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par Xiaodong LÜ et Quanwu MU

Résumé. Un nombre presque premier est un P_r s'il a au plus r facteurs premiers, comptés avec multiplicité. Dans cet article nous montrons que pour tout entier impair N suffisamment grand, l'équation

$$N = x^2 + p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^6 + p_6^7$$

admet une solution avec x un nombre presque premier P_{42} et les autres termes étant des puissances de nombres premiers.

Abstract. Let P_r denote an almost-prime with at most r prime factors, counted according to multiplicity. In this paper it is proved that for every sufficiently large odd integer N, the equation

$$N = x^2 + p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^6 + p_6^7$$

is solvable with x being an almost-prime P_{42} and the other terms powers of primes.

1. Introduction

Let b, c and N be positive integers and define $H_{b,c}(N)$ to be the number of solutions of the diophantine equation

$$N = x_1^2 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^b + x_7^c \tag{1.1}$$

in positive integers x_j. In 1981, C. Hooley [9] obtained an asymptotic formula for $H_{3,5}(N)$. From J. Brüdern's work [2] one can easily get the asymptotic formula for $H_{3,c}(N)$ [12]. Using a sort of pruning technique, Lu [12] established the asymptotic formula for $H_{4,k}(N)(4 \leq k \leq 6)$ and gave the lower bound estimates of the expected order of magnitude for $H_{4,k}(N)(7 \leq k \leq 17), H_{5,j}(N)(5 \leq j \leq 9)$ and $H_{6,l}(N)(6 \leq l \leq 7)$. Motivated by [12], A. M. Dashkevich [5] got the asymptotic formula for $H_{6,8}(N)$.

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In view of C. Hooley, Lu and A. M. Dashkevich's results, it is reasonable to conjecture that for every sufficiently large odd integer N the equation

$$N = p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^b + p_7^c \qquad (3 \le b \le c) \tag{1.2}$$

is solvable, where and below the letter p, with or without subscript, always denotes a prime number. This conjecture is perhaps out of reach at present. However, motivated by [4], the sieve theory and the circle method enable us to obtain the following approximation to it.

Theorem 1.1. *Let b and c be positive integers such that*

$$\frac{5}{18} < \frac{1}{b} + \frac{1}{c} \le \frac{1}{3}.$$

For a sufficiently large odd integer N, let $R_{b,c}(N)$ denote the number of solutions of the equation

$$N = x^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^b + p_7^c$$

with x being an almost-prime P_r and the other variables primes, where r is equal to $\left[\frac{4}{3}\left(\frac{1}{b} + \frac{1}{c} - \frac{5}{18}\right)^{-1}\right]$. Then we have

$$R_{b,c}(N) \gg \frac{N^{\frac{1}{b}+\frac{1}{c}+\frac{13}{18}}}{\log^7 N}.$$

We only provide the proof for the case $b = 6$, $c = 7$ and $r = 42$. The other results can be deduced similarly.

2. Notation and auxiliary lemmas

In this paper, N denotes a sufficiently large odd integer. As usual, $\varphi(n)$ and $\mu(n)$ stand for the Euler totient function and the Möbius function respectively. Denote by $\tau_k(n)$ the k-dimensional divisor function and write $\tau(n) = \tau_2(n)$. By $p^\ell || m$ we mean that $p^\ell | m$ but $p^{\ell+1} \nmid m$. We use (m, n) to stand for the greatest common divisor of m and n. We always denote by χ a Dirichlet character $\pmod q$, and by χ_0 the principal Dirichlet character $\pmod q$. By $\sum_{\chi(q)}$ we denote a sum with χ running over the Dirichlet character $\pmod q$. For positive A and B, $A \asymp B$ stands for $A \ll B \ll A$. We denote by $\sum_{x(q)}$ and $\sum_{x(q)^*}$ sums with x running over a complete system and a reduced system of residues modulo q respectively. We write $e(\alpha) = e^{2\pi\alpha}$. Let $a(m)$ and $b(n)$ be complex numbers with $|a(m)| \le 1$, $|b(n)| \le 1$. The letter ε denotes a sufficiently small positive number. c is some positive constant. Put

$$A = 10^{10}, \quad Q_0 = (\log N)^{20A}, \quad Q_1 = N^{\frac{59}{126}+30\varepsilon}, \quad Q_2 = N^{\frac{1}{2}},$$

$$Q_3 = N^{\frac{11}{21}}, \quad D = N^{\frac{1}{42}-30\varepsilon}, \quad U_j = \frac{1}{j}N^{\frac{1}{j}}, \quad U_3^* = \frac{1}{3}N^{\frac{5}{18}},$$

$$f_2(\alpha, d) = \sum_{U_2 < d\ell \le 2U_2} e\left(\alpha\,(d\ell)^2\right), \quad h(\alpha) = \sum_{m \le D^{\frac{2}{3}}} a(m) \sum_{n \le D^{\frac{1}{3}}} b(n) f_2(\alpha, mn),$$

$$f_j(\alpha) = \sum_{U_j < p \le 2U_j} (\log p) e(\alpha p^j), \quad f_3^*(\alpha) = \sum_{U_3^* < p \le 2U_3^*} (\log p) e(\alpha p^3),$$

$$S_k(q,\, a) = \sum_{r(q)} e\Big(\frac{ar^k}{q}\Big), \quad S_k^*(q,\, a) = \sum_{r(q)^*} e\Big(\frac{ar^k}{q}\Big),$$

$$G_j(\chi,\, a) = \sum_{r(q)} \chi(r) e\Big(\frac{ar^j}{q}\Big), \quad v_j(\beta) = \int_{U_j}^{2U_j} e(\beta u^j)\, \mathrm{d}u,$$

$$v_3^*(\beta) = \int_{U_3^*}^{2U_3^*} e(\beta u^3)\, \mathrm{d}u,$$

$$\mathfrak{J}(N) = \int_{-\infty}^{\infty} v_2(\theta) v_3(\theta)^2 v_3^*(\theta)^2 v_6(\theta) v_7(\theta) e(-\theta N)\, \mathrm{d}\theta,$$

$$A_d(q,\, N) = \frac{1}{q\varphi(q)^6} \sum_{\substack{a=1,\\ (a,\, q)=1}}^{q} S_2(q,\, ad^2) S_3^*(q,\, a)^4 S_6^*(q,\, a) S_7^*(q,\, a) e\Big(-\frac{aN}{q}\Big),$$

$$\mathfrak{S}_d(N) = \sum_{q=1}^{\infty} A_d(q,\, N),$$

Let Q be some positive number satisfying $Q < \sqrt{N}/2$. We define the intervals

$$\mathfrak{M}(q,\, a,\, Q) = \Big\{\alpha \in (0,\ 1] \ :\ \Big|q\alpha - a\Big| \le Q/N\Big\},$$

denote by $\mathfrak{M}(Q)$ the union of all $\mathfrak{M}(q,\, a,\, Q)$ with $1 \le a \le q \le Q$ and $(a,\, q) = 1$. Note that the intervals $\mathfrak{M}(q,\, a,\, Q)$, composing $\mathfrak{M}(Q)$, are mutually disjoint. Let $\mathfrak{B}(Q_1)$ be the union of all $\mathfrak{M}(q,\, a,\, N/Q_3)$ with $1 \le a \le q \le Q_1$ and $(a,\, q) = 1$, and $\mathfrak{b}(Q_1) = (0,\ 1] \setminus \mathfrak{B}(Q_1)$.

Lemma 2.1. *For $\alpha \in \mathfrak{b}(Q_1)$, we have*

$$h(\alpha) \ll N^{\frac{67}{252}-10\varepsilon}.$$

Proof. It follows from (4.5) in [3]. □

Lemma 2.2. *For* $(q,\, a) = 1$*, we have*

(i) $S_k(q,\, a) \ll q^{1-\frac{1}{k}}$;

(ii) $G_j(\chi,\, a) \ll q^{\frac{1}{2}+\varepsilon}$.

In particular, for $(p,\, a) = 1$ *we have*

(iii) $|S_k(p,\, a)| \le ((k, p-1) - 1)p^{\frac{1}{2}}$;

(iv) $|S_k^*(p,\, a)| \le ((k, p-1) - 1)p^{\frac{1}{2}} + 1$;

(v) $S_k^*(p^\ell,\, a) = 0$ *for* $\ell \ge \gamma(p)$,

where $\gamma(p) = \begin{cases} \theta + 2, & \text{if } p^\theta \| k,\ p \ne 2 \ \text{ or } \ p = 2,\ \theta = 0, \\ \theta + 3, & \text{if } p^\theta \| k,\ p = 2,\ \theta > 0. \end{cases}$

Proof. For (i) and (iii)-(iv), see Theorem 4.2 and Lemma 4.3 of [14] respectively. For (ii) and (v), see Lemmas 8.5 and 8.3 of [10]. □

Lemma 2.3. *Let* $2 \le k_1 \le k_2 \le \cdots \le k_s$ *be natural numbers such that*

$$\sum_{i=j+1}^{s} \frac{1}{k_i} \le \frac{1}{k_j}, \qquad 1 \le j \le s-1.$$

Then we have

$$\int_0^1 \Big|\prod_{i=1}^{s} f_{k_i}(\alpha)\Big|^2 \,\mathrm{d}\alpha \ll N^{\frac{1}{k_1}+\cdots+\frac{1}{k_s}+\varepsilon}.$$

Proof. See Lemma 1 of J. Brüdern [1]. □

Lemma 2.4. *We have*

(i) $\int_0^1 \big|f_3(\alpha) f_3^*(\alpha)^2\big|^2 \,\mathrm{d}\alpha \ll N^{\frac{8}{9}+\varepsilon}$,

(ii) $\int_0^1 \big|f_3(\alpha) f_3^*(\alpha)\big|^4 \,\mathrm{d}\alpha \ \ll N^{\frac{13}{9}}\big(\log N\big)^8$.

Proof. See Lemmas 2.2 and 2.4 of Cai [4]. □

For $\alpha = \frac{a}{q} + \beta$, define

$$\text{(2.1)}\quad \mathfrak{N}(q,\, a) = \Big(\frac{a}{q} - \frac{1}{qQ_0^{\frac{1}{2}}},\ \ \frac{a}{q} + \frac{1}{qQ_0^{\frac{1}{2}}}\Big], \qquad V_j(\alpha) = \frac{S_j^*(q,a)}{\varphi(q)} v_j(\beta),$$

$$V_3^*(\alpha) = \frac{S_3^*(q,a)}{\varphi(q)} v_3^*(\beta), \qquad \Delta_6(\alpha) = f_6(\alpha) - \frac{S_6^*(q,a)}{\varphi(q)} \sum_{U_6 < n \le 2U_6} e(\beta n^6)$$

and

$$V_2^*(\alpha) = \sum_{d \le D} \frac{c(d)}{dq} S_2(q, ad^2) v_2(\beta),$$

where

$$c(d) = \sum_{\substack{mn=d \\ m \le D^{2/3}, \\ n \le D^{1/3}}} a(m)b(n) \ll \tau(d).$$

Lemma 2.5. *We have*

$$\text{(i)}\quad \sum_{1\le q\le Q_0^{\frac12}} \sum_{\substack{a=-q\\(a,\,q)=1}}^{2q} \int_{\mathfrak{N}(q,\,a)} \big|V_2^*(\alpha)\Delta_6(\alpha)\big|^2\,\mathrm{d}\alpha \ll N^{\frac13}\big(\log N\big)^{-100A},$$

$$\text{(ii)}\quad \sum_{1\le q\le Q_0^{\frac12}} \sum_{\substack{a=-q\\(a,\,q)=1}}^{2q} \int_{\mathfrak{N}(q,\,a)} |V_2^*(\alpha)|^2\,\mathrm{d}\alpha \ll \big(\log N\big)^{11A}.$$

Proof. By Lemma 2.2 (i), the inequalities $(q,\,d^2)\le(q,\,d)^2,\ \tau(d\ell)\le\tau(d)\tau(\ell)$ and $\sum_{d\le x}\frac{\tau(d)}{d}\ll(\log x)^2$, we get

$$V_2^*(\alpha)\ll\sum_{d\le D}\frac{\tau(d)}{d}(q,\,d^2)^{\frac12}q^{-\frac12}|v_2(\beta)|\ll\tau_3(q)q^{-\frac12}|v_2(\beta)|(\log N)^2. \tag{2.2}$$

Let

$$E(\chi)=\begin{cases}1, & \chi=\chi_0,\\ 0, & \chi\ne\chi_0,\end{cases}$$

$$J_6(\chi,\,\beta)=\sum_{U_6<p\le 2U_6}\chi(p)e(\beta p^6)\log p-E(\chi)\sum_{U_6<n\le 2U_6}e(\beta n^6).$$

Then for $\alpha=\frac{a}{q}+\beta$, we have

$$\Delta_6(\alpha)=\frac{1}{\varphi(q)}\sum_{\chi(q)}G_6(\overline{\chi},a)J_6(\chi,\,\beta)\ll q^{\frac12+\varepsilon}\max_{\chi(q)}|J_6(\chi,\,\beta)|, \tag{2.3}$$

where Lemma 2.2 (ii) is used.

From the standard estimate

$$v_j(\beta)\ll\frac{U_j}{1+|\beta|N} \tag{2.4}$$

which follows from Lemma 4.2 in [13], and (2.2)–(2.3), we get

$$\begin{aligned}
&\sum_{1\le q\le Q_0^{\frac12}} \sum_{\substack{a=-q\\(a,\,q)=1}}^{2q} \int_{\mathfrak{N}(q,\,a)} |V_2^*(\alpha)\Delta_6(\alpha)|^2\,\mathrm{d}\alpha \\
&\ll(\log N)^4\sum_{1\le q\le Q_0^{\frac12}}q^{4\varepsilon}\sum_{a(q)^*}\max_{\chi(q)}\int_{|\beta|\le Q_0^{-\frac12}}\big|v_2(\beta)J_6(\chi,\,\beta)\big|^2\mathrm{d}\beta \\
&\ll(\log N)^4\sum_{1\le q\le Q_0^{\frac12}}q^{4\varepsilon}\sum_{a(q)^*}\max_{\chi(q)}\Bigg(N\int_{|\beta|\le\frac1N}\big|J_6(\chi,\,\beta)\big|^2\mathrm{d}\beta \\
&\qquad+\max_{1\le Z\le NQ_0^{-\frac12}}(\log N)NZ^{-2}\int_{\frac{Z}{N}\le|\beta|\le\frac{2Z}{N}}\big|J_6(\chi,\,\beta)\big|^2\mathrm{d}\beta\Bigg).
\end{aligned} \tag{2.5}$$

By Gallagher's Lemma 1 in [7] and Siegel-Walfisz Theorem (see (4) in Chapter 22 of [6]), we have

$$\tag{2.6} \begin{aligned} &\int_{|\beta|\le\frac{1}{N}} |J_6(\chi,\,\beta)|^2 \mathrm{d}\beta \\ &\qquad \ll \frac{1}{N^2}\int_2^N \Bigg| \sum_{\substack{U_6<p\le 2U_6 \\ \theta<p^6\le\theta+\frac{N}{2}}} \chi(p)\log p - E(\chi) \sum_{\substack{U_6<n\le 2U_6 \\ \theta<n^6\le\theta+\frac{N}{2}}} 1 \Bigg|^2 \mathrm{d}\theta \\ &\qquad \ll \frac{1}{N^2} N U_6^2 \cdot \exp(-\log^{\frac{1}{3}} N) \end{aligned}$$

and similarly,

$$\tag{2.7} \int_{\frac{Z}{N}\le|\beta|\le\frac{2Z}{N}} |J_6(\chi,\,\beta)|^2 \mathrm{d}\beta \ll Z^2 N^{-\frac{2}{3}} \cdot \exp(-\log^{\frac{1}{3}} N).$$

It follows from (2.5)–(2.7) that

$$\tag{2.8} \begin{aligned} &\sum_{1\le q\le Q_0^{\frac{1}{2}}} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathfrak{N}(q,\,a)} |V_2^*(\alpha)\Delta_6(\alpha)|^2 \,\mathrm{d}\alpha \\ &\qquad\qquad \ll (Q_0^{\frac{1}{2}})^{2+4\varepsilon} (\log N)^5 N^{\frac{1}{3}} \cdot \exp(-\log^{\frac{1}{3}} N) \\ &\qquad\qquad \ll N^{\frac{1}{3}} (\log N)^{-100A}. \end{aligned}$$

This proves (i). By (2.2) and (2.4), we have

$$\begin{aligned} &\sum_{1\le q\le Q_0^{\frac{1}{2}}} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathfrak{N}(q,\,a)} |V_2^*(\alpha)|^2 \,\mathrm{d}\alpha \\ &\ll (\log N)^4 \sum_{1\le q\le Q_0^{\frac{1}{2}}} q^{-1+\varepsilon} \sum_{a(q)^*} \int_{|\beta|\le Q_0^{-\frac{1}{2}}} |v_2(\beta)|^2 \mathrm{d}\beta \\ &\ll (\log N)^4 \sum_{1\le q\le Q_0^{\frac{1}{2}}} q^{-1+\varepsilon} \sum_{a(q)^*} \left(N\int_{|\beta|\le\frac{1}{N}} \mathrm{d}\beta + N^{-1}\int_{\frac{1}{N}<|\beta|\le Q_0^{-\frac{1}{2}}} |\beta|^{-2}\mathrm{d}\beta \right) \\ &\ll (\log N)^{11A}. \end{aligned}$$

The proof of Lemma 2.5 is completed. □

Lemma 2.6. *For $\alpha \in \mathfrak{M}(Q_0)$, we have*

$$f_3(\alpha) = V_3(\alpha) + O\Big(U_3 \exp\big(-(\log N)^{\frac{1}{3}}\big)\Big),$$
$$f_3^*(\alpha) = V_3^*(\alpha) + O\Big(U_3^* \exp\big(-(\log N)^{\frac{1}{3}}\big)\Big),$$
$$f_7(\alpha) = V_7(\alpha) + O\Big(U_7 \exp\big(-(\log N)^{\frac{1}{3}}\big)\Big).$$

Proof. By some routine arrangements, it follows from Siegel-Walfisz Theorem and summation by parts. □

3. A mean value theorem

In this section, we prove a mean value theorem for the proof of the theorem.

Lemma 3.1. *Let*

$$J_d(N) = \sum_{\substack{(dx)^2+p_1^3+\cdots+p_4^3+p_5^6+p_6^7=N \\ U_2<dx\le 2U_2,\ U_3<p_1,p_2\le 2U_3, \\ U_3^*<p_3,p_4\le 2U_3^*,\ U_6<p_5\le 2U_6, \\ U_7<p_6\le 2U_7}} \prod_{i=1}^{6} \log p_i.$$

Then we have

$$\sum_{m\le D^{2/3}} a(m) \sum_{n\le D^{1/3}} b(n) \left(J_{mn}(N) - \frac{\mathfrak{S}_{mn}(N)}{mn}\mathfrak{J}(N)\right) \ll N^{\frac{65}{63}} \big(\log N\big)^{-A}.$$

Proof. It is easy to see that

$$(3.1)\qquad \sum_{m\le D^{2/3}} a(m) \sum_{n\le D^{1/3}} b(n) J_{mn}(N)$$
$$= \int_0^1 h(\alpha) f_3(\alpha)^2 f_3^*(\alpha)^2 f_6(\alpha) f_7(\alpha) e(-\alpha N)\,\mathrm{d}\alpha$$
$$= \left(\int_{\mathfrak{b}(Q_1)} + \int_{\mathfrak{B}(Q_1)}\right) h(\alpha) f_3(\alpha)^2 f_3^*(\alpha)^2 f_6(\alpha) f_7(\alpha) e(-\alpha N)\,\mathrm{d}\alpha.$$

By Cauchy's inequality, Lemmas 2.3 and 2.4, we have

$$(3.2)\qquad \int_0^1 \Big|f_3(\alpha)^2 f_3^*(\alpha)^2 f_6(\alpha) f_7(\alpha)\Big|\,\mathrm{d}\alpha$$
$$\le \left(\int_0^1 \Big|f_3(\alpha) f_6(\alpha) f_7(\alpha)\Big|^2 \mathrm{d}\alpha\right)^{\frac{1}{2}} \left(\int_0^1 \Big|f_3(\alpha) f_3^*(\alpha)^2\Big|^2 \mathrm{d}\alpha\right)^{\frac{1}{2}}$$
$$\le \left(N^{\frac{9}{14}+\varepsilon+\frac{8}{9}+\varepsilon}\right)^{\frac{1}{2}} \le N^{\frac{193}{252}+\varepsilon},$$

this with Lemma 2.1 gives

$$\begin{aligned}(3.3)\quad &\int_{\mathfrak{b}(Q_1)} h(\alpha)f_3(\alpha)^2 f_3^*(\alpha)^2 f_6(\alpha)f_7(\alpha)e(-\alpha N)\,\mathrm{d}\alpha\\ &\ll \max_{\alpha\in\mathfrak{b}(Q_1)}|h(\alpha)|\int_0^1 \Big|f_3(\alpha)^2 f_3^*(\alpha)^2 f_6(\alpha)f_7(\alpha)\Big|\mathrm{d}\alpha\\ &\ll N^{\frac{65}{63}-\varepsilon}.\end{aligned}$$

When $\alpha\in\mathfrak{B}(Q_1)$, using Theorem 4.1 of Vaughan [14] we can get

$$h(\alpha)-V_2^*(\alpha)\ll DQ_1^{\frac{1}{2}+\varepsilon}.$$

Thus,

$$\begin{aligned}(3.4)\quad &\int_{\mathfrak{B}(Q_1)}(h(\alpha)-V_2^*(\alpha))f_3(\alpha)^2 f_3^*(\alpha)^2 f_6(\alpha)f_7(\alpha)e(-\alpha N)\,\mathrm{d}\alpha\\ &\ll DQ_1^{\frac{1}{2}+\varepsilon}\int_0^1\Big|f_3(\alpha)^2 f_3^*(\alpha)^2 f_6(\alpha)f_7(\alpha)\Big|\,\mathrm{d}\alpha\ll N^{\frac{65}{63}-\varepsilon},\end{aligned}$$

where (3.2) is used.

We rewrite (3.4) in the form

$$\begin{aligned}(3.5)\quad &\int_{\mathfrak{B}(Q_1)} h(\alpha)f_3(\alpha)^2 f_3^*(\alpha)^2 f_6(\alpha)f_7(\alpha)e(-\alpha N)\,\mathrm{d}\alpha\\ &=\int_{\mathfrak{B}(Q_1)} V_2^*(\alpha)f_3(\alpha)^2 f_3^*(\alpha)^2 f_6(\alpha)f_7(\alpha)e(-\alpha N)\,\mathrm{d}\alpha+O\big(N^{\frac{65}{63}-\varepsilon}\big)\\ &=\bigg(\int_{\mathfrak{M}(U_7)}+\int_{\mathfrak{B}(Q_1)\setminus\mathfrak{M}(U_7)}\bigg)V_2^*(\alpha)f_3(\alpha)^2 f_3^*(\alpha)^2 f_6(\alpha)f_7(\alpha)e(-\alpha N)\,\mathrm{d}\alpha\\ &\quad+O\big(N^{\frac{65}{63}-\varepsilon}\big).\end{aligned}$$

In the following, we will first prove that the second integral in the above brackets only gives a negligible contribution, and then show that the first integral in the above brackets yields the main term.

Let

$$\mathfrak{N}=\bigcup_{1\le q\le Q_0^{\frac{1}{2}}}\ \bigcup_{\substack{a=-q\\(a,\,q)=1}}^{2q}\mathfrak{N}(q,\,a),$$

where $\mathfrak{N}(q,\,a)$ is defined by (2.1). Then we have

$$(3.6)\qquad \mathfrak{B}(Q_1)\subset\Big(-\frac{1}{Q_3},\ 1\Big]\subset\mathfrak{N}.$$

Note that for $\alpha = \frac{a}{q} + \beta \in \mathfrak{B}(Q_1) \setminus \mathfrak{M}(U_7)$, one has

$$q \geq U_7 \qquad \text{or} \qquad |\beta| \geq \frac{U_7}{qN},$$

this with (2.2) and (2.4) yields

$$V_2^*(\alpha) \ll U_2 U_7^{-\frac{1}{2}+\varepsilon}.$$

Thus, by Cauchy's inequality and Hua's inequality, we have

$$\begin{aligned}
(3.7)\quad & \int_{\mathfrak{B}(Q_1)\setminus\mathfrak{M}(U_7)} V_2^*(\alpha) f_3(\alpha)^2 f_3^*(\alpha)^2 f_6(\alpha) f_7(\alpha) e(-\alpha N)\,\mathrm{d}\alpha \\
&\ll \max_{\alpha\in\mathfrak{B}(Q_1)\setminus\mathfrak{M}(U_7)} |V_2^*(\alpha)|^{\frac{1}{32}} \Big(\int_{\mathfrak{B}(Q_1)} |V_2^*(\alpha)|^2\,\mathrm{d}\alpha\Big)^{\frac{31}{64}} \\
&\quad \times \Big(\int_0^1 |f_3(\alpha) f_3^*(\alpha)|^4\,\mathrm{d}\alpha\Big)^{\frac{1}{2}} \Big(\int_0^1 |f_6(\alpha)|^{128}\,\mathrm{d}\alpha\Big)^{\frac{1}{128}} \Big(\int_0^1 |f_7(\alpha)|^{128}\,\mathrm{d}\alpha\Big)^{\frac{1}{128}} \\
&\ll \big(U_2 U_7^{-\frac{1}{2}+\varepsilon}\big)^{\frac{1}{32}} U_6^{\frac{1}{2}} \left(\sum_{1\leq q\leq Q_0^{\frac{1}{2}}} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathfrak{N}(q,a)} |V_2^*(\alpha)|^2\,\mathrm{d}\alpha \right)^{\frac{31}{64}} \\
&\quad \times \Big(\int_0^1 |f_3(\alpha) f_3^*(\alpha)|^4\,\mathrm{d}\alpha\Big)^{\frac{1}{2}} \Big(\int_0^1 |f_6(\alpha)|^{64}\,\mathrm{d}\alpha\Big)^{\frac{1}{128}} \Big(\int_0^1 |f_7(\alpha)|^{128}\,\mathrm{d}\alpha\Big)^{\frac{1}{128}} \\
&\ll \big(N^{\frac{1}{2}-\frac{1}{14}+\varepsilon}\big)^{\frac{1}{32}} \cdot N^{\frac{1}{12}+\frac{13}{18}+\varepsilon} \cdot \big(N^{\frac{29}{3}+\varepsilon}\big)^{\frac{1}{128}} \cdot \big(N^{\frac{121}{7}+\varepsilon}\big)^{\frac{1}{128}} \\
&\ll N^{\frac{593}{576}+4\varepsilon} \ll N^{\frac{65}{63}-\varepsilon}.
\end{aligned}$$

where Lemmas 2.4 and 2.5 (ii) are used.

It follows from [13, Lemma 4.8] that

$$\begin{aligned}
(3.8)\quad & \int_{\mathfrak{M}(U_7)} V_2^*(\alpha) f_3(\alpha)^2 f_3^*(\alpha)^2 f_6(\alpha) f_7(\alpha) e(-\alpha N)\,\mathrm{d}\alpha \\
&= \int_{\mathfrak{M}(U_7)} V_2^*(\alpha) f_3(\alpha)^2 f_3^*(\alpha)^2 V_6(\alpha) f_7(\alpha) e(-\alpha N)\,\mathrm{d}\alpha \\
&\quad + \int_{\mathfrak{M}(U_7)} V_2^*(\alpha) f_3(\alpha)^2 f_3^*(\alpha)^2 \Delta_6(\alpha) f_7(\alpha) e(-\alpha N)\,\mathrm{d}\alpha \\
&\quad + O\Big(\int_{\mathfrak{M}(U_7)} \big|V_2^*(\alpha) f_3(\alpha)^2 f_3^*(\alpha)^2 f_7(\alpha)\big|\,\mathrm{d}\alpha\Big).
\end{aligned}$$

By Cauchy's inequality, Lemmas 2.4 (ii) and 2.5, we deduce that

$$(3.9)\quad \int_{\mathfrak{M}(U_7)} V_2^*(\alpha)f_3(\alpha)^2f_3^*(\alpha)^2\Delta_6(\alpha)f_7(\alpha)e(-\alpha N)\,\mathrm{d}\alpha$$
$$\ll \max_{\alpha\in\mathfrak{M}(U_7)}|f_7(\alpha)|\Big(\int_{\mathfrak{M}(U_7)}|V_2^*(\alpha)\Delta_6(\alpha)|^2\,\mathrm{d}\alpha\Big)^{\frac12}\Big(\int_0^1|f_3(\alpha)f_3^*(\alpha)|^4\,\mathrm{d}\alpha\Big)^{\frac12}$$
$$\ll N^{\frac{65}{63}}(\log N)^{-A}$$

and

$$(3.10)\quad \int_{\mathfrak{M}(U_7)} \big|V_2^*(\alpha)f_3(\alpha)^2f_3^*(\alpha)^2f_7(\alpha)\big|\,\mathrm{d}\alpha$$
$$\ll \max_{\alpha\in\mathfrak{M}(U_7)}|f_7(\alpha)|\Big(\int_{\mathfrak{M}(U_7)}|V_2^*(\alpha)|^2\,\mathrm{d}\alpha\Big)^{\frac12}\Big(\int_0^1|f_3(\alpha)f_3^*(\alpha)|^4\,\mathrm{d}\alpha\Big)^{\frac12}$$
$$\ll N^{\frac{65}{63}}(\log N)^{-A}.$$

By (3.8)–(3.10), we get

$$(3.11)\quad \int_{\mathfrak{M}(U_7)} V_2^*(\alpha)f_3(\alpha)^2f_3^*(\alpha)^2f_6(\alpha)f_7(\alpha)e(-\alpha N)\,\mathrm{d}\alpha$$
$$= \int_{\mathfrak{M}(U_7)} V_2^*(\alpha)f_3(\alpha)^2f_3^*(\alpha)^2V_6(\alpha)f_7(\alpha)e(-\alpha N)\,\mathrm{d}\alpha + O\big(N^{\frac{65}{63}}(\log N)^{-A}\big)$$
$$= \Big(\int_{\mathfrak{M}(Q_0)} + \int_{\mathfrak{M}(U_7)\setminus\mathfrak{M}(Q_0)}\Big)V_2^*(\alpha)f_3(\alpha)^2f_3^*(\alpha)^2V_6(\alpha)f_7(\alpha)e(-\alpha N)\,\mathrm{d}\alpha$$
$$+ O\big(N^{\frac{65}{63}}(\log N)^{-A}\big).$$

Similarly to (3.6), we get

$$\mathfrak{M}(U_7) \subset \Big(-\frac{U_7}{N},\ 1\Big] \subset \mathfrak{N}.$$

For $\alpha = \frac{a}{q} + \beta \in \mathfrak{M}(U_7)\setminus\mathfrak{M}(Q_0)$, it is clear that

$$q \ge Q_0 \quad \text{or} \quad |\beta| \ge \frac{Q_0}{qN}.$$

Lemma 2.2 (ii) and (2.4), we have

$$V_6(\alpha) \ll U_6 Q_0^{-\frac12+\varepsilon}.$$

Hence by Lemmas 2.4 (ii) and 2.5 (ii), we obtain

$$
\begin{aligned}
(3.12)\quad & \int_{\mathfrak{M}(U_7)\setminus\mathfrak{M}(Q_0)} V_2^*(\alpha) f_3(\alpha)^2 f_3^*(\alpha)^2 V_6(\alpha) f_7(\alpha) e(-\alpha N)\,\mathrm{d}\alpha \\
& \ll \max_{\alpha\in\mathfrak{M}(U_7)\setminus\mathfrak{M}(Q_0)} |V_6(\alpha)|\, U_7 \Big(\int_{\mathfrak{M}(U_7)} |V_2^*(\alpha)|^2\,\mathrm{d}\alpha \Big)^{\frac12} \Big(\int_0^1 |f_3(\alpha) f_3^*(\alpha)|^4\,\mathrm{d}\alpha \Big)^{\frac12} \\
& \ll \big(U_6 Q_0^{-\frac12+\varepsilon}\big)\, U_7 \left(\sum_{1\le q\le Q_0^{\frac12}} \sum_{\substack{a=-q\\ (a,\,q)=1}}^{2q} \int_{\mathfrak{N}(q,\,a)} |V_2^*(\alpha)|^2\,\mathrm{d}\alpha \right)^{\frac12} \\
& \qquad \times \Big(\int_0^1 |f_3(\alpha) f_3^*(\alpha)|^4\,\mathrm{d}\alpha \Big)^{\frac12} \\
& \ll N^{\frac{65}{63}} \big(\log N\big)^{-A}.
\end{aligned}
$$

For $\alpha \in \mathfrak{M}(Q_0)$, Lemma 2.6 implies that

$$
\begin{aligned}
& \Big| V_2^*(\alpha) f_3(\alpha)^2 f_3^*(\alpha)^2 V_6(\alpha) f_7(\alpha) e(-\alpha N) \\
& \qquad\qquad - V_2^*(\alpha) V_3(\alpha)^2 V_3^*(\alpha)^2 V_6(\alpha) V_7(\alpha) e(-\alpha N) \Big| \\
& \qquad \ll \Big(\max_{\alpha\in\mathfrak{M}(Q_0)} |V_2^*(\alpha)| \Big) U_3^2 U_3^{*2} U_6 U_7 \exp\big(-(\log N)^{\frac13}\big) \\
& \qquad \ll N^{\frac{128}{63}} \exp\big(-(\log N)^{\frac14}\big),
\end{aligned}
$$

where the trivial bound $|V_2^*(\alpha)| \ll U_2 \log^2 N$ which can be deduced from (2.2) and (2.4) is used. The above estimate gives

$$
\begin{aligned}
(3.13)\quad & \int_{\mathfrak{M}(Q_0)} V_2^*(\alpha) f_3(\alpha)^2 f_3^*(\alpha)^2 V_6(\alpha) f_7(\alpha) e(-\alpha N)\,\mathrm{d}\alpha \\
& = \int_{\mathfrak{M}_0} V_2^*(\alpha) V_3(\alpha)^2 V_3^*(\alpha)^2 V_6(\alpha) V_7(\alpha) e(-\alpha N)\,\mathrm{d}\alpha \\
& \qquad + O\Big(\sum_{q\le Q_0^5} \sum_{\substack{a=1\\ (a,q)=1}}^{q} \int_{|\beta|\le \frac{Q_0}{N}} N^{\frac{128}{63}} \exp\big(-(\log N)^{\frac14}\big)\,\mathrm{d}\alpha \Big) \\
& = \int_{\mathfrak{M}_0} V_2^*(\alpha) V_3(\alpha)^2 V_3^*(\alpha)^2 V_6(\alpha) V_7(\alpha) e(-\alpha N)\,\mathrm{d}\alpha + O\big(N^{\frac{65}{63}} (\log N)^{-A}\big).
\end{aligned}
$$

Now the well-known standard endgame technique in the circle method establishes that

$$\int_{\mathfrak{M}_0} V_2^*(\alpha)V_3(\alpha)^2V_3^*(\alpha)^2V_6(\alpha)V_7(\alpha)e(-\alpha N)\,\mathrm{d}\alpha \\ = \sum_{m\le D^{2/3}} a(m) \sum_{n\le D^{1/3}} b(n)\frac{\mathfrak{S}_{mn}(N)}{mn}\mathfrak{J}(N) + O\big(N^{\frac{65}{63}}(\log N)^{-A}\big), \tag{3.14}$$

and

$$N^{\frac{65}{63}} \ll \mathfrak{J}(N) \ll N^{\frac{65}{63}}. \tag{3.15}$$

From (3.1), (3.3), (3.5) and (3.7)–(3.15), we complete the proof of Lemma 3.1. □

4. On the function $\omega(d)$ and singular seris $\mathfrak{S}_1(N)$

Lemma 4.1. *Let $K(q,N)$ and $L(q,N)$ denote the number of solutions to the congruences*

$$u_1^3+u_2^3+u_3^3+u_4^3+u_5^6+u_6^7 \equiv N(\bmod q), \\ 1\le u_j\le q,\ (u_j,q)=1,\ j=1,\cdots,6$$

and

$$y^2+u_1^3+u_2^3+u_3^3+u_4^3+u_5^6+u_6^7 \equiv N(\bmod q), \\ 1\le y,u_\ell\le q,\ (u_\ell,q)=1,\ \ell=1,\cdots,6$$

respectively. Then we have $L(p,N)>K(p,N)$. Moreover, we have

$$L(p,N)=p^6+O(p^5), \tag{4.1}$$

$$K(p,N)=p^5+O(p^4). \tag{4.2}$$

Proof. Let $L^*(q,N)$ denote the number of solutions to the congruence

$$y^2+u_1^3+u_2^3+u_3^3+u_4^3+u_5^6+u_6^7 \equiv N(\bmod q),\ 1\le y,u_j\le q,\ (yu_j,q)=1.$$

Then by the orthogonality of characters, we have

$$pL^*(p,N) = \sum_{a=1}^{p} S_2^*(p,a)S_3^{*4}(p,a)S_6^*(p,a)S_7^*(p,a)e_p(-aN) \\ = (p-1)^7+E_p, \tag{4.3}$$

where

$$E_p = \sum_{a=1}^{p-1} S_2^*(p,a)S_3^{*4}(p,a)S_6^*(p,a)S_7^*(p,a)e_p(-aN). \tag{4.4}$$

By Lemma 2.2 (iv), we have

$$|E_p| \le (p-1)(\sqrt{p}+1)(2\sqrt{p}+1)^4(5\sqrt{p}+1)(6\sqrt{p}+1).$$

It is easy to see that $|E_p| < (p-1)^7$ for $p \geq 23$, hence we have $L^*(p,N) > 0$. For $p = 2,3,5,7,11,13,17,19$ we can verify by hand that $L^*(p,N) > 0$ and we have $L^*(p,N) > 0$ for every prime, and

$$L(p,N) = L^*(p,N) + K(p,N) > K(p,N). \tag{4.5}$$

By (4.3)–(4.4) we have

$$L^*(p,N) = p^6 + O(p^5)$$

and (4.1)-(4.2) follow from similar arguments. □

Lemma 4.2. *The series* $\mathfrak{S}_1(N)$ *is convergent and* $\mathfrak{S}_1(N) > 0$.

Proof. The convergence of $\mathfrak{S}_1(N)$ follows from Lemma 2.2 (i)-(ii) easily. Note the fact that $A_1(q,N)$ is multiplicative in q and by Lemma 2.2 (v), we have

$$\mathfrak{S}_1(N) = \prod_p \Big(1 + A_1(p,N)\Big). \tag{4.6}$$

By Lemma 2.2 (iii)-(iv), for $p \geq 1000$

$$|A_1(p,N)| \leq \frac{(p-1)\sqrt{p}(2\sqrt{p}+1)^4(5\sqrt{p}+1)(6\sqrt{p}+1)}{p(p-1)^6} \leq \frac{1000}{p^2}.$$

Thus,

$$\prod_{p\geq 1000} (1 + A_1(p,N)) \geq \prod_{p \geq 1000} (1 - \frac{1000}{p^2}) > c > 0. \tag{4.7}$$

It is easy to verify that

$$1 + A_1(p,N) = \frac{L(p,N)}{(p-1)^6}. \tag{4.8}$$

Now by Lemma 4.1 and (4.6)–(4.8) we have $\mathfrak{S}_1(N) > 0$, and the proof of Lemma 4.2 is completed. □

Similar to (4.6), we have

$$\mathfrak{S}_d(N) = \prod_{p \nmid d} \Big(1 + A_d(p,N)\Big) \prod_{p | d} \Big(1 + A_d(p,N)\Big). \tag{4.9}$$

Define

$$\omega(d) = \frac{\mathfrak{S}_d(N)}{\mathfrak{S}_1(N)}.$$

It follows from the facts $S_k(q, ad^k) = S_k(q,a)$ for $(d,q) = 1$, $A_d(p,N) = A_p(p,N)$ for $p \mid d$, (4.6) and (4.9) that

$$\omega(p) = \frac{1 + A_p(p,N)}{1 + A_1(p,N)},$$
$$\omega(d) = \prod_{p \| d} \omega(p). \tag{4.10}$$

Moreover, it is easy to show that

$$1 + A_p(p,N) = \frac{pK(p,N)}{(p-1)^6}. \tag{4.11}$$

From (4.8) and (4.11) we get

$$\omega(p) = \frac{pK(p,N)}{L(p,N)}. \tag{4.12}$$

It follows from (4.10), Lemma 4.1 and (4.12) that

Lemma 4.3. *The function $\omega(d)$ is multiplicative and*

$$0 \le \omega(p) < p, \qquad \omega(p) = 1 + O(p^{-1}).$$

5. Proof of Theorem

In this section, $f(s)$ denotes the classical function in the linear sieve theory, and γ denotes Euler's constant. By (2.8) in [8] we have

$$f(s) = \frac{2e^{\gamma}\log(s-1)}{s}, \qquad 2 \le s \le 4. \tag{5.1}$$

In the proof of the theorem we adopt the following notations:

$$z = D^{\frac{1}{2.01}}, \qquad \mathfrak{P} = \prod_{2<p<z} p, \qquad W(z) = \prod_{p<z}\left(1 - \frac{\omega(p)}{p}\right).$$

Let

$$\mathfrak{R}(N) = \sum_{\substack{x^2+p_1^3+p_2^3+p_3^3+p_4^3+p_5^6+p_6^7=N \\ (x,\mathfrak{P})=1,\ \ U_2<x\le 2U_2, \\ U_3<p_1,p_2\le 2U_3,\ U_3^*<p_3,p_4\le 2U_3^* \\ U_6<p_5\le 2U_6,\ U_7<p_6\le 2U_7}} \prod_{j=1}^{6} \log p_j,$$

$$r(x) = \sum_{\substack{x^2+p_1^3+\cdots+p_4^3+p_5^6+p_6^7=N \\ U_3<p_1,p_2\le 2U_3,\ U_3^*<p_3,p_4\le 2U_3^* \\ U_6<p_5\le 2U_6,\ U_7<p_6\le 2U_7}} \prod_{j=1}^{6} \log p_j.$$

Then we have

$$R_{6,7}(N) \geq \frac{\mathfrak{R}(N)}{(\log N)^6} \quad \text{and} \quad \mathfrak{R}(N) = \sum_{\substack{U_2 < x \leq 2U_2 \\ (x,\mathfrak{P})=1}} r(x). \tag{5.2}$$

By [11, Lemma 9.1], Lemmas 4.3 and 3.1, we get

$$\begin{aligned}\mathfrak{R}(N) &> \mathfrak{S}_1(N)\mathfrak{J}(N)W(z)\left(f\Big(\frac{\log D}{\log z}\Big) + O\left((\log\log N)^{-1/50}\right)\right), \\ &= \mathfrak{S}_1(N)\mathfrak{J}(N)W(z)\left(f\,(2.01) + O\big(\,(\log\log N)^{-1/50}\,\big)\right).\end{aligned}$$

Mertens' third Theorem and Lemma 4.3 imply that

$$\frac{1}{\log N} \ll W(z) \ll \frac{1}{\log N}. \tag{5.3}$$

Then by (5.2)–(5.3) we obtain

$$R_{6,7}(N) \gg \frac{\mathfrak{S}_1(N)\mathfrak{J}(N)}{(\log N)^7} \gg \frac{N^{\frac{65}{63}}}{(\log N)^7},$$

where Lemma 4.2 and (3.15) are used. This completes the proof of the Theorem.

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