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# Waring-Goldbach problem: two squares and higher powers

par Yingchun CAI

Résumé. Soit $P_r$ l'ensemble des nombres presque-premiers avec au plus $r$ facteurs premiers comptés avec avec multiplicité. Dans cet article, on motre que pour tout entier pair $N$ suffisamment grand, l'équation

$$N = x^2 + p^2 + p_1^5 + p_2^5 + p_3^5 + p_4^5 + p_5^5 + p_6^5$$

a des solutions avec $x$ un $P_6$ et les autres étant des nombres premiers. Ceci est une amélioration de résultats antérieurs de C. Hooley.

Abstract. Let $P_r$ denote an almost-prime with at most $r$ prime factors, counted according to multiplicity. In this paper it is proved, that for every sufficiently large even integer $N$, the equation

$$N = x^2 + p^2 + p_1^5 + p_2^5 + p_3^5 + p_4^5 + p_5^5 + p_6^5$$

is solvable with $x$ being a $P_6$ and the other variables primes. This result constitutes an enhancement upon that of C. Hooley.

## 1. Introduction

Waring problem of mixed type concerns the representation of a natural number $N$ as the form

$$N = x_1^{k_1} + \cdots + x_s^{k_s} \quad (k_1 \leq \cdots \leq k_s). \tag{1.1}$$

Not very much is known about results of this kind. For references in this aspect we refer the reader to section P12 of LeVeque's *Reviews in number theory*, the bibliography in [14] and the recent papers by J. Brűdern and by T. D. Wooley.

Hooley [7] introduced divisor sum techniques into the investigation of (1.1). In particular, these techniques provided asymptotic formulae for

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the number of representations of the equations

$$N = x_1^2 + x_2^2 + y_1^3 + y_2^3 + y_3^3 + y_4^3,$$
$$N = x_1^2 + x_2^2 + y_1^4 + y_2^4 + y_3^4 + y_4^4 + y_5^4.$$

Afterwards, Hooley's results were generalized by Brűdern [1]. By introducing a prunning process into the Hardy-Littlewood method, he proved that the equations

$$N = x_1^2 + x_2^2 + y_1^5 + y_2^5 + y_3^5 + y_4^5 + y_5^5 + y_6^5,$$
$$N = x_1^2 + x_2^2 + y_1^4 + y_2^6 + y_3^6 + y_4^6 + y_5^6 + y_6^6 + y_7^6$$

are solvable for sufficiently large integer $N$.

In principle the Hardy-Littlewood method is applicable to problems of this kind, but one has to overcome various difficulties not experienced in the pure Waring problem (1.1) with $k_1 = \cdots = k_s$. In particular, the choice of the relevant parameters in the definition of major and minor arcs tends to become complicated if a deeper representation problem (1.1) is under consideration.

In view of Hooley's and Brűdern's results, it is plausible to expect that for every sufficiently large integer in some congruence classes, the above equations are solvable in primes. But this expectation is probably far out of the reach of modern number theory techniques. However, motivated by [2], [3], the sieve theory and the Hardy-Littlewood method enable us to obtain the following approximation to them.

In this paper, $P_r$ denote an almost-prime with at most $r$ prime factors, counted according to multiplicity.

**Theorem 1.1.** *For every sufficiently large even integer $N$, the number of solutions of the equation*

$$N = x^2 + p^2 + p_1^3 + p_2^3 + p_3^3 + p_4^3 \tag{1.2}$$

*with $x$ being a $P_3$ and the other variables primes, is*

$$\gg N^{\frac{11}{9}} \log^{-6} N.$$

**Theorem 1.2.** *For every sufficiently large integer $N \equiv 7$ or $103 \pmod{240}$, the number of solutions of the equation*

$$N = x^2 + p^2 + p_1^4 + p_2^4 + p_3^4 + p_4^4 + p_5^4 \tag{1.3}$$

*with $x$ being a $P_4$ and the other variables primes, is*

$$\gg N^{\frac{33}{28}} \log^{-7} N.$$

**Theorem 1.3.** *For every sufficiently large even integer $N$, the number of solutions of the equation*

$$N = x^2 + p^2 + p_1^5 + p_2^5 + p_3^5 + p_4^5 + p_5^5 + p_6^5 \tag{1.4}$$

*with $x$ being a $P_6$ and the other variables primes, is*

$$\gg N^{\frac{91}{85}} \log^{-8} N.$$

In this paper we present the detailed proof of Theorem 1.3 only, by the admissible exponents $(1, \frac{5}{6}, \frac{5}{6})$ (see [13]) and $(1, \frac{6}{7}, \frac{6}{7})$ (see [11]), Theorem 1.1 and Theorem 1.2 can be proved along similar line of arguments respectively.

## 2. Notation and Some Preliminary Lemmas

In this paper, $\varepsilon \in (0, 10^{-10})$ and $N$ denotes a sufficiently large even integer in terms of $\varepsilon$. The constants in $O$-term and $\ll$-symbol depend at most on $\varepsilon$. By $A \asymp B$ we mean that $A \ll B$ and $B \ll A$. The letter $p$, with or without subscript, is reserved for a prime number. We denote by $(m, n)$ the greatest common divisor of $m$ and $n$. As usual, $\varphi(n)$ and $\mu(n)$ denote Euler's function and Mőbius function respectively. By $\tau(n)$ we denote the divisor function, and by $a(n)$ we denote an arithmetical function bounded above by $\tau(n)$. We use $e(\alpha)$ to denote $e^{2\pi i\alpha}$ and $e_q(\alpha) = e(\alpha/q)$. We denote by $\sum\limits_{x(q)}$ and $\sum\limits_{x(q)*}$ sums with $x$ running over a complete system and a reduced system of residues modulo $q$ respectively. We always denote by $\chi$ a Dirichlet character $(\bmod q)$, and by $\chi^0$ the principal Dirichlet character $(\bmod q)$. By $\sum\limits_{\chi(q)}$ we denote a sum with $\chi$ running over the Dirichlet characters $(\bmod q)$.
Let

$$A = 10^{10}, \quad Q_0 = \log^{20A} N, \quad Q_1 = N^{\frac{73}{170}+10\varepsilon}, \quad Q_2 = N^{\frac{1}{2}-10\varepsilon},$$

$$D = N^{\frac{9}{170}-10\varepsilon}, \ \ z = D^{\frac{1}{3}}, \ \ U_k = 0.5N^{\frac{1}{k}}, \ \ V_1 = 0.5N^{\frac{72}{425}}, \ \ V_2 = 0.5N^{\frac{64}{425}},$$

$$\mathcal{M}_r = \{\ m \mid U_2 < m \le 2U_2, m = p_1 \cdots p_r, z \le p_1 \le \cdots \le p_{r-1}, \\ p_1 \cdots p_{r-2} p_{r-1}^2 \le 2U_2\} \ \ (7 \le r \le 28),$$

$$G_k(\chi, a) = \sum_{r(q)} \chi(r) e_q(ar^k), \ \ S_k^*(q, a) = G_k(\chi^0, a), \ \ S_k(q, a) = \sum_{r(q)} e_q(ar^k),$$

$$B_d(q, N) = \sum_{a(q)*} S_2(q, ad^2) S_2^*(q, a) S_5^{*6}(q, a) e_q(-aN),$$

$$A_d(q, N) = \frac{B_d(q, N)}{q\varphi^7(q)}, \quad A(q, N) = A_1(q, N),$$

$$\mathfrak{S}_d(N) = \sum_{q=1}^{\infty} A_d(q, N), \quad \mathfrak{S}(N) = \mathfrak{S}_1(N),$$

$$f_k(\alpha)=\sum_{U_k<p\le 2U_k}(\log p)e(\alpha p^k),\quad f_{2,r}(\alpha)=\sum_{m\in\mathcal{M}_r}e(\alpha m^2),$$

$$g_j(\alpha)=\sum_{V_j<p\le 2V_j}(\log p)e(\alpha p^5),\quad u_k(\lambda)=\int_{U_k}^{2U_k}e(\lambda u^k)du,$$

$$v_j(\lambda)=\int_{V_j}^{2V_j}e(\lambda u^5)du,\ \mathfrak{I}(N)=\int_{-\infty}^{\infty}u_2^2(\lambda)u_5^3(\lambda)v_1(\lambda)v_2^2(\lambda)e(-\lambda N)d\lambda.$$

**Lemma 2.1** (See [10]). *Let*

$$F_k(\alpha)=\sum_{U_k<n\le 2U_k}e(\alpha n^k),\ \ G_j(\alpha)=\sum_{V_j<n\le 2V_j}e(\alpha n^5).$$

*Then we have*

$$\int_0^1|F_5(\alpha)G_1(\alpha)G_2^2(\alpha)|^2d\alpha\ll N^{\frac{57}{85}+\varepsilon}.$$

*Proof.* This is the case $j=7$ of Lemma 6.1 in [10]. □

**Lemma 2.2.** *For $(q,a)=1$ we have*

(*i*) $S_k(q,a)\ll q^{1-\frac{1}{k}}$;
(*ii*) $G_k(\chi,a)\ll q^{\frac12+\varepsilon}$.

*In particular, for $(p,a)=1$ we have*

(*iii*) $|S_k(p,a)|\le((k,p-1)-1)p^{\frac12}$;
(*iv*) $|S_k^*(p,a)|\le((k,p-1)-1)p^{\frac12}+1$;
(*v*) $S_k^*(p^l,a)=0$ *for* $l\ge\gamma(p)$,

*where*

$$\gamma(p)=\begin{cases}\theta+2 & \text{if } p^\theta\|k,\ \ p\ne 2 \text{ or } p=2,\theta=0\\ \theta+3 & \text{if } p^\theta\|k,\ \ p=2,\theta>0.\end{cases}$$

*Proof.* For $(i)$, $(iii)$ and $(iv)$, see Theorem 4.2 and Lemma 4.3 in [14] respectively. For $(ii)$, see Chapter VI, problem 14 in [15]. For $(v)$, see Lemma 8.3 in [8]. □

**Lemma 2.3.** *Let $F_k(\alpha)$ and $G_j(\alpha)$ be the functions defined in Lemma 2.1. Then we have*

$$\int_0^1|F_2(\alpha)F_5(\alpha)G_1(\alpha)G_2^2(\alpha)|^2d\alpha\ll N^{\frac{114}{85}}.$$

*Proof.* For $(a,q)=1$, put

$$\mathfrak{I}(q,a)=\left(\frac{a}{q}-\frac{1}{10qN^{\frac{2}{3}}},\ \frac{a}{q}+\frac{1}{10qN^{\frac{2}{3}}}\right],\ \mathfrak{I}_0(q,a)=\left(\frac{a}{q}-\frac{1}{N^{\frac{5}{6}}},\ \frac{a}{q}+\frac{1}{N^{\frac{5}{6}}}\right],$$

$$\mathfrak{I}=\bigcup_{1\le q\le N^{\frac{1}{3}}}\ \bigcup_{\substack{a=1\\(a,q)=1}}^{q}\mathfrak{I}(q,a),\quad \mathfrak{i}=\left(-\frac{1}{10N^{\frac{2}{3}}},\ 1-\frac{1}{10N^{\frac{2}{3}}}\right]\setminus\mathfrak{I},$$

$$\mathfrak{I}_1(q,a)=\left(\frac{a}{q}-\frac{1}{qQ_0},\ \frac{a}{q}+\frac{1}{qQ_0}\right]\setminus\mathfrak{I}_0(q,a).$$

Then we have

$$\begin{aligned}(2.1)\quad \int_0^1 &|F_2(\alpha)F_5(\alpha)G_1(\alpha)G_2^2(\alpha)|^2d\alpha\\ &=\int_{-\frac{1}{10N^{\frac{2}{3}}}}^{1-\frac{1}{10N^{\frac{2}{3}}}}|F_2(\alpha)F_5(\alpha)G_1(\alpha)G_2^2(\alpha)|^2d\alpha\\ &=\int_{\mathfrak{I}}|F_2(\alpha)F_5(\alpha)G_1(\alpha)G_2^2(\alpha)|^2d\alpha\\ &\qquad+\int_{\mathfrak{i}}|F_2(\alpha)F_5(\alpha)G_1(\alpha)G_2^2(\alpha)|^2d\alpha.\end{aligned}$$

It follows from Weyl's inequality that for $\alpha\in\mathfrak{i}$

$$F_2(\alpha)\ll N^{\frac{1}{3}+\varepsilon},$$

and hence we get

$$\begin{aligned}(2.2)\quad \int_{\mathfrak{i}}|F_2(\alpha)F_5(\alpha)G_1(\alpha)G_2^2(\alpha)|^2d\alpha &\ll N^{\frac{2}{3}+2\varepsilon}\int_0^1|F_5(\alpha)G_1(\alpha)G_2^2(\alpha)|^2d\alpha\\ &\ll N^{\frac{114}{85}},\end{aligned}$$

where Lemma 2.1 is used.

For $\alpha=\frac{a}{q}+\beta$, let

$$W_k(\alpha)=\frac{S_k(q,a)}{q}u_k(\beta).$$

Then for $\alpha=\frac{a}{q}+\beta\in\mathfrak{I}$, by Theorem 4.1 in Vaughan [14], we obtain

$$\begin{aligned}F_2(\alpha)&=W_2(\alpha)+O(N^{\frac{1}{6}+\varepsilon}),\\ F_2^2(\alpha)&\ll|W_2(\alpha)|^2+O(N^{\frac{2}{3}+\varepsilon})\end{aligned}$$

and so we have

$$
\begin{aligned}
(2.3)\quad & \int_{\mathfrak{I}} |F_2(\alpha)F_5(\alpha)G_1(\alpha)G_2^2(\alpha)|^2 d\alpha \\
&\ll \int_{\mathfrak{I}} |W_2(\alpha)F_5(\alpha)G_1(\alpha)G_2^2(\alpha)|^2 d\alpha \\
&\quad + O\left(N^{\frac{2}{3}+\varepsilon}\int_0^1 |F_5(\alpha)G_1(\alpha)G_2^2(\alpha)|^2 d\alpha\right) \\
&\ll \sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{I}\cap\mathfrak{I}_0(q,a)} |W_2(\alpha)F_5(\alpha)G_1(\alpha)G_2^2(\alpha)|^2 d\alpha \\
&\quad + \sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{I}\cap\mathfrak{I}_1(q,a)} |W_2(\alpha)F_5(\alpha)G_1(\alpha)G_2^2(\alpha)|^2 d\alpha + O(N^{\frac{114}{85}}),
\end{aligned}
$$

where Lemma 2.1 is used.

For $\alpha = \frac{a}{q} + \beta \in \mathfrak{I}_1(q,a)$, by Lemma 4.2 in [12], we get

$$
W_2(\alpha) \ll N^{\frac{1}{3}}, \tag{2.4}
$$

from which and Lemma 2.1, we have

$$
\begin{aligned}
(2.5)\quad & \sum_{q\le Q_0}\sum_{\substack{a=-q\\a,q)=1}}^{2q}\int_{\mathfrak{I}\cap\mathfrak{I}_1(q,a)} |W_2(\alpha)F_5(\alpha)G_1(\alpha)G_2^2(\alpha)|^2 d\alpha \\
&\ll N^{\frac{2}{3}}\int_0^1 |F_5(\alpha)G_1(\alpha)G_2^2(\alpha)|^2 d\alpha \\
&\ll N^{\frac{114}{85}}.
\end{aligned}
$$

For $\alpha = \frac{a}{q} + \beta \in \mathfrak{I}_0(q,a)$, by Theorem 4.1 in [14] and Lemma 4.2 in [12], we obtain

$$
F_5(\alpha) = W_5(\alpha) + O(q^{\frac{1}{2}+\varepsilon}) \ll \frac{U_5}{q^{\frac{1}{5}}(1+|\beta|N)}, \tag{2.6}
$$

$$
G_j(\alpha) = \frac{S_5(q,a)}{q}v_j(\beta) + O(q^{\frac{1}{2}+\varepsilon}) \ll \frac{V_j}{q^{\frac{1}{5}}}, \quad j=1,2, \tag{2.7}
$$

where Lemma 2.2 $(i)$ is used.

From (2.6) and (2.7), we have

$$\begin{aligned}(2.8)\quad &\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{I}\cap\mathfrak{I}_0(q,a)}|W_2(\alpha)F_5(\alpha)G_1(\alpha)G_2^2(\alpha)|^2d\alpha\\ &\ll \sum_{1\le q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q} q^{-\frac{13}{5}}\int_{|\beta|\le N^{-\frac{5}{6}}}\frac{N^{\frac{199}{85}}}{(1+|\beta|N)^4}d\beta\\ &\ll N^{\frac{114}{85}}.\end{aligned}$$

Now by (2.1)–(2.3), (2.5) and (2.8), the proof of Lemma 2.3 is completed. □

**Lemma 2.4.** *For $\alpha=\frac{a}{q}+\beta$, let*

$$W(\alpha)=\sum_{d\le D}\frac{a(d)}{dq}S_2(q,ad^2)u_2(\beta), \tag{2.9}$$

$$\Delta_k(\alpha)=f_k(\alpha)-\frac{S_k^*(q,a)}{\varphi(q)}\sum_{U_k<n\le 2U_k}e(\beta n^k), \tag{2.10}$$

$$\mathcal{I}(q,a)=\left(\frac{a}{q}-\frac{1}{qQ_0},\frac{a}{q}+\frac{1}{qQ_0}\right]. \tag{2.11}$$

*Then we have*

$$\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}(q,a)}|W^2(\alpha)\Delta_k^2(\alpha)|d\alpha\ll N^{\frac{2}{k}}\log^{-100A}N.$$

*Proof.* From Lemma 2.2 (*i*) we get

$$\begin{aligned}\left|W\left(\frac{a}{q}+\beta\right)\right| &\ll \sum_{d\le D}\frac{\tau(d)}{d}(q,d^2)^{\frac12}q^{-\frac12}|u_2(\beta)|\\ &\ll q^{-\frac12+\varepsilon}|u_2(\beta)|\log^2N,\end{aligned} \tag{2.12}$$

where the inequalities $(q,d^2)\le(q,d)^2$ and $\tau(dl)\le\tau(d)\tau(l)$ are used.

On the other hand, let

$$E(\chi)=\begin{cases}1, & \chi=\chi^0,\\ 0, & \chi\ne\chi^0,\end{cases}$$

$$J_k(\chi,\beta)=\sum_{U_k<p\le 2U_k}\chi(p)e(\beta p^k)\log p-E(\chi)\sum_{U_k<n\le 2U_k}e(\beta n^k).$$

Then for $\alpha = \frac{a}{q} + \beta$, we have

$$\begin{aligned}\Delta_k(\alpha) &= \frac{1}{\varphi(q)} \sum_{\chi(q)} G_k(\bar{\chi}, a) J_k(\chi, \beta) \\ &\ll q^{-\frac{1}{2}+\varepsilon} \sum_{\chi(q)} |J_k(\chi, \beta)| \\ &\ll q^{\varepsilon} \left( \sum_{\chi(q)} |J_k(\chi, \beta)|^2 \right)^{\frac{1}{2}},\end{aligned} \tag{2.13}$$

where Lemma 2.2 $(ii)$ and Cauchy's inequality are used.

From the standard estimate

$$u_k(\beta) \ll \frac{U_k}{1 + N|\beta|}, \tag{2.14}$$

which follows from Lemma 4.2 in [14], and (2.12)–(2.13), we get

$$\begin{aligned}&\sum_{q \le Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathcal{I}(q,a)} |W^2(\alpha) \Delta_k^2(\alpha)| d\alpha \\ &\ll (\log^4 N) \sum_{q \le Q_0} q^{-1+4\varepsilon} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \sum_{\chi(q)} \int_{|\beta| \le \frac{1}{Q_0}} |u_2(\beta)|^2 |J_k(\chi, \beta)|^2 d\beta \\ &\ll (\log^4 N) \sum_{q \le Q_0} q^{-1+4\varepsilon} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \sum_{\chi(q)} \left( N \int_{|\beta| \le \frac{1}{N}} |J_k(\chi, \beta)|^2 d\beta \right. \\ &\qquad \left. + \max_{1 \le Z \le N Q_0^{-1}} (\log N) N Z^{-2} \int_{\frac{Z}{N} \le |\beta| \le \frac{2Z}{N}} |J_k(\chi, \beta)|^2 d\beta \right).\end{aligned} \tag{2.15}$$

By Gallagher's Lemma in [5] and Siegel's Lemma in prime number theory, we have

$$\begin{aligned}&\int_{|\beta| \le \frac{1}{N}} |J_k(\chi, \beta)|^2 d\beta \\ &\ll \frac{1}{N^2} \int_2^N \left| \sum_{\substack{U_k < p \le 2U_k \\ \theta < p^k \le \theta + \frac{N}{2}}} \chi(p) \log p - E(\chi) \sum_{\substack{U_k < n \le 2U_k \\ \theta < n^k \le \theta + \frac{N}{2}}} 1 \right|^2 d\theta \\ &\ll \frac{1}{N^2} N U_k^2 \exp(-\log^{\frac{1}{3}} N),\end{aligned} \tag{2.16}$$

and

$$\int_{\frac{Z}{N}\le|\beta|\le\frac{2Z}{N}} |J_k(\chi,\beta)|^2 d\beta \ll Z^2 N^{\frac{2}{k}-1}\exp(-\log^{\frac{1}{3}} N). \tag{2.17}$$

It follows from (2.15)–(2.17) that

$$\begin{aligned}\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}(q,a)} |W^2(\alpha)\Delta_k^2(\alpha)|d\alpha &\ll Q_0^{2+4\varepsilon}N^{\frac{2}{k}}\exp(-\log^{\frac{1}{4}} N)\\ &\ll N^{\frac{2}{k}}\log^{-100A} N.\end{aligned} \tag{2.18}$$

Now the proof of Lemma 2.4 is completed. □

**Lemma 2.5.** *Let*

$$U_k(\alpha)=\frac{S_k^*(q,a)}{\varphi(q)}u_k(\beta). \tag{2.19}$$

*Then we have*

(*i*) $\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}(q,a)}|U_k(\alpha)|^2d\alpha \ll N^{\frac{2}{k}-1}\log^{21A} N$,

(*ii*) $\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}(q,a)}|W(\alpha)|^2d\alpha \ll \log^{21A} N$,

*where* $W(\alpha)$ *and* $\mathcal{I}(q,a)$ *are defined by* (2.9) *and* (2.11) *respectively.*

*Proof.* It follows from Lemma 2.2 (*ii*) and (2.14) that

$$\begin{aligned}&\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}(q,a)}|U_k^2(\alpha)|d\alpha\\ &\quad\ll \sum_{q\le Q_0} q^{-1+\varepsilon}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{|\beta|\le\frac{1}{Q_0}}|u_k(\beta)|^2d\beta\\ &\quad\ll \sum_{q\le Q_0} q^{-1+\varepsilon}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\left(N^{\frac{2}{k}}\int_{|\beta|\le\frac{1}{N}}d\beta + N^{\frac{2}{k}-2}\int_{\frac{1}{N}\le|\beta|\le\frac{1}{Q_0}}|\beta|^{-2}d\beta\right)\\ &\quad\ll N^{\frac{2}{k}-1}\log^{21A} N,\end{aligned}$$

and Lemma 2.5 (*i*) is proved. By (2.12) we can prove Lemma 2.5 (*ii*) in a similar way. □

For $(a,q)=1, 1\le a\le q$, put

$$\mathfrak{M}_0(q,a)=\left(\frac{a}{q}-\frac{Q_0}{N},\frac{a}{q}+\frac{Q_0}{N}\right],\ \mathfrak{M}_0=\bigcup_{1\le q\le Q_0^5}\bigcup_{\substack{a=1\\(a,q)=1}}^{q}\mathfrak{M}_0(q,a),$$

$$\mathfrak{M}(q,a)=\left(\frac{a}{q}-\frac{1}{qQ_2},\frac{a}{q}+\frac{1}{qQ_2}\right],\ \mathfrak{M}=\bigcup_{1\le q\le Q_0^5}\bigcup_{\substack{a=1\\(a,q)=1}}^{q}\mathfrak{M}(q,a),$$

$$\mathfrak{I}_0=\left(-\frac{1}{Q_2},1-\frac{1}{Q_2}\right],\ \mathfrak{m}_0=\mathfrak{M}\setminus\mathfrak{M}_0,$$

$$\mathfrak{m}_1=\bigcup_{Q_0^5<q\le Q_1}\bigcup_{\substack{a=1\\(a,q)=1}}^{q}\mathfrak{M}(q,a),\ \mathfrak{m}_2=\mathfrak{I}_0\setminus\left(\mathfrak{M}\bigcup\mathfrak{m}_1\right).$$

Then we have the Farey dissection

$$\mathfrak{I}_0=\mathfrak{M}_0\bigcup\mathfrak{m}_0\bigcup\mathfrak{m}_1\bigcup\mathfrak{m}_2. \tag{2.20}$$

**Lemma 2.6.** *For $\alpha=\frac{a}{q}+\beta$, let*

$$V_j(\alpha)=\frac{S_5^*(q,a)}{\varphi(q)}v_j(\beta),\qquad j=1,2.$$

*Then for $\alpha=\frac{a}{q}+\beta\in\mathfrak{M}_0$, we have*

(*i*) $f_k(\alpha)=U_k(\alpha)+O(U_k\exp(-\log^{\frac13}N)),\qquad k=2,5$

(*ii*) $g_j(\alpha)=V_j(\alpha)+O(V_j\exp(-\log^{\frac13}N)),\qquad j=1,2$

(*iii*) $f_{2,r}(\alpha)=\frac{c_rU_2(\alpha)}{\log U_2}+O(U_2\exp(-\log^{\frac13}N)),\qquad 7\le r\le 28$

*where $U_k(\alpha)$ is defined by* (2.19), *and*

$$c_r=(1+O(\varepsilon))\int_{r-1}^{\frac{82}{3}}\frac{dt_1}{t_1}\int_{r-2}^{t_1-1}\frac{dt_2}{t_2}\cdots\int_{3}^{t_{r-4}-1}\frac{dt_{r-3}}{t_{r-3}}\int_{2}^{t_{r-3}-1}\frac{\log(t_{r-2}-1)dt_{r-2}}{t_{r-2}}$$

*Proof.* By some routine arrangements and summation by parts, (*i*), (*ii*) and (*iii*) follow from Siegel-Walfisz theorem and Prime number theorem. □

## 3. Mean value theorems

In this section we prove two mean value theorems for the proof of Theorem 1.3.

**Proposition 3.1.** *Let*

$$J_d(N)=\sum_{\substack{(dl)^2+p^2+p_1^5+\cdots+p_6^5=N\\U_2<dl,p\le 2U_2,\,U_5<p_1,p_2,p_3\le 2U_5\\V_1<p_4\le 2V_1,\,V_2<p_5,p_6\le 2V_2}}(\log p)(\log p_1)\cdots(\log p_6).$$

*Then for* $|u(m)| \le 1, |v(n)| \le 1$, *we have*

$$\sum_{m \le D^{\frac{2}{3}},\, n \le D^{\frac{1}{3}}} u(m)v(n)\left(J_{mn}(N) - \frac{\mathfrak{S}_{mn}(N)}{mn}\mathfrak{J}(N)\right) \ll N^{\frac{91}{85}} \log^{-A} N.$$

*Proof.* Let

$$h(\alpha) = \sum_{m \le D^{\frac{2}{3}},\, n \le D^{\frac{1}{3}}} u(m)v(n) \sum_{\frac{U_2}{mn} < l \le \frac{2U_2}{mn}} e(\alpha(mnl)^2),$$

$$K(\alpha) = h(\alpha) f_2(\alpha) f_5^3(\alpha) g_1(\alpha) g_2^2(\alpha) e(-\alpha N).$$

Then by the Farey dissection (2.20), we have

$$\begin{aligned}\sum_{m \le D^{\frac{2}{3}},\, n \le D^{\frac{1}{3}}} u(m)v(n)J_{mn}(N) &= \int_{\mathfrak{J}_0} K(\alpha)d\alpha \\ &= \left(\int_{\mathfrak{M}_0} + \int_{\mathfrak{m}_0} + \int_{\mathfrak{m}_1} + \int_{\mathfrak{m}_2}\right) K(\alpha)d\alpha.\end{aligned} \tag{3.1}$$

From Cauchy's inequality, Lemma 1 in [1] and Lemma 2.1, we obtain

$$\begin{aligned}&\int_0^1 |f_2(\alpha) f_5^3(\alpha) g_1(\alpha) g_2^2(\alpha)| d\alpha \\ &\ll \left(\int_0^1 |f_2(\alpha) f_5^2(\alpha)|^2 d\alpha\right)^{\frac{1}{2}} \left(\int_0^1 |f_5(\alpha) g_1(\alpha) g_2^2(\alpha)|^2 \alpha\right)^{\frac{1}{2}} \\ &\ll N^{\frac{267}{340}+\varepsilon}.\end{aligned} \tag{3.2}$$

By (3.2), we get

$$\begin{aligned}\int_{\mathfrak{m}_2} K(\alpha)d\alpha &\ll \max_{\alpha \in \mathfrak{m}_2} |h(\alpha)| \left(\int_0^1 |f_2(\alpha) f_5^3(\alpha) g_1(\alpha) g_2^2(\alpha)| d\alpha\right) \\ &\ll N^{\frac{91}{85}-\varepsilon},\end{aligned} \tag{3.3}$$

where the bound $h(\alpha) \ll N^{\frac{97}{340}-3\varepsilon}$ for $\alpha \in \mathfrak{m}_2$, which follows from (4.6) in [4], is used.

Write

$$a(d) = \sum_{\substack{m \le D^{\frac{2}{3}},\, n \le D^{\frac{1}{3}} \\ mn = d}} u(m)v(n), \quad h(\alpha) = \sum_{d \le D} a(d) \sum_{\frac{U_2}{d} < l \le \frac{2U_2}{d}} e(\alpha(dl)^2).$$

Then from Theorem 4.1 in Vaughan [14], for $\alpha \in \mathfrak{m}_1$, we obtain

$$h(\alpha) = W(\alpha) + O(DQ_1^{\frac{1}{2}+\varepsilon}) = W(\alpha) + O(N^{\frac{91}{340}+\varepsilon}), \tag{3.4}$$

where $W(\alpha)$ is defined by (2.9). Let

$$K_1(\alpha) = W(\alpha) f_2(\alpha) f_5^3(\alpha) g_1(\alpha) g_2^2(\alpha) e(-\alpha N).$$

Then by (3.2) and (3.4), we have

$$\int_{\mathfrak{m}_1} K(\alpha)d\alpha = \int_{\mathfrak{m}_1} K_1(\alpha)d\alpha + O(N^{\frac{91}{85}-\varepsilon}). \tag{3.5}$$

Let

$$\mathcal{I}_0(q,a) = \left(\frac{a}{q} - \frac{1}{N^{\frac{39}{50}}}, \frac{a}{q} + \frac{1}{N^{\frac{39}{50}}}\right], \quad \mathcal{I}_1(q,a) = \mathcal{I}(q,a) \setminus \mathcal{I}_0(q,a),$$

where $\mathcal{I}(q,a)$ is defined by (2.11). Then we have

$$\begin{aligned}\int_{\mathfrak{m}_1} K_1(\alpha)d\alpha \le \sum_{q\le Q_0} \sum_{\substack{a=-q\\(a,q)=1}}^{2q} \int_{\mathfrak{m}_1 \bigcap \mathcal{I}_0(q,a)} | K_1(\alpha) | d\alpha \\ + \sum_{q\le Q_0} \sum_{\substack{a=-q\\(a,q)=1}}^{2q} \int_{\mathfrak{m}_1 \bigcap \mathcal{I}_1(q,a)} | K_1(\alpha) | d\alpha.\end{aligned} \tag{3.6}$$

From (3.2), we have

$$\begin{aligned}\sum_{q\le Q_0} \sum_{\substack{a=-q\\(a,q)=1}}^{2q} \int_{\mathfrak{m}_1 \bigcap \mathcal{I}_1(q,a)} | K_1(\alpha) | d\alpha \\ \ll N^{\frac{97}{340}-2\varepsilon} \int_0^1 |f_2(\alpha)f_5^3(\alpha)g_1(\alpha)g_2^2(\alpha)|d\alpha \\ \ll N^{\frac{91}{85}-\varepsilon},\end{aligned} \tag{3.7}$$

where the bound $W(\alpha) \ll N^{\frac{97}{340}-2\varepsilon}$ for $\alpha \in \mathcal{I}_1(q,a)$, which follows from (2.12) and (2.14), is used.

By Lemma 4.7 and Lemma 4.5 in [12], we obtain

$$\begin{aligned}&\int_{\mathfrak{m}_1 \bigcap \mathcal{I}_0(q,a)} | K_1(\alpha) | d\alpha \\ &= \int_{\mathfrak{m}_1 \bigcap \mathcal{I}_0(q,a)} | W(\alpha)U_5(\alpha)f_2(\alpha)f_5^2(\alpha)g_1(\alpha)g_2^2(\alpha) | d\alpha \\ &\quad + \int_{\mathfrak{m}_1 \bigcap \mathcal{I}_0(q,a)} | W(\alpha)\Delta_5(\alpha)f_2(\alpha)f_5^2(\alpha)g_1(\alpha)g_2^2(\alpha) | d\alpha \\ &\quad + O\left(N^{\frac{11}{100}} \int_{\mathfrak{m}_1 \bigcap \mathcal{I}_0(q,a)} |W(\alpha)f_2(\alpha)f_5^2(\alpha)g_1(\alpha)g_2^2(\alpha)|d\alpha\right),\end{aligned} \tag{3.8}$$

where $\Delta_5(\alpha), U_5(\alpha)$ are defined by (2.10) and (2.19) respectively.

From Cauchy's inequality, Lemma 2.3 and Lemma 2.4, we get

$$
\begin{aligned}
(3.9)\quad \sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{m}_1\bigcap\mathcal{I}_0(q,a)} &\mid W(\alpha)\Delta_5(\alpha)f_2(\alpha)f_5^2(\alpha)g_1(\alpha)g_2^2(\alpha)\mid d\alpha \\
&\ll N^{\frac15}\left(\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}(q,a)}|W(\alpha)\Delta_5(\alpha)|^2d\alpha\right)^{\frac12} \\
&\quad\times\left(\int_0^1|f_2(\alpha)f_5(\alpha)g_1(\alpha)g_2^2(\alpha)|^2d\alpha\right)^{\frac12} \\
&\ll N^{\frac{91}{85}}\log^{-10A}N.
\end{aligned}
$$

It follows from Cauchy's inequality, Lemma 2.3 and Lemma 2.5 ($i$) that

$$
\begin{aligned}
(3.10)\quad \sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{m}_1\bigcap\mathcal{I}_0(q,a)} &\mid W(\alpha)U_5(\alpha)f_2(\alpha)f_5^2(\alpha)g_1(\alpha)g_2^2(\alpha)\mid d\alpha \\
&\ll N^{\frac15+\frac12}\log^{-33A}N\left(\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}_0(q,a)}|U_5(\alpha)|^2d\alpha\right)^{\frac12} \\
&\quad\times\left(\int_0^1|f_2(\alpha)f_5(\alpha)g_1(\alpha)g_2^2(\alpha)|^2d\alpha\right)^{\frac12} \\
&\ll N^{\frac{91}{85}}\log^{-10A}N,
\end{aligned}
$$

where the bound $W(\alpha)\ll N^{\frac12}\log^{-33A}N$ for $\alpha\in\mathfrak{m}_1$, which follows from (2.12) and (2.14), is used.

By Cauchy's inequality, Lemma 2.3 and Lemma 2.5 ($ii$), we have

$$
\begin{aligned}
(3.11)\quad N^{\frac{11}{100}}\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{m}_1\bigcap\mathcal{I}_0(q,a)} &|W(\alpha)f_2(\alpha)f_5^2(\alpha)g_1(\alpha)g_2^2(\alpha)|d\alpha \\
&\ll N^{\frac{31}{100}}\left(\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathcal{I}_0(q,a)}|W(\alpha)|^2d\alpha\right)^{\frac12} \\
&\quad\times\left(\int_0^1|f_2(\alpha)f_5(\alpha)g_1(\alpha)g_2^2(\alpha)|^2d\alpha\right)^{\frac12} \\
&\ll N^{\frac{91}{85}}\log^{-10A}N.
\end{aligned}
$$

It follows from (3.8)–(3.11) that

$$\sum_{q\le Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{m}_1\bigcap\mathcal{I}_0(q,a)}K_1(\alpha)d\alpha\ll N^{\frac{91}{85}}\log^{-10A}N. \tag{3.12}$$

From (3.5)–(3.7) and (3.12), we get

$$\int_{\mathfrak{m}_1}K(\alpha)d\alpha\ll N^{\frac{91}{85}}\log^{-10A}N. \tag{3.13}$$

By arguments similar to but simpler than that leading to (3.13), we obtain

$$\int_{\mathfrak{m}_0}K(\alpha)d\alpha\ll N^{\frac{91}{85}}\log^{-10A}N. \tag{3.14}$$

For $\alpha\in\mathfrak{M}_0$, let

$$K_0(\alpha)=W(\alpha)U_2(\alpha)U_5^3(\alpha)V_1(\alpha)V_2^2(\alpha)e(-\alpha N). \tag{3.15}$$

Then it follows from Lemma 2.6 and (3.4) which holds for $\alpha\in\mathfrak{M}_0$ also, that for $\alpha\in\mathfrak{M}_0$, we have

$$K(\alpha)-K_0(\alpha)\ll N^{\frac{176}{85}}\exp(-\log^{\frac{1}{4}}N). \tag{3.16}$$

By (3.16), we get

$$\int_{\mathfrak{M}_0}K(\alpha)d\alpha=\int_{\mathfrak{M}_0}K_0(\alpha)d\alpha+O\left(N^{\frac{91}{85}}\log^{-A}N\right). \tag{3.17}$$

Now the well-known standard endgame technique in the Hardy-Littlewood method establishes that

$$\int_{\mathfrak{M}_0}K_0(\alpha)d\alpha=\sum_{\substack{m\le D^{\frac{2}{3}},\\ n\le D^{\frac{1}{3}}}}u(m)v(n)\frac{\mathfrak{S}_{mn}(N)}{mn}\mathfrak{I}(N)+O\left(N^{\frac{91}{85}}\log^{-A}N\right), \tag{3.18}$$

$$\mathfrak{I}(N)\asymp N^{\frac{91}{85}}. \tag{3.19}$$

Now upon combining (3.1), (3.3), (3.13), (3.14) and (3.18), the proof of Proposition 3.1 is completed. □

By the same method, we have

**Proposition 3.2.** *For* $7\le r\le 28$ *let*

$$J_d^{(r)}(N)=\sum_{\substack{(dl)^2+m^2+p_1^5+\cdots+p_6^5=N\\ U_2<dl\le 2U_2,\,U_5<p_1,p_2,p_3\le 2U_5\\ m\in\mathcal{M}_r,\,V_1<p_4\le 2V_1,\,V_2<p_5,p_6\le 2V_2}}(\log p_1)\cdots(\log p_6).$$

*Then for* $|u(m)| \leq 1, |v(n)| \leq 1$, *we have*

$$\sum_{m \leq D^{\frac{2}{3}},\, n \leq D^{\frac{1}{3}}} u(m)v(n)\left(J_{mn}^{(r)}(N) - c_r \frac{\mathfrak{S}_{mn}(N)}{mn \log U_2}\mathfrak{I}(N)\right) \ll N^{\frac{91}{85}} \log^{-A} N,$$

*where* $c_r$ *is defined in Lemma 2.6.*

## 4. On the function $\omega(d)$

In this section we investigate the function $\omega(d)$ which is defined in (4.8) and is required in the proof of Theorem 1.3.

**Lemma 4.1.** *Let* $K(q,N)$ *and* $L(q,N)$ *denote the number of solutions to the congruences*

$$y^2 + \sum_{j=1}^{6} u_j^5 \equiv N (\mathrm{mod} q), \;\; 1 \leq y, u_j \leq q, \;\; (yu_j, q) = 1$$

*and*

$$x^2 + y^2 + \sum_{j=1}^{6} u_j^5 \equiv N (\mathrm{mod} q), \;\; 1 \leq x, y, u_j \leq q, \;\; (yu_j, q) = 1$$

*respectively. Then we have* $L(p,N) > K(p,N)$. *Moreover, we have*

$$L(p,N) = p^7 + O(p^6), \tag{4.1}$$

$$K(p,N) = p^6 + O(p^5). \tag{4.2}$$

*Proof.* Let $L^*(q,N)$ denote the number of solutions to the congruence

$$x^2 + y^2 + \sum_{j=1}^{6} u_j^5 \equiv N (\mathrm{mod} q), \;\; 1 \leq x, y, u_j \leq q, \;\; (xyu_j, q) = 1.$$

Then by the orthogonality of additive characters, we have

$$\begin{aligned} pL^*(p,N) &= \sum_{a=1}^{p} S_2^{*2}(p,a) S_5^{*6}(p,a) e_p(-aN) \\ &= (p-1)^8 + E_p, \end{aligned} \tag{4.3}$$

where

$$E_p = \sum_{a=1}^{p-1} S_2^{*2}(p,a) S_5^{*6}(p,a) e_p(-aN).$$

By Lemma 2.3 ($iv$), we obtain

$$
\begin{aligned}
|E_p| &\le (\sqrt{p}+1)^2 \sum_{a=1}^{p-1} |S_5^{*6}(p,a)| \\
&= (\sqrt{p}+1)^2 \left( \sum_{a=1}^{p} |S_5^{*6}(p,a)| - (p-1)^6 \right) \\
&\le (\sqrt{p}+1)^2 \left( 5p(p-1)^5 - (p-1)^6 \right) \\
&= (p-1)^5 (\sqrt{p}+1)^2 \,(4p+1)\,.
\end{aligned} \tag{4.4}
$$

It is easy to verify that $|E_p| < (p-1)^8$ for $p \ge 11$, hence we have $L^*(p,N) > 0$ for $p \ge 11$.

On the other hand, when $p = 2,3,5,7$, we can check by hand that $L^*(p,N) > 0$.

In summary, we have $L(p,N) = L^*(p,N) + K(p,N) > K(p,N)$.

By (4.3) and (4.4), we have

$$L^*(p,N) = p^7 + O(p^6),$$

and (4.1) and (4.2) follow from similar arguments.

Now Lemma 4.1 is proved. □

**Lemma 4.2.** *The series* $\mathfrak{S}(N)$ *is convergent and* $\mathfrak{S}(N) > 0$.

*Proof.* The convergence of $\mathfrak{S}(N)$ follows from Lemma 2.3 ($i$)–($ii$) easily. Note the fact that $A(q,N)$ is multiplicative in $q$, by Lemma 2.3 ($v$), we know that

$$\mathfrak{S}(N) = \prod_{p} (1 + A(p,N)). \tag{4.5}$$

For $p > 101$, by Lemma 2.3 ($iii$)–($iv$), we have

$$|A(p,N)| \le \frac{(p-1)p^{\frac12}(p^{\frac12}+1)^2(4p^{\frac12}+1)^6}{p(p-1)^7} \le \frac{400}{p^2}.$$

So we get

$$\prod_{p>101} (1 + A(p,N)) \ge \prod_{p>101} \left(1 - \frac{400}{p^2}\right) > c > 0. \tag{4.6}$$

It is easy to show that

$$1 + A(p,N) = \frac{L(p,N)}{p(p-1)^7}. \tag{4.7}$$

Now by Lemma 4.1 and (4.5)–(4.7), we have $\mathfrak{S}(N) > 0$, and the proof of Lemma 4.2 is completed. □

In view of Lemma 4.2, for natural number $d$, we may define

$$\omega(d) = \frac{\mathfrak{S}_d(N)}{\mathfrak{S}(N)}. \tag{4.8}$$

Similar to (4.5), we have

$$\mathfrak{S}_d(N) = \prod_{p\nmid d}(1 + A_d(p, N)) \times \prod_{p|d}(1 + A_d(p, N)). \tag{4.9}$$

It follows from the facts that $S_k(q, ad^k) = S_k(q, a)$ for $(d, q) = 1$, $A_d(p, N) = A_p(p, N)$ for $p|d$ and (4.8)–(4.9) that

$$\omega(p^l) = \frac{1 + A_p(p, N)}{1 + A(p, N)}, \tag{4.10}$$

$$\omega(d) = \prod_{p|d}\omega(p) \tag{4.11}$$

for each prime $p$ and natural number $l$. It is easy to show that

$$1 + A_p(p, N) = \frac{K(p, N)}{(p-1)^7}. \tag{4.12}$$

From (4.7), (4.10) and (4.12), we have

$$\omega(p^l) = \frac{pK(p, N)}{L(p, N)} \tag{4.13}$$

for each prime $p$ and natural number $l$.

It follows from Lemma 4.1 and (4.13) that

**Lemma 4.3.** *The function $\omega(d)$ is multiplicative, and*

$$0 \le \omega(p) < p, \quad \omega(p^l) = 1 + O(p^{-1})$$

*for each prime $p$ and natural number $l$.*

## 5. Proof of Theorem 1.3

In this section $f(s)$ and $F(s)$ denote the classical functions in the linear sieve theory, and $\gamma = 0.577\cdots$ denotes Euler's constant. Then by (8.2.8) and (8.2.9) in [6], we have

$$f(s) = \frac{2e^\gamma \log(s-1)}{s}, \qquad 2 \le s \le 4;$$
$$F(s) = \frac{2e^\gamma}{s}, \qquad 1 \le s \le 3.$$

In the proof of Theorem 1.3 we adopt the following notation:

$$\mathfrak{P} = \prod_{2<p<z} p, \quad \log 2\mathbf{U} = (\log 2U_2)(\log^3 2U_5)(\log 2V_1)(\log^2 2V_2),$$
$$\log \mathbf{U} = (\log U_2)(\log^3 U_5)(\log V_1)(\log^2 V_2).$$

Let

$$\mathfrak{N}(z) = \prod_{2<p<z} \left(1 - \frac{\omega(p)}{p}\right).$$

Then by Lemma 4.3 and Merten's prime number theorem, we obtain

$$\mathfrak{N}(z) \asymp \frac{1}{\log N}. \tag{5.1}$$

Let $R(N)$ denote the number of solutions of the equation (1.4) with $x$ being a $P_6$ and the other variables primes. Then we have

$$\begin{aligned} R(N) \geq & \sum_{\substack{l^2+p^2+p_1^5+\cdots+p_6^5=N \\ U_2<l,p\leq 2U_2,\, U_5<p_1,p_2,p_3\leq 2U_5 \\ (l,\mathfrak{P})=1, V_1<p_4\leq 2V_1,\, V_2<p_5,p_6\leq 2V_2}} 1 \\ & - \sum_{r=7}^{28} \sum_{\substack{m^2+p^2+p_1^5+\cdots+p_6^5=N \\ U_2<p\leq 2U_2,\, U_5<p_1,p_2,p_3\leq 2U_5 \\ m\in\mathcal{M}_r,\, V_1<p_4\leq 2V_1,\, V_2<p_5,p_6\leq 2V_2}} 1 \\ = & \mathcal{R}(N) - \sum_{r=7}^{28} \mathcal{R}_r(N), \quad \text{say.} \end{aligned} \tag{5.2}$$

In the following we shall give a non-trivial lower bound for $R(N)$ by the linear sieve theory with the bilinear error term in [9].

**5.1. The lower bound for $\mathcal{R}(N)$.** Let

$$\mathcal{N}(l) = \sum_{\substack{l^2+p^2+p_1^5+\cdots+p_6^5=N \\ U_2<p\leq 2U_2,\, U_5<p_1,p_2,p_3\leq 2U_5 \\ V_1<p_4\leq 2V_1,\, V_2<p_5,p_6\leq 2V_2}} (\log p)(\log p_1)\cdots(\log p_6),$$

$$\mathcal{E}(d) = \sum_{\substack{U_2<l\leq 2U_2 \\ l\equiv 0(\bmod d)}} \mathcal{N}(l) - \frac{\omega(d)}{d}\mathfrak{S}(N)\mathfrak{J}(N).$$

Then by Theorem 1 in [9] (see also Lemma 9.1 in [10]) and Proposition 3.1, we get

$$\begin{aligned} \mathcal{R}(N) &\geq \frac{1}{\log 2\mathbf{U}} \sum_{\substack{U_2<l\leq 2U_2 \\ (l,\mathfrak{P})=1}} \mathcal{N}(l) \\ &\geq \left(1 + O\left(\log^{-\frac{1}{3}} D\right)\right) \frac{f(3)\mathfrak{S}(N)\mathfrak{J}(N)\mathfrak{N}(z)}{\log \mathbf{U}} \\ &\quad + O\left(N^{\frac{91}{85}} \log^{-100} N\right). \end{aligned} \tag{5.3}$$

**5.2. The upper bound for $\mathcal{R}_r(N)$.** Let

$$\mathcal{N}_r(l) = \sum_{\substack{l^2+m^2+p_1^5+\cdots+p_6^5=N \\ m\in\mathcal{M}_r,\ U_5<p_1,p_2,p_3\le 2U_5 \\ V_1<p_4\le 2V_1,\ V_2<p_5,p_6\le 2V_2}} 1$$

$$\mathcal{E}_r(d) = \sum_{\substack{U_2<l\le 2U_2 \\ l\equiv 0(\bmod d)}} \mathcal{N}_r(l) - \frac{c_r\omega(d)}{d}\mathfrak{S}(N)\mathfrak{J}(N),$$

where $c_r$ is defined in Lemma 2.6. Then by Theorem 1 in [9] (see also Lemma 9.1 in [10]) and Proposition 3.2, we have

$$\begin{aligned}\mathcal{R}_r(N) &\le \frac{\log U_2}{\log \mathbf{U}} \sum_{\substack{U_2<l\le 2U_2 \\ (l,\mathfrak{P})=1}} \mathcal{N}_r(l) \\ &\le \left(1+O\left(\log^{-\frac{1}{3}} D\right)\right)\frac{F(3)c_r\mathfrak{S}(N)\mathfrak{J}(N)\mathfrak{N}(z)}{\log \mathbf{U}} \\ &\quad + O\left(N^{\frac{91}{85}}\log^{-100} N\right).\end{aligned} \tag{5.4}$$

**5.3. Proof of Theorem 1.3.** By numerical integration, we get

$$\begin{gathered} c_7 < 0.2093,\ \ c_8 < 0.0427,\ \ c_r < 0.0067 \text{ for } 9\le r\le 28 \\ \sum_{r=7}^{28} c_r < 0.3861. \end{gathered} \tag{5.5}$$

From (5.1)–(5.5), we have

$$\begin{aligned}R(N) &> 0.3\frac{2e^\gamma}{3}\frac{\mathfrak{S}(N)\mathfrak{J}(N)\mathfrak{N}(z)}{\log \mathbf{U}} + O\left(N^{\frac{91}{85}}\log^{-100} N\right) \\ &\gg \frac{N^{\frac{91}{85}}}{\log^8 N},\end{aligned}$$

where (3.19) and Lemma 4.2 are employed. The proof of Theorem 1.3 is completed.

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