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par Luis ARENAS-CARMONA

Résumé. Nous calculons le corps de classes spinoriel pour un genre d'ordres qui sont des intersections de deux ordres maximaux, dans une algèbre centrale simple de dimension 9 ou plus. Autrement dit, nous calculons le nombre des classes de conjugaison dans un genre de tels ordres, en termes du degré d'une extension des corps de classes. Nous donnons des applications à l'étude des groupes d'automorphismes de ces ordres et à l'étude des représentations d'ordres commutatifs.

Abstract. We compute the spinor class field for a genus of orders, in a central simple algebra of dimension 9 or higher, that are intersections of two maximal orders, i.e., we compute the number of conjugacy classes in a genus of such orders, as the degree of an explicit extension of class fields. We give applications to the study of the automorphism groups of these orders and to the study of representations of commutative orders.

1. Introduction

Let K be a number field with ring of integers $\mathcal{O}$. Let $\mathfrak{A}$ be a central simple K-algebra (K-CSA or CSA over K) of dimension $n^2 \geq 4$. All lattices and orders are assumed to be $\mathcal{O}$-modules.

In recent years there has been an increasing interest in the so called selectivity problem for maximal orders: Understanding the set of maximal orders in $\mathfrak{A}$ containing an isomorphic copy of a given suborder $\mathfrak{H}$, usually a commutative order. This has been partly motivated by the role played by quaternion orders in the construction of isospectral but non-isometric hyperbolic manifolds [13]. However, these results have also applications to the arithmetic of both, the order $\mathfrak{H}$ and the set of maximal orders containing $\mathfrak{H}$. A couple of questions that easily reduce to selectivity problems [6, §5] are the following:

(1) When does the order $\mathfrak{H}$ contains an ideal isomorphic, as an $\mathcal{O}$-module, to the lattice $\mathcal{O} \times \cdots \times \mathcal{O} \times J$ for a given ideal $J \subseteq \mathcal{O}$?

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(2) If $\mathfrak{A}=\mathbb{M}_r(B)$ is a matrix algebra over a division algebra B, which maximal orders $\mathfrak{D}\subseteq\mathfrak{A}$ have the form $\mathfrak{D}\cong\mathbb{M}_r(\mathfrak{B})$ for some order $\mathfrak{B}\subseteq B$?

The selectivity problem extends naturally to orders of maximal rank, or as we say in all that follows, full orders in $\mathfrak{A}$. In this setting, the problem is stated in terms of genera. A genus is a maximal set of full orders whose completions are conjugate locally at all places. The selectivity problem in this setting consists in studying the set of orders in a genus containing a copy of a given order $\mathfrak{H}$. This is a natural generalization for two reasons:

(1) The set of maximal orders is a genus.
(2) The existence of one order in a given genus containing a copy of $\mathfrak{H}$ can be determined by purely local computations.

This extended problem has been studied for some particular types of full orders in quaternion algebras, like Eichler orders, but aside from a few existential results little is known for non-maximal orders in the higher dimensional case. Mainly because we lack a simple description of the set of conjugacy classes in a genus.

In this work we give this description for the simplest type of non-maximal full orders, intersections of two maximal orders, also called generalized Eichler orders (or GEOs) in all that follows. These are split orders in the sense defined by T. Shemanske in [11], while the set of conjugacy classes of such orders is simpler to describe, especially for algebras of odd dimension:

Theorem 1.1. *The number $|\mathfrak{A}^*\backslash\mathbb{O}|$ of conjugacy classes in a genus $\mathbb{O}$ of GEOs in an odd-dimensional CSA is independent of $\mathbb{O}$. More precisely, there exists a non-cannonical bijective map ψ from the set $\mathfrak{A}^*\backslash\mathbb{O}_0$ of conjugacy classes of maximal orders onto $\mathfrak{A}^*\backslash\mathbb{O}$ of the form $\psi(\mathrm{cls}(\mathfrak{D}))=\mathrm{cls}(\mathfrak{D}\cap\Delta(\mathfrak{D}))$, for some map $\Delta:\mathbb{O}_0\to\mathbb{O}_0$.*

A fundamental tool for the study of genera is the spinor class field, an explicit abelian extension Σ/K that classifies conjugacy classes in a genus $\mathbb{O}$ [3, §3], in the sense that we can construct an explicit map $\rho:\mathbb{O}\times\mathbb{O}=\mathbb{O}^2\longrightarrow\mathrm{Gal}(\Sigma/K)$, with the following properties (§2):

(1) $\mathfrak{D}$ is conjugate to $\mathfrak{D}'$ if and only if $\rho(\mathfrak{D},\mathfrak{D}')=\mathrm{Id}_\Sigma$,
(2) $\rho(\mathfrak{D},\mathfrak{D}'')=\rho(\mathfrak{D},\mathfrak{D}')\rho(\mathfrak{D}',\mathfrak{D}''),$ for all $(\mathfrak{D},\mathfrak{D}',\mathfrak{D}'')\in\mathbb{O}^3$.

This holds for a CSA of dimension 3^2 or larger, and also for a quaternion algebra satisfying Eichler's condition (§2). The class group defining Σ, for any genus $\mathbb{O}$ of Eichler orders, is already implicit in [12, Cor. III.5.7]. However, for a CSA $\mathfrak{A}$ of dimension $n\geq 3^2$, only the spinor class field for maximal orders was previously known explicitly. If $f_\wp(L/K)$ denotes the inertia degree of the field extension L/K at any place $\wp$, while we write $f_\wp(\mathfrak{A}/K)=f$ whenever $\mathfrak{A}_\wp\cong\mathbb{M}_f(E_\wp)$ for a central division algebra $E_\wp$ over $K_\wp$, this field can be described as follows: [3, §2]:

> The spinor class field for maximal orders in $\mathfrak{A}$ is the maximal exponent-n sub-extension Σ_0 of the wide Hilbert class field of K satisfying the following conditions:
> (1) $f_\wp(\Sigma_0/K)$ divides $f_\wp(\mathfrak{A}/K)$ at all finite places.
> (2) Σ_0/K splits completely at every real place $\wp$ of K where $f_\wp(\mathfrak{A}/K) = n$.

In the same language, the corresponding result for Eichler orders in quaternion algebras is as follows [7, Theorem 1.2]:

> The spinor class field for Eichler orders of level $I = \prod_\wp \wp^{\alpha(\wp)}$ in a quaternion algebra $\mathfrak{A}$ over K is the maximal subfield Σ, of the spinor class field Σ_0 of maximal orders, such that Σ/K splits at all places where $\alpha(\wp)$ is odd.

The purpose of the current work is to give a similar result for GEOs in a K-CSA $\mathfrak{A}$ of arbitrary dimension. This is done in Theorem 3.1. The splitting condition on places where $\alpha(\wp)$ is odd is replaced by a local condition depending on the definition on some technical *type distance*, and a notion of local symmetry. Here, a local order is called symmetric if a higher dimensional analog of the branch $S_0(\mathfrak{H})$ defined in [7, §2], has a nontrivial symmetry (§3-4). For locally symmetric orders, the correspondence described in Theorem 1.1 is canonical (Cor. 6.6). This machinery is applied in §5 to the selectivity problem. An additional application is the following:

Theorem 1.2. *Let $\mathfrak{D}$ be a GEO in the matrix algebra $\mathfrak{A} = \mathbb{M}_n(K)$, and let $\mathfrak{N}$ be the normalizer of $\mathfrak{D}$ in $\mathfrak{A}^*$. Then*

$$|\mathfrak{N}/K^*\mathfrak{D}^*| = \frac{2^t|\mathfrak{g}(n)|}{[\Sigma_0 : \Sigma]},$$

where t is the number of finite places $\wp$ of K, such that $\mathfrak{D}_\wp$ is a non-maximal symmetric GEO, $\mathfrak{g}(n)$ is the maximal exponent-n subgroup of the ideal class group of K, while Σ and Σ_0 are the spinor class fields for the genus of $\mathfrak{D}$ and the genus of maximal orders, respectively.

Example 1.3. Let $\mathfrak{D}$ be an Eichler order in $\mathbb{M}_2(\mathbb{Q})$, and let $\mathfrak{N}$ be as in Theorem 1.2. Then $|\mathfrak{N}/\mathbb{Q}^*\mathfrak{D}^*| = 2^t$, where t is the number of places dividing the level of $\mathfrak{D}$. When $\mathfrak{D} = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, representatives of all clases in the quotient $\mathfrak{N}/\mathbb{Q}^*\mathfrak{D}^*$ are given by the Atkin-Lehner involutions (see [8] or [9]).

We would not be overly surprised if the techniques presented in this work can be extended to chain orders [2], or even general split orders. For this, we need to understand the symmetry group of the convex polytopes described in [11].

2. The theory of spinor class field

In this section we review the basic facts about spinor class fields of orders. All results in this sections can be proved in the more general context of S-orders in CSA's over global fields, See [3] or [5] for details. In all that follows $\mathfrak{A}$ and K are as in the introduction and $\mathfrak{D}$ is a full order in $\mathfrak{A}$. Let $\mathfrak{H}$ be an arbitrary suborder of $\mathfrak{D}$ (full or otherwise). Completions at a finite place $\wp$ are denoted $\mathfrak{D}_\wp$ or $\mathfrak{H}_\wp$, and a similar convention applies to algebras. Let $\Pi = \Pi(K)$ be the set of all places in K finite or otherwise, let $\mathbb{A} \subseteq \prod_{\wp\in\Pi} K_\wp$ be the adele ring of K, and let $J_K = \mathbb{A}^*$ be its idele group. Let $\mathfrak{A}_\mathbb{A} = \mathfrak{A}\otimes_K \mathbb{A}$ be the adelization of the algebra. If $a = (a_\wp)_\wp \in \mathfrak{A}_\mathbb{A}$ is an adelic element, we let $a\mathfrak{D}a^{-1}$ denote the order $\mathfrak{D}'$ defined locally by $\mathfrak{D}'_\wp = a_\wp \mathfrak{D}_\wp a_\wp^{-1}$ at all finite places $\wp$. By convention, we set $\mathfrak{D}_\wp = \mathfrak{A}_\wp$ at infinite places, if $\mathfrak{D}$ is full.

Since any two maximal orders are locally conjugate at all places, if we fix a maximal order $\mathfrak{D}$, any other maximal order in $\mathfrak{A}$ has the form $\mathfrak{D}' = a\mathfrak{D}a^{-1}$ for some adelic element $a \in \mathfrak{A}_\mathbb{A}^*$. More generally, it is said that two full orders $\mathfrak{D}$ and $\mathfrak{D}'$, in $\mathfrak{A}$, are in the same genus if $\mathfrak{D}' = a\mathfrak{D}a^{-1}$ for some adelic element a. The spinor class field $\Sigma = \Sigma(\mathfrak{D})$ is defined as the class field corresponding to the group $K^*H(\mathfrak{D}) \subseteq J_K$, where

$$H(\mathfrak{D}) = \{N(a) | a \in \mathfrak{A}_\mathbb{A}^*,\ a\mathfrak{D}a^{-1} = \mathfrak{D}\}.$$

Let $t \mapsto [t, \Sigma/K]$ denote the Artin map on ideles. The distance between the orders $\mathfrak{D}$ and $\mathfrak{D}' \in \mathrm{gen}(\mathfrak{D})$ is the element $\rho(\mathfrak{D},\mathfrak{D}') \in \mathrm{Gal}(\Sigma/K)$ defined by $\rho(\mathfrak{D},\mathfrak{D}') = [N(a), \Sigma/K]$, for any adelic element $a \in \mathfrak{A}_\mathbb{A}^*$ satisfying $\mathfrak{D}' = a\mathfrak{D}a^{-1}$. This implies the multiplicative property $\rho(\mathfrak{D},\mathfrak{D}'') = \rho(\mathfrak{D},\mathfrak{D}')\rho(\mathfrak{D}',\mathfrak{D}'')$, as in §1. We say that two orders $\mathfrak{D}$ and $\mathfrak{D}'$ are in the same spinor genus whenever their distance is the identity Id_Σ. Two conjugate orders are in the same spinor genus, and the converse is a consequence of the strong approximation property for the group $\mathrm{SL}_1(\mathfrak{A})$. In the present setting, strong approximation is equivalent to **EC** below, which is assumed throughout this paper, except in the last example in §6.

Eichler Condition (EC): Either $n > 2$ or $\mathfrak{A}$ is unramified at some archimedean place.

Note for future reference that $H(\mathfrak{D}) = J_K \cap \prod_{\wp\in\Pi(K)} H_\wp(\mathfrak{D})$, where

$$H_\wp(\mathfrak{D}) = \{N(a) | a \in \mathfrak{A}_\wp^*,\ a\mathfrak{D}_\wp a^{-1} = \mathfrak{D}_\wp\}.$$

The sets $H(\mathfrak{D})$ and $H_\wp(\mathfrak{D})$ are called global and local spinor image, respectively. When k is an arbitrary local field, we also write $H_k(\mathfrak{E})$ for the spinor image of a local order $\mathfrak{E}$ in a k-CSA $\mathfrak{A}$, which is defined analogously. If $\mathfrak{E}$ is maximal, and if $\mathfrak{A} \cong \mathbb{M}_f(B)$ for a division algebra B, it is known that $H_k(\mathfrak{E}) = k^{*f}\mathcal{O}_k^*$, see the continuation of Example 1 in §2 of [3].

3. Locally symmetric GEOs

Let $k = K_\wp$ be a local field and let $\mathfrak{A} = \mathbb{M}_f(B)$ be a k-CSA, where B is a division algebra. Recall that B^f, the space of column vectors, is naturally a left $\mathbb{M}_f(B)$-module and a right B-module, and this bi-module structure is the one considered throughout this paper. Every maximal order in $\mathfrak{A}$ has the form $\mathfrak{D}_\Lambda = \{a \in \mathfrak{A} | a\Lambda \subseteq \Lambda\}$, for some full lattice $\Lambda \subseteq B^f$ satisfying $\Lambda\mathcal{O}_B = \Lambda$, where $\mathcal{O}_B$ is the maximal order of B. Such lattices are called $\mathcal{O}_B$-lattices. Note that, for $\lambda \in B^*$, the map $x \mapsto x\lambda$ is not a B-module homomorphism unless λ is central, but $\Lambda \mapsto \Lambda\lambda$ defines an action of B^* on the set of $\mathcal{O}_B$-lattices since $\lambda\mathcal{O}_B = \mathcal{O}_B\lambda$. In these notations, $\mathfrak{D}_\Lambda = \mathfrak{D}_M$ if and only if $M = \Lambda\lambda$ for some $\lambda \in B^*$.

Let Λ and M be two full $\mathcal{O}_B$-lattices in B^f and let π be a uniformizing parameter of B. By the theory of invariant factors, there exists a B-basis $\{e_1, \dots, e_f\}$ of B^f, such that $\Lambda = \sum_{i=1}^f e_i\mathcal{O}_B$, and $M = \sum_{i=1}^f e_i\pi^{r_i}\mathcal{O}_B$, where $r_1 \leq r_2 \leq \cdots \leq r_f$. The elements $\pi^{r_1}, \cdots, \pi^{r_f}$ are call the invariant factors of the pair (Λ, M). If $\overrightarrow{u} = (1, 1, \dots, 1)$, the element

$$\mathcal{T}(\mathfrak{D}_\Lambda, \mathfrak{D}_M) = (r_1, \dots, r_f) + \langle\overrightarrow{u}\rangle \in \Gamma := \mathbb{Z}^f/\langle\overrightarrow{u}\rangle$$

is called the type distance of the pair of orders $(\mathfrak{D}_\Lambda, \mathfrak{D}_M)$, and is well defined by the discussion in last paragraph. A local GEO $\mathfrak{D} = \mathfrak{D}_\Lambda \cap \mathfrak{D}_M$ is said to be symmetric if $\mathcal{T}(\mathfrak{D}_\Lambda, \mathfrak{D}_M) = \mathcal{T}(\mathfrak{D}_M, \mathfrak{D}_\Lambda)$, or equivalently: $r_{i+1} - r_i = r_{f-i+1} - r_{f-i}$ for every element $i \in \{1, \dots, f-1\}$.

The class $\rho_\wp(\mathfrak{D}_\Lambda, \mathfrak{D}_M) = \overline{r_1 + \cdots + r_f} \in \mathbb{Z}/f\mathbb{Z}$ is called the total distance between the local maximal orders $\mathfrak{D}_\Lambda$ and $\mathfrak{D}_M$. This distance is a conjugation invariant of the pair $(\mathfrak{D}_\Lambda, \mathfrak{D}_M)$, and is related to the global distance ρ_0 between maximal orders by the formula

$$\rho_0(\mathfrak{D}, \mathfrak{D}') = \prod_{\wp \in R} |[\wp, \Sigma_0/K]|^{\rho_\wp(\mathfrak{D}_\wp, \mathfrak{D}'_\wp)} \in \mathrm{Gal}(\Sigma_0/K),$$

where $R = R(\mathfrak{D}, \mathfrak{D}')$ is the set of finite places $\wp$ of K satisfying $\mathfrak{D}_\wp \neq \mathfrak{D}'_\wp$, and $J \mapsto |[J, \Sigma_0/K]|$ is the Artin map on ideals. Theorem 3.1 bellow follows from Lemma 4.11 in §4.

Theorem 3.1. *The spinor class field for a global GEO $\mathfrak{D} = \mathfrak{D}_1 \cap \mathfrak{D}_2$ in a K-CSA $\mathfrak{A}$ is the maximal subextension Σ, of the spinor class field Σ_0 for maximal orders, whose local inertia degree $f_\wp(\Sigma/K)$ divides the total distance $\rho_\wp(\mathfrak{D}_{1\wp}, \mathfrak{D}_{2\wp})$ at every place $\wp$ where $\mathfrak{D}_\wp$ is symmetric.*

A global GEO is locally symmetric if every completion is symmetric. The type distance at $\wp$ of the pair of completions $(\mathfrak{D}_\wp, \mathfrak{D}'_\wp)$ is denoted $\mathcal{T}_\wp(\mathfrak{D}, \mathfrak{D}')$. If $\mathfrak{D} = \mathfrak{D}_1 \cap \mathfrak{D}_2$, it is proved in Lemma 4.8 below that the local pair $\{\mathfrak{D}_{1\wp}, \mathfrak{D}_{2\wp}\}$ is unique. This, together with the theory of invariant factors, implies that two global GEOs $\mathfrak{D} = \mathfrak{D}_1 \cap \mathfrak{D}_2$ and $\mathfrak{D}' = \mathfrak{D}'_1 \cap \mathfrak{D}'_2$

are in the same genus if and only if, at every finite place $\wp$, we have either $\mathcal{T}_\wp(\mathfrak{D}_1,\mathfrak{D}_2)=\mathcal{T}_\wp(\mathfrak{D}'_1,\mathfrak{D}'_2)$ or $\mathcal{T}_\wp(\mathfrak{D}_1,\mathfrak{D}_2)=\mathcal{T}_\wp(\mathfrak{D}'_2,\mathfrak{D}'_1)$.

4. Blocks in Weil apartments.

In all of this section, let k be a non-archimedean local field, and let B be a central division k-algebra with uniformizing parameter π. Let $\mathfrak{B}$ be the Bruhat-Tits building (or BT-building) associated to $\mathrm{PGL}_n(B)$, as defined in [1] or [2]. Recall that the vertices of $\mathfrak{B}$ are in one to one correspondence with the maximal orders in $\mathbb{M}_n(B)$. An apartment is the maximal subcomplex whose vertices correspond to maximal orders containing a fixed conjugate of the order $\mathfrak{P}=\bigoplus_{i=1}^n \mathcal{O}_B E_{i,i}$ of integral diagonal matrices, where $\{E_{i,j}\}_{i,j}$ is the canonical B-basis of $\mathbb{M}_n(B)$. Consider the apartment A_0 corresponding to $\mathfrak{P}$, which we call the standard apartment. Note that the set of maximal orders in A_0 is in correspondence with the homothety classes of left fractional $\mathfrak{P}$-ideals in $k\mathfrak{P}$. In other words they are the stabilizers $\mathfrak{D}_{\vec{a}}$ of the lattices of the form $\bigoplus_{i=1}^n e_i\pi^{a_i}\mathcal{O}_B$, where $\{e_1,\dots,e_n\}$ is the cannonical basis of the column space B^n, and $\vec{a}=(a_1,\dots,a_n)\in\mathbb{Z}^n$.

Let $\vec{u}=(1,\dots,1)\in\mathbb{Z}^n$. Elements of $\Gamma=\mathbb{Z}^n/\langle\vec{u}\rangle$ are denoted in brackets, e.g., $[b]$ and $[d]$, in all that follows. Furthermore, for every element $\vec{a}=(a_1,\dots,a_n)\in\mathbb{Z}^n$, we write

$$[b]=[b_1,\dots,b_{n-1}]=[a_2-a_1,\dots,a_n-a_{n-1}]:=\vec{a}+\langle\vec{u}\rangle,$$

for the corresponding coset. An expression like $[d]=\vec{c}+\langle\vec{u}\rangle$ must be interpreted analogously. As $\mathfrak{D}_{\vec{a}+m\vec{u}}=\mathfrak{D}_{\vec{a}}$ for any $\vec{a}\in\mathbb{Z}^n$ and any $m\in\mathbb{Z}$, this order is denoted $\mathfrak{D}_{[b]}$ in the sequel. Furthermore, the permutation group S_n acts naturally on the order $\mathfrak{P}$ and its generated K-algebra. This induces an action of S_n on the group of fractional ideals of $\mathfrak{P}$ that can be interpreted as either, an action on the vertices of the chamber, or an action on Γ.

Example 4.1. If $n=5$, the permutation $\sigma=(12)(345)$ satisfies

$$\sigma(\mathfrak{D}_{[1,2,3,4]})=\sigma(\mathfrak{D}_{(0,1,3,6,10)})=\mathfrak{D}_{(1,0,10,3,6)}=\mathfrak{D}_{[-1,10,-7,3]}.$$

Example 4.2. If $n=3$, the orbit of $\mathfrak{D}_{[2,1]}$ is the set

$$\left\{\mathfrak{D}_{[2,1]},\mathfrak{D}_{[3,-1]},\mathfrak{D}_{[-2,3]},\mathfrak{D}_{[1,-3]},\mathfrak{D}_{[-3,2]},\mathfrak{D}_{[-1,-2]}\right\}.$$

Next result is immediate from the definition:

Lemma 4.3. *$\mathfrak{D}_0$ is the only vertex in the standard apartment stabilized by the whole of S_n. Every S_n-orbit in the standard apartment contains a unique order of the form $\mathfrak{D}_{[b_1,\dots,b_{n-1}]}$ with $b_1,\dots,b_{n-1}\geq 0$.*

We call either, an element $[b]=[b_1,\dots,b_{n-1}]$ with $b_1,\dots,b_{n-1}\geq 0$, or the corresponding order $\mathfrak{D}_{[b_1,\dots,b_{n-1}]}$, totally positive. Note that the coset

$[b]=\overrightarrow{a}+\langle\overrightarrow{u}\rangle$ is totally positive if and only if $\overrightarrow{a}=(a_1,\dots,a_n)$ is an increasing sequence.

Lemma 4.4. *Assume $[b]$ is totally positive. Then the maximal orders containing the GEO $\mathfrak{D}=\mathfrak{D}_0\cap\mathfrak{D}_{[b]}$ are exactly the orders $\mathfrak{D}_{[c]}$ with $[c]=[c_1,\dots,c_{n-1}]$ and $0\leq c_i\leq b_i$ for every $i=1,\dots,n-1$.*

Proof. First note that both $\mathfrak{D}_0$ and $\mathfrak{D}_{[b]}$ contain the order $\mathfrak{P}$ of integral diagonal matrices, so the same hold for every maximal order containing their intersection. We conclude that every such order is in the standard apartment. Next, set $[b]=\overrightarrow{a}+\langle\overrightarrow{u}\rangle$ and $[d]=\overrightarrow{c}+\langle\overrightarrow{u}\rangle$ as before. Then the inequality $b_i\geq c_i$ for $i=1,\dots,n-1$, implies $a_j-a_i\geq d_j-d_i$ for every pair (i,j) with $1\leq i<j\leq n$. The result follows if we observe that

$$\mathfrak{D}_{[b]}=\begin{pmatrix} \mathcal{O}_K & \pi^{a_1-a_2}\mathcal{O}_K & \pi^{a_1-a_3}\mathcal{O}_K & \cdots & \pi^{a_1-a_n}\mathcal{O}_K \\ \pi^{a_2-a_1}\mathcal{O}_K & \mathcal{O}_K & \pi^{a_2-a_3}\mathcal{O}_K & \cdots & \pi^{a_2-a_n}\mathcal{O}_K \\ \pi^{a_3-a_1}\mathcal{O}_K & \pi^{a_3-a_2}\mathcal{O}_K & \mathcal{O}_K & \cdots & \pi^{a_3-a_n}\mathcal{O}_K \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \pi^{a_n-a_1}\mathcal{O}_K & \pi^{a_n-a_2}\mathcal{O}_K & \pi^{a_n-a_3}\mathcal{O}_K & \cdots & \mathcal{O}_K \end{pmatrix},$$

and a similar formula holds for every order in the apartment. □

In what follows we denote by $\mathfrak{S}_0(\mathfrak{H})$, for every order $\mathfrak{H}$, the maximal subcomplex $\mathfrak{S}$ of the BT-building $\mathfrak{B}$ such that every vertex of $\mathfrak{S}$ corresponds to a maximal order containing $\mathfrak{H}$, and call it the block of $\mathfrak{H}$. Note that if $\mathfrak{H}'$ is the intersection of all maximal orders containing $\mathfrak{H}$, then $\mathfrak{S}_0(\mathfrak{H})=\mathfrak{S}_0(\mathfrak{H}')$. We let $S_0(\mathfrak{H})$ denote the set of vertices of $\mathfrak{S}_0(\mathfrak{H})$. On Γ we define the total length function $\big|\big|[b]\big|\big|=\sum_{i=1}^{n-1}|b_i|$.

Example 4.5. The cell-complexes $\mathfrak{S}_0(\mathfrak{D}')$, for $\mathfrak{D}'$ in the S_3-orbit of $\mathfrak{D}=\mathfrak{D}_0\cap\mathfrak{D}_{[2,1]}$, are shown in Figure 4.1.

Lemma 4.6. *If $[b]\in\Gamma$ is totally positive, for any $\sigma\in S_n$ we have $\big|\big|[b]\big|\big|\leq\big|\big|\sigma[b]\big|\big|$, with equality if and only if $\sigma[b]\in\{[b],[b]^*\}$, where*

$$[b]^*=[-b_{n-1},\dots,-b_2,-b_1]. \tag{4.1}$$

Furthermore, the complex $\mathfrak{S}_0\big(\sigma(\mathfrak{D})\big)$, where $\mathfrak{D}=\mathfrak{D}_0\cap\mathfrak{D}_{[b]}$, is a paralelotope whose edges are parallel to the axes if and only if $\sigma[b]\in\{[b],[b]^*\}$.

Proof. Note that if $[b]=\overrightarrow{a}+\langle\overrightarrow{u}\rangle$, the total length of $[b]$ is the total variation of the sequence $\overrightarrow{a}=(a_1,\dots,a_n)$. The first inequality follows. Furthermore, the orbit of $[b]$ contains exactly two vectors of minimal length, namely the ones corresponding to an increasing and a decreasing sequence in the orbit of $\overrightarrow{a}$. The S_n-action is linear on the vector space Γ, whence it takes parallelotopes onto parallelotopes and edges onto edges. Note that the length of $[b]$ can be described as the number of edges in a path going from 0 to $[b]$,

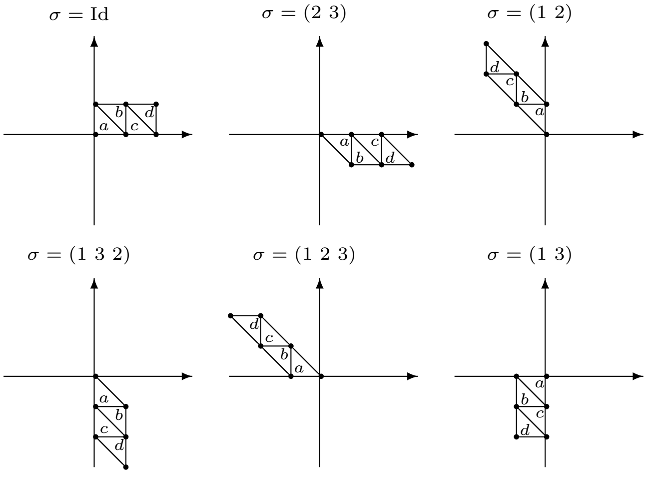

FIGURE 4.1. The block $S_0\Big(\sigma(\mathfrak{D})\Big)$ for each element σ in the symmetric group S_3.

running along the edges of a parallelotope whose edges are parallel to the axes. The last statement follows. ☐

Note that the correspondence $[b] \mapsto [b]^*$, as in (4.1), has the following properties:

- $\mathfrak{D}_{[b]^*} = \tau(\mathfrak{D}_{[b]})$, where $\tau = (1\ n)(2\ n-1)\cdots$ is the permutation reversing the n-tuple $(1, 2, \ldots, n)$,
- $\mathcal{T}(\mathfrak{D}_{[b]}, \mathfrak{D}_0) = -[b]^*$, in particular the order $\mathfrak{D}_0 \cap \mathfrak{D}_{[b]}$ is symmetric if and only if $[b]^* = -[b]$.

Example 4.7. Recall that a chain order $\mathfrak{C}$ is the intersection of all orders in a simplex of the BT-building [2, §2]. For such an order, the block $\mathfrak{S}_0(\mathfrak{C})$ is a simplex, as follows from Definition 3.1 and Proposition 3.3 in [11]. In particular, the only GEOs that are chain orders are the conjugates of $\mathfrak{D}_0 \cap \mathfrak{D}_{[e(i)]}$, for $i = 1, \ldots, n-1$, where the i-th coordinate of $[e(i)]$ is 1 and all the others are 0. This order is symmetric only when $i = n/2$, so in particular n is even.

The correspondence $\mathfrak{H} \mapsto S_0(\mathfrak{H})$ reverses inclusions, so that for every pair of elements $[c]$ and $[d]$ in Γ, with $\mathfrak{D}_{[c]}, \mathfrak{D}_{[d]} \in S_0(\mathfrak{D})$, the intersection $\mathfrak{D}' = \mathfrak{D}_{[c]} \cap \mathfrak{D}_{[d]}$ satisfies $S_0(\mathfrak{D}) \supseteq S_0(\mathfrak{D}')$. In fact, a stronger statement is true.

Lemma 4.8. *Let $\mathfrak{D} = \mathfrak{D}_0 \cap \mathfrak{D}_{[b]}$ be as above. If $\mathfrak{D}_{[c]}, \mathfrak{D}_{[d]} \in S_0(\mathfrak{D})$, satisfy $S_0(\mathfrak{D}) = S_0\big(\mathfrak{D}_{[c]} \cap \mathfrak{D}_{[d]}\big)$, then $([c], [d]) = (0, [b])$ or $([d], [c]) = (0, [b])$.*

Proof. Without loss of generality, we can assume that $[b]$ is totally positive. Observe that $||[c]-[d]|| \leq ||[b]||$, with equality if and only if $\mathfrak{D}_{[c]}$ and $\mathfrak{D}_{[d]}$ are opposite vertices of $S_0(\mathfrak{D})$. Conjugation by the diagonal matrix $\mathrm{diag}(1,\pi^{c_1},\dots,\pi^{c_1+\cdots+c_{n-1}})$ takes $\mathfrak{D}_{[t]}$ to $\mathfrak{D}_{[t]-[c]}$ for every $[t]\in\Gamma$. Note that there exists a permutation $\sigma\in S_n$ taking $[d]-[c]$ to a totally positive element $[r]$. Since S_n acts linearly on Γ, the cell-complex $\mathfrak{S}_0\left(\mathfrak{D}_{[c]}\cap\mathfrak{D}_{[d]}\right)$ is a parallelotope having $[c]$ and $[d]$ as opposite vertices. Furthermore, by hypotheses $\mathfrak{S}_0(\mathfrak{D}_{[c]}\cap\mathfrak{D}_{[d]})$ is a parallelotope whose edges are parallel to the coordinate axes. It follows from Lemma 4.6 that $[d]-[c]\in\{[r],[r]^*\}$. Since $[r]$ is totally positive, we conclude that either $[d]-[c]$ or $[c]-[d]$ is totally positive. The result follows. □

Lemma 4.9. *Let $\mathfrak{D}$ be as above. Let μ be an automorphism of $\mathfrak{A}$ satisfying $\mu(S_0(\mathfrak{D}))=S_0(\mathfrak{D})$. Then*

$$\left\{\mu(\mathfrak{D}_0),\mu\left(\mathfrak{D}_{[b]}\right)\right\}=\left\{\mathfrak{D}_0,\mathfrak{D}_{[b]}\right\}.$$

Proof. Note that $\mu(\mathfrak{D})$ is contained in exactly the same maximal orders as $\mathfrak{D}$, and furthermore $\mu(\mathfrak{D})=\mu(\mathfrak{D}_0)\cap\mu(\mathfrak{D}_{[b]})$, whence the result follows from the previous lemma. □

Lemma 4.10. *There exists an automorphism of $\mathfrak{A}$, satisfying $\mu(\mathfrak{D}_0)=\mathfrak{D}_{[b]}$ and $\mu(\mathfrak{D}_{[b]})=\mathfrak{D}_0$, if and only if $[b]^*=-[b]$.*

Proof. If $[b]^*=-[b]$, the permutation $\tau\in S_n$ defined by $\tau(i)=n-i$ takes $[b]$ to $-[b]$, so we can define $\mu=\rho\circ\tau$, where ρ is conjugation by the diagonal matrix $\mathrm{diag}(\pi^{a_1},\dots,\pi^{a_n})$ with $[b]=\overrightarrow{a}+\langle\overrightarrow{u}\rangle$. Sufficiency follows.

We denote by δ the canonical graph-distance on the 1-skeleton of the BT-building. We say that a pair $(\mathfrak{D},\mathfrak{D}')$ of maximal orders is of line-type if there are exactly $\delta(\mathfrak{D},\mathfrak{D}')+1$ maximal orders containing $\mathfrak{D}\cap\mathfrak{D}'$. A pair $([c],[d])\in\Gamma^2$ is of line-type when $(\mathfrak{D}_{[c]},\mathfrak{D}_{[d]})$ is of line-type. Automorphisms of $\mathfrak{A}$ necessarily take pairs of line-type to pairs of line-type. Note that, if $[b]=[b_1,\dots,b_{n-1}]$ is totally positive, then

$$1+\delta\left(\mathfrak{D}_0,\mathfrak{D}_{[b]}\right)\leq 1+\sum_{i=1}^{n-1}b_i\leq\prod_{i=1}^{n-1}(1+b_i),$$

and equality between the second and third expressions imply that at most one b_i in non-zero. Note that the expression on the right of the preceding chain of inequalities is actually the number of vertices in $S_0(\mathfrak{D})$. Hence, if $([c],[d])$ is of line type and $[d]-[c]$ is totally positive, then $S_0\left(\mathfrak{D}_{[c]}\cap\mathfrak{D}_{[d]}\right)$ is a line parallel to one of the axes. We conclude that any automorphism preserving $\mathfrak{D}_0$, and mapping $\mathfrak{D}_{[b]}$ to another totally positive order must preserve the set of axes of the polytope $S_0(\mathfrak{D}_0\cap\mathfrak{D}_{[b]})$. No automorphism can take a line parallel to one axis to a line parallel to a different axis, since the total distance between consecutive elements in the line is different. Applying this result to $(\rho\circ\tau)\circ\mu$ with $\rho\circ\tau$ defined as before, we show that an automorphism interchanging the orders $\mathfrak{D}_0$ and $\mathfrak{D}_{[b]}$ must replace a line where consecutive vertices have a total distance $a\in\mathbb{Z}/n\mathbb{Z}$ with a line where consecutive vertices have a total distance $-a$. This can happen only if $-[b]=[b]^*$. Necessity follows. □

For simplicity, if $\mathfrak{D}=\mathfrak{D}'\cap\mathfrak{D}''$, for local maximal orders $\mathfrak{D}'$ and $\mathfrak{D}''$ satisfying $[b]=\mathcal{T}(\mathfrak{D}',\mathfrak{D}'')$, we say that $\mathfrak{D}$ is a local GEO of type $[b]$. By the theory developed in §2, Theorem 3.1 follows from the following lemma:

Lemma 4.11. *For any GEO $\mathfrak{D}$ of type $[b]$, we have $H_k(\mathfrak{D})=H_k(\mathfrak{D}_0)=\mathcal{O}_k^*k^{*f}$ unless the following conditions hold:*

(1) *f is even.*
(2) *$[b]=-[b]^*$.*
(3) *$\sum_{i=1}^f a_i\equiv\frac{f}{2}\pmod f$, where $\overrightarrow{a}+\langle\overrightarrow{u}\rangle=[b]$.*

In the latter case $H_k(\mathfrak{D})=\mathcal{O}_k^*k^{*(f/2)}$.

Proof. For any element $\lambda\in B$, we have $\lambda\mathfrak{D}_{[b]}\lambda^{-1}=\mathfrak{D}_{[b]}$, and the same holds for a diagonal matrix whose diagonal entries are units. We conclude that $\mathcal{O}_k^*k^{*f}\subseteq H_k(\mathfrak{D})$. On the other hand, $H_k(\mathfrak{D}_0)=\mathcal{O}_k^*k^{*f}$, whence, unless condition 2 holds, we have equality because of Lemma 4.9 and Lemma 4.10. In the exceptional case, we conclude again from Lemma 4.9, that $H(\mathfrak{D})$ is generated by $\mathcal{O}_k^*k^{*f}$ and the norm of an arbitrary element u satisfying $u\mathfrak{D}_0u^{-1}=\mathfrak{D}_{[b]}$ and $u\mathfrak{D}_{[b]}u^{-1}=\mathfrak{D}_0$. The norm of such an element must satisfy $N(u)\mathcal{O}_k^*k^{*f}=\pi^d\mathcal{O}_k^*k^{*f}$, where d is the total distance $r_1+\cdots+r_f$ between $\mathfrak{D}_0$ and $\mathfrak{D}_{[b]}$. The symmetry condition implies that r_m+r_{f-m+1} is independent of m. It follows that either $d\equiv 0\pmod f$ or $d\equiv\frac{f}{2}\pmod f$. Certainly, this distinction is meaningless if f is odd. The result follows. □

5. Representations

Let $\mathfrak{H}$ be a suborder of a full order $\mathfrak{D}\subseteq\mathfrak{A}$, let $\Sigma=\Sigma(\mathfrak{D})$ be the spinor class field, and consider the set

$$\Phi=\{\rho(\mathfrak{D},\mathfrak{D}')|\mathfrak{D}'\in\mathrm{gen}(\mathfrak{D}),\ \mathfrak{H}\subseteq\mathfrak{D}'\}\subseteq\mathrm{Gal}(\Sigma/K).$$

When Φ is a group, the fixed subfield $F(\mathfrak{D}|\mathfrak{H})=\Sigma^{\Phi}$ is called the representation field. More generally, the field $F_-(\mathfrak{D}|\mathfrak{H})=\Sigma^{\langle\Phi\rangle}$, which is usually easy to compute, is called the lower representation field, while the fixed field $F^-(\mathfrak{D}|\mathfrak{H})=\Sigma^{\Gamma}$, where $\Gamma=\{\gamma\in\mathrm{Gal}(\Sigma/K)|\gamma\Phi=\Phi\}$, the upper representation field, has the trivial bound $F^-(\mathfrak{D}|\mathfrak{H})\subseteq L$ when $\mathfrak{H}$ is an order contained in the maximal subfield L (see the discussion preceding [4, Prop. 4.1]). Note that the representation field is defined if and only if $\Gamma=\langle\Phi\rangle$, i.e., $F_-(\mathfrak{D}|\mathfrak{H})=F^-(\mathfrak{D}|\mathfrak{H})$.

The field $F_-(\mathfrak{D}|\mathfrak{H})$ is the class field corresponding to the class group $K^*\langle I(\mathfrak{D}|\mathfrak{H})\rangle \subseteq J_K$, where $I(\mathfrak{D}|\mathfrak{H}) = \{N(a)|a \in \mathfrak{A}_{\mathbb{A}},\ \mathfrak{H} \subseteq a\mathfrak{D}a^{-1}\}$ is the relative spinor image. We conclude that the function $\mathfrak{D} \mapsto F_-(\mathfrak{D}|\mathfrak{H})$ reverses inclusions. In particular, if $\mathfrak{H}$ is an order in a maximal subfield L, and if we have $F_-(\mathfrak{D}|\mathfrak{H}) = L$ for some full order $\mathfrak{D}$ containing $\mathfrak{H}$, the same holds for every full order $\mathfrak{D}'$ with $\mathfrak{H} \subseteq \mathfrak{D}' \subseteq \mathfrak{D}$, and the representation field is defined for any such order. For the ring of integers $\mathcal{O}_L$ of the maximal subfield L, we can give a more precise result. We say that a full order $\mathfrak{D}$ is strongly unramified if $\Sigma(\mathfrak{D})$ is contained in the spinor class field for maximal orders Σ_0. Recall that a K-CSA has no partial ramification if it is locally a matrix or a division algebra at all finite places [3].

Proposition 5.1. *Assume $\mathfrak{D}$ is a strongly unramified order, and $\mathfrak{A}$ has no partial ramification. Then for every maximal subfield $L \subseteq \mathfrak{A}$, such that $\mathcal{O}_L \subseteq \mathfrak{D}$, the representation field $F(\mathfrak{D}|\mathcal{O}_L)$ is defined and in fact $F(\mathfrak{D}|\mathcal{O}_L) = \Sigma(\mathfrak{D}) \cap L$.*

Proof. It follows from [3, Prop. 4.3.4] than $F(\mathfrak{D}_0|\mathcal{O}_L) = \Sigma_0 \cap L$, if $\mathcal{O}_L \subseteq \mathfrak{D}_0$ and $\mathfrak{D}_0$ is maximal. The monotonicity implies that $F_-(\mathfrak{D}|\mathcal{O}_L) \supseteq \Sigma_0 \cap L$ for any full order $\mathfrak{D}$ containing $\mathcal{O}_L$. On the other hand, we always have the trivial bound $F^-(\mathfrak{D}|\mathcal{O}_L) \subseteq L$. Since $\Sigma(\mathfrak{D}) \subseteq \Sigma_0$, we have

$$\Sigma(\mathfrak{D}) \cap L \subseteq \Sigma_0 \cap L \subseteq F_-(\mathfrak{D}|\mathcal{O}_L) \subseteq F^-(\mathfrak{D}|\mathcal{O}_L) \subseteq \Sigma(\mathfrak{D}) \cap L,$$

whence equality follows. □

The preceding proposition applies in particular to GEOs. The hypothesis on $\mathfrak{A}$ is necessary, even for $\mathfrak{D} = \mathfrak{D}_0$, as shown by the counter-example in [3, §4.3]. This result does not helps us to know whether $\mathcal{O}_L$ embeds into some order in the genus of $\mathfrak{D}$ or not. This is a local problem and can be answered in some cases by Proposition 5.2 below.

Proposition 5.2. *Let $\mathfrak{H}$ be a local order such that $\mathfrak{S}_0(\mathfrak{H})$ is contained in the standard apartment. Assume $\mathfrak{D}$ is a local GEO of type $[b] \in \Gamma$. Then the following statements are equivalent:*

1. *$\mathfrak{S}_0(\mathfrak{H})$ has two vertices whose type difference is $[b]$.*
2. *There exist two vertices $\mathfrak{D}_{[c]}$ and $\mathfrak{D}_{[d]}$ in $S_0(\mathfrak{H})$, such that $[d]-[c]$ is in the S_n-orbit of $[b]$.*
3. *$\mathfrak{H}$ is contained in a conjugate of $\mathfrak{D}$.*

Proof. The equivalence between (1) and (3) follows from the well known fact that the automorphism group of $\mathfrak{A}$ acts transitively on pairs of lattices with the same invariant factors (§3). It is immediate that (2) implies (1), so we prove the converse. Assume $[b] = \mathcal{T}(\mathfrak{D}_{[c]}, \mathfrak{D}_{[d]})$ for two maximal orders $\mathfrak{D}_{[c]}$ and $\mathfrak{D}_{[d]}$ in $S_0(\mathfrak{H})$. By applying the action of S_n on the standard apartment (§3), we can assume that $[d]-[c]$ is in the first quadrant. We conclude that

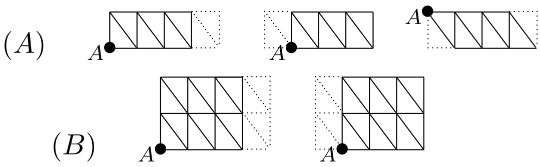

Figure 5.1. Two examples of embeddings of one GEO into another.

$[b]$ and $[d]-[c]$ are in the same orbit under the normalizer $\mathfrak{N}(\mathfrak{D}_0) \subseteq \mathfrak{A}^*$. Now we can prove that $[d]-[c]=[b]$, reasoning as in the proof of Lemma 4.10. The result follows. □

Example 5.3. Let $\mathfrak{D}$ and $\mathfrak{H}$ be local GEOs of type $[3,1]$ and $[4,1]$ respectively. Then $\mathfrak{S}_0(\mathfrak{D})$ can be embedded into $\mathfrak{S}_0(\mathfrak{H})$ in three different ways, corresponding to three embeddings of $\mathfrak{H}$ into $\mathfrak{D}$, as shown in Figure 5.1(A). Note that A denotes the image of one fixed maximal order in $\mathfrak{S}_0(\mathfrak{D})$. In this case the relative spinor image is $I_k(\mathfrak{D}|\mathfrak{H}) = k^*$, since the images of the vertex A, under each of the three embeddings, are at different total distances from the origin.

Example 5.4. Let $\mathfrak{D}$ and $\mathfrak{H}$ be local GEOs of type $[3,2]$ and $[4,2]$ respectively. Then $\mathfrak{S}_0(\mathfrak{D})$ can be embedded into $\mathfrak{S}_0(\mathfrak{H})$ in just two different ways as in Figure 5.1(B). In this case the relative spinor image $I_k(\mathfrak{D}|\mathfrak{H}) = \mathcal{O}_k^* k^{*3} \cup \pi^2 \mathcal{O}_k^* k^{*3}$ is not a group.

In order to apply this result to commutative orders, we need an explicit description of the cell complex $\mathfrak{S}_0(\mathfrak{H})$. We can do this for the order $\mathfrak{H} = \mathcal{O}_E$, for a maximal subfield $E \subseteq \mathfrak{A}$, when $\mathfrak{A}$ has no partial ramification. Recall that, for an extension of number fields E/K, the local completion $E_\wp = E \otimes_K K_\wp$ is a product of fields, and therefore we need a description of $\mathfrak{S}_0(\mathcal{O}_L)$ for a semisimple commutative algebra $L = E_\wp$. This is provided by next result. Here we identify the set of vertices of an n–dimensional apartment with $\mathbb{Z}^n$, and all cartesian products must be understood in this context.

Proposition 5.5. *Assume that the n-dimensional local semisimple algebra $L = \prod_{i=1}^r L_i \subseteq \mathbb{M}_n(k)$ is a product of the fields L_i. Then $\mathfrak{S}_0(\mathcal{O}_L)$ is contained in an apartment A and its set of vertices has a decomposition of the form*

$$S_0(\mathcal{O}_L) = S_1 \times \mathbb{Z} \times S_2 \times \cdots \times \mathbb{Z} \times S_r,$$

where every $S_i \subseteq \mathbb{Z}^{[L_i:k]-1}$ is the set of vertices in a simplex of dimension $e(L_i/k)-1$.

Proof. Note that the regular representation $\phi : L \to \mathbb{M}_n(k)$ is, up to conjugacy, the only faithful n-dimensional representation of the k-algebra L. We

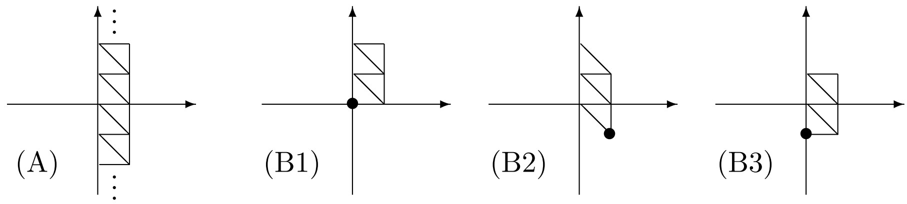

FIGURE 5.2. Embedding a maximal separable conmutative order in a GEO of type $[1,2]$.

conclude that the maximal orders containing $\mathcal{O}_L$ are in correspondence with the classes of fractional ideals in $\mathcal{O}_L$ up to k^*-multiplication. By choosing a suitable basis, we can assume that

$$L = \left\{ \left(\begin{array}{cccc} \phi_1(\lambda_1) & 0 & \cdots & 0 \\ 0 & \phi_2(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_r(\lambda_r) \end{array} \right) \middle| \lambda_i \in L_i \right\} \subseteq \mathbb{M}_n(k),$$

where $\phi_i : L_i \to \mathbb{M}_{[L_i:k]}(k)$ is the regular representation with respect to a basis of the form $S_i \cup \pi_i S_i \cup \cdots \cup \pi_i^{e(L_i/k)-1} S_i$ of L_i, for an arbitrary k-basis S_i of the largest unramified subfield of L_i, and an arbitrary uniformizing parameter π_i of L_i. Then all fractional ideals in L have the form $I_1 \times \cdots \times I_r$, where I_i is a fractional ideal in L_i. In particular, I_i is is homothetic to one of the ideals $(1), (\pi_i^1), \ldots, \left(\pi_i^{e(L_i/k)-1}\right)$. The fact that the corresponding vertices form a simplex is immediate from the definition of the BT-building in [2, §3]. $\square$

Example 5.6. Consider the algebra $L = F \times k$, where F is a ramified quadratic extension of k, identified with the set of matrices of the form $\left(\begin{smallmatrix} \phi(f) & 0 \\ 0 & a \end{smallmatrix} \right)$, for $f \in F$, $a \in k$, and $\phi : F \to \mathbb{M}_2(k)$ the regular representation. Then $\mathfrak{S}_0(\mathcal{O}_L)$ is as shown in Figure 5.2(A).

The picture already tells us that $\mathcal{O}_L$ is contained in a local GEO $\mathfrak{D}$ of type $[1,2]$, namely the one corresponding to the block in Figure 5.2(B1). The blocks of the orders $E\mathfrak{D}E^{-1}$ and $E^2\mathfrak{D}E^{-2}$, where $E = \left(\begin{smallmatrix} \phi(\pi_F) & 0 \\ 0 & 1 \end{smallmatrix} \right)$ and π_F is a uniformizing parameter of F, are shown in Figure 5.2(B2) and Figure 5.2(B3) respectively. Note that the multiplicative group of L acts transitively on the set of fractional ideals, whence conjugating by such elements, the block of any GEO representing $\mathcal{O}_L$ can be moved inside $\mathfrak{S}_0(\mathcal{O}_L)$, taking a given vertex to any prescribed possition in this block. This can be used to give a second proof of Proposition 5.1 for GEOs.

6. Global cell blocks for GEOs

For a global algebra satisfying **EC**, there is a simple way to describe the conjugacy classes in a genus of GEOs in terms of maximal orders. First we consider an indefinite quaternion algebra $\mathfrak{A}$ and an Eichler order $\mathfrak{D} \subset \mathfrak{A}$ whose level has only two prime divisors, say $\mathcal{L}(\mathfrak{D}) = \wp_1^{\alpha_1}\wp_2^{\alpha_2}$. Note that there exists $\alpha_i + 1$ local maximal orders containing $\mathfrak{D}_{\wp_i}$ and they lie on a path of the BT-tree at $\wp_i$ for $i \in \{1,2\}$. It follows that the global maximal orders containing $\mathfrak{D}$ correspond to the vertices of a rectangular grid with α_1+1 columns and α_2+1 rows. If we label these vertices alternating labels on each row and column as shown in Figure 6.1(A), each label correspond to a onjugacy class.

Let $\mathbb{O}_0$ be the genus of maximal orders in $\mathfrak{A}$ and let $\rho_0 : \mathbb{O}_0 \times \mathbb{O}_0 \to \mathrm{Gal}(\Sigma_0/K)$ be the distance map on maximal orders. The maximal orders $\mathfrak{D}_1$ and $\mathfrak{D}_2$, corresponding to any pair of horizontally (resp. vertically) adjacent vertices, satisfy $\rho_0(\mathfrak{D}_1,\mathfrak{D}_2) = |[\wp_1,\Sigma_0/K]|$ (resp. $|[\wp_2,\Sigma_0/K]|$), where $I \mapsto |[I,\Sigma_0/K]|$ is the Artin map on ideals. It follows that the isomorphism class of every maximal order in the grid depends only on the isomorphism class of the order in the lower-left corner. This gives a simple way to describe which Eichler orders embed into which others whose precise formulation is left to the reader. Observe that the orders in the two corners of the lower edge are conjugate when the length of that edge is even, while otherwise, their distance is $|[\wp_1,\Sigma_0/K]|$ (See Figure 6.1). A similar result holds for the other edges, and for Eichler orders of arbitrary level using higher dimensional grids. According to the description of the spinor class field for Eichler orders of a given level quoted in the introduction, the spinor class field $\Sigma(\mathfrak{D})$ is the fixed field of the group generated by the Frobenius elements $|[\wp,\Sigma_0/K]|$, for all places $\wp$ for which the valuation $v_\wp(\mathcal{L})$ of the level $\mathcal{L}$ of $\mathfrak{D}$ is odd. By a simple counting argument (see the proof of Proposition 6.5 below), next result follows:

Proposition 6.1. *Let $\mathfrak{D} = \mathfrak{D}_1 \cap \mathfrak{D}_2$ and $\mathfrak{D}' = \mathfrak{D}_3 \cap \mathfrak{D}_4$ be Eichler orders of the same level, in a quaternion algebra $\mathfrak{A}$ satisfying **EC**, and let $\Sigma \subseteq \Sigma_0$ be the corresponding spinor class field. Let $\mathbb{O} = \mathrm{gen}(\mathfrak{D})$, while $\mathbb{O}_0$ is the genus of maximal orders. Let $\rho : \mathbb{O} \times \mathbb{O} \to \mathrm{Gal}(\Sigma/K)$ and $\rho_0 : \mathbb{O}_0 \times \mathbb{O}_0 \to \mathrm{Gal}(\Sigma_0/K)$ be the corresponding distance functions. Then $\mathfrak{D}$ and $\mathfrak{D}'$ are conjugate if and only if $\rho_0(\mathfrak{D}_1,\mathfrak{D}_3) \in \mathrm{Gal}(\Sigma_0/\Sigma)$.*

More generally, in the hypotheses of the previous lemma, for any idelic element $a \in \mathfrak{A}_\mathbb{A}$ satisfying $\mathfrak{D}_3 = a\mathfrak{D}_1a^{-1}$, we have

$$\rho(\mathfrak{D},\mathfrak{D}') = \rho(\mathfrak{D},a\mathfrak{D}a^{-1})\rho(a\mathfrak{D}a^{-1},\mathfrak{D}') = \rho_0(\mathfrak{D}_1,\mathfrak{D}_3)\big|_\Sigma,$$

since, by the preceeding result, $a\mathfrak{D}a^{-1}$ and $\mathfrak{D}'$ are conjugate.

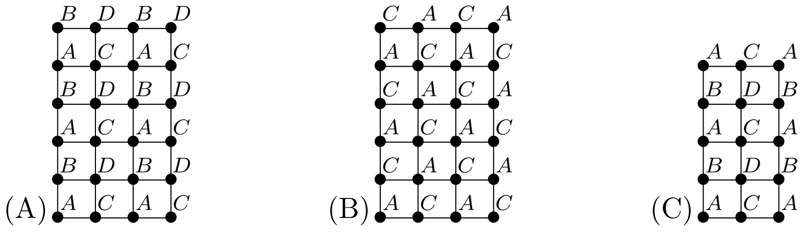

FIGURE 6.1. Maximal orders containing an Eichler order.

Example 6.2. Assume $|[\wp_1, \Sigma_0/K]|$ and $|[\wp_2, \Sigma_0/K]|$ are non-trivial and different. If $\alpha_1 = 3$ and $\alpha_2 = 5$ as in Figure 6.1(A), we have $[\Sigma_0 : \Sigma] = 4$, and $\mathfrak{D}_1 \cap \mathfrak{D}_2$ is conjugate to $\mathfrak{D}_3 \cap \mathfrak{D}_4$, as soon as $\mathfrak{D}_1$ is in the class A, while $\mathfrak{D}_3 \in A \cup B \cup C \cup D$. In Figure 6.1(C), the level $\wp_1^2 \wp_2^4$ of the Eichler order is a square, so there is a unique class of maximal orders, namely A, in the corners of the grid corresponding to this Eichler order. Such an order can only be the intersection of two maximal orders in A. Figure 6.1(B) shows an example where $|[\wp_1, \Sigma_0/K]| = |[\wp_2, \Sigma_0/K]|$ is non-trivial, whence $[\Sigma_0 : \Sigma] = 2$. Here the same Eichler order can be defined as the intersection of two orders in A or two orders in C.

For a GEO $\mathfrak{D}$, we can give a similar labeling of vertices to the product cell complex $\mathbb{S}(\mathfrak{D}) = \prod_\wp \mathfrak{S}_0(\mathfrak{D}_\wp)$ where the product is taken over all places at which $\mathfrak{D}$ is not maximal, and each factor of this product is the block corresponding to a local GEO as described in §3-§4. When $\mathfrak{D}_\wp = \mathfrak{D}_0 \cap \mathfrak{D}_{[r]}$ for a totally positive element $[r] \in \Gamma$, the pair of vertices $(\mathfrak{D}_0, \mathfrak{D}_{[r]})$ is completely determined up to reversion by Lemma 4.8. We call such a pair, a pair of opposite vertices, while $\mathfrak{D}_0$ and $\mathfrak{D}_{[r]}$ are called extreme vertices. A vertex in the product cell complex $\mathbb{S}(\mathfrak{D})$ is called extreme if each of its coordinates is extreme. We adopt a similar convention for pairs of opposite vertices. Locally, there are two ordered pairs of opposite vertices, whenever the order is not maximal. Globally, there are 2^t pairs of such vertices, where t is the number of places where the order is not maximal.

Proposition 6.3. *Let $\mathfrak{A}$ be a CSA over a number field K satisfying* ***EC****. Let $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, and $\mathfrak{D}_4$ be 4 maximal orders. Then there exists an element $a \in \mathfrak{A}$ satisfying both $\mathfrak{D}_3 = a\mathfrak{D}_1 a^{-1}$ and $\mathfrak{D}_4 = a\mathfrak{D}_2 a^{-1}$ if and only if the following conditions hold:*

- *$\mathcal{T}_\wp(\mathfrak{D}_1, \mathfrak{D}_2) = \mathcal{T}_\wp(\mathfrak{D}_3, \mathfrak{D}_4)$ for all finite places $\wp$.*
- *$\mathfrak{D}_1$ and $\mathfrak{D}_3$ are isomorphic.*

Proof. The conditions are obviously necessary, so we prove the sufficiency. Without loss of generality, we can assume that $\mathfrak{D}_1 = \mathfrak{D}_3$. In particular, there exists an adelic element $b \in \mathfrak{A}_{\mathbb{A}}^*$ satisfying both $\mathfrak{D}_1 = b\mathfrak{D}_1 b^{-1}$ and

$\mathfrak{D}_4 = b\mathfrak{D}_2 b^{-1}$. Fix a finite place $\wp$, and write $\mathfrak{A}_\wp = \mathbb{M}_f(B_\wp)$ where $B_\wp$ is a local division algebra. By a change of basis, if needed, we can assume also that $\mathfrak{D}_{2\wp}$ and $\mathfrak{D}_{1\wp}$ are both in the standard apartment and $\mathfrak{D}_{1\wp} = \mathbb{M}_f(\mathcal{O}_B)$. Since $\mathfrak{D}_{1\wp} = b_\wp \mathfrak{D}_{1\wp} b_\wp^{-1}$, we conclude that $b_\wp \in B^*\mathfrak{D}_{1\wp}^*$, and therefore its reduced norm satisfies $N(b_\wp) \in K_\wp^{*f}\mathcal{O}_{K_\wp}^*$. Since conjugation by a diagonal matrix of the form $\mathrm{diag}(\beta_1, \dots \beta_f)$, where $\beta_1, \dots \beta_f \in B$ are elements with the same absolute value, stabilizes every point in the standard apartment, we can replace b by an idele whose reduced norm is 1. Now the result is a consequence of the Strong Approximation Theorem for the group $\mathrm{SL}_1(\mathfrak{A})$. □

Proof of Theorem 1.1. Let Σ_0 and Σ be as in Theorem 3.1. As noted in the proof of Lemma 4.11, the symmetry condition implies that $2(r_1 + \cdots + r_f)$ is always a multiple of $f = f_\wp(\mathfrak{A}/K)$. We conclude that Σ_0/Σ is always an exponent-2 extension. On the other hand, Σ_0/K has exponent n where $n^2 = \dim_K \mathfrak{A}$. If n is odd we conclude $\Sigma = \Sigma_0$, and the first statement follows.

To prove the last statement, we fix an order $\mathfrak{D}$ in the genus, and an expression $\mathfrak{D} = \mathfrak{D}_1 \cap \mathfrak{D}_2$. Then for any maximal order $\mathfrak{D}'_1$ we choose $a \in \mathfrak{A}^*_{\mathbb{A}}$ satisfying $\mathfrak{D}'_1 = a\mathfrak{D}_1 a^{-1}$ and define $\Delta(\mathfrak{D}'_1) = a\mathfrak{D}_2 a^{-1}$. □

Remark 6.4. Note that $\Delta(\mathfrak{D}'_1)$ above depends on the choice of a but the conjugacy class of $\mathfrak{D}' = a\mathfrak{D}a^{-1}$ does not, by the preceding result. The map ψ is, however, non canonical, as it depends on the choice of the pair $(\mathfrak{D}_1, \mathfrak{D}_2)$ of opposite vertices in $\mathbb{S}(\mathfrak{D})$.

Proposition 6.5. *Let $\mathfrak{A}$ be a CSA over a number field K satisfying **EC**. Let $\mathfrak{D} = \mathfrak{D}_1 \cap \mathfrak{D}_2$ and $\mathfrak{D}' = \mathfrak{D}_3 \cap \mathfrak{D}_4$ be locally symmetric GEOs of $\mathfrak{A}$ in the same genus $\mathbb{O}$, while $\mathbb{O}_0$ is the genus of maximal orders. Let $\Sigma \subseteq \Sigma_0$ be the corresponding spinor class fields, and let $\rho : \mathbb{O} \times \mathbb{O} \to \mathrm{Gal}(\Sigma/K)$ and $\rho_0 : \mathbb{O}_0 \times \mathbb{O}_0 \to \mathrm{Gal}(\Sigma_0/K)$ be the corresponding distance functions. Then $\rho(\mathfrak{D}, \mathfrak{D}')$ is the restriction of $\rho_0(\mathfrak{D}_1, \mathfrak{D}_3)$ to Σ, in particular $\mathfrak{D}$ and $\mathfrak{D}'$ are conjugate if and only if $\rho_0(\mathfrak{D}_1, \mathfrak{D}_3)$ is trivial on Σ.*

Proof. The symmetry condition implies $\mathcal{T}_\wp(\mathfrak{D}_1, \mathfrak{D}_2) = \mathcal{T}_\wp(\mathfrak{D}_3, \mathfrak{D}_4)$ for every finite place $\wp$. It follows that $\mathfrak{D}$ and $\mathfrak{D}'$ are conjugate if and only if there is an extreme vertex in $\mathbb{S}(\mathfrak{D})$ conjugate to $\mathfrak{D}_3$. By Lemma 4.11, the Galois group $\mathrm{Gal}(\Sigma_0/\Sigma)$ is generated by the elements $\lambda(\wp) = |[\wp, \Sigma_0/K]|^{u(\wp)}$, where $\wp$ runs over the set T of places at which $\mathfrak{D}$ is not maximal and $u(\wp)$ is the total distance between opposite vertices in $S_0(\mathfrak{D}_\wp)$, as defined in §3. Furthermore, $\mathrm{Gal}(\Sigma_0/\Sigma)$ is a group of exponent 2, so every element has the form $\prod_{\wp \in T'} \lambda(\wp)$ for some $T' \subseteq T$. In particular, for any $\sigma \in \mathrm{Gal}(\Sigma_0/\Sigma)$, there exists an extreme vertex in $\mathbb{S}(\mathfrak{D})$ whose distance to $\mathfrak{D}_1$ is σ. If $\rho_0(\mathfrak{D}_1, \mathfrak{D}_3) \in \mathrm{Gal}(\Sigma_0/\Sigma)$, the last statement follows. The general case is proved reasoning as in the paragraph following Proposition 6.1. □

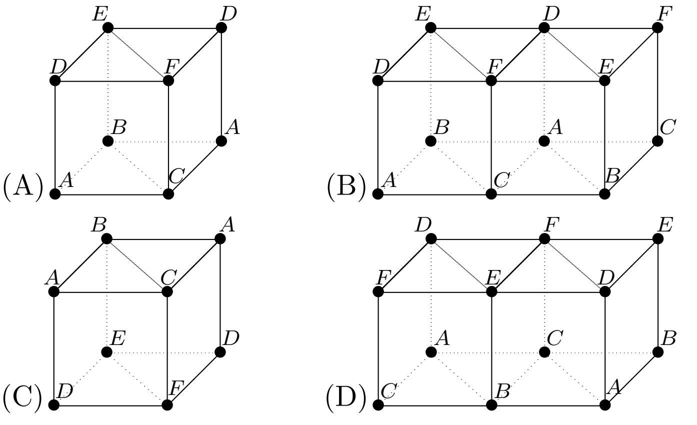

FIGURE 6.2. Maximal orders containing a GEO.

Corollary 6.6. *In the hypotheses o the previous proposition, if* $\Sigma = \Sigma_0$ *all extreme maximal orders in* $\mathbb{S}(\mathfrak{D})$ *are conjugate. This is the case, in particular, when* $\mathfrak{A}$ *is odd dimensional.*

Example 6.7. Assume $\mathfrak{A}$ is a 6^2-dimensional central division algebra having ramification degree 2 at $\wp_1$ and 3 at $\wp_2$, and let $\mathfrak{D}$ be a GEO having type $[1,1]$ at $\wp_1$ and $[1]$ at $\wp_2$. Then the complex $\mathbb{S}(\mathfrak{D})$ is a cube as shown in Figure 6.2(A). This is a locally symmetric GEO, whence the order $\mathfrak{D}'$ whose complex $\mathbb{S}(\mathfrak{D}')$ has an order in the class D at the lower left corner, as in Figure 6.2(C), is isomorphic to $\mathfrak{D}$. On the other hand, if $\mathfrak{D}''$ is a GEO in the same algebra having type $[2,1]$ at $\wp_1$ and $[1]$ at $\wp_2$, the complex $\mathbb{S}(\mathfrak{D}'')$ is as shown in Figure 6.2(B). This is not locally symetric at $\wp_1$, and in fact the order $\mathfrak{D}'''$ in the same genus whose complex $\mathbb{S}(\mathfrak{D}''')$ has an order in the class C in the lower left corner is, in general, not isomorphic to $\mathfrak{D}''$, as a quick glance to Figure 6.2(D) shows.

Proof of Theorem 1.2. Denote by $\mathfrak{N}(\mathfrak{H})$ the normalizer of every order $\mathfrak{H}$. Let $\mathfrak{D} = \mathfrak{D}_1 \cap \mathfrak{D}_2 = \mathfrak{D}_3 \cap \mathfrak{D}_4$. Let $\wp_1, \dots, \wp_r$ be the places at which $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are different. Assume $\mathfrak{D}$ is symmetric exactly at the places $\wp_1, \dots, \wp_t$. By Proposition 6.3, there exists an element $\phi \in \mathfrak{N} = \mathfrak{N}(\mathfrak{D})$ satisfying $\phi\mathfrak{D}_1\phi^{-1} = \mathfrak{D}_3$ and $\phi\mathfrak{D}_2\phi^{-1} = \mathfrak{D}_4$ if and only if $\mathcal{T}_\wp(\mathfrak{D}_1, \mathfrak{D}_2) = \mathcal{T}_\wp(\mathfrak{D}_3, \mathfrak{D}_4)$ for every finite place $\wp$, while $\mathfrak{D}_1$ and $\mathfrak{D}_3$ are isomorphic. The first condition is equivalent to $\mathfrak{D}_{1\wp} = \mathfrak{D}_{3\wp}$ for $\wp \in \{\wp_{t+1}, \dots, \wp_r\}$. There are 2^t such vertices, and they are the extreme vertices in $\mathbb{S}(\widetilde{\mathfrak{D}})$ for a locally symmetric GEO $\widetilde{\mathfrak{D}}$ with $\Sigma(\mathfrak{D}) = \Sigma(\widetilde{\mathfrak{D}})$. Note that $\mathfrak{N}_0 = \mathfrak{N}(\mathfrak{D}_1) \cap \mathfrak{N}(\mathfrak{D}_2)$ is the stabilizer of $\mathfrak{D}_1$ in $\mathfrak{N}$. Reasoning as in the proof of Proposition 6.5 we prove $[\mathfrak{N}(\mathfrak{D}) : \mathfrak{N}_0] = 2^t[\Sigma_0 : \Sigma]^{-1}$.

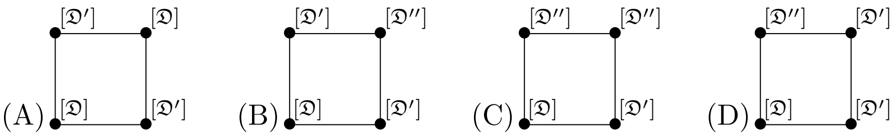

FIGURE 6.3. Some labelings of definite orders in the last example of §6.

Now, let $\Lambda_i \subseteq K^n$ be a lattice satisfying $\mathfrak{D}_i = \mathrm{End}_{\mathcal{O}_K}(\Lambda_i)$, for $i = 1, 2$. If $a \in \mathfrak{N}_0$, then $a\Lambda_i = b_i\Lambda_i$ for some idele $b_i \in J_K$. Note that the ideal $b_i^n\mathcal{O}_K$ is the principal ideal generated by $\det(a)$, whence $b_1\mathcal{O}_K = b_2\mathcal{O}_K$, so we can assume $b_1 = b_2$, and omit the subindex. We define $\Phi(a)$ as the class of the ideal $b\mathcal{O}_K$ in the ideal class group $\mathfrak{g}$ of K. In particular, $\Phi(a)^n \in \mathfrak{g}$ is the identity. If $\Phi(a)$ is the identity, we can assume $b \in K^*$, so $ab^{-1} \in \mathrm{Aut}_{\mathcal{O}_K}(\Lambda_i) = \mathfrak{D}_i^*$, or $a \in K^*\mathfrak{D}^*$. It remains to prove that $\Phi : \mathfrak{N}_0 \mapsto \mathfrak{g}(n)$ is surjective. For this, let b be any idele such that $b^n\mathcal{O}_K$ is a principal ideal, say $b^n\mathcal{O}_K = \lambda\mathcal{O}_K$ with $\lambda \in K^*$. Then choose any global matrix c with determinant λ^{-1}, and any adelic matrix $d = (d_\wp)_\wp$ with determinant $\lambda b^{-n} \in \prod_\wp \mathcal{O}_\wp^*$ satisfying $d\Lambda_i = \Lambda_i$ for $i = 1, 2$. In fact, we can assume, locally, that $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are in the standard apartment and choose a diagonal matrix whose diagonal entries are units. Let $\hat{g} = bcd \in \mathrm{SL}_n(\mathbb{A})$. The Strong Approximation Theorem implies the existence of a global matrix $g \in \mathrm{SL}_n(K)$, such that $g\Lambda_i = \hat{g}\Lambda_i = bc\Lambda_i$. Now we take $a = c^{-1}g$. The result follows. ☐

We conclude by giving an example of a definite quaternion algebra, and two Eichler orders in it that are not isomorphic, but the orders in the lower left (resp. the upper right) corners of the corresponding grids are isomorphic.

Example 6.8. Set $\mathfrak{A} = \left(\frac{-3,-23}{\mathbb{Q}}\right)$, the rational quaternion algebra ramifying only at ∞ and 23. Note that there is a unique spinor genera of maximal orders in $\mathfrak{A}$, since the wide class number of $\mathbb{Q}$ is 1, while there are 3 conjugacy classes [10, Eqn. (8)]. In particular, any two maximal orders in $\mathfrak{A}$ generate isomorphic $\mathbb{Z}[1/p]$-algebras for any prime $p \neq 23$.

As usual, we denote by i and j generators of $\mathfrak{A}$ satisfying $i^2 = -23$, $j^2 = -3$, and $ij = -ji$. The order $\mathbb{Z}[i, j]$ is maximal at any finite place outside $\{2, 3\}$, so the orders $\mathfrak{D} = \mathbb{Z}[i, \omega, \nu]$ and $\mathfrak{D}' = \mathbb{Z}[\eta, j, \nu]$, where $j = 2\omega - 1$, $i = 2\eta - 1$, and $3\nu = ij + j = 2\eta j$, are maximal. These two orders coincide outside the place 2. They are neighbors at 2, since $N(i + j)$ is a uniformizing parameter, so that $\mathbb{Z}_2[i + j] = \mathcal{O}_{\mathbb{Q}_2[i+j]}$ is contained in exactly 2 maximal orders [7, Prop. 4.2]. Note that ω is a unit in $\mathfrak{D}$, and conjugation by ω permutes cyclically the three neighbors at 2 of $\mathfrak{D}$. We conclude that these neighbors are all isomorphic to $\mathfrak{D}'$. In particular $\mathfrak{D}$ and $\mathfrak{D}'$ must fall in different conjugacy classes, since the Bruhat-Tits tree is connected and all conjugacy classes must be represented among the orders that coincide with $\mathfrak{D}$ outside $\{2\}$. For the same reason, $\mathfrak{D}'$ must have a neighbor $\mathfrak{D}''$ in the remaining class.

Fix any finite place q different from 2 and 23. For any Eichler order $\mathfrak{H}$ of level $2q$, consider the square $\mathbb{S}(\mathfrak{H})$, horizontal edges denoting neighbors at 2, with each vertex labeled with the isomorphism class of the corresponding order. If $\mathfrak{D}$ has a neighbor at q that is isomorphic to $\mathfrak{D}'$, we must have two orders $\mathfrak{H}_1$ and $\mathfrak{H}_2$ whose corresponding squares are labeled as in Figure 6.3(A) and 6.3(B). If not, then $\mathfrak{D}$ has a neighbor at q isomorphic to $\mathfrak{D}''$. If $\mathfrak{D}''$ has a neighbor at 2 isomorphic to itself, we have two orders whose corresponding squares are labeled as in Figure 6.3(C) and 6.3(D). Otherwise[1], every neighbor at 2 of $\mathfrak{D}''$ is isomorphic to $\mathfrak{D}'$, and we can interchange $\mathfrak{D}$ and $\mathfrak{D}''$ in Figures 6.3(A) and 6.3(B). In every case we have two squares with two opposite vertices labeled similarly, but not the others, so they must correspond to non-isomorphic orders.

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