3)q^4\tfrac{P(2)}{P(1)} + (1+\zeta_5-2\zeta_5^3)q^5\right)\\ &\quad + T(2,2,5)\left(-(2+2\zeta_5+\zeta_5^3)q^7 + (1+\zeta_5-2\zeta_5^3)q^8\tfrac{P(1)}{P(2)}\right)\\ &\quad + (2+2\zeta_5+\zeta_5^3)q\tfrac{P(2)E(25)^2}{P(1)^2} - (1+\zeta_5-2\zeta_5^3)q^2\tfrac{E(25)^2}{P(1)}\\ &= (1+\zeta_5)\left(q,\zeta_5,\zeta_5^{-1};q\right)_\infty \left(q\tfrac{E(25)}{P(1)^2} - \tfrac{q^7}{E(25)P(2)}T(2,2,5)\right.\\ &\quad \left. -\tfrac{q^4}{E(25)P(1)}T(2,1,5)\right),\end{aligned}$$

where the last equality follows from using (2.5) to get that

$$\begin{aligned}&(1+\zeta_5)\left(q,\zeta_5,\zeta_5^{-1};q\right)_\infty\\ &\qquad = E(25)((2+2\zeta_5+\zeta_5^3)P(2) - (1+\zeta_5-2\zeta_5^3)qP(1)).\end{aligned}$$

We see (3.1) now immediately implies (1.3). □

Proof of (1.4). By (2.2) we have

$$\begin{aligned}&RV(\zeta_5,q)\\ &= \frac{1}{(1+\zeta_5)\left(q,\zeta_5,\zeta_5^{-1};q\right)_\infty}\sum_{j=-\infty}^{\infty}\frac{(1-\zeta_5^j)(1-\zeta_5^{j-1})\zeta_5^{1-j}(-1)^j q^{\frac{j(j+1)}{2}}}{(1-\zeta_5^2q^j)(1-\zeta_5^{-2}q^j)}\\ &= \frac{1}{(1+\zeta_5)\left(q,\zeta_5,\zeta_5^{-1};q\right)_\infty}\sum_{k=2}^{4}(-1)^k(1-\zeta_5^k)(1-\zeta_5^{k-1})\zeta_5^{1-k}q^{\frac{k(k+1)}{2}}\\ &\qquad \times\sum_{j=-\infty}^{\infty}\frac{(-1)^j q^{\frac{25j^2+5j}{2}+5jk}}{(1-\zeta_5^2q^{5j+k})(1-\zeta_5^{-2}q^{5j+k})}\\ &= \frac{1}{(1+\zeta_5)\left(q,\zeta_5,\zeta_5^{-1};q\right)_\infty}\sum_{k=2}^{4}(-1)^k(1-\zeta_5^k)(1-\zeta_5^{k-1})\zeta_5^{1-k}q^{\frac{k(k+1)}{2}}\\ &\qquad \times\sum_{j=-\infty}^{\infty}\frac{(-1)^j q^{\frac{25j(j+1)}{2}-10j+5jk}(1-q^{5j+k})(1-\zeta_5q^{5j+k})(1-\zeta_5^{-1}q^{5j+k})}{(1-q^{25j+5k})}\end{aligned}$$

$$= \frac{1}{(1+\zeta_5)\left(q,\zeta_5,\zeta_5^{-1};q\right)_\infty}\sum_{k=2}^{4}(-1)^k(1-\zeta_5^k)(1-\zeta_5^{k-1})\zeta_5^{1-k}q^{\frac{k(k+1)}{2}}$$
$$\times\Big(T(k,k-2,5)-(1+\zeta_5+\zeta_5^4)q^kT(k,k-1,5)$$
$$+(1+\zeta_5+\zeta_5^4)q^{2k}T(k,k,5)-q^{3k}T(k,k+1,5)\Big).$$

In (2.3) we set $\ell=5$, $a=3+k+c$, $b_1=3$, and $b_2=k$ to get

$$T(k,k+c,5)=q^{15-5k}\frac{P(k+c)}{P(3+c)}T(3,3+c,5)$$
$$-q^{15-5k}\frac{P(3+k+c)P(k-3)E(25)^2}{P(3)P(k)P(3+c)},$$

for $k=2,4$ and $c=-2,-1,0,1$. We set $\ell=5$, $a=1$, and $b=3$ in (2.4) and simplify the products to get

$$T(3,4,5)=q^{-5}\frac{P(1)}{P(2)}T(3,2,5)+q^{-10}\frac{P(1)^2E(25)^2}{P(2)^3}.$$

With these identities we write each of the $T(a,b,5)$ in terms of $T(3,1,5)$ and $T(3,3,5)$ and carefully simplify to find that

$$(3.2)\quad (1+\zeta_5)\left(q,\zeta_5,\zeta_5^{-1};q\right)_\infty RV(\zeta_5,q)$$
$$=T(3,1,5)\left((-2-2\zeta_5-\zeta_5^3)q^5\tfrac{P(2)}{P(1)}+(1+\zeta_5-2\zeta_5^3)q^6\right)$$
$$+T(3,3,5)\left((2+2\zeta_5+\zeta_5^3)q^{12}+(-1-\zeta_5+2\zeta_5^3)q^{13}\tfrac{P(1)}{P(2)}\right)$$
$$+(2+2\zeta_5+\zeta_5^3)q^2\tfrac{E(25)^2}{P(1)}+(-3-3\zeta_5+\zeta_5^3)q^3\tfrac{E(25)^2}{P(2)}$$
$$+(1+\zeta_5-2\zeta_5^3)q^4\tfrac{P(1)E(25)^2}{P(2)^2}$$
$$=(1+\zeta_5)\left(q,\zeta_5,\zeta_5^{-1};q\right)_\infty\Big(-\tfrac{q^5}{P(1)E(49)}T(3,1,5)$$
$$+\tfrac{q^{12}}{P(2)E(49)}T(3,3,5)+q^2\tfrac{E(25)}{P(1)P(2)}-q^3\tfrac{E(25)}{P(2)^2}\Big),$$

where the last equality also follows from

$$(1+\zeta_5)\left(q,\zeta_5,\zeta_5^{-1};q\right)_\infty$$
$$=E(25)((2+2\zeta_5+\zeta_5^3)P(2)-(1+\zeta_5-2\zeta_5^3)qP(1)).$$

We see (3.2) now immediately implies (1.4). □

Proof of (1.5). By (2.1) we have that

$$
\begin{aligned}
&RU(\zeta_7,q)\\
&= \frac{1}{(1+\zeta_7)\left(q,\zeta_7,\zeta_7^{-1};q\right)_\infty}\sum_{j=-\infty}^{\infty}\frac{(1-\zeta_7^j)(1-\zeta_7^{j-1})\zeta_7^{1-j}(-1)^j q^{\frac{j(j+3)}{2}}}{(1-\zeta_7^2q^j)(1-\zeta_7^{-2}q^j)}\\
&= \frac{1}{(1+\zeta_7)\left(q,\zeta_7,\zeta_7^{-1};q\right)_\infty}\sum_{k=2}^{6}(-1)^k(1-\zeta_7^k)(1-\zeta_7^{k-1})\zeta_7^{1-k}q^{\frac{k(k+3)}{2}}\\
&\qquad\times\sum_{j=-\infty}^{\infty}\frac{(-1)^j q^{\frac{49j^2+21j}{2}+7jk}}{(1-q^{49j+7k})}(1-q^{7j+k})(1-\zeta_7q^{7j+k})(1-\zeta_7^3q^{7j+k})\\
&\qquad\times(1-\zeta_7^4q^{7j+k})(1-\zeta_7^6q^{7j+k})\\
&= \frac{1}{(1+\zeta_7)\left(q,\zeta_7,\zeta_7^{-1};q\right)_\infty}\sum_{k=2}^{6}(-1)^k(1-\zeta_7^k)(1-\zeta_7^{k-1})\zeta_7^{1-k}q^{\frac{k(k+3)}{2}}\\
&\qquad\times\Big(T(k,k-2,5)+(\zeta_7^2+\zeta_7^5)q^kT(k,k-1,7)\\
&\qquad\quad+(1+\zeta_7^3+\zeta_7^4)q^{2k}T(k,k,7)-(1+\zeta_7^3+\zeta_7^4)q^{3k}T(k,k+1,7)\\
&\qquad\quad-(\zeta_7^2+\zeta_7^5)q^{4k}T(k,k+2,7)-q^{5k}T(k,k+3,7)\Big).
\end{aligned}
$$

In (2.3) we set $\ell=7$, $a=3+k+c$, $b_1=3$, and $b_2=k$ to get

$$
T(k,k+c,7)=q^{21-7k}\frac{P(k+c)}{P(3+c)}T(3,3+c,7)-q^{21-7k}\frac{P(3+k+c)P(k-3)E(49)^2}{P(3)P(k)P(3+c)},
$$

for $k=2,4,5,6$ and $c=-2,\ldots,3$. We set $\ell=7$, $a=1,2$, and $b=3$ in (2.4) and simplify the products to get

$$
T(3,4,7)=q^{-7}\frac{P(3)}{P(2)}T(3,2,7)-q^{-7}\frac{P(1)^2E(49)^2}{P(3)^2P(2)},
$$

$$
T(3,5,7)=q^{-14}\frac{P(2)}{P(1)}T(3,1,7)-q^{-14}\frac{P(2)E(49)^2}{P(3)^2}.
$$

With these identities we write each of the $T(a,b,7)$ in terms of $T(3,1,7)$, $T(3,2,7)$, and $T(3,3,7)$ and carefully simplify to find that

(3.3)

$$
\begin{aligned}
&(1+\zeta_7)\left(q,\zeta_7,\zeta_7^{-1};q\right)_\infty RU(\zeta_7,q)\\
&= T(3,1,7)\Big((-3-3\zeta_7-2\zeta_7^3-4\zeta_7^4-2\zeta_7^5)q^6\tfrac{P(3)}{P(1)}\\
&\qquad\qquad + (1+\zeta_7+\zeta_7^3+3\zeta_7^4+\zeta_7^5)q^7\tfrac{P(2)}{P(1)}\\
&\qquad\qquad + (4+4\zeta_7+\zeta_7^3+4\zeta_7^4+\zeta_7^5)q^9\Big)\\
&\quad + T(3,2,7)\Big((-1-\zeta_7-2\zeta_7^3-\zeta_7^4-2\zeta_7^5)q^{11}\tfrac{P(3)}{P(2)} + (-1-\zeta_7+2\zeta_7^4)q^{12}\\
&\qquad\qquad + (1+\zeta_7-\zeta_7^3-\zeta_7^5)q^{14}\tfrac{P(1)}{P(2)}\Big)\\
&\quad + T(3,3,7)\Big((2+2\zeta_7+3\zeta_7^4)q^{15} + (-2-2\zeta_7-\zeta_7^3-\zeta_7^4-\zeta_7^5)q^{16}\tfrac{P(2)}{P(3)}\\
&\qquad\qquad + (-3-3\zeta_7-2\zeta_7^3-4\zeta_7^4-2\zeta_7^5)q^{18}\tfrac{P(1)}{P(3)}\Big)\\
&\quad + (2+2\zeta_7+\zeta_7^3+\zeta_7^4+\zeta_7^5)q\tfrac{P(2)E(49)^2}{P(1)^2}\\
&\quad - (2+2\zeta_7+3\zeta_7^4)q^2\tfrac{P(3)E(49)^2}{P(2)P(1)} + (2+2\zeta_7+3\zeta_7^3+4\zeta_7^4+3\zeta_7^5)q^3\tfrac{E(49)^2}{P(1)}\\
&\quad - (3+3\zeta_7+2\zeta_7^3+4\zeta_7^4+2\zeta_7^5)q^4\tfrac{P(2)E(49)^2}{P(3)P(1)} + (2+2\zeta_7+3\zeta_7^4)q^5\tfrac{E(49)^2}{P(2)}\\
&\quad + (2+2\zeta_7+\zeta_7^3+\zeta_7^4+\zeta_7^5)q^6\tfrac{E(49)^2}{P(3)}\\
&\quad - (2+2\zeta_7+2\zeta_7^3+6\zeta_7^4+2\zeta_7^5)q^7\tfrac{P(2)E(49)^2}{P(3)^2}\\
&\quad + (2+2\zeta_7+3\zeta_7^4)q^7\tfrac{P(1)E(49)^2}{P(2)^2}\\
&\quad - (2+2\zeta_7+\zeta_7^3+\zeta_7^4+\zeta_7^5)q^8\tfrac{P(1)E(49)^2}{P(3)P(2)}\\
&\quad - (4+4\zeta_7+\zeta_7^3+4\zeta_7^4+\zeta_7^5)q^9\tfrac{P(1)E(49)^2}{P(3)^2}\\
&\quad + (2+2\zeta_7+3\zeta_7^3+4\zeta_7^4+3\zeta_7^5)q^{11}\tfrac{P(1)^2E(49)^2}{P(3)^2P(2)}\\
&\quad + (1+\zeta_7+\zeta_7^3+3\zeta_7^4+\zeta_7^5)q^{14}\tfrac{P(1)^3E(49)^2}{P(3)^3P(2)}.
\end{aligned}
$$

We slightly alter (3.3) before proceeding. By Lemma 4 of [4] with $b=3$, $c=2$, and $d=1$, we know

$$P(3)^3P(1) - P(2)^3P(3) + q^7P(1)^3P(2) = 0,$$

which yields

$$q\frac{P(2)}{P(1)^2} - q^8\frac{P(1)}{P(2)P(3)} = q\frac{P(3)^2}{P(1)P(2)^2},$$
$$q^{11}\frac{P(1)^2}{P(2)P(3)^2} = q^4\frac{P(2)}{P(1)P(3)} - q^4\frac{P(3)}{P(2)^2},$$
$$q^{14}\frac{P(1)^3}{P(2)P(3)^3} = -q^7\frac{P(1)}{P(2)^2} + q^7\frac{P(2)}{P(3)^2}.$$

These identities allow us to rewrite (3.3) as

(3.4)

$$\begin{aligned}
&(1+\zeta_7)\left(q,\zeta_7,\zeta_7^{-1};q\right)_\infty RU(\zeta_7,q) \\
&= T(3,1,7)\Big((-3-3\zeta_7-2\zeta_7^3-4\zeta_7^4-2\zeta_7^5)q^6\frac{P(3)}{P(1)} \\
&\qquad + (1+\zeta_7+\zeta_7^3+3\zeta_7^4+\zeta_7^5)q^7\frac{P(2)}{P(1)} \\
&\qquad + (4+4\zeta_7+\zeta_7^3+4\zeta_7^4+\zeta_7^5)q^9\Big) \\
&\quad + T(3,2,7)\Big((-1-\zeta_7-2\zeta_7^3-\zeta_7^4-2\zeta_7^5)q^{11}\frac{P(3)}{P(2)} + (-1-\zeta_7+2\zeta_7^4)q^{12} \\
&\qquad + (1+\zeta_7-\zeta_7^3-\zeta_7^5)q^{14}\frac{P(1)}{P(2)}\Big) \\
&\quad + T(3,3,7)\Big((2+2\zeta_7+3\zeta_7^4)q^{15} - (2+2\zeta_7+\zeta_7^3+\zeta_7^4+\zeta_7^5)q^{16}\frac{P(2)}{P(3)} \\
&\qquad - (3+3\zeta_7+2\zeta_7^3+4\zeta_7^4+2\zeta_7^5)q^{18}\frac{P(1)}{P(3)}\Big) \\
&\quad + (2+2\zeta_7+\zeta_7^3+\zeta_7^4+\zeta_7^5)q\frac{P(3)^2E(49)^2}{P(1)P(2)^2} \\
&\quad - (2+2\zeta_7+3\zeta_7^4)q^2\frac{P(3)E(49)^2}{P(2)P(1)} + (2+2\zeta_7+3\zeta_7^3+4\zeta_7^4+3\zeta_7^5)q^3\frac{E(49)^2}{P(1)} \\
&\quad + (-1-\zeta_7+\zeta_7^3+\zeta_7^5)q^4\frac{P(2)E(49)^2}{P(3)P(1)} \\
&\quad - (2+2\zeta_7+3\zeta_7^3+4\zeta_7^4+3\zeta_7^5)q^4\frac{P(3)E(49)^2}{P(2)^2} + (2+2\zeta_7+3\zeta_7^4)q^5\frac{E(49)^2}{P(2)} \\
&\quad + (2+2\zeta_7+\zeta_7^3+\zeta_7^4+\zeta_7^5)q^6\frac{E(49)^2}{P(3)} + (1+\zeta_7-\zeta_7^3-\zeta_7^5))q^7\frac{P(1)E(49)^2}{P(2)^2} \\
&\quad - (1+\zeta_7+\zeta_7^3+3\zeta_7^4+\zeta_7^5))q^7\frac{P(2)E(49)^2}{P(3)^2} \\
&\quad - (4+4\zeta_7+\zeta_7^3+4\zeta_7^4+\zeta_7^5)q^9\frac{P(1)E(49)^2}{P(3)^2} \\
&= (1+\zeta_7)\left(q,\zeta_7,\zeta_7^{-1};q\right)_\infty\Big(-(\zeta_7^2+\zeta_7^5)\frac{q^{15}}{P(3)E(49)}T(3,3,7) \\
&\qquad + (\zeta_7+\zeta_7^6)\frac{q^{11}}{P(2)E(49)}T(3,2,7) - (1+\zeta_7^3+\zeta_7^4)\frac{q^6}{P(1)E(49)}T(3,1,7) \\
&\qquad + q\frac{E(49)P(3)}{P(2)^2P(1)} - (\zeta_7^3+\zeta^4)q^2\frac{E(49)}{P(1)P(2)} + q^3\frac{E(49)}{P(1)P(3)} \\
&\qquad + (\zeta_7+\zeta_7^6)q^4\frac{E(49)}{P(2)^2} + (1+\zeta_7^3+\zeta_7^4)q^6\frac{E(49)}{P(3)^2}\Big),
\end{aligned}$$

where the last equality follows from using (2.5) to get that

$$\begin{aligned}
&(1+\zeta_7)\left(q,\zeta_7,\zeta_7^{-1};q\right)_\infty \\
&\quad = E(49)((2+2\zeta_7+\zeta_7^3+\zeta_7^4+z^5)P(3)+(-1-\zeta_7+\zeta_7^3+\zeta_7^5)qP(2) \\
&\qquad\qquad - (1+\zeta_7+\zeta_7^3+3\zeta_7^4+\zeta_7^5)q^3P(1)).
\end{aligned}$$

We see (3.4) immediately implies (1.5). □

4. Proof of Theorem 1.3

Our proof is similar to the rearrangements and interpretations used by Andrews and Garvan in [3] to interpret the vector crank of [7] as the crank of ordinary partitions. We need only the q-binomial theorem in the form of

$$\frac{(tw;q)_\infty}{(t,w;q)_\infty} = \sum_{m=0}^{\infty} \frac{t^m}{(wq^m;q)_\infty (q;q)_m}.$$

With $t = z^{-1}q^n$ and $w = zq^{n+1}$ this gives us that

$$\begin{aligned}
RU(z,q) &= \sum_{n=1}^{\infty} \frac{q^n\left(q^{2n+1};q\right)_\infty}{(zq^n,z^{-1}q^n,z^2q^n,z^{-2}q^n;q)_\infty} \\
&= \sum_{n=1}^{\infty} \frac{q^n}{(1-zq^n)(z^2q^n,z^{-2}q^n;q)_\infty} \sum_{m=0}^{\infty} \frac{z^{-m}q^{nm}}{(zq^{n+m+1};q)_\infty (q;q)_m} \\
&= \sum_{n=1}^{\infty} \frac{q^n}{(zq^n,z^2q^n,z^{-2}q^n;q)_\infty} \\
&\qquad + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{z^{-m}q^{nm+n}}{(1-zq^n)(zq^{n+m+1},z^2q^n,z^{-2}q^n;q)_\infty (q;q)_m} \\
&= \sum_{n=1}^{\infty} \frac{q^n}{(zq^n,z^2q^n,z^{-2}q^n;q)_\infty} \\
&\qquad + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{q^n}{(1-zq^n)(q^{n+1};q)_m (zq^{n+m+1},z^2q^n,z^{-2}q^n;q)_\infty} \\
&\qquad\qquad \times \frac{z^{-m}q^{nm}(q;q)_{n+m}}{(q;q)_n (q;q)_m}.
\end{aligned}$$

Loosely speaking, z counts certain parts in π_1, z^2 counts $\#(\pi_2)$, z^{-2} counts $\#(\pi_3)$, and z^{-1} counts $\#(\pi_4)$. We see the first sum is the generating function for the partition quadruples $(\pi_1,\pi_2,\pi_3,\pi_4)$ from U when π_4 is the empty partition, where the power of q is the sum of the parts of the π_i and the power of z is $\#(\pi_1)-1+2\#(\pi_2)-2\#(\pi_3)$. That is to say the first sum is the generating function for the number of partition quadruples of n from U with u-rank equal to m when π_4 is the empty partition. For the second sum we first note that $\frac{(q;q)_{n+m}}{(q;q)_n(q;q)_m}$ is well known to be the generating function for partitions with parts at most n in size and at most m parts in total [1, Theorem 3.1], so that $\frac{q^{nm}(q;q)_{n+m}}{(q;q)_n(q;q)_m}$ is the generating function for partitions into exactly m parts and with all parts between n and $2n$. We see then the second sum is the generating function for the partition quadruples $(\pi_1,\pi_2,\pi_3,\pi_4)$ from U when π_4 is non-empty, where the power of q is the sum of the parts of the π_i and the power of z is $\omega(\pi_1,\pi_2,\pi_3,\pi_4)-1+2\#(\pi_2)-2\#(\pi_3)-\#(\pi_4)$. That is to say the second sum is the generating function for the number of partition quadruples of n from U with u-rank equal to m when π_4 is non-empty. Thus $ru(m,n)$ is the number of partition quadruples of n from U with u-rank equal to m.

In the same fashion we find $rv(m,n)$ to be the number of partition quadruples of n from V with v-rank equal to m. The remainder of Theorem 1.3 follows from the fact that $ru(k,\ell,\ell n{+}a)=\frac{u(\ell n+a)}{\ell}$ for $(\ell,a)=(3,0)$, $(5,0)$, $(5,3)$, $(7,0)$, and $(7,5)$ and $rv(k,\ell,\ell n{+}a)=\frac{v(\ell n+a)}{\ell}$ for $(\ell,a)=(3,1)$, $(5,1)$, and $(5,4)$ for all k as established by Theorem 1.2.

5. Remarks

It is somewhat surprising that the dissections of $RU(\zeta_\ell,q)$ and $RV(\zeta_\ell,q)$ are easier to handle than the modulo ℓ dissections of $U(q)$ and $V(q)$. However, the formulas for $U(q)$ and $V(q)$ modulo ℓ are still necessary, as $RU(z,q)$ and $RV(z,q)$ do not also explain the modulo 13 congruences. In particular one can check that the coefficient of q^{13} in $RU(\zeta_{13},q)$ is non-zero. Additionally one can check that the coefficient of q^{13} is non-zero in $F(\zeta_{13}^a,\zeta_{13}^b,\zeta_{13}^c;q)$ for all choices of a, b, and c. We leave it as an open problem to find a statistic to explain the modulo 13 congruences for $u(n)$ and $v(n)$.

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