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Florin NICOLAE

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On the determination of number fields by L-functions

par Florin NICOLAE

Résumé. Nous montrons qu'une extension galoisienne du corps des nombres rationnels est déterminée par toute fonction L d'Artin corespondant à un caractère du groupe de Galois qui contient tous les caractères irréductibles.

Abstract. We prove that a Galois number field is determined by any Artin L-function corresponding to a character of the Galois group which contains all irreducible characters.

Let $K/\mathbb{Q}$ be a finite Galois extension with the Galois group G, $\chi_1, \dots, \chi_r$ the irreducible characters of G with the dimensions $d_1 := \chi_1(1), \dots, d_r := \chi_r(1)$, and $L(s, \chi_1, K/\mathbb{Q}), \dots, L(s, \chi_r, K/\mathbb{Q})$ the corresponding Artin L-functions. The Dedekind zeta function $\zeta_K(s)$ is the Artin L-function corresponding to the regular character $\mathrm{Reg}_G = \sum_{j=1}^{r} d_j \chi_j$:

$$\begin{aligned}\zeta_K(s) &= L(s, \mathrm{Reg}_G, K/\mathbb{Q}) \\ &= L(s, \chi_1, K/\mathbb{Q})^{d_1} \cdot \ldots \cdot L(s, \chi_r, K/\mathbb{Q})^{d_r}.\end{aligned}$$

It is known that $\zeta_K(s)$ determines the field K: if $L/\mathbb{Q}$ is a finite normal extension of $\mathbb{Q}$ such that

$$\zeta_L(s) = \zeta_K(s)$$

then

$$L = K.$$

Since $d_j > 0$, the regular character Reg_G contains all irreducible characters χ_j, $j = 1, \dots, r$. Starting from this observation, we prove that the field K is determined by any Artin L-function corresponding to a character of the Galois group G which contains all irreducible characters.

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Theorem 1. *Let $K/\mathbb{Q}$, $L/\mathbb{Q}$ be finite Galois extensions, G the Galois group of $K/\mathbb{Q}$, $\chi_1, \ldots, \chi_r$ the irreducible characters of G, $k_1 > 0, \ldots, k_r > 0$ integers, $\chi = \sum_{j=1}^r k_j\chi_j$, H the Galois group of $L/\mathbb{Q}$, $\varphi_1, \ldots, \varphi_t$ the irreducible characters of H, $l_1 > 0, \ldots, l_t > 0$ integers, $\varphi = \sum_{j=1}^t l_j\varphi_j$. If*

$$L(s, \chi, K/\mathbb{Q}) = L(s, \varphi, L/\mathbb{Q})$$

then

$$K = L.$$

Proof. Let $M/\mathbb{Q}$ be a finite Galois extension which contains both K and L. For $j = 1, \ldots, r$ let

$$\tilde{\chi_j} : \mathrm{Gal}(M/\mathbb{Q}) \to \mathbb{C}, \tilde{\chi_j}(\sigma) := \chi_j(\sigma\mathrm{Gal}(M/K)),$$

where we identify G with the factor group $\mathrm{Gal}(M/\mathbb{Q})/\mathrm{Gal}(M/K)$. Then $\tilde{\chi_j}$ is an irreducible character of $\mathrm{Gal}(M/\mathbb{Q})$ and

$$\mathrm{Gal}(M/K) = \cap_{j=1}^r \mathrm{Ker}\,(\tilde{\chi_j}), \tag{1}$$

where $\mathrm{Ker}\,(\tilde{\chi_j})$ is the kernel of the character $\tilde{\chi_j}$. For $j = 1, \ldots, t$ let

$$\tilde{\varphi_j} : \mathrm{Gal}(M/\mathbb{Q}) \to \mathbb{C}, \tilde{\varphi_j}(\sigma) := \varphi_j(\sigma\mathrm{Gal}(M/L)),$$

where we identify H with the factor group $\mathrm{Gal}(M/\mathbb{Q})/\mathrm{Gal}(M/L)$. Then $\tilde{\varphi_j}$ is an irreducible character of $\mathrm{Gal}(M/\mathbb{Q})$ and

$$\mathrm{Gal}(M/L) = \cap_{j=1}^t \mathrm{Ker}\,(\tilde{\varphi_j}). \tag{2}$$

It is a fundamental property of Artin L-functions that

$$L(s, \chi_j, K/\mathbb{Q}) = L(s, \tilde{\chi_j}, M/\mathbb{Q}), j = 1, \ldots, r$$

and that

$$L(s, \varphi_j, L/\mathbb{Q}) = L(s, \tilde{\varphi_j}, M/\mathbb{Q}), j = 1, \ldots, t$$

hence

$$L(s, \chi, K/\mathbb{Q}) = \prod_{j=1}^r L(s, \chi_j, K/\mathbb{Q})^{k_j} = \prod_{j=1}^r L(s, \tilde{\chi_j}, M/\mathbb{Q})^{k_j},$$

and

$$L(s, \varphi, L/\mathbb{Q}) = \prod_{j=1}^t L(s, \varphi_j, L/\mathbb{Q})^{l_j} = \prod_{j=1}^t L(s, \tilde{\varphi_j}, M/\mathbb{Q})^{l_j},$$

therefore

$$\prod_{j=1}^r L(s, \tilde{\chi_j}, M/\mathbb{Q})^{k_j} = \prod_{j=1}^t L(s, \tilde{\varphi_j}, L/\mathbb{Q})^{l_j}. \tag{3}$$

Artin proved that the L-functions corresponding to irreducible characters are multiplicatively independent ([1], Satz 5, P. 106). Thus, from the multiplicative relation (3) we deduce that

$$\{\tilde{\chi_1}, \ldots, \tilde{\chi_r}\} = \{\tilde{\varphi_1}, \ldots, \tilde{\varphi_t}\}.$$

Together with (1) and (2), this implies that

$$\mathrm{Gal}(M/K) = \mathrm{Gal}(M/L),$$

so

$$K = L. \qquad \square$$

References

[1] E. Artin, "Über eine neue Art vonL-Reihen", *Abh. Math. Sem. Univ. Hamburg* **3** (1924), no. 1, p. 89-108.

Florin Nicolae
Technische Universität Berlin
Institut für Mathematik
Strasse des 17. Juni 136
D-10623 Berlin, Germany
and
Institute of Mathematics of the Romanian Academy
P.O.BOX 1-764
RO-014700 Bucharest, Romania
E-mail: Florin.Nicolae@imar.ro
URL: http://www.imar.ro/