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# Diophantine approximation with restricted numerators and denominators on semisimple groups

par ALEXANDER GORODNIK et SHIRALI KADYROV

RÉSUMÉ. Nous considérons le problème de l'approximation diophantienne sur les groupes algébriques semi-simples par des points rationnels avec numérateurs et dénominateurs restreints. Nous établissons un résultat quantitatif d'approximation pour tous les points réels du groupe par des points rationnels avec un dénominateur spécifié et un numérateur presque premier.

ABSTRACT. We consider the problem of Diophantine approximation on semisimple algebraic groups by rational points with restricted numerators and denominators and establish a quantitative approximation result for all real points in the group by rational points with a prescribed denominator and an almost prime numerator.

## 1. Introduction

In this paper we are interested in the problem of Diophantine approximation of real points  $x$  by rational points  $\frac{u}{v}$ , where the numerator  $u$  and the denominator  $v$  are restricted to interesting arithmetic sets; for instance, when  $u, v$  are primes or  $r$ -primes. Recall that an integer is called  $r$ -prime if it is a product of at most  $r$  prime factors counted with multiplicities.

Starting from the work of Vinogradov [36], the question for which  $\alpha > 1$  the inequality

$$(1.1) \quad \left| x - \frac{u}{v} \right| \leq v^{-\alpha}, \quad x \in \mathbb{R},$$

has infinitely many solutions with integral  $u$  and prime  $v$  attracted significant attention (see [1, 14, 16, 17, 19, 23, 34]). When  $u$  is  $r$ -prime and  $v$  is prime, this question has been investigated in [15, 33], but it still seems open when both  $u$  and  $v$  are assumed to be prime (see [29, 31]).

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Here we consider an analogous question for the Diophantine approximation on semisimple algebraic groups. For instance, let us consider a special linear group  $\mathrm{SL}_N = \{x \in \mathrm{Mat}_N(\mathbb{C}) : \det(x) = 1\}$ . It is well known that  $\mathrm{SL}_N(\mathbb{Q})$  is dense in  $\mathrm{SL}_N(\mathbb{R})$ , and explicit quantitative density estimates have been established in [9]. Now it is natural to ask whether we can approximate any  $x \in \mathrm{SL}_N(\mathbb{R})$  by rational points  $z \in \mathrm{SL}_N(\mathbb{Q})$  whose coordinates have prescribed arithmetic properties. In particular,

**Question 1.1.** Is the set of points in  $\mathrm{SL}_N(\mathbb{Q})$  with denominators which are powers of one fixed prime (or of finitely many primes) and  $r$ -prime numerators dense in  $\mathrm{SL}_N(\mathbb{R})$ ?

As we shall show, this is indeed the case, and moreover, a quantitative estimate similar to (1.1) holds.

In full generality, our result deals with a simply connected semisimple algebraic  $\mathbb{Q}$ -group  $G \subset \mathrm{GL}_N$  which is split over  $\mathbb{Q}$  and  $\mathbb{Q}$ -simple. It is known that  $G(\mathbb{Q})$  is dense in  $G(\mathbb{R})$ . Moreover, it follows from the strong approximation property [28, §7.4] that for every  $n \geq 2$ , the set  $G(\mathbb{Z}[1/n])$  is dense in  $G(\mathbb{R})$ . Every  $z \in G(\mathbb{Q})$  can be uniquely written as  $z = v^{-1}u$  with  $v \in \mathbb{N}$  and  $u \in \mathrm{Mat}_N(\mathbb{Z})$  such that  $\gcd(u_{11}, \dots, u_{NN}, v) = 1$ . We use the denominator  $\mathrm{den}(z) := v$  to measure complexity of rational points and quantify their density in  $G(\mathbb{R})$  with respect to a right invariant Riemannian metric  $d$  on  $G(\mathbb{R})$ .

We note that it might happen that the set of  $z \in G(\mathbb{Q})$  such that  $\mathrm{den}(z) = n$  could be empty, so that the problem of Diophantine approximation does not make sense for general denominators. We say that an integer  $n$  is *admissible* if there exists  $z \in G(\mathbb{Q})$  such that  $\mathrm{den}(z) = n$ . Because of the weak approximation property the set of admissible denominators can be described in terms of local obstructions. Namely,  $n = \prod_p p^{\alpha_p}$  is admissible if for every  $p$ , there exists  $g \in G(\mathbb{Q}_p)$  such that  $\|g\|_p = p^{\alpha_p}$ , where  $\|\cdot\|_p$  denotes the maximal norm.

Let  $f_1, \dots, f_t$  be a collection of polynomials on  $\mathrm{Mat}_N(\mathbb{C})$  with integral coefficients. We assume that  $f_i$ 's considered as elements of the coordinate ring  $\mathbb{Q}[G]$  are nonzero, coprime, and absolutely irreducible. We say that an element  $z \in \mathrm{Mat}_N(\mathbb{Z}[1/n])$  is  *$r$ -prime*, with respect to the family of polynomials  $f_1, \dots, f_t$ , if  $f_1(z) \cdots f_t(z)$  is a product of at most  $r$  prime factors in the ring  $\mathbb{Z}[1/n]$ .

Our main result is the following

**Theorem 1.2.** *Given a simply connected semisimple algebraic group  $G \subset \mathrm{GL}_N$  defined over  $\mathbb{Q}$ , which is split over  $\mathbb{Q}$  and  $\mathbb{Q}$ -simple, and a collection of polynomials  $f_1, \dots, f_t$  as above, there exist explicit  $\alpha > 0$  and  $r \in \mathbb{N}$  such that for every  $x \in G(\mathbb{R})$  and admissible  $n \geq n_0(x)$ , one can find  $z \in G(\mathbb{Q})$*

satisfying

$$\begin{aligned} d(x, z) &\leq n^{-\alpha}, \\ \text{den}(z) &= n, \end{aligned}$$

and  $r$ -prime in  $\text{Mat}_n(\mathbb{Z}[1/n])$ . Moreover, the constant  $n_0(x)$  is uniform over  $x$  in bounded subsets of  $G(\mathbb{R})$ .

**Remark 1.3.** Let  $\|\cdot\|_\infty$  be the Euclidean norm on  $\text{Mat}_N(\mathbb{R})$ . Using that  $G(\mathbb{R})$  is a submanifold of  $\text{Mat}_N(\mathbb{R})$ , one can check that

$$\|x_1 - x_2\|_\infty \ll d(x_1, x_2), \quad x_1, x_2 \in G(\mathbb{R}),$$

where the implied constant is uniform over  $x_1, x_2$  in bounded subset of  $G(\mathbb{R})$ . Hence, Theorem 1.2 also implies a Diophantine approximation result with respect to the Euclidean norm.

Next we consider the case when  $G$  is not necessarily split over  $\mathbb{Q}$ . Then the set  $G(\mathbb{Z}[1/p])$ , where  $p$  is prime, might be discrete in  $G(\mathbb{R})$ . In fact,  $G(\mathbb{Z}[1/p])$  is dense in  $G(\mathbb{R})$  if and only if the group  $G$  is isotropic over the  $p$ -adic field  $\mathbb{Q}_p$  (see [28, Th. 7.12]). Under this assumption we prove a weaker version of Theorem 1.2 for  $G(\mathbb{Z}[1/p]) \subset G(\mathbb{R})$  where the parameters  $\alpha, r, n_0(x)$  might depend on  $p$ .

**Theorem 1.4.** *Given a simply connected  $\mathbb{Q}$ -simple algebraic group  $G \subset \text{GL}_N$  defined over  $\mathbb{Q}$ , a collection of polynomials  $f_1, \dots, f_t$  as above, and a finite collection  $\mathcal{P}$  of primes such that  $G$  is isotropic over  $\mathbb{Q}_p$  for all  $p \in \mathcal{P}$ , there exist explicit  $\alpha > 0$  and  $r \in \mathbb{N}$  such that for every  $x \in G(\mathbb{R})$  and admissible  $n \geq n_0(x)$  whose prime divisors are in  $\mathcal{P}$ , one can find  $z \in G(\mathbb{Q})$  satisfying*

$$\begin{aligned} d(x, z) &\leq n^{-\alpha}, \\ \text{den}(z) &= n, \end{aligned}$$

and  $r$ -prime in  $\text{Mat}(\mathbb{Z}[1/n])$ . Moreover, the constant  $n_0(x)$  is uniform over  $x$  in bounded subsets of  $G(\mathbb{R})$ .

More explicit statements of Theorems 1.2 and 1.4 are given in Section 6 below.

The proof of the main theorems is based on the uniform spectral gap property for the automorphic unitary representations and the asymptotic analysis of suitable averaging operators combined with standard number-theoretic sieving arguments. In the following section we introduce essential notation and outline the strategy of the proof in more details.

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## 2. Initial set-up

Throughout the paper,  $p$  always denotes a prime number.

Let  $G \subset \mathrm{GL}_N$  be a simply connected  $\mathbb{Q}$ -simple algebraic group defined over  $\mathbb{Q}$ . We also use the same notation for the corresponding integral model of  $G$  defined by the embedding of  $G$  into  $\mathrm{GL}_N$ .

For  $n \in \mathbb{N}$ , we set

$$G_n^f := \prod_{p|n} G(\mathbb{Q}_p) \quad \text{and} \quad G_n := G(\mathbb{R}) \times G_n^f.$$

Let

$$\Gamma_n := G(\mathbb{Z}[1/n]).$$

We consider  $\Gamma_n$  as a subgroup of  $G_n$  embedded in  $G_n$  diagonally. Then  $\Gamma_n$  is a discrete subgroup with finite covolume (see [28, §5.4]).

For every prime  $p$ , we fix a special maximal compact open subgroup  $U_p$  of  $G(\mathbb{Q}_p)$  (as defined in [4, 32]), so that  $U_p = G(\mathbb{Z}_p)$  for almost all  $p$ .

We denote by  $m_{G(\mathbb{Q}_p)}$  the Haar measure on  $G(\mathbb{Q}_p)$  normalised so that

$$m_{G(\mathbb{Q}_p)}(U_p) = 1.$$

The Haar measure  $m_{G_n^f}$  on  $G_n^f$  is the product of the measures  $m_{G(\mathbb{Q}_p)}$  over  $p$  dividing  $n$ . The Haar measure  $m_{G_n}$  on  $G_n$  is the product of a Haar measure  $m_{G(\mathbb{R})}$  on  $G(\mathbb{R})$  and the measure  $m_{G_n^f}$ .

For  $q \in \mathbb{N}$ , coprime to  $n$ , we define the congruence subgroups

$$\Gamma_n(q) := \{\gamma \in \Gamma_n : \gamma = \mathrm{id} \bmod q\}.$$

Clearly,  $\Gamma_n(q)$  is a finite index normal subgroup of  $\Gamma_n$ , and the space

$$Y_{n,q} := G_n / \Gamma_n(q)$$

has finite volume. For simplicity, we also set  $Y_n := G_n / \Gamma_n$ . We denote by  $m_{Y_{n,q}}$  the invariant measure on  $Y_{n,q}$  induced by  $m_{G_n}$  and by  $\mu_{Y_{n,q}}$  the probability invariant measure on  $Y_{n,q}$ , so that  $\mu_{Y_{n,q}} = \frac{m_{Y_{n,q}}}{m_{Y_{n,q}}(Y_{n,q})}$ .

We denote by  $\|\cdot\|_p$  the maximum norm on  $\mathrm{Mat}_N(\mathbb{Q}_p)$ . Given  $n \in \mathbb{N}$  with prime decomposition  $n = \prod_p p^{\alpha_p}$ , we set

$$(2.1) \quad B_{n,p} := \{g \in G(\mathbb{Q}_p) : \|g\|_p = p^{\alpha_p}\} \quad \text{and} \quad B_n^f := \prod_{p|n} B_{n,p}.$$

Throughout the paper we always assume that the integer  $n$  is admissible, namely, the set  $B_n^f$  is not empty. Note that  $B_{n,p}$  is a compact subset of  $G(\mathbb{Q}_p)$ , which is invariant under the compact open subgroup  $G(\mathbb{Z}_p)$ . In particular,  $B_{n,p}$  is invariant under  $U_p$  for almost all  $p$ . We fix a right-invariant

Riemannian metric on  $G(\mathbb{R})$ . For  $x \in G(\mathbb{R})$  and  $\epsilon > 0$ , we set

$$B^\infty(x, \epsilon) := \{g \in G(\mathbb{R}) : d(g, x) \leq \epsilon\},$$

$$B_n(x, \epsilon) := B^\infty(x, \epsilon) \times B_n^f.$$

We denote by  $L^2(Y_{n,q}) = L^2(Y_{n,q}, \mu_{Y_{n,q}})$  the Hilbert space of square-integrable functions on  $Y_{n,q}$ , and by  $L_0^2(Y_{n,q})$  the subspace of functions with zero integral. For  $p = \infty$  and a prime  $p$  dividing  $n$ , the group  $G(\mathbb{Q}_p)$  naturally acts on  $Y_{n,q}$ , and we denote by  $\pi_{Y_{n,q},p}$  the corresponding unitary representation of  $G(\mathbb{Q}_p)$  on  $L^2(Y_{n,q})$ . We denote by  $\pi_{Y_{n,q}}$  and  $\pi_{Y_{n,q},f}$  the unitary representations of  $G_n$  and  $G_n^f$  on  $L^2(Y_{n,q})$  respectively. It would be also convenient to denote by  $\pi_{Y_{n,q},p}^0$ ,  $\pi_{Y_{n,q}}^0$ ,  $\pi_{Y_{n,q},f}^0$  the restrictions of the above representations to  $L_0^2(Y_{n,q})$ .

Given a unitary representation  $\pi$  of a locally compact group  $G$  on a Hilbert space  $\mathcal{H}$  and a finite Borel measure  $\beta$  on  $G$ , we denote by  $\pi(\beta) : \mathcal{H} \rightarrow \mathcal{H}$  the corresponding averaging operator defined by

$$\pi(\beta)v = \int_G \pi(g)v d\beta(g), \quad v \in \mathcal{H}.$$

We note that if  $\beta$  is a probability measure, then  $\|\pi(\beta)\| \leq 1$ .

The crucial ingredient of our argument is the study of suitable averaging operators on the space  $L^2(Y_{n,q})$ . Namely, we denote by  $\beta_{n,x,\epsilon}$  the uniform probability measure on  $G_n$  supported on the set  $B_n(x, \epsilon)$ . This defines an averaging operator

$$(2.2) \quad \pi_{Y_{n,q}}(\beta_{n,x,\epsilon}) : L^2(Y_{n,q}) \rightarrow L^2(Y_{n,q}) :$$

$$f \mapsto \frac{1}{m_{G_n}(B_n(x, \epsilon))} \int_{B_n(x, \epsilon)} \pi_{Y_{n,q}}(g)f dm_{G_n}(g).$$

We note that this operator is well defined only for admissible  $p$ .

A unitary representation  $\pi$  of a locally compact group  $G$  on a Hilbert space  $\mathcal{H}$  is called  $L^{r+}$  integrable if there exists a dense family of vectors  $v_1, v_2 \in \mathcal{H}$  such that the function

$$g \mapsto \langle \pi(g)v_1, v_2 \rangle, \quad g \in G,$$

is in  $L^{r+\delta}(G)$  for every  $\delta > 0$ . It will be crucial for our argument that the representations  $\pi_{Y_{n,q},p}$ , restricted to  $L_0^2(Y_{n,q})$ , are uniformly integrable. Namely, there exists  $r \geq 2$ , independent of  $n, q, p$ , such that all the representations  $\pi_{Y_{n,q},p}$ , restricted to  $L_0^2(Y_{n,q})$ , are  $L^{r+}$  integrable. We denote by  $r(G) \geq 2$  the least real number with this property. Let  $\iota(G)$  be the least even integer greater than or equal to  $r(G)/2$  if  $r(G) > 2$ , and  $\iota(G) = 1$  if  $r(G) = 2$ .

The uniform integrability of the representations  $\pi_{Y_{n,q},p}^0$  follows from the property  $\tau$  (see [5, 7, 21]) which states that the family of representations of every noncompact simple factor of  $G(\mathbb{Q}_p)$  on  $L_0^2(Y_{n,q})$  is uniformly isolated

from the trivial representations. This in turn implies that the representations  $\pi_{Y_{n,q,p}}^0$  are  $L^{r^+}$  integrable for some uniform  $r$  (see [3, 8, 18]).

**Outline of the proof.** In Section 3, we analyse the asymptotic behaviour of the averaging operators  $\pi_{Y_{n,q}}(\beta_{n,x,\epsilon})$  and establish a quantitative mean ergodic theorem for them, namely, an estimate of the form

$$(2.3) \quad \left\| \pi_{Y_{n,q}}(\beta_{n,x,\epsilon}) \Big|_{L_0^2(Y_{n,q})} \right\| \ll m_{G_n^f}(B_n^f)^{-\theta},$$

where  $\theta > 0$  is determined by the integrability exponent  $r(G)$ . This argument is based on the techniques developed in [10, 11], but it is crucial for our application that the implied constant in (2.3) is uniform on  $n$ , and this requires additional considerations.

Section 4 plays auxiliary role. In this section we establish several volume estimates which might be of independent interest. We use these estimates in the later sections to guarantee uniformity in the parameter  $n$ .

In Section 5, we use (2.3) to estimate the cardinality of elements in  $\bar{\gamma}\Gamma_n(q)$  lying in the regions  $B_n(x,\epsilon)$ , for admissible  $n$ . Typically, such a counting estimate requires that the regions are well-rounded in the sense of [11, Def. 1.1], but the regions  $B_n(x,\epsilon)$  are not well-rounded as  $\epsilon \rightarrow 0^+$ . Nonetheless, we shall establish a quantitative estimate for  $|B_n(x,\epsilon) \cap \bar{\gamma}\Gamma_n(q)|$  as  $n \rightarrow \infty$ .

Finally, in Section 6, we use a combinatorial sieving argument as in [12, 13, 26] to estimate the cardinality of almost prime points lying in the regions  $B_n(x,\epsilon)$ . This leads to the proof of the main theorems.

### 3. Averaging operators

In this section, we study the asymptotic behaviour of the averaging operators  $\pi_{Y_{n,q}}(\beta_{n,x,\epsilon})$  defined in (2.2) for admissible  $n$ . Similar problem has been previously investigated in [10, 11]. In particular, the following theorem can be proved by adopting the methods of [11, Th. 4.5].

**Theorem 3.1.** *For every  $\eta > 0$  and  $f \in L^2(Y_{n,q})$ ,*

$$\left\| \pi_{Y_{n,q}}(\beta_{n,x,\epsilon})f - \int_{Y_{n,q}} f d\mu_{Y_{n,q}} \right\|_2 \ll_{n,\eta} m_{G_n^f}(B_n^f)^{-(2\iota(G))^{-1}+\eta} \|f\|_2,$$

where the implied constant depends only on the set of prime divisors of  $n$ .

*Proof.* We recall that  $\pi_{Y_{n,q}}^0$  and  $\pi_{Y_{n,q,p}}^0$  denote the representations of  $G_n$  and  $G(\mathbb{Q}_p)$  on  $L_0^2(Y_{n,q})$  respectively. The probability measure  $\beta_{n,x,\epsilon}$  decomposes as a product

$$\beta_{n,x,\epsilon} = \beta_{x,\epsilon}^\infty \otimes \left( \bigotimes_{p|n} \beta_{n,p} \right),$$

where  $\beta_{x,\epsilon}^\infty$  is the uniform probability measure on  $G(\mathbb{R})$  supported on the set  $B^\infty(x, \epsilon)$ , and  $\beta_{n,p}$ 's are the uniform probability measures on  $G(\mathbb{Q}_p)$  supported on  $B_{n,p}$ . This implies that  $\pi_{Y_{n,q}}^0(\beta_{n,x,\epsilon})$  can be written as a product of commuting operators

$$\pi_{Y_{n,q}}^0(\beta_{n,x,\epsilon}) = \pi_{Y_{n,q},\infty}^0(\beta_{x,\epsilon}^\infty) \prod_{p|n} \pi_{Y_{n,q,p}}^0(\beta_{n,p}).$$

The statement of the theorem is equivalent to the estimate

$$\left\| \pi_{Y_{n,q}}^0(\beta_{n,x,\epsilon}) \right\| \ll_{n,\eta} m_{G_n^f}(B_n^f)^{-(2\iota(G))^{-1}+\eta}, \quad \eta > 0.$$

Since  $\|\pi_{Y_{n,q},\infty}^0(\beta_{x,\epsilon}^\infty)\| \leq 1$ , we obtain

$$(3.1) \quad \left\| \pi_{Y_{n,q}}^0(\beta_{n,x,\epsilon}) \right\| \leq \prod_{p|n} \left\| \pi_{Y_{n,q,p}}^0(\beta_{n,p}) \right\|.$$

To estimate the norms of  $\pi_{Y_{n,q,p}}^0(\beta_{n,p})$ , we use the argument as in [11, Th. 4.5] (which is based on Nevo's spectral transfer principle [25, Th. 1]). Explicitly, for real-valued functions  $f_1, f_2$  in  $L_0^2(Y_{n,q})$ , it follows from the Jensen's inequality that

$$\begin{aligned} \left\langle \pi_{Y_{n,q,p}}^0(\beta_{n,p}) f_1, f_2 \right\rangle^{\iota(G)} &= \left( \int_{G(\mathbb{Q}_p)} \left\langle \pi_{Y_{n,q,p}}^0(g) f_1, f_2 \right\rangle d\beta_{n,p}(g) \right)^{\iota(G)} \\ &\leq \int_{G(\mathbb{Q}_p)} \left\langle \pi_{Y_{n,q,p}}^0(g) f_1, f_2 \right\rangle^{\iota(G)} d\beta_{n,p}(g) \\ &= \int_{G(\mathbb{Q}_p)} \left\langle (\pi_{Y_{n,q,p}}^0)^{\otimes \iota(G)}(g) f_1^{\otimes \iota(G)}, f_2^{\otimes \iota(G)} \right\rangle d\beta_{n,p}(g) \\ &= \left\langle (\pi_{Y_{n,q,p}}^0)^{\otimes \iota(G)}(\beta_{n,p}) f_1^{\otimes \iota(G)}, f_2^{\otimes \iota(G)} \right\rangle \\ &\leq \left\| (\pi_{Y_{n,q,p}}^0)^{\otimes \iota(G)}(\beta_{n,p}) \right\| \|f_1\|^{\otimes \iota(G)} \|f_2\|^{\otimes \iota(G)}. \end{aligned}$$

Hence,

$$(3.2) \quad \left\| \pi_{Y_{n,q,p}}^0(\beta_{n,p}) \right\| \leq \left\| (\pi_{Y_{n,q,p}}^0)^{\otimes \iota(G)}(\beta_{n,p}) \right\|^{1/\iota(G)}.$$

Since  $\iota(G) \geq r(G)/2$ , the representation  $(\pi_{Y_{n,q,p}}^0)^{\otimes \iota(G)}$  is  $L^{2+}$  integrable, and by [6, Th. 1],

$$(3.3) \quad \left\| (\pi_{Y_{n,q,p}}^0)^{\otimes \iota(G)}(\beta_{n,p}) \right\| \leq \left\| \lambda_{G(\mathbb{Q}_p)}(\beta_{n,p}) \right\|,$$

where  $\lambda_{G(\mathbb{Q}_p)}$  denotes the regular representation of  $G(\mathbb{Q}_p)$  on  $L^2(G(\mathbb{Q}_p))$ . By the Kunze–Stein inequality, for every  $f \in L^2(G(\mathbb{Q}_p))$  and  $r \in [1, 2)$ ,

$$\|\beta_{n,p} * f\|_2 \ll_{p,q} \|\beta_{n,p}\|_q \|f\|_2 = m_{G(\mathbb{Q}_p)}(B_{n,q})^{1/r-1} \|f\|_2.$$



For semisimple simply connected groups over non-Archimedean fields, this inequality was proven in [35]. Then it follows that

$$\left\| \lambda_{\mathbb{G}(\mathbb{Q}_p)}(\beta_{n,p}) \right\| \ll_{p,\eta} m_{\mathbb{G}(\mathbb{Q}_p)}(B_{n,q})^{-1/2+\eta}, \quad \eta > 0,$$

and

$$(3.4) \quad \left\| \pi_{Y_{n,q,p}}^0(\beta_{n,p}) \right\| \ll_{p,\eta} m_{\mathbb{G}(\mathbb{Q}_p)}(B_{n,p})^{-(2\iota(\mathbb{G}))^{-1}+\eta}, \quad \eta > 0,$$

which completes the proof of the theorem.  $\square$

We note that the crucial estimate (3.4) can be deduced from the spherical Kunze–Stein inequality (see, for instance, [10, Prop. 5.9]), and the implied constant in (3.4) can be estimated explicitly, but unfortunately it blows up as  $p \rightarrow \infty$ .

For our purposes we need the following uniform version of Theorem 3.1.

**Theorem 3.2.** *For every  $\eta > 0$  and  $f \in L^2(Y_{n,q})$ ,*

$$\left\| \pi_{Y_{n,q}}(\beta_{n,x,\epsilon})f - \int_{Y_{n,q}} f d\mu_{Y_{n,q}} \right\|_2 \ll_{\eta} m_{G_n^f}(B_n^f)^{-(4\iota(\mathbb{G}))^{-1}+\eta} \|f\|_2.$$

*Proof.* As in the above proof, we need to estimate  $\|\pi_{Y_{n,q}}^0(\beta_{n,x,\epsilon})\|$ , and because of (3.1), it is sufficient to give a bound on the norms of  $\pi_{Y_{n,q,p}}^0(\beta_{n,p})$ . We claim that

$$(3.5) \quad \left\| \pi_{Y_{n,q,p}}^0(\beta_{n,p}) \right\| \leq c_{p,\eta} m_{\mathbb{G}(\mathbb{Q}_p)}(B_{n,p})^{-(4\iota(\mathbb{G}))^{-1}+\eta}, \quad \eta > 0,$$

where the constant  $c_{p,\eta} \geq 1$  satisfies

$$\prod_p c_{p,\eta} < \infty.$$

Clearly, (3.1) combined with (3.5) implies the theorem.

To prove (3.5), we use that by the estimates (3.2)–(3.3),

$$(3.6) \quad \left\| \pi_{Y_{n,q,p}}^0(\beta_{n,p}) \right\| \leq \left\| \lambda_{\mathbb{G}(\mathbb{Q}_p)}(\beta_{n,p}) \right\|^{1/\iota(\mathbb{G})}.$$

Let  $\tilde{B}_{n,p} := U_p B_{n,p} U_p$  and  $\tilde{\beta}_{n,p}$  be the uniform probability measure supported on  $\tilde{B}_{n,p}$ . Recall that for almost all  $p$ , we have  $U_p = \mathbb{G}(\mathbb{Z}_p)$ . For those  $p$ , we have  $\tilde{B}_{n,p} = B_{n,p}$  and

$$(3.7) \quad \left\| \lambda_{\mathbb{G}(\mathbb{Q}_p)}(\beta_{n,p}) \right\| = \left\| \lambda_{\mathbb{G}(\mathbb{Q}_p)}(\tilde{\beta}_{n,p}) \right\|.$$

To deal with the remaining finite set of primes, we observe that  $B_{n,p} \subset \tilde{B}_{n,p}$ , and hence for every  $f \in L^2(G(\mathbb{Q}_p))$ ,

$$\begin{aligned} \left\| \lambda_{G(\mathbb{Q}_p)}(\beta_{n,p})f \right\| &\leq \left\| \lambda_{G(\mathbb{Q}_p)}(\beta_{n,p})|f| \right\| \\ &\leq \frac{m_{G(\mathbb{Q}_p)}(\tilde{B}_{n,p})}{m_{G(\mathbb{Q}_p)}(B_{n,p})} \cdot \left\| \lambda_{G(\mathbb{Q}_p)}(\tilde{\beta}_{n,p})|f| \right\|. \end{aligned}$$

Hence,

$$(3.8) \quad \left\| \lambda_{G(\mathbb{Q}_p)}(\beta_{n,p}) \right\| \leq \frac{m_{G(\mathbb{Q}_p)}(\tilde{B}_{n,p})}{m_{G(\mathbb{Q}_p)}(B_{n,p})} \cdot \left\| \lambda_{G(\mathbb{Q}_p)}(\tilde{\beta}_{n,p}) \right\|.$$

Since  $U_p$  and  $G(\mathbb{Z}_p)$  are both compact open subgroups in  $G(\mathbb{Q}_p)$ , they are commensurable, and it follows that for some  $c_p \geq 1$ ,

$$m_{G(\mathbb{Q}_p)}(\tilde{B}_{n,p}) \leq c_p m_{G(\mathbb{Q}_p)}(B_{n,p}).$$

Hence, it follows from (3.7) and (3.8) that

$$(3.9) \quad \left\| \lambda_{G(\mathbb{Q}_p)}(\beta_{n,p}) \right\| \leq c_p \left\| \lambda_{G(\mathbb{Q}_p)}(\tilde{\beta}_{n,p}) \right\|,$$

where  $c_p = 1$  for almost all  $p$ .

Since  $\tilde{\beta}_{n,p}$  is bi-invariant under  $U_p$ , using Herz' majoration argument (see [10, Prop. 5.9]), we can estimate the norm  $\left\| \lambda_{G(\mathbb{Q}_p)}(\tilde{\beta}_{n,p}) \right\|$ . Indeed, since  $U_p$ 's are special subgroups, it follows from the structure theory [32, 3.3.2] that there exists a closed amenable subgroup  $Q_p$  such that

$$(3.10) \quad G(\mathbb{Q}_p) = U_p Q_p,$$

i.e.,  $G(\mathbb{Q}_p)$  is an Iwasawa group in the sense of [10, Def. 5.1(1)]. By [10, Prop. 5.9],

$$\left\| \lambda_{G(\mathbb{Q}_p)}(\tilde{\beta}_{n,p}) \right\| = \int_{G(\mathbb{Q}_p)} \Xi_p d\beta_{n,p},$$

where  $\Xi_p$  is the Harish-Chandra function  $\Xi_p$  (see [10, Def. 5.1(2)]). We recall that  $\Xi_p$  belongs to  $L^{2+\epsilon}(G(\mathbb{Q}_p))$  for every  $\epsilon > 0$ , so that by Hölder inequality, for every  $s \in [1, 2)$  and  $t = (1 - 1/s)^{-1}$ ,

$$(3.11) \quad \left\| \lambda_{G(\mathbb{Q}_p)}(\tilde{\beta}_{n,p}) \right\| \leq \|\Xi_p\|_t \cdot \|\tilde{\beta}_{n,p}\|_s = \|\Xi_p\|_t \cdot m_{G(\mathbb{Q}_p)}(\tilde{B}_{n,p})^{-1+1/s}.$$

We set  $a_{p,s} = \|\Xi_p\|_t$ .

To estimate  $a_{p,s}$ , we recall the explicit formula for the Harish-Chandra function. Let  $\Delta_p$  denote the modular function on the group  $Q_p$ . For  $g \in G(\mathbb{Q}_p)$ , we denote by  $q(g)$  the  $Q_p$ -component of  $g$  with respect to the decomposition (3.10). The Harish-Chandra function on  $G(\mathbb{Q}_p)$  is defined by

$$\Xi_p(g) = \int_{U_p} \Delta_p(q(gu))^{-1/2} dm_{G(\mathbb{Q}_p)}(u).$$

We have

$$a_{p,s} \geq \Xi_p(e) m_{G(\mathbb{Q}_p)}(U_p)^{1/t} = 1,$$

and by [11, Prop. 6.3], when  $t > 4$  (that is, when  $s < 4/3$ ),

$$\prod_p a_{p,s} < \infty.$$

Combining (3.6), (3.9), and (3.11), we conclude that

$$\left\| \pi_{Y_{n,q,p}}^0(\beta_{n,p}) \right\| \leq (a_{p,s} c_p)^{1/\iota(G)} m_{G(\mathbb{Q}_p)}(B_{n,p})^{(-1+1/s)/\iota(G)},$$

where

$$\prod_p (a_{p,s} c_p)^{1/\iota(G)} < \infty$$

for  $s < 4/3$ . This implies (3.5) and completes the proof of the theorem.  $\square$

#### 4. Volume estimates

This section plays auxiliary role and can be skipped for the first reading. Here we prove uniform estimates for the volumes of the sets  $Y_n$  and the sets  $B_n^f$ .

##### Proposition 4.1.

$$\inf_{n \in \mathbb{N}} m_{Y_n}(Y_n) > 0 \quad \text{and} \quad \sup_{n \in \mathbb{N}} m_{Y_n}(Y_n) < \infty.$$

*Proof.* For  $n \in \mathbb{N}$  we define

$$(4.1) \quad \mathcal{O}_n^f := \prod_{p|n} G(\mathbb{Z}_p),$$

which is a compact open subgroup of  $G_n^f$ .

To prove the first claim, we fix a sufficiently small open subset  $\mathcal{O}^\infty$  of  $G(\mathbb{R})$  and set  $\mathcal{O}_n := \mathcal{O}^\infty \times \mathcal{O}_n^f$ . We claim that  $\mathcal{O}_n$  injects into  $Y_n = G_n/\Gamma_n$  under the projection map  $G_n \rightarrow G_n/\Gamma_n$ . Indeed, if for some  $\gamma \in \Gamma_n$ , we have  $\mathcal{O}_n \gamma \cap \mathcal{O}_n \neq \emptyset$ , then it follows that  $\gamma \in G(\mathbb{Z})$ . Therefore, it is sufficient to take  $\mathcal{O}^\infty$  which injects into  $G(\mathbb{R})/G(\mathbb{Z})$ . Then

$$m_{Y_n}(Y_n) \geq m_{G(\mathbb{R})}(\mathcal{O}^\infty) \prod_{p|n} m_{G(\mathbb{Q}_p)}(G(\mathbb{Z}_p)).$$

Since  $m_{G(\mathbb{Q}_p)}(G(\mathbb{Z}_p)) = m_{G(\mathbb{Q}_p)}(U_p) = 1$  for almost all  $p$ , this proves the first claim.

Now we turn to the proof of the second claim. We fix a prime  $p_0$  such that  $G$  is isotropic over  $\mathbb{Q}_{p_0}$  and  $U_{p_0} = G(\mathbb{Z}_{p_0})$  (such a prime exists by [28, Th. 6.7]). We consider the two separate cases depending on whether  $p_0$  divides  $n$  or not.

Suppose that  $p_0$  divides  $n$  and write  $n = p_0^\alpha n_0$  with  $n_0$  coprime to  $p_0$ . We identify  $G_{p_0}$  and  $G_{n_0}^f$  with the corresponding subgroups of  $G_n$ , so that

$$(4.2) \quad G_n = G_{p_0} G_{n_0}^f.$$

Since  $G$  is isotropic over  $\mathbb{Q}_{p_0}$ , it follows that  $G_{p_0}$  is not compact, and by the strong approximation property [28, §7.4],  $G_{p_0} \Gamma_n$  is dense in  $G_n$ . In particular, it follows that for every  $g \in G_n$ ,

$$G_{p_0} \Gamma_n \cap O_{n_0}^f g \neq \emptyset.$$

This proves that

$$G_n = G_{p_0} O_{n_0}^f \Gamma_n.$$

Let  $\Omega_{p_0}$  be a measurable fundamental set in  $G_{p_0}$  for the right action of  $\Gamma_{p_0}$ . Then  $G_{p_0} = \Omega_{p_0} \Gamma_{p_0}$ , and every element  $g \in G_n$  can be written with respect to the decomposition (4.2) as

$$g = (\omega \delta, o) \cdot (\gamma, \gamma) = (\omega, o \delta^{-1}) \cdot (\delta \gamma, \delta \gamma),$$

where  $\omega \in \Omega_{p_0}$ ,  $\delta \in \Gamma_{p_0}$ ,  $o \in O_{n_0}^f$ , and  $\gamma \in \Gamma_n$ . Since  $o \delta^{-1} \in O_{n_0}^f$  and  $\delta \gamma \in \Gamma_n$ , this shows that

$$G_n = \Omega_{p_0} O_{n_0}^f \Gamma_n.$$

Therefore,

$$(4.3) \quad m_{Y_n}(Y_n) \leq m_{G_n}(\Omega_{p_0} O_{n_0}^f) = m_{G_{p_0}}(\Omega_{p_0}) \cdot m_{G_{n_0}^f}(O_{n_0}^f).$$

We observe that

$$m_{G_{n_0}^f}(O_{n_0}^f) = \prod_{p|n_0} m_{G(\mathbb{Q}_p)}(G(\mathbb{Z}_p)).$$

Since  $G(\mathbb{Z}_p)$  is compact,  $m_{G(\mathbb{Q}_p)}(G(\mathbb{Z}_p)) < \infty$ . Moreover, for almost  $p$ , we have  $G(\mathbb{Z}_p) = U_p$  and  $m_{G(\mathbb{Q}_p)}(G(\mathbb{Z}_p)) = 1$ . Hence, the second claim of the lemma follows from (4.3).

Now suppose that  $p_0$  does not divide  $n$ . In this case, we identify  $G_n$  and  $G(\mathbb{Q}_{p_0})$  with the corresponding subgroups of  $G_{np_0}$ , so that

$$(4.4) \quad G_{np_0} = G_n G(\mathbb{Q}_{p_0}).$$

Let  $\Omega_n$  be a measurable fundamental set in  $G_n$  for the right action of  $\Gamma_n$ . We claim that the natural projection map

$$(4.5) \quad \Omega_n \times G(\mathbb{Z}_{p_0}) \rightarrow Y_{np_0} = G_{np_0} / \Gamma_{np_0}$$

defined by the decomposition (4.4) is one-to-one. Indeed, suppose that for some  $g_1, g_2 \in \Omega_n$  and  $u_1, u_2 \in G(\mathbb{Z}_{p_0})$ , there exists  $\gamma \in \Gamma_{np_0}$  such that  $g_1 \gamma = g_2$  and  $u_1 \gamma = u_2$ . Then

$$\gamma = u_1^{-1} u_2 \in \Gamma_{np_0} \cap G(\mathbb{Z}_{p_0}) = \Gamma_n$$

and since  $\Omega_n$  is a fundamental set, it follows that  $\gamma = e$ . This proves the claim. It is clear the map (4.5) sends the product measure  $m_{G_n} \otimes m_{G(\mathbb{Q}_{p_0})}$  to the measure  $m_{Y_{np_0}}$ . Hence, we obtain

$$m_{G_n}(\Omega_n)m_{G(\mathbb{Q}_{p_0})}(G(\mathbb{Z}_{p_0})) \leq m_{Y_{np_0}}(Y_{np_0}).$$

Since it follows from the previous paragraph that the right hand side is uniformly bounded, we conclude that  $m_{Y_n}(Y_n) = m_{G_n}(\Omega_n)$  is uniformly bounded as well. This completes the proof of the proposition.  $\square$

We say that a number  $n$  is *isotropic* if for every prime divisor  $p$  of  $n$ , the group  $G$  is isotropic over  $\mathbb{Q}_p$ . In particular, if  $G$  is isotropic over  $\mathbb{Q}$ , then every number is isotropic.

**Proposition 4.2.** *There exists  $a > 0$  such that for every isotropic admissible number  $n$ ,*

$$m_{G_n^f}(B_n^f) \geq c(n)n^a,$$

where  $c(n) > 0$  depends only on the set of prime divisors of  $n$ .

Moreover, if  $G$  is split over  $\mathbb{Q}$ , then  $c(n)$  is independent of  $n$ .

*Proof.* Let  $n = \prod_p p^{\alpha_p}$  be the prime decomposition of  $n$ . Since  $B_n^f$  is the product of the sets  $B_{n,p}$  (see (2.1)), it is sufficient to prove that there exists  $c_p > 0$  such that

$$m_{G(\mathbb{Q}_p)}(B_{n,p}) \geq c_p(p^{\alpha_p})^a,$$

When  $G$  is split over  $\mathbb{Q}$ , we show that  $c_p = 1$  for almost all  $p$ .

Recall that  $U_p$  is a special maximal compact subgroup. Therefore, by [32, 3.3.3], we have a Cartan decomposition

$$G(\mathbb{Q}_p) = U_p Z_p(\mathbb{Q}_p) U_p,$$

where  $Z_p$  is the centraliser of a maximal  $\mathbb{Q}_p$ -split torus  $S_p$  in  $G$ . We fix a set  $\Pi_p$  of (restricted) simple roots for  $G$  with respect to  $S_p$ , and set

$$S_p^+ = \{s \in S(\mathbb{Q}_p) : |\chi(s)|_p \geq 1 \text{ for } \chi \in \Pi_p\}.$$

Then the Cartan decomposition takes form

$$(4.6) \quad G(\mathbb{Q}_p) = U_p S_p^+ \Theta_p U_p,$$

where  $\Theta_p$  is a finite subset of  $Z_p(\mathbb{Q}_p)$ . Since  $U_p$  is compact and  $\Theta_p$  is finite, there exists  $c'_p > 0$  such that for  $g = u_1 s \omega u_2 \in U_p S_p^+ \Theta_p U_p$ , we have

$$(4.7) \quad \|g\| \leq c'_p \|s\|_p.$$

Consider the representation  $\rho : G \rightarrow \mathrm{GL}_N$  corresponding to the embedding  $G \subset \mathrm{GL}_N$  and denote by  $\Phi_p$  the set of weights of this representation with respect to  $S_p$ . Since  $S_p$  is split over  $\mathbb{Q}_p$ , the action of  $S_p(\mathbb{Q}_p)$  on  $\mathbb{Q}_p^N$  is completely reducible, and

$$(4.8) \quad \|s\|_p \leq c''_p \cdot \max_{\xi \in \Phi_p} |\xi(s)|_p$$

for some  $c''_p > 0$ .

Let  $\Pi_p^\vee$  denote the set of fundamental weights corresponding to  $\Pi_p$ . A weight  $\xi$  is called dominant if

$$\xi = \prod_{\psi \in \Pi_p^\vee} \psi^{n_\psi}$$

for some nonnegative integers  $n_\psi$ . It follows from [27, Ch. 3, §1.9] that there exists  $k_0 \in \mathbb{N}$  such that every weight  $\xi$  can be written as

$$(4.9) \quad \xi^{k_0} = \prod_{\chi \in \Pi_p} \chi^{m_\chi}$$

with  $m_\chi \in \mathbb{Z}$ . Moreover, if  $\xi$  is dominant, then the integers  $m_\chi$  are non-negative.

By [2, Th. 7.2], there exists a semisimple subgroup  $\tilde{G}_p$  of  $G$  which is split over  $\mathbb{Q}_p$ , contains  $S_p$  as a maximal torus, and the set  $\Pi_p$  forms the set of simple roots for  $S_p$  in  $\tilde{G}_p$ . The linear representations of  $\tilde{G}_p$  defined over  $\mathbb{Q}_p$  are described by the theory of highest weights. In particular, it follows from the description of possible weights (see [27, Ch. 3, §2.2]) that the maximum in (4.8) can be taken over the dominant weights in  $\Phi_p$ . Moreover, it follows from the classification of semisimple groups and their representations that the set of all possible weights appearing in  $\rho|_{S_p}$  for some  $p$  is finite. Let  $\Delta_p$  be the product of all positive roots of  $S_p$  in  $G$  counted with multiplicities. Then we deduce from (4.9) that there exists  $\ell \in \mathbb{N}$ , independent of  $p$ , such that

$$(4.10) \quad \xi \leq \Delta_p^\ell$$

for all dominant weights  $\xi$  appearing in  $\rho|_{S_p}$ .

Now combining (4.7), (4.8) and (4.10), we deduce that for all  $g = u_1 s \omega u_2 \in U_p S_p^+ \Theta_p U_p$ , we have

$$\|g\|_p \leq (c'_p c''_p) |\Delta_p(s)|^\ell,$$

and when  $g \in B_{n,p}$ , we obtain

$$(4.11) \quad |\Delta_p(s)| \geq (c'_p c''_p)^{-1/\ell} (p^{\alpha_p})^{1/\ell}.$$

By [22, 3.2.15],

$$(4.12) \quad m_{G(\mathbb{Q}_p)}(U_p s U_p) \geq |\Delta_p(s)|.$$

Since both  $G(\mathbb{Z}_p)$  and  $U_p$  are compact open subgroups,  $G(\mathbb{Z}_p) \cap U_p$  has finite index in  $U_p$ , and there exists an open normal subgroup  $V_p$  of  $U_p$  contained in  $G(\mathbb{Z}_p)$ . Then for  $g = u_1 s \omega u_2 \in U_p S_p^+ \Theta_p U_p$ , we obtain

$$(4.13) \quad \begin{aligned} m_{G(\mathbb{Q}_p)}(V_p g V_p) &= m_{G(\mathbb{Q}_p)}(u_1 V_p s (\omega V_p \omega^{-1}) \omega u_2) \\ &= m_{G(\mathbb{Q}_p)}(V_p s (\omega V_p \omega^{-1})). \end{aligned}$$

By compactness, both  $V_p$  and  $V_p \cap \omega V_p \omega^{-1}$  have finite index in  $U_p$ . Therefore, it follows that

$$(4.14) \quad m_{G(\mathbb{Q}_p)}(U_p s U_p) \leq c_p''' m_{G(\mathbb{Q}_p)}(V_p s (\omega V_p \omega^{-1}))$$

for some  $c_p''' > 0$ . Hence, when  $g \in B_{n,p}$ , we deduce from (4.11)–(4.12) combined with (4.13)–(4.14) that

$$m_{G(\mathbb{Q}_p)}(V_p g V_p) \geq (c_p' c_p'')^{-1/\ell} (c_p''')^{-1} (p^{\alpha_p})^{1/\ell}.$$

Since  $n$  is admissible, there exists  $g \in B_{n,p}$ , and since  $V_p \subset G(\mathbb{Z}_p)$ , we have  $V_p g V_p \subset B_{n,p}$ . This implies the first claim of the proposition.

To prove the second claim, we assume that  $G$  is split over  $\mathbb{Q}$ . Then the Cartan decomposition (4.6) is of the form  $G(\mathbb{Q}_p) = U_p S_p^+ U_p$ . For almost all  $p$ , we can take  $S_p$  to be a maximal  $\mathbb{Q}$ -split torus  $S$ . Then the action of  $S(\mathbb{Q})$  on  $\mathbb{Q}^N$  is completely reducible. Since for almost all  $p$ , we have  $U_p = G(\mathbb{Z}_p)$ , the estimate (4.7) holds with  $c_p' = 1$  for almost all  $p$ . Since the action of  $S$  is completely reducible over  $\mathbb{Q}$ , for  $s \in S$ ,

$$\rho(s) = \sum_{\xi \in \Phi_p} \xi(s) v_\xi$$

for some  $v_\xi \in \text{Mat}_N(\mathbb{Q})$ . Hence, in the estimate (4.8),

$$c_p'' = \max_{\xi} \|v_\xi\|_p,$$

and it is clear that for almost all  $p$ ,  $c_p'' = 1$ . Finally, in the argument (4.13)–(4.14), we can take  $V_p = U_p = G(\mathbb{Z}_p)$  for almost all  $p$ . Therefore, for almost all  $p$ , we obtain

$$m_{G(\mathbb{Q}_p)}(B_{n,p}) \geq (p^{\alpha_p})^{1/\ell},$$

which completes the proof of the proposition.  $\square$

## 5. Counting estimates

In this section we prove an estimate on the number of lattice points in the regions  $B_n(x, \epsilon)$  with admissible  $n$ .

**Theorem 5.1.** *For every coprime  $n, q \in \mathbb{N}$  with admissible  $n$ ,  $\bar{\gamma} \in \Gamma_n$ ,  $x \in G(\mathbb{R})$ ,  $\kappa, \eta > 0$ , and  $\epsilon \in (0, \epsilon_0(\kappa, x)]$ , the following estimate holds*

$$\begin{aligned} |B_n(x, \epsilon) \cap \bar{\gamma} \Gamma_n(q)| &= \frac{m_{G_n}(B_n(x, \epsilon))}{m_{Y_{n,q}}(Y_{n,q})} + O_x \left( \epsilon^{\kappa + \dim(G)} m_{G_n^f}(B_n^f) \right) \\ &\quad + O_{x,\eta} \left( \epsilon^{-\kappa \dim(G)} m_{G_n^f}(B_n^f)^{1 - (4\ell(G))^{-1} + \eta} \right), \end{aligned}$$

where  $\epsilon_0(\kappa, x)$  and the implied constants are uniform over  $x$  in bounded sets.

This result will be deduced from Theorem 3.2 following the strategy of [11].

We set

$$(5.1) \quad \mathcal{O}_n(\epsilon) := B^\infty(e, \epsilon) \times \prod_{p|n} G(\mathbb{Z}_p),$$

where  $B^\infty(e, \epsilon)$  is the  $\epsilon$ -neighbourhood of identity in  $G(\mathbb{R})$  with respect to the Riemannian metric  $d$  on  $G(\mathbb{R})$ . Since the metric  $d$  is right invariant,  $\mathcal{O}_n(\epsilon)$  is a symmetric neighbourhood of identity in  $G_n$ .

**Lemma 5.2.** *For every  $n \in \mathbb{N}$  and  $\epsilon \in (0, 1]$ ,*

$$\epsilon^{\dim(G)} \ll m_{G_n}(\mathcal{O}_n(\epsilon)) \ll \epsilon^{\dim(G)}.$$

*Proof.* Writing  $B^\infty(e, \epsilon)$  in the exponential coordinates for the Riemannian metric, we deduce that

$$(5.2) \quad \epsilon^{\dim(G)} \ll m_{G(\mathbb{R})}(B^\infty(e, \epsilon)) \ll \epsilon^{\dim(G)}.$$

We recall that the measures  $m_{G(\mathbb{Q}_p)}$  are normalised so that  $m_{G(\mathbb{Q}_p)}(U_p) = 1$ , and  $U_p = G(\mathbb{Z}_p)$  for all but finitely many primes  $p$ . For the remaining primes, we observe that since  $U_p$  and  $G(\mathbb{Z}_p)$  are compact open subgroups in  $G(\mathbb{Q}_p)$ , it follows that  $U_p \cap G(\mathbb{Z}_p)$  has finite index in both  $U_p$  and  $G(\mathbb{Z}_p)$ . Therefore,

$$1 \ll \prod_{p|n} m_{G(\mathbb{Q}_p)}(G(\mathbb{Z}_p)) \ll 1.$$

Combining this estimate with (5.2), we deduce the lemma.  $\square$

**Lemma 5.3.** *We have*

(i) *For every  $n \in \mathbb{N}$ ,  $x \in G(\mathbb{R})$ , and  $\epsilon, \epsilon' \in (0, 1]$ ,*

$$\mathcal{O}_n(\epsilon') B_n(x, \epsilon) \mathcal{O}_n(\epsilon') \subset B_n(x, \epsilon + c_1(x)\epsilon'),$$

*where  $c_1(x)$  is uniform over  $x$  in bounded sets.*

(ii) *For every  $n \in \mathbb{N}$ ,  $x \in G(\mathbb{R})$ , and  $\epsilon, \epsilon' \in (0, \epsilon_0(x)]$ ,*

$$m_{G_n}(B_n(x, \epsilon + \epsilon')) \leq m_{G_n}(B_n(x, \epsilon)) + c_2(x)\epsilon' \epsilon^{\dim(G)-1} m_{G_n^f}(B_n^f),$$

*where  $\epsilon_0(x)$  and  $c_2(x)$  are uniform over  $x$  in bounded sets.*

*Proof.* To prove (i), we observe that  $B_n(x, \epsilon) = B^\infty(x, \epsilon) \times \prod_{p|n} B_{n,p}$  and the sets  $B_{n,p}$  are invariant under  $G(\mathbb{Z}_p)$ . Thus, it suffices to prove that for every  $u_1, u_2 \in B^\infty(e, \epsilon')$  and  $b \in B^\infty(x, \epsilon)$ , we have

$$(5.3) \quad d(x, u_1 b u_2) \leq \epsilon + c_1(x)\epsilon'.$$



Using the right invariance of the Riemannian metric on  $G(\mathbb{R})$ , we obtain

$$\begin{aligned} d(x, u_1 b u_2) &= d(x(bu_2)^{-1}, u_1) \leq d(x(bu_2)^{-1}, e) + d(e, u_1) \\ &\leq d(x, bu_2) + \epsilon' \leq d(xb^{-1}, bu_2 b^{-1}) + \epsilon' \\ &\leq d(xb^{-1}, e) + d(e, bu_2 b^{-1}) + \epsilon' \leq \epsilon + d(e, bu_2 b^{-1}) + \epsilon'. \end{aligned}$$

Since  $d(e, bu_2 b^{-1}) \ll_b d(e, u_2) \leq \epsilon'$  where the implied constant is uniform over  $b$  in bounded sets, we deduce (5.3).

To prove (ii), it is sufficient to show that

$$m_{G(\mathbb{R})}(B^\infty(x, \epsilon + \epsilon')) - m_{G(\mathbb{R})}(B^\infty(x, \epsilon)) \leq c_2(x) \epsilon' \epsilon^{\dim(G)-1}.$$

This follows from the disintegration formula for the measure  $m_{G(\mathbb{R})}$  as in [30, p. 66].  $\square$

Let  $\chi_{n,\epsilon}$  be the constant multiple of the characteristic function of the set  $\mathcal{O}_n(\epsilon)$  which is normalised so that  $\int_{G_n} \chi_{n,\epsilon} dm_{G_n} = 1$ . For  $\bar{\gamma} \in \Gamma_n$ , we also define a function  $\phi_{n,q,\epsilon}^{\bar{\gamma}}$  on  $Y_{n,q} = G_n/\Gamma_n(q)$  by

$$\phi_{n,q,\epsilon}^{\bar{\gamma}}(g\Gamma_n(q)) := \sum_{\gamma \in \Gamma_n(q)} \chi_{n,\epsilon}(g\gamma\bar{\gamma}) = \sum_{\delta \in \bar{\gamma}\Gamma_n(q)} \chi_{n,\epsilon}(g\delta).$$

Note that  $\bar{\gamma}\Gamma_n(q) = \Gamma_n(q)\bar{\gamma}$  because  $\Gamma_n(q)$  is normal in  $\Gamma_n$ . Clearly,  $\phi_{n,q,\epsilon}^{\bar{\gamma}}$  is a bounded measurable function on  $Y_{n,q}$  with compact support.

The following lemma shows that averages of this function can be used to approximate the cardinality  $|B_n(x, \epsilon) \cap \bar{\gamma}\Gamma_n(q)|$ .

**Lemma 5.4.** *For every coprime  $n, q \in \mathbb{N}$ ,  $\bar{\gamma} \in \Gamma_n$ ,  $x \in G(\mathbb{R})$ ,  $\epsilon, \epsilon' \in (0, 1]$ , and  $h \in \mathcal{O}_n(\epsilon')$ ,*

$$\begin{aligned} |B_n(x, \epsilon) \cap \bar{\gamma}\Gamma_n(q)| &\leq \int_{B_n(x, \epsilon + c_1(x)\epsilon')} \phi_{n,q,\epsilon'}^{\bar{\gamma}}(g^{-1}h\Gamma_n(q)) dm_{G_n}(g), \\ |B_n(x, \epsilon) \cap \bar{\gamma}\Gamma_n(q)| &\geq \int_{B_n(x, \epsilon - c_1(x)\epsilon')} \phi_{n,q,\epsilon'}^{\bar{\gamma}}(g^{-1}h\Gamma_n(q)) dm_{G_n}(g). \end{aligned}$$

*Proof.* Suppose that for some  $\delta \in B_n(x, \epsilon) \cap \bar{\gamma}\Gamma_n(q)$  and  $h \in \mathcal{O}_n(\epsilon')$ , we have  $\chi_{n,\epsilon'}(g^{-1}h\delta) \neq 0$ . Then by Lemma 5.3(i),

$$(5.4) \quad g \in h\delta\mathcal{O}_n(\epsilon')^{-1} \subset \mathcal{O}_n(\epsilon')B_n(\epsilon)\mathcal{O}_n(\epsilon') \subset B_n(x, \epsilon + c_1(x)\epsilon').$$

Since the function  $\chi_{n,\epsilon'}$  is non-negative, and

$$(5.5) \quad \int_{G_n} \chi_{n,\epsilon'}(g^{-1}) dm_{G_n}(g) = \int_{G_n} \chi_{n,\epsilon'} dm_{G_n} = 1,$$

it follows from (5.4) that

$$\begin{aligned}
 & \int_{B_n(x, \epsilon + c_1(x)\epsilon')} \phi_{n, q, \epsilon'}^{\bar{\gamma}}(g^{-1}h\Gamma_n(q)) dm_{G_n}(g) \\
 & \geq \sum_{\delta \in B_n(x, \epsilon) \cap \bar{\gamma}\Gamma_n(q)} \int_{B_n(x, \epsilon + c_1(x)\epsilon')} \chi_{n, \epsilon'}(g^{-1}h\delta) dm_{G_n}(g) \\
 & = \sum_{\delta \in B_n(x, \epsilon) \cap \bar{\gamma}\Gamma_n(q)} \int_{G_n} \chi_{n, \epsilon'}(g^{-1}h\delta) dm_{G_n}(g) \geq |B_n(x, \epsilon) \cap \bar{\gamma}\Gamma_n(q)|.
 \end{aligned}$$

This proves the first inequality.

To prove the second inequality, we observe that if  $\chi_{n, \epsilon'}(g^{-1}h\delta) \neq 0$  for some  $g \in B_n(x, \epsilon - c_1(x)\epsilon')$ ,  $h \in \mathcal{O}_n(\epsilon')$  and  $\delta \in \bar{\gamma}\Gamma_n(q)$ , then by Lemma 5.3(i),

$$\delta \in h^{-1}g\mathcal{O}_n(\epsilon') \subset \mathcal{O}_n(\epsilon')B_n(\epsilon - c_1(x)\epsilon')\mathcal{O}_n(\epsilon') \subset B_n(x, \epsilon).$$

This implies that

$$\begin{aligned}
 & \int_{B_n(x, \epsilon - c_1(x)\epsilon')} \phi_{n, q, \epsilon'}^{\bar{\gamma}}(g^{-1}h\Gamma_n(q)) dm_{G_n}(g) \\
 & = \sum_{\delta \in B_n(x, \epsilon) \cap \bar{\gamma}\Gamma_n(q)} \int_{B_n(x, \epsilon - c_1(x)\epsilon')} \chi_{n, \epsilon'}(g^{-1}h\delta) dm_{G_n}(g) \\
 & \leq \sum_{\delta \in B_n(x, \epsilon) \cap \bar{\gamma}\Gamma_n(q)} \int_{G_n} \chi_{n, \epsilon'}(g^{-1}h\delta) dm_{G_n}(g) = |B_n(x, \epsilon) \cap \bar{\gamma}\Gamma_n(q)|,
 \end{aligned}$$

where we used that  $\chi_{n, \epsilon'} \geq 0$  and (5.5). Hence, the second inequality also holds.  $\square$

*Proof of Theorem 5.1.* We first show that there exists  $\theta_0 > 0$  such that for any distinct  $\gamma_1, \gamma_2 \in \Gamma_n$ ,

$$(5.6) \quad \mathcal{O}_n(\theta_0)\gamma_1 \cap \mathcal{O}_n(\theta_0)\gamma_2 = \emptyset.$$

Indeed, suppose that for some

$$(g_\infty, g_f), (h_\infty, h_f) \in \mathcal{O}_n(\theta) = B^\infty(e, \theta) \times \prod_{p|n} G(\mathbb{Z}_p) \quad \text{and} \quad \gamma \in \Gamma_n,$$

we have

$$(g_\infty, g_f) = (h_\infty, h_f)(\gamma, \gamma).$$

Then

$$(\gamma, \gamma) = (g_\infty h_\infty^{-1}, g_f h_f^{-1}) \in B^\infty(e, \theta)^2 \times \prod_{p|n} G(\mathbb{Z}_p).$$

In particular, we conclude that  $\gamma \in G(\mathbb{Z})$ , and hence  $\gamma = e$  if  $\theta$  is sufficiently small.

It follows from (5.6) for every  $\theta \in (0, \theta_0]$ ,

$$(5.7) \quad \mu_{Y_{n,q}}(\mathcal{O}_n(\theta)\Gamma_n(q)) = \frac{m_{G_n}(\mathcal{O}_n(\theta))}{m_{Y_{n,q}}(Y_{n,q})}.$$

Moreover,

$$\int_{Y_{n,q}} \phi_{n,q,\theta}^{\bar{\gamma}} d\mu_{Y_{n,q}} = \int_{G_n} \chi_{n,\theta}(g) \frac{dm_{G_n}(g)}{m_{Y_{n,q}}(Y_{n,q})} = \frac{1}{m_{Y_{n,q}}(Y_{n,q})},$$

and similarly,

$$\|\phi_{n,q,\theta}^{\bar{\gamma}}\|_2^2 = \int_{G_n} \chi_{n,\theta}^2(g) \frac{dm_{G_n}(g)}{m_{Y_{n,q}}(Y_{n,q})} = \frac{m_{G_n}(\mathcal{O}_n(\theta))^{-1}}{m_{Y_{n,q}}(Y_{n,q})}.$$

By Theorem 3.2, for every  $\rho, \eta > 0$ , there exists  $c_\eta > 0$  such that

$$\begin{aligned} \left\| \pi_{Y_{n,q}}(\beta_{n,x,\rho}) \phi_{n,q,\theta}^{\bar{\gamma}} - \int_{Y_{n,q}} \phi_{n,q,\theta}^{\bar{\gamma}} d\mu_{Y_{n,q}} \right\|_2 \\ \leq c_\eta m_{G_n^f}(B_n^f)^{-(4\iota(G))^{-1}+\eta} \|\phi_{n,q,\theta}^{\bar{\gamma}}\|_2. \end{aligned}$$

Therefore, we deduce that for every  $\delta > 0$ ,

$$\begin{aligned} \mu_{Y_{n,q}} \left( \left\{ h\Gamma_n(q) : \left| \pi_{Y_{n,q}}(\beta_{n,x,\rho}) \phi_{n,q,\theta}^{\bar{\gamma}}(h\Gamma_n(q)) - \frac{1}{m_{Y_{n,q}}(Y_{n,q})} \right| > \delta \right\} \right) \\ \leq c_\eta^2 \delta^{-2} \frac{m_{G_n}(\mathcal{O}_n(\theta))^{-1}}{m_{Y_{n,q}}(Y_{n,q})} m_{G_n^f}(B_n^f)^{-(2\iota(G))^{-1}+2\eta}. \end{aligned}$$

Let us take  $\delta > 0$  such that

$$(5.8) \quad \mu_{Y_{n,q}}(\mathcal{O}_n(\theta)\Gamma_n(q)) > c_\eta^2 \delta^{-2} \frac{m_{G_n}(\mathcal{O}_n(\theta))^{-1}}{m_{Y_{n,q}}(Y_{n,q})} m_{G_n^f}(B_n^f)^{-(2\iota(G))^{-1}+2\eta}.$$

Note that it follows from (5.7) that we may choose  $\delta$  so that

$$\begin{aligned} \delta &= O_\eta \left( m_{G_n}(\mathcal{O}_n(\theta))^{-1} m_{G_n^f}(B_n^f)^{-(4\iota(G))^{-1}+\eta} \right) \\ &= O_\eta \left( \theta^{-\dim(G)} m_{G_n^f}(B_n^f)^{-(4\iota(G))^{-1}+\eta} \right), \end{aligned}$$

where we used Lemma 5.2. Then we deduce from (5.8) that there exists  $h \in \mathcal{O}_n(\theta)$  satisfying

$$\left| \pi_{Y_{n,q}}(\beta_{n,x,\rho}) \phi_{n,q,\theta}^{\bar{\gamma}}(h\Gamma_n(q)) - \frac{1}{m_{Y_{n,q}}(Y_{n,q})} \right| \leq \delta,$$

which gives

$$\left| \int_{B_n(x,\rho)} \phi_{n,q,\theta}^{\bar{\gamma}}(g^{-1}h\Gamma_n(q)) dm_{G_n}(g) - \frac{m_{G_n}(B_n(x,\rho))}{m_{Y_{n,q}}(Y_{n,q})} \right| \leq \delta m_{G_n}(B_n(x,\rho)).$$

Therefore,

$$(5.9) \quad \int_{B_n(x, \rho)} \phi_{n, q, \theta}^{\bar{\gamma}}(g^{-1}h\Gamma_n(q)) dm_{G_n}(g) \\ = \frac{m_{G_n}(B_n(x, \rho))}{m_{Y_{n, q}}(Y_{n, q})} + O_\eta \left( \theta^{-\dim(G)} m_{G_n}(B_n(x, \rho)) m_{G_n^f}(B_n^f)^{-(4\iota(G))^{-1} + \eta} \right).$$

Now to finish the proof of the theorem, we observe that according to Lemma 5.4,

$$|B_n(x, \epsilon) \cap \bar{\gamma}\Gamma_n(q)| \leq \int_{B_n(x, \epsilon + c_1(x)\epsilon^{\kappa+1})} \phi_{n, q, \epsilon^{\kappa+1}}^{\bar{\gamma}}(g^{-1}h\Gamma_n(q)) dm_{G_n}(g).$$

Combining this estimate with (5.9), we obtain

$$(5.10) \quad |B_n(x, \epsilon) \cap \bar{\gamma}\Gamma_n(q)| \leq \frac{m_{G_n}(B_n(x, \epsilon + c_1(x)\epsilon^{\kappa+1}))}{m_{Y_{n, q}}(Y_{n, q})} \\ + O_\eta \left( \epsilon^{-(\kappa+1)\dim(G)} m_{G_n}(B_n(x, \epsilon + c_1(x)\epsilon^{\kappa+1})) m_{G_n^f}(B_n^f)^{-(4\iota(G))^{-1} + \eta} \right).$$

By Lemma 5.3(ii), for sufficiently small  $\epsilon > 0$ ,

$$m_{G_n}(B_n(x, \epsilon + c_1(x)\epsilon^{\kappa+1})) = m_{G_n}(B_n(x, \epsilon)) + O_x \left( \epsilon^{\kappa + \dim(G)} m_{G_n^f}(B_n^f) \right),$$

where the implied constants are uniform over  $x$  in bounded sets. Also, we have

$$m_{G_n}(B_n(x, \epsilon + c_1(x)\epsilon^{\kappa+1})) = O_x \left( \epsilon^{\dim(G)} m_{G_n^f}(B_n^f) \right).$$

Therefore, (5.10) implies that

$$|B_n(x, \epsilon) \cap \bar{\gamma}\Gamma_n(q)| \leq \frac{m_{G_n}(B_n(x, \epsilon))}{m_{Y_{n, q}}(Y_{n, q})} + O_x \left( \epsilon^{\kappa + \dim(G)} m_{G_n^f}(B_n^f) \right) \\ + O_{x, \eta} \left( \epsilon^{-\kappa \dim(G)} m_{G_n^f}(B_n^f)^{1 - (4\iota(G))^{-1} + \eta} \right).$$

Here we used that  $m_{Y_{n, q}}(Y_{n, q}) \gg 1$  which follows from Proposition 4.1. This proves the required upper bound for  $|B_n(x, \epsilon) \cap \bar{\gamma}\Gamma_n(q)|$ .

The proof of the lower bound is similar, and we use the second estimate from Lemma 5.4. Note that in this proof we need to arrange that  $\epsilon - c_1(x)\epsilon^{\kappa+1} > 0$  which holds for sufficiently small  $\epsilon$ , depending on  $\kappa, x$ .  $\square$

## 6. Completion of the proof

In this section, we finish the proof of our main results stated in the Introduction. It would be convenient to introduce a parameter

$$(6.1) \quad a(G) := \limsup_{n \rightarrow \infty} \frac{\log m_{G_n^f}(B_n^f)}{\log n},$$

where the lim sup is taken over all admissible integers  $n$ . Note that according to Theorem 5.1, the quantity  $a(G)$  measures the polynomial growth rate of

the number of rational points in  $G$  with given denominator and lying in a bounded subset of  $G(\mathbb{R})$ . By Proposition 4.2,  $a(G) > 0$  if  $G$  is split over  $\mathbb{Q}$ . Given a finite set  $\mathcal{P}$  of prime numbers, we also set

$$a(G, \mathcal{P}) := \limsup_{n \rightarrow \infty}' \frac{\log m_{G_n^f}(B_n^f)}{\log n},$$

where the lim sup is taken over all admissible integers  $n$  with prime divisors in  $\mathcal{P}$ . If the group  $G$  is isotropic over  $\mathbb{Q}_p$  for every  $p \in \mathcal{P}$ , then  $a(G, \mathcal{P}) > 0$  by Proposition 4.2.

Throughout this section we use the following simplified notation:

$$d := \dim(G), \quad a := a(G), \quad a_{\mathcal{P}} := a(G, \mathcal{P}), \quad \iota := \iota(G).$$

Recall that  $\iota(G)$  is computed in terms of the integrability exponent of automorphic representations (see Section 2).

For an integral polynomial  $f$ , we denote by  $\Delta_n(f)$  the positive integer, coprime to  $n$ , representing the greatest common divisor of  $f(\gamma)$ ,  $\gamma \in \Gamma_n$ , in the ring  $\mathbb{Z}[1/n]$ . We denote by  $\delta_n(f)$  the number of prime factors of  $\Delta_n(f)$ . Note that  $\Delta_n(f)$  divides  $\Delta_1(f)$  and  $\delta_n(f) \leq \delta_1(f)$ .

The following result is a more precise version of Theorem 1.2.

**Theorem 6.1.** *Let  $G \subset \mathrm{GL}_N$  be a simply connected  $\mathbb{Q}$ -simple algebraic group defined over  $\mathbb{Q}$ , which is split over  $\mathbb{Q}$  and  $f_1, \dots, f_t$  a collection of polynomials as in the Introduction. Then for every  $x \in G(\mathbb{R})$ ,  $\alpha \in (0, \alpha_0)$  with  $\alpha_0 := d^{-1}a(4\iota)^{-1}$  and admissible  $n \geq n_0(\alpha, x)$ , there exists  $z \in G(\mathbb{Q})$  satisfying*

$$\begin{aligned} d(x, z) &\leq n^{-\alpha}, \\ \mathrm{den}(z) &= n, \end{aligned}$$

and  $r$ -prime in  $\mathrm{Mat}_N(\mathbb{Z}[1/n])$ , where

$$r = \delta_n(f_1 \cdots f_t) + \left\lceil \frac{9t \deg(f_1 \cdots f_t)(d+1)^2}{a(4\iota)^{-1} - \alpha d} \right\rceil.$$

The constant  $n_0(\alpha, x)$  is uniform over  $x$  in bounded subsets of  $G(\mathbb{R})$ .

*Proof.* By Theorem 5.1, for every coprime  $n, q \in \mathbb{N}$  with admissible  $n$ ,  $\bar{\gamma} \in \Gamma_n$ ,  $x \in G(\mathbb{R})$ ,  $\kappa, \eta > 0$ , and  $\epsilon \in (0, \epsilon_0(\kappa, x))$ ,

$$(6.2) \quad |B_n(x, \epsilon) \cap \bar{\gamma}\Gamma_n(q)| = \frac{m_{G_n}(B_n(x, \epsilon))}{m_{Y_{n,q}}(Y_{n,q})} + O_x\left(\epsilon^{\kappa+d} m_{G_n^f}(B_n^f)\right) \\ + O_{x,\eta}\left(\epsilon^{-\kappa d} m_{G_n^f}(B_n^f)^{1-(4\iota(G))^{-1}+\eta}\right),$$

where the implied constants are uniform over  $x$  in bounded sets. We apply this estimate with  $\epsilon_n = m_{G_n^f}(B_n^f)^{-\alpha'}$  where  $n$  is sufficiently large,  $\alpha' \in$

$(0, \alpha'_0)$ , and  $\alpha'_0 := \alpha_0/a = d^{-1}(4t)^{-1}$ . To optimise the error term in (6.2), we choose

$$(6.3) \quad \kappa = \frac{(4t)^{-1} - \alpha'd}{\alpha'(d+1)}.$$

Note that the parameter  $\alpha'_0$  is chosen to guarantee that  $\kappa > 0$ . Then (6.2) becomes

$$(6.4) \quad |B_n(x, \epsilon_n) \cap \bar{\gamma}\Gamma_n(q)| = \frac{m_{G_n}(B_n(x, \epsilon_n))}{m_{Y_{n,q}}(Y_{n,q})} + O_{x,\eta} \left( m_{G_n^f}(B_n^f)^{1-\alpha'(\kappa+d)+\eta} \right).$$

It would be convenient to set

$$T_n(x) := |B_n(x, \epsilon_n) \cap \Gamma_n|.$$

Since

$$m_{G(\mathbb{R})}(B^\infty(x, \epsilon)) = m_{G(\mathbb{R})}(B^\infty(e, \epsilon)) \gg \epsilon^d,$$

we have

$$(6.5) \quad m_{G_n}(B_n(x, \epsilon_n)) \gg m_{G_n^f}(B_n^f)^{1-\alpha'd}.$$

Using (6.5) and Proposition 4.1, we deduce from (6.4) with sufficiently small  $\eta > 0$  that

$$(6.6) \quad \begin{aligned} T_n(x) &\gg \frac{m_{G_n^f}(B_n^f)^{1-\alpha'd}}{m_{Y_n}(Y_n)} + O_{x,\eta} \left( m_{G_n^f}(B_n^f)^{1-\alpha'(\kappa+d)+\eta} \right) \\ &\gg m_{G_n^f}(B_n^f)^{1-\alpha'd} \end{aligned}$$

when  $n$  is sufficiently large. In particular, it follows from the definition of  $a = a(G)$  that for every  $b \in (0, a)$  and sufficiently large  $n$ ,

$$(6.7) \quad T_n(x) \gg n^{b(1-\alpha'd)}.$$

Since

$$m_{Y_{n,q}}(Y_{n,q}) = m_{Y_n}(Y_n) \cdot [\Gamma_n : \Gamma_n(q)],$$

it follows from (6.4) that

$$(6.8) \quad |B_n(x, \epsilon_n) \cap \bar{\gamma}\Gamma_n(q)| = \frac{T_n(x)}{[\Gamma_n : \Gamma_n(q)]} + O_{x,\eta} \left( m_{G_n^f}(B_n^f)^{1-\alpha'(\kappa+d)+\eta} \right).$$

Every  $w \in \mathbb{Z}[1/n]$  can be uniquely written as  $w = u \cdot [w]_n$  where  $u$  is a unit in  $\mathbb{Z}[1/n]$  and  $[w]_n \in \mathbb{Z}_{\geq 0}$  is coprime to  $n$ . Let  $f = f_1 \cdots f_t$  and  $\mathcal{P}_{n,z}$  be the set of prime numbers which are coprime to  $\Delta_n(f)n$  and bounded by  $z$ . We denote by  $S_{n,z}(x)$  the cardinality of the set of  $\gamma \in B_n(x, \epsilon_n) \cap \Gamma_n$  such that  $[f(\gamma)]_n$  is coprime to  $\mathcal{P}_{n,z}$  (equivalently,  $f(\gamma)$  is coprime to  $\mathcal{P}_{n,z}$  in the ring  $\mathbb{Z}[1/n]$ ).

We will apply the combinatorial sieve as in [13, Th. 7.4] (see also [26, Sec. 2]) to estimate the quantity  $S_{n,z}(x)$ . For this we let

$$a_k := |\{\gamma \in B_n(x, \epsilon_n) \cap \Gamma_n : [f(\gamma)]_n = k\}|.$$

Then  $T_n(x) = \sum_{k \geq 0} a_k$ . To apply the combinatorial sieve we need to verify the following three conditions:

(A<sub>0</sub>) For every square-free  $q$  divisible only by primes in  $\mathcal{P}_{n,z}$ ,

$$(6.9) \quad \sum_{k=0 \pmod q} a_k = \frac{\rho(q)}{q} T_n(x) + R_q,$$

where  $\rho(q)$  is a nonnegative multiplicative function such that for primes  $p \in \mathcal{P}_{n,z}$ , there exists  $c_1 < 1$  satisfying

$$(6.10) \quad \frac{\rho(p)}{p} \leq c_1.$$

(A<sub>1</sub>) Summing over square-free  $q$  divisible only by primes in  $\mathcal{P}_{n,z}$ ,

$$\sum'_{q \leq T_n(x)^\tau} |R_q| \leq c_2 T_n(x)^{1-\zeta}$$

for some  $c_2, \tau, \zeta > 0$ .

(A<sub>2</sub>) For some  $w \in [2, z]$ ,

$$(6.11) \quad -l \leq \sum_{p \in \mathcal{P}_{n,z}: w \leq p < z} \frac{\rho(p) \log p}{p} - t \log \frac{z}{w} \leq c_3$$

for some  $c_3, l, t > 0$ .

Once the conditions (A<sub>0</sub>), (A<sub>1</sub>), and (A<sub>2</sub>) are verified, by [13, Th. 7.4], for  $z = T_n(x)^{\tau/s}$  with  $s > 9t$ , we have the following estimate

$$(6.12) \quad S_{n,z}(x) \geq T_n(x) W(z) \left( C_1 - C_2 l \frac{(\log \log 3T_n(x))^{3t+2}}{\log T_n(x)} \right),$$

where

$$W(z) = \prod_{p \in \mathcal{P}_{n,z}: p \leq z} \left( 1 - \frac{\rho(p)}{p} \right),$$

and the constants  $C_1, C_2 > 0$  are determined by  $c_1, c_2, c_3, \tau, \zeta, t$ .

We denote by  $\Gamma_n^{(q)}$  the image of  $\Gamma_n$  in  $\mathrm{GL}_N(\mathbb{Z}[1/n]/q\mathbb{Z}[1/n])$ . Note that

$$\Gamma_n^{(q)} \simeq \Gamma_n / \Gamma_n(q).$$

To verify  $(A_0)$ , we observe that (6.8) implies that

$$\begin{aligned}
 & \sum_{k=0 \bmod q} a_k \\
 &= |\{\gamma \in B_n(x, \epsilon_n) \cap \Gamma_n : f(\gamma) = 0 \bmod q\}| \\
 &= \sum_{\bar{\gamma} \in \Gamma_n^{(q)} : f(\bar{\gamma}) = 0 \bmod q} |B_n(x, \epsilon_n) \cap \bar{\gamma}\Gamma_n(q)| \\
 &= |\Gamma_n^{(q)} \cap \{f = 0\}| \left( \frac{T_n(x)}{[\Gamma_n : \Gamma_n(q)]} + O_{x,\eta} \left( m_{G_n^f}(B_n^f)^{1-\alpha'(\kappa+d)+\eta} \right) \right) \\
 &= \frac{\rho(q)}{q} T_n(x) + O_{x,\eta} \left( |\Gamma_n^{(q)} \cap \{f = 0\}| \cdot m_{G_n^f}(B_n^f)^{1-\alpha'(\kappa+d)+\eta} \right),
 \end{aligned}$$

where

$$\rho(q) := \frac{q|\Gamma_n^{(q)} \cap \{f = 0\}|}{[\Gamma_n : \Gamma_n(q)]}.$$

Since  $q$  is coprime to  $\Delta_n(f)$ , the polynomial  $f$  is not identically zero on  $\Gamma_n^{(q)}$  and  $\rho(q) < q$ . Moreover, it follows from the strong approximation property that

$$(6.13) \quad \Gamma_n^{(q)} \simeq \prod_{p|q} \Gamma_n^{(p)},$$

and for all but finitely many primes  $p$ ,

$$\Gamma_n^{(p)} \simeq G^{(p)}(\mathbb{F}_p),$$

where  $G^{(p)}$  denotes the reduction of  $G$  modulo  $p$ . It follows from (6.13) that the function  $\rho$  is multiplicative

We observe that

$$G^{(p)} \cap \{f = 0\} = X_1^{(p)} \cup \dots \cup X_t^{(p)}$$

where the varieties  $X_i^{(p)} = G^{(p)} \cap \{f_i = 0\}$  are absolutely irreducible for all but finitely many  $p$  by the Noether theorem. Therefore, using the Lang-Weil estimate [20], we deduce that

$$|X_i^{(p)}(\mathbb{F}_p)| = p^{\dim(G)-1} + O(p^{\dim(G)-3/2}).$$

Moreover, since  $f_i$ 's are coprime, for  $i \neq j$ ,  $\dim(X_i^{(p)} \cap X_j^{(p)}) \leq \dim(G) - 2$  and

$$|(X_i^{(p)} \cap X_j^{(p)})(\mathbb{F}_p)| = O(p^{\dim(G)-2}).$$

Hence,

$$|G^{(p)}(\mathbb{F}_p) \cap \{f = 0\}| = tp^{\dim(G)-1} + O(p^{\dim(G)-3/2}).$$

Since

$$|G^{(p)}(\mathbb{F}_p)| = p^{\dim(G)} + O(p^{\dim(G)-1/2}),$$



we deduce that

$$(6.14) \quad \rho(p) = t + O(p^{-1/2}).$$

Since  $\rho(p) < p$  for all primes  $p \in \mathcal{P}_{n,z}$ , this also implies that (6.10) holds. This establishes  $(A_0)$  with

$$R_q = O_{x,\eta} \left( |\Gamma_n^{(q)} \cap \{f = 0\}| \cdot m_{G_n^f}(B_n^f)^{1-\alpha'(\kappa+d)+\eta} \right).$$

It follows from (6.6) that

$$\begin{aligned} \sum'_{q \leq T_n(x)^\tau} |R_q| &\ll_{x,\eta} \sum_{q \leq T_n(x)^\tau} q^d m_{G_n^f}(B_n^f)^{1-\alpha'(\kappa+d)+\eta} \\ &\ll_{x,\eta} (T_n(x)^\tau)^{d+1} T_n(x)^{(1-\alpha'(\kappa+d)+\eta)/(1-\alpha'd)}. \end{aligned}$$

Since  $\eta$  can be taken to be arbitrary positive number, we conclude that

$$\sum'_{q \leq T_n(x)^\tau} |R_q| \ll_{x,\eta} T_n(x)^{1-\zeta}$$

with some  $\zeta > 0$ , when

$$(6.15) \quad \tau < \tau_0 := (1 - (1 - \alpha'(\kappa + d))/(1 - \alpha'd))/(d + 1).$$

Note that since  $\kappa > 0$ , it follows that  $\tau_0 > 0$ . This proves  $(A_1)$ . From (6.3),

$$\tau_0 = \frac{(4t)^{-1} - \alpha'd}{(d+1)^2(1-\alpha'd)}.$$

It follows from (6.14) and [24, Th. 2.7(b)] that

$$-c_3 \leq \sum_{w \leq p < z} \frac{\rho(p) \log p}{p} - t \log \frac{z}{w} \leq c_3$$

for some  $c_3 > 0$ . In particular, this implies the upper bound in (6.11). To establish the lower bound, we use the estimate

$$\sum_{p|q} \frac{\log p}{p} = O(\log \log q)$$

(see, for instance, [12, Lem. 5.2]). This implies that condition  $(A_2)$  holds with

$$l = O(\log \log(\Delta_n(f)n)) = O(\log \log n).$$

Now we are in position to apply the main sieving argument (6.12). Note that for  $z = T_n(x)^{\tau/s}$ , it follows from (6.14) that

$$W(z) \gg (\log z)^{-t}.$$

Therefore, (6.12) gives

$$S_{n, T_n(x)^{\tau/s}}(x) \geq \frac{T_n(x)}{(\log T_n(x))^t} \left( C_1 - C_2'(\log \log n) \frac{(\log \log 3T_n(x))^{3t+2}}{\log T_n(x)} \right),$$

Using (6.7), we deduce that for sufficiently large  $n$ ,

$$S_{n, T_n(x)^{\tau/s}}(x) \gg_x \frac{T_n(x)}{(\log T_n(x))^t}.$$

Every  $\gamma$  counted in  $S_{n, T_n(x)^{\tau/s}}(x)$  satisfies

$$(6.16) \quad d(x, \gamma) \leq \epsilon_n \quad \text{and} \quad \gamma \in B_n^f.$$

This implies that  $\text{den}(\gamma) = n$ , and the numerator of  $\gamma$  is  $O_x(n)$ . In particular,

$$[f(\gamma)]_n \ll_x n^{\deg(f)}.$$

On the other hand, for any  $\gamma$  counted in  $S_{n, T_n(x)^{\tau/s}}(x)$ , all prime numbers  $p$  which are coprime to  $\Delta_n(f)$  and divide  $[f(\gamma)]_n$  must satisfy

$$p > z = T_n(x)^{\tau/s}.$$

Thus, using (6.7), we deduce that the number of such prime factors is bounded from above by

$$\frac{\log(n^{\deg(f)} + O_x(1))}{\log(T_n(x)^{\tau/s})} = \frac{s \deg(f)}{\tau b(1 - \alpha'd)} + o_x(1).$$

Since this estimate holds for all  $s > 9t$ ,  $b < a$  and  $\tau < \tau_0$ , the number of such prime factors is at most

$$\left\lceil \frac{9t \deg(f)}{\tau_0 a(1 - \alpha'd)} \right\rceil = \left\lceil \frac{9t \deg(f)(d+1)^2}{a((4t)^{-1} - \alpha'd)} \right\rceil$$

provided that  $n$  is sufficiently large. Therefore, we conclude that the element  $\gamma$  is  $r$ -prime in  $\text{Mat}_N(\mathbb{Z}[1/n])$  with

$$r = \delta_n(f) + \left\lceil \frac{9t \deg(f)(d+1)^2}{a((4t)^{-1} - \alpha'd)} \right\rceil.$$

For every  $b \in (0, a)$  and sufficiently large  $n$ ,

$$\epsilon_n = m_{G_n^f}(B_n^f)^{-\alpha'} \leq n^{-b\alpha'}.$$

Therefore, it follows from (6.16) that for every  $\alpha < a\alpha'_0 = \alpha_0$  and sufficiently large  $n$ ,

$$d(x, \gamma) \leq n^{-\alpha}.$$

This completes the proof of the theorem.  $\square$

When  $G$  is not assumed to be split over  $\mathbb{Q}$ , we have the following version of Theorem 6.1.

**Theorem 6.2.** *Let  $G \subset \mathrm{GL}_N$  be a simply connected  $\mathbb{Q}$ -simple algebraic group defined over  $\mathbb{Q}$ ,  $f_1, \dots, f_t$  a collection of polynomials as in the Introduction, and  $\mathcal{P}$  a finite collection of prime number such that  $G$  is isotropic over  $\mathbb{Q}_p$  for all  $p \in \mathcal{P}$ . Then for every  $x \in G(\mathbb{R})$ ,  $\alpha \in (0, \alpha_0)$  with  $\alpha_0 := d^{-1}a_{\mathcal{P}}(2t)^{-1}$  and admissible  $n \geq n_0(\alpha, x)$  whose prime divisors are in  $\mathcal{P}$ , there exists  $z \in G(\mathbb{Q})$  satisfying*

$$\begin{aligned} d(x, z) &\leq n^{-\alpha}, \\ \mathrm{den}(z) &= n, \end{aligned}$$

and  $r$ -prime in  $\mathrm{Mat}_N(\mathbb{Z}[1/n])$ , where

$$r = \delta_n(f_1 \cdots f_t) + \left\lceil \frac{9t \deg(f_1 \cdots f_t)(d+1)^2}{a_{\mathcal{P}}(2t)^{-1} - \alpha d} \right\rceil.$$

The constant  $n_0(\alpha, x)$  is uniform over  $x$  in bounded subsets of  $G(\mathbb{R})$ .

The proof of Theorem 6.2 goes along the same lines as the proof of Theorem 6.1, but instead of the estimate on the averaging operators given by Theorem 3.2, we use Theorem 3.1. This leads to slightly better estimates for the parameters  $\alpha$  and  $r$ , but the parameter  $n_0(\alpha, x)$  now might depend on the set  $\mathcal{P}$ .

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