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Lower Bounds on the Dimension of the Cohomology of Bianchi Groups via Sczech Cocyles

par Mehmet Haluk ŞENGÜN et Seyfi TÜRKELLI

Résumé. En utilisant les cocyles de Sczech, nous calculons les traces de certaines involutions sur la cohomologie d'Eisenstein pour les sous-groupes de congruence principaux de groupes de Bianchi. Ces traces, combinées avec les résultats de [13, 14, 2], donnent des bornes inférieures explicites pour la cohomologie cuspidale de ces groupes. Les bornes asymptotiques inférieures qui découlent de nos résultats complètent les bornes asymptotiques supérieures récemment obtenues dans [4, 5, 12].

Abstract. Using Sczech cocyles, we compute the traces of certain involutions on the Eisenstein cohomology of principal congruence subgroups of Bianchi groups. These traces, combined with results of [13, 14, 2], give explicit lower bounds for the cuspidal cohomology of these groups. The asymptotic lower bounds that follow from our results complement the recent asymptotic upper bounds found in [4, 5, 12].

1. Introduction

Let K be an imaginary quadratic field with ring of integers $\mathcal{O}$ and let Γ denote a principal congruence subgroup of the Bianchi group $\mathrm{SL}_2(\mathcal{O})$ associated to K. In this paper, we compute the trace of certain involutions on the Eisenstein cohomology of Γ. A novel aspect of our approach is the use of the so-called *Sczech cocycles* whose traditional application has been, mainly, to the computation of the special values of L–functions. Our trace computations, when combined with well-known work of Rohlfs on the Lefschetz numbers of finite order automorphism of arithmetic groups, give rise to explicit lower bounds for the cuspidal cohomology of Γ. The asymptotic lower bounds that follows complement the asymptotic upper bounds obtained in [4, 5, 12] nicely.

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More precisely, fix a square-free negative integer $d \neq -1, -3$, let K be the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ with ring of integers $\mathcal{O}$. Let G be the associated Bianchi group $\mathrm{SL}_2(\mathcal{O})$ and $\Gamma(N)$ denote the principal congruence subgroup of G of level $(N) \lhd \mathcal{O}$ for any given rational integer N. Let σ denote involution induced on the Eisentein cohomology of $\Gamma(N)$ by the nontrivial automorphism of K.

Theorem 1.1. *Assume that K is of class number one and let p be a rational prime that is inert in K. Then we have*

$$\mathrm{tr}(\sigma \mid H^1_{Eis}(\Gamma(p^n), \mathbb{C})) = \begin{cases} -(p^2+1), & \text{if } n = 1, \\ -(p^{2n} - p^{2n-2}), & \text{if } n > 1. \end{cases}$$

As is well-known, there are only nine imaginary quadratic fields of class number one and thus our result has a limited scope. A complete result can be obtained[1] by employing Harder's theory of "Eisenstein Cohomology" [8]. Nevertheless, we believe that our result has merit in the simplicity and novelty of its proof. The proof, which is in Section 4.2.4, employs the explicit cocycles of Sczech [17] (defined by means of certain elliptic analogues of classical Dedekind sums) and utilizes results of Ito [9, 10] (see Section 4.2.3).

Dealing with the second degree Eisenstein cohomology is easier as the whole discussion can be transferred to an investigation of the cohomology of 2-tori. In particular, we are able relax the hypothesis greatly and also work with non-trivial coefficient modules of arithmetic interest. Given any nonnegative integer k, let E_k be the space of homogeneous polynomials over $\mathbb{C}$ in two variables of degree k with the following $\mathrm{SL}_2(\mathbb{C})$–action: given a polynomial $p(x, y) \in E_k$,

$$p(x, y) \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = p(ax + by, cx + dy).$$

Consider the $\mathrm{SL}_2(\mathbb{C})$–module $E_{k,k} := E_k \otimes_{\mathbb{C}} \overline{E_k}$ where the action of $\mathrm{SL}_2(\mathbb{C})$ on the second factor is twisted by the conjugation. The proof of the following result is contained in Section 4.2.2.

Theorem 1.2. *Let t be the number of distinct prime divisors of the discriminant of $K/\mathbb{Q}$. Let $N = p_1^{n_1} \dots p_r^{n_r}$ be a positive integer whose prime divisors p_i are unramified in K.*

We have

$$\mathrm{tr}(\sigma \mid H^2_{Eis}(\Gamma(N), E_{k,k})) = -2^{t-r-1} \cdot \prod_{i=1}^{r} (p_i^{2n_i} - p_i^{2(n_i-1)}) + \delta(0, k),$$

[1] At the end of [6], Harder reports the trace for general class number K. He does not provide a proof however informs us that he uses the adelic setting and representation theoretic approach for his computations and that his final result depends on certain factors in the functional equation of associated Hecke L–series.

where δ is the Kronecker δ–function, in other words, $\delta(0,k) = 0$ unless $k = 0$ in which case $\delta(0,k) = 1$. In particular,

$$\operatorname{tr}(\sigma \mid H^2_{Eis}(\mathrm{SL}_2(\mathcal{O}), E_{k,k})) = -2^{t-1} + \delta(0,k).$$

Once the traces are computed, one can use, as explained in Section 2.1, results of Rohlfs [13] and Blume-Nienhaus [2] to obtain explicit lower bounds for the cuspidal cohomology of Γ. All that is missing is a Lefschetz number computation which we do in Section 3.2.

Proposition 1.3. *Let $N > 2$. Then*

$$L(\sigma, \Gamma(N), E_{k,k}) = \begin{cases} (A+2B)\dfrac{-N^3}{12}\displaystyle\prod_{p|N}(1-p^{-2})\cdot(k+1) & \text{if } N \text{ is even,} \\ (A+3B)\dfrac{-N^3}{12}\displaystyle\prod_{p|N}(1-p^{-2})\cdot(k+1) & \text{if } N \text{ is odd.} \end{cases}$$

where A, B are explicit constants depending on the ramification data of $K/\mathbb{Q}$.

The explicit lower bounds lead to the following asymptotic bounds which are discussed in Section 5. For a related result, see [16].

Corollary 1.4. *Let p be a rational prime that is unramified in K. Then, as k increases and n is fixed*

$$\dim H^1_{cusp}(\Gamma(p^n), E_{k,k}) \gg k$$

where the implicit constant depends on the level $\Gamma(p^n)$ and the field K. Assume further that K is of class number one and that p is inert in K. Then, as n increases

$$\dim H^1_{cusp}(\Gamma(p^n), \mathbb{C}) \gg p^{3n}$$

where the implicit constant depends on the field K.

We also consider the Lefschetz numbers and the Eisenstein traces for the involution given by the $\mathrm{GL}_2/\mathrm{SL}_2$–twist of σ. The results, when combined with those for σ, give a closed formula for the trace of σ on the first cohomology of $\mathrm{GL}_2(\mathcal{O})$, see Theorem 5.3. This implies the following asymptotics for the cohomology of $\mathrm{GL}_2(\mathcal{O})$, see Corollary 5.4.

Corollary 1.5. *Let D be the discriminant of $K/\mathbb{Q}$ and $\mathcal{O}_K$ be its ring of integers. As $K/\mathbb{Q}$ is fixed and $k \to \infty$, we have*

$$\dim H^1(\mathrm{GL}_2(\mathcal{O}_K), E_{k,k}) \gg k$$

where the implicit constant depends on the discriminant D. As k is fixed and $|D| \to \infty$, we have

$$\dim H^1(\mathrm{GL}_2(\mathcal{O}_K), E_{k,k}) \gg \varphi(D)$$

where φ is the Euler φ function and the implicit constant depends on the weight k.

As $H^1(\mathrm{GL}_2(\mathcal{O}), E_{k,k})$ embeds into $H^1_{cusp}(\mathrm{SL}_2(\mathcal{O}), E_{k,k})$, the asymptotic lower bounds of the above corollary also applies to $H^1_{cusp}(\mathrm{SL}_2(\mathcal{O}), E_{k,k})$. Rohlfs showed in [14] that $H^1_{cusp}(\mathrm{SL}_2(\mathcal{O}), \mathbb{C}) \gg \varphi(D)$ as $|D| \to \infty$, yielding the same asymptotic as ours. We note that Krämer [11] produces the upper bound

$$\dim H^1_{cusp}(\mathrm{SL}_2(\mathcal{O}), \mathbb{C}) \ll |D|^{3/2}.$$

For pedagogical purposes, the contents of the paper proceed in the reverse order the way they were summarized here.

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2. A Lefschetz fixed point theorem

As above, let K be an imaginary quadratic field with ring of integers $\mathcal{O}$ and G be the associated Bianchi group $\mathrm{SL}_2(\mathcal{O})$. We consider G as a lattice inside the real Lie group $\mathrm{SL}_2(\mathbb{C})$ and thus view it as a discrete group of isometries of the hyperbolic 3–space $\mathbb{H}$. Let $\rho \in \mathrm{Aut}(G)$ be an involution and $\mathfrak{g} = \{1, \rho\}$ be the subgroup of the automorphism group $\mathrm{Aut}(G)$ (note that $\mathrm{Out}(G)$ is finite elementary abelian 2-group which is explicitly determined by Smillie and Vogtmann in [20]). Let Γ be a $\mathfrak{g}$–stable torsion-free finite index subgroup of G considered as a normal subgroup of the semidirect product $\tilde{\Gamma} = \Gamma \rtimes \mathfrak{g}$. The group $\tilde{\Gamma}$ has a natural action on $\mathbb{H}$ that extends the action of Γ. Thus, $\tilde{\Gamma}$ acts on hyperbolic 3–manifold $Y_\Gamma = \Gamma\backslash\mathbb{H}$.

Let E be a Γ–module with a $\mathfrak{g}$–action such that this action is compatible with the action on Γ, that is, ${}^\rho(g \cdot e) = {}^\rho g \cdot {}^\rho e$. Then $\mathfrak{g}$ acts on the cohomology groups $H^i(Y_\Gamma, \mathcal{E})$ where $\mathcal{E}$ is the locally constant sheaf on Y_Γ induced by E. Therefore, we can define the *Lefschetz number*

$$L(\rho, \Gamma, E) = \sum_i (-1)^i \mathrm{tr}(\rho \mid H^i(Y_\Gamma, \mathcal{E})).$$

Let Y_Γ^ρ be the set of fixed points of the ρ–action on Y_Γ. Let $\mathcal{E}^\rho$ denote the restriction of the sheaf $\mathcal{E}$ to Y_Γ^ρ. Then ρ acts on the stalk of $\mathcal{E}^\rho$ and $L(\rho, Y_\Gamma^\rho, \mathcal{E}^\rho)$ is defined. As shown in [15, p.152], one has that

$$L(\rho, \Gamma, E) = L(\rho, Y_\Gamma^\rho, \mathcal{E}^\rho).$$

The connected components of Y_Γ^ρ can be parametrized by the first non-abelian (Galois) cohomology $H^1(\mathfrak{g}, \Gamma)$. If γ is a cocycle for $H^1(\mathfrak{g}, \Gamma)$, we have a γ–twisted ρ–action on $\mathbb{H}$ given by $x \mapsto {}^{\rho}x\gamma^{-1}$. The fixed point set $\mathbb{H}(\gamma)$ of the γ–twisted action on $\mathbb{H}$ is non-empty and its image in Y_Γ, denoted $F(\gamma)$, is a locally symmetric subspace of Y_Γ^ρ.

There is also a γ–twisted ρ–action on Γ given by $g \mapsto \gamma\ {}^{\rho}g\gamma^{-1}$ for $g \in \Gamma$. Let $\Gamma(\gamma)$ denote the set of fixed points of this action. When Γ is torsion-free, the canonical map

$$\pi_\gamma : \Gamma(\gamma)\backslash\mathbb{H}(\gamma) \to Y_\Gamma$$

is injective. The image of π_γ is homeomorphic to $F(\gamma)$.

There is a twisted ρ–action on E as well, given by $e \mapsto {}^{\rho}e\gamma$ for $e \in E$. The trace of this action on E does not depend on the choice of the cocycle γ in its class and therefore will be written as $\mathrm{tr}(\rho_\gamma \mid E)$. We have the following geometric reformulation of the Lefschetz trace formula for the torsion-free case.

Theorem 2.1 (Rohlfs)**.** *Assume that* Γ *is torsion-free. Then*

$$L(\rho, \Gamma, E) = \sum_{\gamma \in H^1(\mathfrak{g},\Gamma)} \chi(F(\gamma))\mathrm{tr}(\rho_\gamma \mid E).$$

A more general version of the above theorem that treats Γ with possibly torsion elements is given by Blume-Nienhaus in [2, I.1.6].

2.1. Lower bounds for the cohomology via Lefschetz numbers. For the rest of the section, assume that ρ is *orientation-reversing*, as it will be the case with the specific involutions that we will work with in Section 3. In this section, we want to give a lower bound for the dimension of the cuspidal cohomology in terms of the Lefschetz number of ρ.

Let X_Γ denote the Borel-Serre compactification of Y_Γ. This is a compact manifold with boundary whose interior is homeomorphic to Y_Γ. Moreover, the embedding $Y_\Gamma \hookrightarrow X_\Gamma$ is homotopy equivariant, giving an isomorphism

$$H^i(Y_\Gamma, \mathcal{E}) \simeq H^i(X_\Gamma, \bar{\mathcal{E}})$$

where $\bar{\mathcal{E}}$ is a certain sheaf on X_Γ that extends $\mathcal{E}$.

Consider the long exact sequence

$$\ldots \to H^i_c(X_\Gamma, \bar{\mathcal{E}}_n) \to H^i(X_\Gamma, \bar{\mathcal{E}}_n) \to H^i(\partial X_\Gamma, \bar{\mathcal{E}}_n) \to \ldots$$

where H^i_c denotes the compactly supported cohomology.

The *cuspidal cohomology* H^i_{cusp} is defined as the image of the compactly supported cohomology. The *Eisenstein cohomology* H^i_{Eis} is a certain complement of the cuspidal cohomology inside H^i and it is isomorphic to the image of the restriction map inside the cohomology of the boundary. Assume that the action of ρ on Y_Γ extends to X_Γ, which will be the case for our specific involutions of Section 3. This induces involutions on the terms of the above long exact sequence. We therefore have, in the obvious notation, that

$$\mathrm{tr}(\rho^i) = \mathrm{tr}(\rho^i_{cusp}) + \mathrm{tr}(\rho^i_{Eis}).$$

Poincaré duality implies that $H^1_{cusp} \simeq H^2_{cusp}$. Since ρ is an orientation reversing involution, it follows that $\mathrm{tr}(\rho^1_{cusp}) = -\mathrm{tr}(\rho^2_{cusp})$. Hence we get

$$L(\rho, \Gamma, E) = \mathrm{tr}(\rho^0) - 2\mathrm{tr}(\rho^1_{cusp}) - \mathrm{tr}(\rho^1_{Eis}) + \mathrm{tr}(\rho^2_{Eis}),$$

and this implies the following proposition.

Proposition 2.2. *With the above notation, we have*

$$\dim H^1_{cusp}(\Gamma, E) \geq \frac{1}{2}\Big| L(\rho, \Gamma, E) + \mathrm{tr}(\rho^1_{Eis}) - \mathrm{tr}(\rho^2_{Eis}) - \mathrm{tr}(\rho^0)\Big|.$$

Proof. Since ρ is an involution, the eigenvalues of ρ^1_{cusp} are ± 1, and so

$$\dim H^1(\Gamma, E) \geq |\mathrm{tr}(\rho^1_{cusp})|.$$

The result now follows from the identity above. □

Note that when $E = E_{k,k}$ with $k > 0$, $\mathrm{tr}(\rho^0) = 0$ as E is an irreducible Γ–representation.

3. Lefschetz numbers for specific involutions

Let σ be complex conjugation. Its action on $\mathbb{H}$ is defined by $(z, r) \mapsto (\bar{z}, r)$. It also acts on $\mathrm{SL}_2(\mathbb{C})$ by acting on the entries of a matrix in the obvious way. If $M \in \mathrm{SL}_2(\mathbb{C})$, then we write ${}^{\sigma}M$, or simply $\bar{M}$, for the image of M under the action of σ.

Below, we will also consider *twisted complex conjugation*, which will be denoted by τ. It acts on $\mathbb{H}$ via $(z, r) \mapsto (-\bar{z}, r)$ where $\bar{z}$ denotes the complex conjugate of z. Its action on $\mathrm{SL}_2(\mathbb{C})$ is defined as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} \bar{a} & -\bar{b} \\ -\bar{c} & \bar{d} \end{smallmatrix}\right)$ where the bar in the notation denotes the complex conjugation. It is convenient to regard τ as the composition $\alpha \circ \sigma = \sigma \circ \alpha$ where $\alpha\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right) = \beta\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\beta$ where $\beta := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, for every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$ and $\alpha(z, r) = (-z, r)$ for every $(z, r) \in \mathbb{H}$.

Both σ and τ are *orientation-reversing* and they can be extended to the Borel-Serre compactification naturally (see [14] Section 1.4). The action of σ on $E_{k,k}$ can be described as follows: $\sigma(P \otimes Q) = Q \otimes P$. Similarly, we have $\tau(P \otimes Q) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) Q \otimes \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) P$. These actions are compatible with those on $\mathrm{SL}_2(\mathbb{C})$.

In this section, we discuss the Lefschetz numbers for these two involutions. We will use the symbol ρ when we want to state results which are true for both of them. We start with a useful lemma (see [2, I.4.3] for a proof).

Lemma 3.1. *Let $\gamma \in \Gamma$ and $x = (\gamma\rho)^2$. Then* $\mathrm{tr}(\gamma\rho \mid E_{k,k}) = \mathrm{tr}(x \mid E_k)$.

3.1. Lefschetz numbers for Bianchi groups. Let Γ denote the full Bianchi group $\mathrm{SL}_2(\mathcal{O})$. For $k = 0$, that is $E_{k,k} = \mathbb{C}$, the Lefschetz numbers for σ and τ were computed by Krämer [11]. For general $E_{k,k}$, these numbers were computed by Blume-Neinhaus [2].

For a rational prime p which ramifies in K and an integer a, let $(a|p)$ denote the Hilbert symbol. By definition, $(a|p)$ is equal 1 if there is an element in some finite extension of K_p, the completion of K at the unique prime ideal over p, whose norm is equal to a, and is equal to -1 otherwise. Equivalently, $(a|p)$ is the value at a of the quadratic character associated to the local extension $\mathbb{Q}_p(\sqrt{d})/\mathbb{Q}_p$. Note that if $p \neq 2$, then $(a|p)$ is equal to the Legendre symbol $\left(\frac{a}{p}\right)$.

Theorem 3.2 (Blume-Nienhaus, [2])**.** *Let D be the discriminant of $K/\mathbb{Q}$ with D_2 its 2-part. Let ρ represent either τ or σ. Also, put $q = 1$ or $q = -1$ depending on whether $\rho = \tau$ or $\rho = \sigma$, respectively.*

$$\begin{aligned}
(-1)^k L(\rho, \Gamma, E_{k,k}) \\
&= \frac{-q}{12} \prod_{\substack{p|D \\ p\neq 2}} (p + \left(\frac{q}{p}\right)) \prod_{\substack{p|D \\ p=2}} (D_2 + (q|2))(k+1) \\
&\quad + \frac{q}{12} \prod_{\substack{p|D \\ p\neq 2}} (1 + \left(\frac{-q}{p}\right)) \prod_{\substack{p|D \\ p=2}} (4 + (-q|2))(-1)^k(k+1) \\
&\quad + \frac{1}{2} \prod_{\substack{p|D \\ p\neq 2}} (1 + \left(\frac{-2q}{p}\right)) \left(\frac{k+1}{4}\right) \\
&\quad + \frac{1}{3} \left(\prod_{\substack{p|D \\ p\neq 3}} (1 + (-3q|p)) + (-1)^k \prod_{p|D} (1 + (-q|p)) \right) \left(\frac{k+1}{3}\right).
\end{aligned}$$

Here products over empty sets are understood to be equal to 1.

Proof. Observe that in Blume-Nienhaus' notation, $\Gamma(1) = \Gamma(-1) = \mathrm{SL}_2(\mathcal{O})$. For $q = 1, -1$ respectively, his involutions $\left(\begin{smallmatrix} 0 & 1 \\ q & 0 \end{smallmatrix}\right)\sigma$ (see Theorem V.5.3. of [2]) differ from our $\tau = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\sigma$ and σ by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. However at the level of cohomology, his involutions induce the same action as ours: conjugation by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is an inner-automorphism of $\mathrm{SL}_2(\mathcal{O})$ and hence induces trivial action on the cohomology, see [3, p.79]. $\square$

3.2. Lefschetz number of σ for principal congruence subgroups. Let $\Gamma = \Gamma(N) \subseteq \mathrm{SL}_2(\mathcal{O})$ be a principal congruence subgroup of level $(N) \lhd \mathcal{O}$. Denote its image in $\mathrm{PSL}_2(\mathcal{O})$ by $\bar{\Gamma}$. Then for $N > 2$, Γ (and so $\bar{\Gamma}$) is torsion-free. As before, we put $Y_\Gamma = \bar{\Gamma}\backslash\mathbb{H}$.

In this section, we will use Theorem 2.1 to calculate the Lefschetz numbers $L(\sigma, \Gamma(N), E_{k,k})$. First, we need to analyze the fixed point set Y_Γ^σ.

Let $H(1)$ be the subset of $H^1(\sigma, \bar{\Gamma})$ consisting of the cocycles $\gamma \in \bar{\Gamma}$ with $\det(\gamma\, {}^\sigma\gamma) = 1$. And, let $H(2)$ be the subset consisting of the cocycles γ with $\det(\gamma\, {}^\sigma\gamma) = -1$. We have $H^1(\sigma, \bar{\Gamma}) = H(1) \cup H(2)$. If Γ is torsion free, then $H^1(\sigma, \bar{\Gamma}) = H(1)$.

Let $\gamma_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\gamma_1' = \left(\begin{smallmatrix} 1 & \sqrt{d} \\ 0 & 1 \end{smallmatrix}\right)$ and $\gamma_2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Let γ_2' be $\left(\begin{smallmatrix} 1+\sqrt{d} & (2-d)/2 \\ -2 & -1+\sqrt{d} \end{smallmatrix}\right)$ if $d \equiv 2 \mod 4$, and $\left(\begin{smallmatrix} \sqrt{d} & (d-1)/2 \\ 2 & \sqrt{d} \end{smallmatrix}\right)$ if $d \equiv 1 \mod 4$. Notice that $\gamma_1, \gamma_1' \in H(1)$ and $\gamma_2, \gamma_2' \in H(2)$.

The locally symmetric space $F(\gamma)$, defined in section 2, is a surface if $\gamma \in H(1)$ and is a point if $\gamma \in H(2)$. In [13], Rohlfs gives the number of translations of the surfaces corresponding to γ_1, γ_1' and the number of translations of the points corresponding to γ_2, γ_2'.

Theorem 3.3 (Rohlfs, Theorem 4.1. of [13])**.** *Let D be the discriminant of $K/\mathbb{Q}$ and t be the number of distinct prime divisors of D. Let $(N) = \prod_{p|D} \mathfrak{p}_p^{j_p} \prod_{p\nmid D} (p)^{j_p}$ be an ideal with $N > 2$, and let $\Gamma = \Gamma(N)$ be the principal congruence subgroup of level (N). Let $s = \#\{p \text{ prime} \mid p|D, p \neq 2 \text{ and } j_p \neq 0\}$.*

Then Y_Γ^σ consists of translations of surfaces $F(\gamma_1)$ and $F(\gamma_1')$ and the number of translations of these surfaces are denoted by A and B respectively in the table below.

d	j_2	A	B
$d \equiv 1\ (4)$	≥ 0	2^{t-s}	0
$d \equiv 2\ (4)$	0	2^{t-s}	2^{t-s-1}
	1	2^{t-s}	2^{t-s-1}
	2	$8 \cdot 2^{t-s}$	0
	≥ 3	$8 \cdot 2^{t-s-1}$	0
$d \equiv 3\ (4)$	0	2^{t-s}	2^{t-s-1}
	1	2^{t-s}	0
	2	$8 \cdot 2^{t-s}$	0
	$j_2 = 2n+1 \geq 3$	2^{t-s-1}	0
	$j_2 = 2n \geq 4$	$8 \cdot 2^{t-s-1}$	0

Now, using Theorem 2.1 and Theorem 3.3, we want to calculate the Lefschetz number for $\Gamma(N)$.

Proposition 3.4. *Let* $\Gamma(N), A, B$ *be as in the theorem above. Then*

$$L(\sigma, \Gamma(N), E_{k,k}) = \begin{cases} (A+2B)\dfrac{-N^3}{12}\displaystyle\prod_{p|N}(1-p^{-2})\cdot(k+1) & \textit{if } N \textit{ is even,} \\ (A+3B)\dfrac{-N^3}{12}\displaystyle\prod_{p|N}(1-p^{-2})\cdot(k+1) & \textit{if } N \textit{ is odd.} \end{cases}$$

Proof. For each $\gamma \in H(1)$, by Lemma 3.1, $\mathrm{tr}(\gamma\sigma \mid E_{k,k}) = \mathrm{tr}(\mathbf{1} \mid E_{k,k}) = (k+1)$. Therefore, by Theorem 2.1, we just need to calculate the Euler-Poincare characteristics $\chi(\Gamma^{\gamma\sigma})$ for γ_1 and γ_1'.

An easy calculation shows that $\Gamma^{\gamma_1\sigma} = \Gamma_N$, the principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level N. Let Y_N denote the hyperbolic surface associated to Γ_N. It is well-known that Y_N has $\frac{1}{2}N^2\prod_{p|N}(1-p^{-2})$ cusps. If X_N denotes the compact surface obtained from Y_N by adding the cusps, then by [19, 1.6.4], we have $\chi(X_N) = (-1/12)N^2(N-6)\prod_{p|N}(1-p^{-2})$. Therefore $\chi(\Gamma_N) = \chi(Y_N) = \chi(X_N) - \#\{\text{cusps of } Y_N\} = (-1/12)N^3\prod_{p|N}(1-p^{-2})$.

If $d \equiv 1 \pmod 4$, the number B of translations of the surfaces $F(\gamma_1')$ is 0 as noted in the theorem above. So, below, we only need to consider the cases $d \equiv 2, 3 \pmod 4$.

Let $h = \left(\begin{smallmatrix} 1 & \sqrt{d} \\ 0 & 2 \end{smallmatrix}\right)$. One can see that $\Gamma^{\gamma_1'\sigma}$ is equal to

$$\left\{ \begin{pmatrix} x+z\sqrt{d} & y+\frac{w-x}{2}\sqrt{d} \\ 2z & w-z\sqrt{d} \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}) \mid \right.$$
$$\left. x-1 \equiv w-1 \equiv y \equiv z \equiv 0 \pmod N \text{ and } w \equiv x \pmod{2N} \right\}.$$

Suppose that $d \equiv 2 \pmod 4$. In this case, the condition that $w \equiv x \pmod{2N}$ is redundant as it automatically follows from the condition that the determinant of a matrix in $\mathrm{SL}_2(\mathcal{O})$ is equal to 1. Therefore, in this case

$$\Gamma^{\gamma_1'\sigma} = \left\{ \begin{pmatrix} x+z\sqrt{d} & y+\frac{w-x}{2}\sqrt{d} \\ 2z & w-z\sqrt{d} \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}) \mid x \equiv w \equiv 1,\ y \equiv z \equiv 0 \pmod N \right\}.$$

Now, an easy calculation shows that

$$h^{-1}\Gamma^{\gamma_1'\sigma}h = \left\{ \left(\begin{smallmatrix} x & 2y+zd \\ z & w \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid x \equiv w \equiv 1, \quad y \equiv z \equiv 0 \pmod N \right\}.$$

Thus,

$$h^{-1}\Gamma^{\gamma_1'\sigma}h = \Gamma_N \cap \Gamma^0(2N)$$

where Γ_N is the principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level N, and $\Gamma^0(2N) \subset \mathrm{SL}_2(\mathbb{Z})$ is the group of matrices that are lower-triangular modulo $2N$. The index $[\Gamma_N : \Gamma_N \cap \Gamma^0(2N)]$ is equal to 3 if N is odd, and it is equal to 2 if N is even. This is a straightforward calculation; if N is odd, then the cosets are represented by the matrices

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right), \begin{pmatrix} N^2+1 & N^3+2N \\ N & N^2+1 \end{pmatrix},$$

and when N is even, the third matrix above is in the coset of the identity matrix.

If $d \equiv 3 \pmod 4$ and N is even then, as above, the condition that $w \equiv x \pmod{2N}$ is redundant because it automatically follows from the condition that $det = 1$. Having noted this, now one can see that $h^{-1}\Gamma^{\gamma_1'\sigma}h = \Gamma_N'$ where

$$\Gamma_N' = \{\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \Gamma_N \mid y \equiv z \pmod{2N}\}.$$

In this case, $[\Gamma_N : \Gamma_N'] = 2$ and the cosets are represented by the matrices

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right).$$

If $d \equiv 3 \pmod 4$ and N is odd, then the condition that $w \equiv x \pmod{2N}$ does not follow automatically. One can see that $h^{-1}\Gamma^{\gamma_1'\sigma}h = \Gamma_N''$ where

$$\Gamma_N'' = \{\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \Gamma_N \mid x \equiv w \text{ and } y \equiv z \pmod{2N}\}.$$

In this case, $[\Gamma_N : \Gamma_N''] = 3$ and the cosets are represented by the matrices

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right).$$

Hence, when $d \equiv 2, 3 \pmod 4$, the index $[\Gamma_N : h^{-1}\Gamma^{\gamma_1'\sigma}h]$ is 2 if N is even, and it is 3 if N is odd. This implies that

$$\chi(\Gamma^{\gamma_1'\sigma}) = \begin{cases} 2\chi(\Gamma_N) & \text{if } N \text{ is even,} \\ 3\chi(\Gamma_N) & \text{if } N \text{ is odd.} \end{cases}$$

This completes the proof. □

Corollary 3.5. *Let p be an odd rational prime that is unramified over K. Let t be the number of distinct prime divisors of D. Then, for $n > 0$ we have*

$$L(\sigma, \Gamma(p^n), E_{k,k}) = \begin{cases} -2^t \cdot \dfrac{p^{3n} - p^{3n-2}}{12} \cdot (k+1) & \textit{if } d \equiv 1 \pmod 4, \\ -5 \cdot 2^{t-1} \cdot \dfrac{p^{3n} - p^{3n-2}}{12} \cdot (k+1) & \textit{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

Proof. Since p is odd, $j_2 = 0$. Moreover since p is the only divisor of the level, and since it is unramified, we have $s = 0$. Thus in the case $d \equiv 1 \pmod 4$, we have $A+3B = 2^t$. If $d \equiv 2, 3 \pmod 4$, $A+3B = 2^t+3\cdot 2^{t-1} = 5\cdot 2^{t-1}$. □

4. Trace on the Eisenstein cohomology

Proposition 2.2 shows that in order to obtain lower bounds for the cuspidal cohomology using the Lefschetz numbers, one needs to compute the trace of the involution on the Eisenstein part of the cohomology in all degrees. The latter is certainly no simple task and is the topic of this section of the paper.

In this section we study the trace of involutions induced by σ and τ on the Eisenstein part of the cohomology.

4.1. Bianchi groups. The boundary ∂X_Γ is a disjoint union of 2-tori, each closing a cusp of Y_Γ. The set of cusps of Γ can be identified with the orbit space $\Gamma\backslash\mathbb{P}^1(K)$. It is well-known that the number of cusps is $h(K)$, the class number of K, when Γ is the full Bianchi group.

The fundamental group of a 2-torus is a free abelian group on two generators and it is easy to compute the size of its cohomology.

Proposition 4.1. *Let Γ be a congruence subgroup of a Bianchi group and let $\mathcal{C}_\Gamma$ denote the set of cusps of Γ. Then*

$$\dim H^0(\partial X_\Gamma, \bar{\mathcal{E}}_k) = \dim H^2(\partial X_\Gamma, \bar{\mathcal{E}}_k) = \#\mathcal{C}_\Gamma$$
$$\dim H^1(\partial X_\Gamma, \bar{\mathcal{E}}_k) = 2\cdot\#\mathcal{C}_\Gamma.$$

The long exact sequence associated to the pair $(X_\Gamma, \partial X_\Gamma)$ is compatible with the action of the involution τ. It follows from algebraic topology that for $k > 0$, the image of the restriction map

$$H^i(X_\Gamma, \bar{\mathcal{E}}_k) \to H^i(\partial X_\Gamma, \bar{\mathcal{E}}_k)$$

is onto when $i = 2$ and its image has half the rank of the target space when $i = 1$. Hence we have the following.

Corollary 4.2. *Let $k > 0$ and Γ as above. Then*

$$\dim H^0_{Eis}(\Gamma, E_{k,k}) = 0, \quad \textit{and} \quad \dim H^i_{Eis}(\Gamma, E_{k,k}) = \#\mathcal{C}_\Gamma \quad \textit{for} \;\; i = 1, 2.$$

In particular, for $i = 0, 1, 2$,

$$|\mathrm{tr}(\rho^i_{Eis})| \leq \#\mathcal{C}_\Gamma$$

for any involution ρ.

The following is direct consequence a result of Serre (see [18, Thm 9]).

Proposition 4.3. *Let $\Gamma = \mathrm{SL}_2(\mathcal{O})$. Then the image of the restriction map*

$$H^1(X_\Gamma, \mathbb{C}) \to H^1(\partial X_\Gamma, \mathbb{C})$$

is inside the -1–eigenspace of complex conjugation acting on $H^1(\partial X_\Gamma, \mathbb{C})$.

Let us note that this result is extended to all maximal orders of $M_2(K)$ (with complex conjugation twisted accordingly) by Blume-Nienhaus [2, V.5.7.] and by Berger [1, Section 5.2.].

Corollary 4.4. *Let σ^i_{Eis} be the involution on $H^i_{Eis}(\mathrm{SL}_2(\mathcal{O}),\mathbb{C})$ given by complex conjugation. Then*

$$\mathrm{tr}(\sigma^0_{Eis}) = 1, \quad \mathrm{tr}(\sigma^1_{Eis}) = -h(K), \quad \mathrm{tr}(\sigma^2_{Eis}) = -2^{t-1}+1$$

where t is the number of primes that ramify in K and $h(K)$ is the class number of K.

Proof. For convenience, put $X = X_{\mathrm{SL}_2(\mathcal{O})}$. The claim for σ^0_{Eis} follows from the fact that $H^0_{Eis}(X,\mathbb{C}) = H^0(X,\mathbb{C}) = \mathbb{C}$. The action of σ on the latter is trivial. The claim for σ^1_{Eis} follows immediately from Serre's result above. It is well-known that the set of cusps of $\mathrm{SL}_2(\mathcal{O})$ is in bijection with the class group of K and the action of complex conjugation σ on the cusps translates to taking inverse in the class group. Hence an element of the class group is fixed by σ if it is of order 2. Genus Theory tells us that the number of elements of order 2 in the class group is 2^{t-1}, implying that the trace of the involution induced by σ on $H^0(\partial X,\mathbb{C})$ is 2^{t-1}. See [18, Section 9] for more details. It follows from Poincaré duality and the orientation-reversing nature of complex conjugation that the trace of the involution induced by σ on $H^2(\partial X,\mathbb{C})$ is -2^{t-1}. The long exact sequence associated to the pair $(X,\partial X)$ tells us that that $H^2(\partial X,\mathbb{C}) \simeq H^2_{Eis}(X,\mathbb{C}) \oplus H^3(X,\partial X,\mathbb{C})$. Here the last summand is isomorphic to $\mathbb{C}$ and σ acts on it as -1, which follows from the fact that the action of σ on $H^0(X,\mathbb{C})$ is trivial. This gives the claim for σ^2_{Eis}. □

4.2. Principal congruence subgroups. We will now consider the case of principal congruence subgroups, starting with a brief discussion of their cusps.

4.2.1. The cusps. Given an ideal $\mathfrak{a}$ of $\mathcal{O}$, let $\Gamma = \Gamma(\mathfrak{a})$ denote the principal congruence subgroup of level $\mathfrak{a}$. Given $x \in \mathbb{P}^1(K)$, let G_x, Γ_x denote the stabilizer of x in G and Γ respectively. Moreover, let $\mathcal{C}_G, \mathcal{C}_\Gamma$ denote the set of cusps[2] of G and Γ respectively. Then we have the following relationship

$$\mathcal{C}_\Gamma = \bigsqcup_{x\in\mathcal{C}_G} \Gamma\backslash Gx.$$

Since Γ is normal in G, $\#(\Gamma\backslash Gx)$ is the same for any $x \in \mathcal{C}_G$. Thus using the bijection

$$\Gamma\backslash G/G_x \xrightarrow{g\mapsto gx} \Gamma\backslash Gx,$$

we obtain the formula

$$\#\mathcal{C}_\Gamma = h(K)\cdot\#(\Gamma\backslash G/G_\infty)$$

where ∞ denotes the cusp at infinity, represented by $(1:0) \in \mathbb{P}^1(K)$.

[2] We will identify a cusp with any of its representatives whenever it is convenient.

Observe that $G_\infty = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \cap G = \left(\begin{smallmatrix} \pm 1 & \mathcal{O} \\ 0 & \pm 1 \end{smallmatrix}\right)$. Identifying the coset space $\Gamma\backslash G$ with the finite group $\mathrm{SL}_2(\mathcal{O}/\mathfrak{a})$, the double coset space $\Gamma\backslash G/G_\infty$ can be viewed as the coset space of $\left(\begin{smallmatrix} \pm 1 & \mathcal{O}/\mathfrak{a} \\ 0 & \pm 1 \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathcal{O}/\mathfrak{a})$.

The above discussion shows that when the norm $\mathbf{N}\mathfrak{a} > 2$, the number of cusps of Γ is given by the formula

$$\#\mathcal{C}_\Gamma = h(K) \cdot [\mathrm{SL}_2(\mathcal{O}/\mathfrak{a}) : \left(\begin{smallmatrix} \pm 1 & \mathcal{O}/\mathfrak{a} \\ 0 & \pm 1 \end{smallmatrix}\right)] = \frac{h(K)}{2} \cdot \mathbf{N}\mathfrak{a}^2 \prod_{\mathfrak{p}|\mathfrak{a}} (1 - \mathbf{N}\mathfrak{p}^{-2}). \tag{4.1}$$

Here the product runs over the prime factors $\mathfrak{p}$ of the ideal $\mathfrak{a}$ and $\mathbf{N}$ denotes the norm.

We end with a lemma that will be useful in the next section.

Lemma 4.5. *Let ρ denote either σ or τ and let $c \in (\mathcal{C}_G)^\rho$, i.e. a cusp of G that is ρ–invariant. Then*

$$\#(\Gamma\backslash Gc)^\rho = \#(\Gamma\backslash G\infty)^\rho.$$

Proof. Let $A \in \mathrm{SL}_2(K)$ be such that $\infty \cdot A = c$. As c is fixed by ρ, it corresponds to an element in the ideal class group of K which is of order 2 (here we use the fact that τ is a twisted form of σ). Work of Smillie and Vogtmann [20] shows us that conjugation by A gives rise to an orientation-preserving involutary automorphism ϕ_A of Y_G which *commutes* with the orientation-reversing involutary automorphism of Y_G induced by ρ. The automorphism ϕ_A extends to an automorphism of X_G, inducing an isomorphism (still denoted ϕ_A) between the boundary component 2-torus at the cusp ∞ and the boundary component 2-torus at the cusp c. The isomorphism ϕ_A commutes with the automorphisms induced by ρ on these two boundary components. It follows easily that ϕ_A gives a bijection between $\Gamma\backslash Gc$ and $\Gamma\backslash G\infty$ that commutes with the action of ρ. $\square$

4.2.2. Trace on $\boldsymbol{H^2_{Eis}}$. The following theorem generalizes part of the above Corollary 4.2 and part of the results announced by Harder at the very end of [6] (where there is a factor of 2^{-r} missing).

Theorem 4.6. *Let K be an imaginary quadratic field and t be the number of rational primes ramifying in K. Let $N = p_1^{n_1} \dots p_r^{n_r}$ be a positive odd number whose prime divisors p_i are unramified in K and let $\Gamma(N)$ be the principal congruence subgroup of the Bianchi group $\mathrm{SL}_2(\mathcal{O})$ of level (N). Let ρ denote either σ or τ. Then*

$$\mathrm{tr}(\rho \mid H^2_{Eis}(\Gamma(N), E_{k,k})) = -2^{t-r-1} \cdot \prod_{i=1}^{r} (p_i^{2n_i} - p_i^{2n_i-2}) + \delta(0,k),$$

where δ is the Kronecker δ–function, in other words, $\delta(0,k) = 0$ unless $k = 0$ in which case $\delta(0,k) = 1$. In particular, the trace of ρ^2_{Eis} on

$H^2(\mathrm{SL}_2(\mathcal{O}), E_{k,k})$ is

$$-2^{t-1} + \delta(0,k).$$

Proof. Assume until the very end of the proof that $k > 0$. In this case the restriction map $H^2(X_\Gamma, \mathcal{M}_k) \to H^2(\partial X_\Gamma, \mathcal{M}_k)$ is onto and thus it suffices to compute the trace of ρ^2 on $H^2(\partial X_\Gamma, \mathcal{M}_k)$. As before, Poincaré duality together with the fact that ρ reverses the orientation reduce the problem to computing the trace of ρ^0 on $H^0(\partial X_\Gamma, \mathcal{M}_k)$ instead.

The cohomology of the boundary can be expressed as a direct sum of the cohomology of the boundary components X_c, which are 2-tori;

$$H^0(\partial X_\Gamma, \mathcal{M}_k) \simeq \bigoplus_{c \in \mathcal{C}_\Gamma} H^0(X_c, \mathcal{M}_k) \simeq \bigoplus_{c \in \mathcal{C}_\Gamma} H^0(\Gamma_c, E_{k,k}). \tag{4.2}$$

The last isomorphism follows from the fact that for each X_c is a $K(\Gamma_c, 1)$–space for the stabilizer group Γ_c which consist of unipotent elements in Γ stabilizing the cusp c (Here we used the fact the $N > 2$ and thus $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is not in Γ).

If c is a cusp, then ρ takes Γ_c to $\Gamma_{\rho(c)}$. If $c \neq \rho(c)$, then

$$H^0(\Gamma_c, E_{k,k}) \oplus H^0(\Gamma_{\sigma(c)}, E_{k,k})$$

is a ρ^0–invariant subspace of the right hand side of (4.2). As ρ^0 takes the basis of the first summand to the basis of the second summand, the trace of ρ^0 on this subspace is 0.

If $c = \rho(c)$, then ρ^0 acts on the one-dimensional space $H^0(\Gamma_c, E_{k,k})$ and it is easy to see that the action has trace 1. Thus

$$\mathrm{tr}(\sigma \mid H^0(\partial X_\Gamma, \mathcal{M}_k)) = \#(\mathcal{C}_\Gamma)^\sigma,$$

that is, the trace of σ^0 is equal to the number of cusps of Γ that are invariant under the action of σ.

Recall from Section 4.2.1 that $\mathcal{C}_\Gamma = \bigsqcup_{x \in \mathcal{C}_G} \Gamma \backslash Gx$. Clearly a cusp $c = gx\Gamma \in \mathcal{C}_\Gamma$ is ρ–invariant only if $x \in \mathcal{C}_G$ is so. So we have

$$(\mathcal{C}_\Gamma)^\rho = \bigsqcup_{x \in (\mathcal{C}_G)^\rho} (\Gamma \backslash Gx)^\rho.$$

We already know that $\#(\mathcal{C}_G)^\rho = 2^{t-1}$. By Lemma 4.5, all that is left is to compute $\#(\Gamma \backslash G\infty)^\rho$.

For the rest of the proof, *we put*

$$R := \mathcal{O}/(N) \qquad U^+(R) := \begin{pmatrix} \pm 1 & R \\ 0 & \pm 1 \end{pmatrix}, \qquad U(R) := \left(\begin{smallmatrix} 1 & R \\ 0 & 1 \end{smallmatrix}\right).$$

Following our discussion in Section 4.2.1, we recast our goal as to compute

$$\#(U^+(R) \backslash \mathrm{SL}_2(R))^\rho.$$

Observe that σ fixes the ideal (N) and thus its action on $\mathcal{O}$ descends down to an action on R. The action of σ and τ on $\mathrm{SL}_2(R)$ are as follows

$$\sigma\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} {}^{\sigma}a & {}^{\sigma}b \\ {}^{\sigma}c & {}^{\sigma}d \end{smallmatrix}\right), \qquad \tau\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \begin{pmatrix} {}^{\sigma}a & -{}^{\sigma}b \\ -{}^{\sigma}c & {}^{\sigma}d \end{pmatrix}.$$

We first treat the *special situation* where $N = p^n$ with p a rational prime. Consider the bijections

$$U^+(R)\backslash \mathrm{SL}_2(R) \simeq U^+(R)\backslash B(R) \ \times \ B(R)\backslash \mathrm{SL}_2(R)$$

where $B(R)$ is the subgroup of upper-triangular matrices. There are well-known bijections

$$B(R)\backslash \mathrm{SL}_2(R) \leftrightarrow \mathbb{P}^1(R), \quad \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto (a : c)$$

where $\mathbb{P}^1(R)$ denotes the projective line over R, and

$$U(R)\backslash B(R) \leftrightarrow R^*, \quad \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \mapsto a.$$

These bijections lead to the identification

$$U^+(R)\backslash \mathrm{SL}_2(R) \simeq R^*/\{\pm 1\} \times \mathbb{P}^1(R).$$

It is straightforward to transfer the action of σ and τ to the right hand side. We immediately see that

$$(U^+(R)\backslash \mathrm{SL}_2(R))^\rho \simeq (R^*/\{\pm 1\})^\rho \times \mathbb{P}^1(R)^\rho.$$

Let us start with computing $\#\mathbb{P}^1(R)^\rho$. It can be seen that $\mathbb{P}^1(R^\sigma) \hookrightarrow \mathbb{P}^1(R)$ and in fact $\mathbb{P}^1(R^\sigma) = \mathbb{P}^1(R)^\sigma$. Note that $\mathbb{P}^1(R^\sigma) \simeq \mathbb{P}^1(\mathbb{Z}/p^n\mathbb{Z})$ and thus has cardinality $p^n + p^{n-1}$. Computation shows that $\mathbb{P}^1(R)^\tau$ has the same number of elements.

Let us now consider $\#(R^*)^\rho$. The action of σ and τ is the same on R^*. Clearly we have $\#(R^*/\{\pm 1\})^\rho = (1/2) \cdot \#(R^*)^\sigma$. When p is split in K, we have $R \simeq (\mathbb{Z}/p^n\mathbb{Z})^2$ and ρ acts by swapping the two coordinates. Thus $\#(R^*)^\rho = \#(\mathbb{Z}/p^n\mathbb{Z})^* = p^n - p^{n-1}$. When p is inert in K, we can view R as the quadratic extension $(\mathbb{Z}/p^n\mathbb{Z})[\omega]$ of the ring $\mathbb{Z}/p^n\mathbb{Z}$ and ${}^{\rho}(a + b\cdot\omega) = a - b\cdot\omega$. It follows that $R^* = \{a + b\cdot\omega \mid p \nmid a \ \text{ or } \ p \nmid b\}$ and $(R^*)^\rho$ is given by $\{a + b\cdot\omega \in T \mid p \nmid a, \quad b = 0\}$ which is of cardinality $p^n - p^{n-1}$. Putting things together, in both inert and split cases, we get the quantity $\#(U^+(R)\backslash \mathrm{SL}_2(R))^\rho = (1/2)\cdot\left(p^{2n} - p^{2(n-1)}\right)$.

To finish the proof, let us assume that $N = p_1^{n_1}\dots p_r^{n_r}$ is positive number whose prime divisors p_i are unramified in K. The result in this general case follows from the simple fact that

$$\mathrm{SL}_2(\mathcal{O}/(N)) \simeq \mathrm{SL}_2(\mathcal{O}/(p_1)^{n_1}) \times \dots \mathrm{SL}_2(\mathcal{O}/(p_r)^{n_r}).$$

The case $k = 0$ follows from the basic observations that were employed at the end of the proof of Corollary 4.4. □

4.2.3. Szech Cocyles. In the next subsection, we will compute the trace of σ on $H^1_{Eis}(\Gamma,\mathbb{C})$. Our strategy will be to use the explicit 1–cocycles defined by Sczech in [17] which produce a basis for $H^1_{Eis}(\Gamma(N),\mathbb{C})$. In this subsection, we review the results we need on Sczech cocycles.

Consider $\mathcal{O}$ as a lattice in $\mathbb{C}$. For $k=0,1,2$ and $u\in\mathbb{C}$ put

$$E_k(u)=E_k(u,\mathcal{O})=\sum_{\substack{w\in\mathcal{O}\\ w\neq -u}}(w+u)^{-k}|w+u|^{-s}\ |_{s=0}$$

where $\dots|_{s=0}$ means that the value is defined by analytic continuation to $s=0$. Moreover define $E(u)$ by setting

$$2E(u)=\begin{cases}2E_2(0), & u\in\mathcal{O}\\ \wp(u)-E_1(u)^2, & u\notin\mathcal{O}\end{cases}$$

where $\wp(u)$ denotes the Weierstrass $\wp$–function.

Let N be a positive integer. Given $u,v\in\frac{1}{N}\mathcal{O}$, Sczech forms homomorphisms

$$\Psi(u,v):\Gamma(N)\to\mathbb{C}$$

which depend only on the classes of u and v in $\frac{1}{N}\mathcal{O}/\mathcal{O}$. For $A=\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)\in\Gamma(N)$, we have the simple description

$$\Psi(u,v)(A)=-\left(\tfrac{\bar b}{d}\right)E(u)-\tfrac{b}{d}E_0(u)E_2(v)$$

where

$$\left(\tfrac{t}{s}\right)=-1+\#\{y \mod s\mathcal{O}\mid y^2\equiv t \mod s\mathcal{O}\}$$

is the Legendre symbol. For non-parabolic $A\in\Gamma(N)$ there is a similar but more complicated description which uses finite sums that involve the E_k's, generalizing the classical Dedekind sums. Note that $E(-u)=E(u)$ and $E_k(-u)=E_k(u)$ for even k. So, $\Psi(-u,-v)=\Psi(u,v)$ on parabolic elements. In fact, using the definition in [17, Section 4], one can see that $\Psi(-u,-v)=\Psi(u,v)$.

It is shown by Sczech that the collection $\Psi(u,v)$ with $(u,v)\in(\frac{1}{N}\mathcal{O}/\mathcal{O})^2$ live in the Eisenstein part of the cohomology and that the number of linearly independent such homomorphisms is equal to the number of cusps of Γ. Thus they generate $H^1_{Eis}(\Gamma(N),\mathbb{C})$.

Ito showed in [9] that, see also Weselmann [21], up to a coboundary, the cocycles of Sczech are integrals of closed harmonic differential forms given by certain Eisenstein series defined on the hyperbolic 3-space $\mathbb{H}$. Following Ito, we can form an Eisenstein series $E_{(u,v)}(\tau,s)$ for $(\tau,s)\in\mathbb{H}\times\mathbb{C}$ with values in $\mathbb{C}^3$ associated to each cusp of $\Gamma(N)$. As a function of s, $E_{(u,v)}(\tau,s)$ can be analytically continued to all of $\mathbb{C}$ and work of Harder [7] shows that differential 1-form on the hyperbolic 3-space induced by $E_{(u,v)}(\tau,s)$ is closed for $s=0$. Ito showed that the cocycle given by the integral of this closed differential 1-form differs from the cocycle $\Psi(u,v)$ of Sczech by a coboundary.

The fact that the above Eisenstein series associated to different cusps are linearly independent (they are non-vanishing only at their associated cusp) implies that the cohomology classes of Sczech cocycles which are associated to the cusps of $\Gamma(N)$ form a *basis* of $H^1_{Eis}(\Gamma(N), \mathbb{C})$.

In another paper [10], Ito provides us the following results:

$$\Psi(0,0)(\bar{A}) = -\Psi(0,0)(A)$$

where bar means that we take the complex conjugates of the entries of the matrix A. More generally, he proves that

$$\Psi(u,v)(\bar{A}) = \frac{-1}{N^2} \sum_{s,t \in \frac{1}{N}\mathcal{O}/\mathcal{O}} \phi(s\bar{v} - t\bar{u})\Psi(s,t)(A)$$

where $\phi(z) := exp(2\pi i(z - \bar{z})/D)$ with D denoting the discriminant of K. Observe that when $(s,t) = (u,v)$ or $(s,t) = (0,0)$, we have $\phi(s\bar{v} - t\bar{u}) = 1$. Using this, let us write this summation in a more suggestive way:

$$\Psi(u,v)(\bar{A}) = \frac{-1}{N^2}\Bigg[\Bigg(\sum_{\substack{s,t \in \frac{1}{N}\mathcal{O}/\mathcal{O} \\ (s,t) \neq (u,v) \\ (s,t) \neq (0,0)}} \phi(s\bar{v} - t\bar{u})\Psi(s,t)(A)\Bigg) + \Psi(u,v)(A) + \Psi(0,0)(A)\Bigg]$$

The latter formula sheds light onto the action of complex conjugation σ on the Sczech cocycles which is given by

$$\sigma(\Psi(u,v))(A) := \Psi(u,v)(\bar{A}).$$

We see that $\sigma(\Psi(u,v))$ is expressed as summation over *all* the Sczech cocycles. We will regard σ as a linear operator on the formal space $\mathbb{C}[\Psi_N]$ for which the Sczech cocycles are taken as basis.

The pair $(0,0)$ in $(\frac{1}{N}\mathcal{O}/\mathcal{O})^2$ never corresponds to a cusp of $\Gamma(N)$, so let us eliminate the term $\Psi(0,0)$ from the big summation. Using Ito's summation formula for the case $(u,v) = (0,0)$, we get

$$\Psi(0,0)(\bar{A}) = \frac{-1}{N^2}\Bigg[\Bigg(\sum_{\substack{s,t \in \frac{1}{N}\mathcal{O}/\mathcal{O} \\ (s,t) \neq (0,0)}} \Psi(s,t)(A)\Bigg) + \Psi(0,0)(A)\Bigg]$$

Now plug in the identity $\Psi(0,0)(\bar{A}) = -\Psi(0,0)(A)$, we get

$$\Psi(0,0)(A) = \frac{1}{N^2 - 1} \sum_{\substack{s,t \in \frac{1}{N}\mathcal{O}/\mathcal{O} \\ (s,t) \neq (0,0)}} \Psi(s,t)(A).$$

Now for $(u,v) \neq (0,0)$, we have

$$\Psi(u,v)(\bar{A}) = \frac{-1}{N^2}\Bigg[\Bigg(\sum_{\substack{s,t\in\frac{1}{N}\mathcal{O}/\mathcal{O}\\(s,t)\neq(0,0)}} \phi(s\bar{v}-t\bar{u})\Psi(s,t)(A)\Bigg) + \Psi(0,0)(A)\Bigg].$$

Substituting $\Psi(0,0)(A)$, we get the following expression for $\Psi(u,v)(\bar{A})$

$$\frac{-1}{(N^2)(N^2-1)} \sum_{\substack{s,t\in\frac{1}{N}\mathcal{O}/\mathcal{O}\\(s,t)\neq(0,0)}} \Psi(s,t)(A) + \frac{-1}{N^2} \sum_{\substack{s,t\in\frac{1}{N}\mathcal{O}/\mathcal{O}\\(s,t)\neq(0,0)}} \phi(s\bar{v}-t\bar{u})\Psi(s,t)(A). \tag{4.3}$$

Since $\Psi(-u,-v) = \Psi(u,v)$, we can define $\mathbb{C}[\Psi_N^*]$ to be the formal vector space generated by the basis

$$\{\Psi(u,v) \mid (u,v) \in (\tfrac{1}{N}\mathcal{O}/\mathcal{O})^2/\pm 1 \quad \text{and } (u,v) \neq (0,0)\}.$$

Having eliminated $\Psi(0,0)$ in Equation (4.3), we can regard σ as a linear operator on the formal space $\mathbb{C}[\Psi_N^*]$. After noting that $\Psi(-u,-v) = \Psi(u,v)$ and $\phi(-u\bar{v}+v\bar{u}) = \phi(u\bar{v}-v\bar{u}) = 1$, we see that the coefficient of the summand $\Psi(u,v)(A)$ on the right hand side of the equality (4.3) is

$$\tfrac{-\xi(u,v)}{(N^2)(N^2-1)} + \tfrac{-\xi(u,v)}{N^2} = \tfrac{-\xi(u,v)}{N^2-1}$$

where $\xi(u,v) = \sharp\{\pm(u,v)\} = 1$ or 2. This implies that the trace of σ on $\mathbb{C}[\Psi_N^*]$ is

$$(N^4-1)\tfrac{-1}{N^2-1} = -(N^2+1). \tag{4.4}$$

4.2.4. Trace on H^1_{Eis}. Our goal is to apply the above results to the computation of the trace of σ on $H^1_{Eis}(\Gamma(p^n),\mathbb{C})$ for some rational prime p which is unramified in K. We were able to do this only when K is of class number one and p is inert in K. In this case, the cusps of $\Gamma(p^n)$ are in bijection with the sets $\{\pm(\bar{x},\bar{y})\} \subset (\mathcal{O}/(p^n))^2$ such that the order of $(\bar{x},\bar{y})$ is p^n (via the map $\frac{x}{y} \mapsto (y,-x)$). One can see that $c(\Gamma(p^n)) = \frac{1}{2}(p^{2n})^2 - (p^{2n-2})^2$ if $p^n > 2$, and that $c(\Gamma(p^n)) = (p^{2n})^2 - (p^{2n-2})^2$ if $p^n = 2$ (because $(\bar{x},\bar{y})$ has order 2 and so $(\bar{x},\bar{y}) = -(\bar{x},\bar{y})$).

In the rest of this subsection, we will prove the following result which is a partial generalization of a result announced by Harder in [6].

Theorem 4.7. *Assume that K is of class number one and let p be a rational prime that is inert in K. Then we have*

$$\mathrm{tr}(\sigma \mid H^1_{Eis}(\Gamma(p^n),\mathbb{C})) = \begin{cases} -(p^2+1), & \textit{if } n=1 \\ -(p^{2n}-p^{2n-2}), & \textit{if } n>1. \end{cases}$$

Proof. We will proceed by induction. Let $n = 1$. Then by a comparison with the number of cusps of $\Gamma(p)$, we see that the Sczech cocyles in the set

$$\{\Psi(u,v) \mid (u,v) \in (\tfrac{1}{p}\mathcal{O}/\mathcal{O})^2/\pm 1 \quad \text{and } (u,v) \neq (0,0)\}$$

form a basis of $H^1_{Eis}(\Gamma(p),\mathbb{C})$. Thus the trace of σ on $\mathbb{C}[\Psi^*_{p^n}]$ is equal to the trace of σ on $H^1_{Eis}(\Gamma(p),\mathbb{C})$. By our observation above, we get the claim for $n = 1$.

Before we proceed with the inductive step, let us discuss the structure of cusps. The following diagram is commutative.

$$\begin{tikzcd}
\mathcal{O}/(p) \arrow[r, "\varepsilon"] & \mathcal{O}/(p^2) \arrow[r, "\varepsilon"] & \mathcal{O}/(p^3) \arrow[r, "\varepsilon"] & \dots \\
\frac{1}{p}\mathcal{O}/\mathcal{O} \arrow[r, "\varepsilon'"] \arrow[u] & \frac{1}{p^2}\mathcal{O}/\mathcal{O} \arrow[r, "\varepsilon'"] \arrow[u] & \frac{1}{p^3}\mathcal{O}/\mathcal{O} \arrow[r, "\varepsilon'"] \arrow[u] & \dots
\end{tikzcd}$$

The maps ε are the natural inclusion maps $[x] \mapsto [px]$. the vertical arrows are the natural bijections that we mentioned above and the maps ε' are induced by the natural inclusions $\frac{1}{p}\mathcal{O} \subset \frac{1}{p^2}\mathcal{O}$. The crucial observation is that the set of elements of order p^n in $(\mathcal{O}/(p^n))^2$ is exactly $(\mathcal{O}/(p^n))^2 \backslash (\varepsilon(\mathcal{O}/(p^{n-1})))^2$. Hence in order to find the trace of σ on the Sczech cocyles which are associated to the cusps of $\Gamma(p^n)$, all we need to do is to compute the difference between the traces of σ on $\mathbb{C}[\Psi_{p^n}]$ and $\mathbb{C}[\Psi_{p^{n-1}}]$. This is the same as the difference between the traces of σ on $\mathbb{C}[\Psi^*_{p^n}]$ and $\mathbb{C}[\Psi^*_{p^{n-1}}]$ which we already computed in Equation 4.4:

$$-(p^{2n}+1) - (-(p^{2n-2}+1)) = -(p^{2n} - p^{2n-2})$$

as claimed. $\square$

Remark 4.8.

(1) The above proof does not carry over to the case where p is split. To see this, put $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. Then the set of cusps of $\Gamma(p^n)$, which has cardinality $(p^{2n}(1-p^{-2}))^2$, is in bijection with the Cartesian product of the set of cusps of $\Gamma(\mathfrak{p}^n)$ and the set of cusps of $\Gamma(\bar{\mathfrak{p}}^n)$. Both of the latter sets are in bijection with the set of elements of order p^n of $\mathbb{Z}/p^n\mathbb{Z}$. In a very similar way to the one in the proof, we have a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & (\mathcal{O}/\mathfrak{p}^{n-1})^2 \times (\mathcal{O}/\bar{\mathfrak{p}}^{n-1})^2 \arrow[r, "\varepsilon_{\mathfrak{p}}\times\varepsilon_{\bar{\mathfrak{p}}}"] & (\mathcal{O}/\mathfrak{p}^{n})^2 \times (\mathcal{O}/\bar{\mathfrak{p}}^{n})^2 \arrow[r] & \dots \\
\dots \arrow[r] & (\frac{1}{p^{n-1}}\mathcal{O}/\mathcal{O})^2 \arrow[r, "\varepsilon'"] \arrow[u] & (\frac{1}{p^{n}}\mathcal{O}/\mathcal{O})^2 \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

However, unlike in the inert case, the subset of elements in $(\frac{1}{p^n}\mathcal{O}/\mathcal{O})^2$ which correspond to the cusps of $\Gamma(p^n)$ is *not* the complement of the image of $(\frac{1}{p^{n-1}}\mathcal{O}/\mathcal{O})^2$ in $(\frac{1}{p^n}\mathcal{O}/\mathcal{O})^2$. This obstructs the recursive use of Equation 4.4 in this case. In fact, the subset of elements in $(\frac{1}{p^n}\mathcal{O}/\mathcal{O})^2$ which correspond to the cusps of $\Gamma(p^n)$ is given by

$$\Big((\mathcal{O}/\mathfrak{p}^n)^2\backslash \mathrm{Im}(\varepsilon_{\mathfrak{p}})\Big) \times \Big((\mathcal{O}/\bar{\mathfrak{p}}^n)^2\backslash \mathrm{Im}(\varepsilon_{\bar{\mathfrak{p}}})\Big)\,.$$

However we do not see a way to isolate this set in a recursive way.

(2) In order to treat $H^1_{Eis}(\Gamma(N), E_{k,k})$ using our approach, a vector-valued version of Sczech's cocycles should be developed. We do not attempt to do this here.

These Eisenstein trace results together with Lefschetz number computations of previous sections can be plugged in the formula of Proposition 2.2, giving *explicit* lower bounds for the cuspidal cohomology of Bianchi groups. We leave such tasks to the interested reader as the formulas will be quite complicated.

5. Asymptotic lower bounds

In [4], Calegari and Emerton considered how the size of the cohomology, with fixed coefficient module, varied in a tower of arithmetic groups. Their general result when applied to our situation gives the following.

Theorem 5.1 (Calegari-Emerton [4]). *Let $\Gamma(\mathfrak{p}^n)$ denote the principal congruence subgroup of level $\mathfrak{p}^n$ of a Bianchi group $\mathrm{SL}_2(\mathcal{O})$ where $\mathfrak{p}$ is an unramified prime ideal of $\mathcal{O}$. Fix E. Then*

(1) *if the residue degree of $\mathfrak{p}$ is one, then*

$$\dim H^1(\Gamma(\mathfrak{p}^n), E) \ll p^{2n},$$

(2) *if the residue degree of $\mathfrak{p}$ is two, then*

$$\dim H^1(\Gamma(\mathfrak{p}^n), E) \ll p^{5n}$$

as n increases.

Note that the trivial upper bounds are p^{3n} and p^{6n} respectively. It is natural to look at these asymptotics from the perspective of the volume which is a topological invariant in our setting. Observe that the volume of $Y_{\Gamma(\mathfrak{p}^n)}$ is given by a constant times the index of $\Gamma(\mathfrak{p}^n)$ in the Bianchi group $\mathrm{SL}_2(\mathcal{O})$. Thus asymptotically, the trivial asymptotic upper bound for the above cohomology groups is *linear* in the volume and the above upper bounds of Calegari and Emerton can be interpreted as *sublinear*.

Using the techniques discussed in this paper, we can derive the following lower bounds.

Proposition 5.2. *Let p be a rational prime that is unramified in K and let $\Gamma(p^n)$ denote the principal congruence subgroup of level $(p)^n$ of a Bianchi group $\mathrm{SL}_2(\mathcal{O})$.*

(1) *Then*
$$\dim H^1_{cusp}(\Gamma(p^n), E_{k,k}) \gg k$$
as k increases and n is fixed,

(2) *Assume further that K is of class number one. Then*
$$\dim H^1_{cusp}(\Gamma(p^n), \mathbb{C}) \gg p^{3n}$$
as n increases.

Proof. Recall from Proposition 2.2 that
$$\dim H^1_{cusp}(\Gamma, E_{k,k}) \geq \frac{1}{2}\Big(L(\sigma, \Gamma, k) + \mathrm{tr}(\sigma^1_{Eis}, \Gamma, k) - \mathrm{tr}(\sigma^2_{Eis}, \Gamma, k)\Big).$$
When Γ is fixed, by Corollary 4.2 the dimension of the Eisenstein part of the cohomology is the same for every weight $k > 0$. Hence, the asymptotic for (1) is given by Corollary 3.5.

The claim in (2) follows directly from Theorems 4.7 and 4.6, together with the Lefschetz number formula provided in Corollary 3.5. $\square$

5.1. Lower bounds for $\mathbf{GL_2}$. In this section we will discuss the trace of σ on the cohomoogy of $\mathrm{GL}_2(\mathcal{O})$. For convenience let us put $\Gamma = \mathrm{SL}_2(\mathcal{O})$ and $G = \mathrm{GL}_2(\mathcal{O})$.

Let us start with a couple of observations. As $G = \Gamma \rtimes \langle\beta\rangle$ with $\beta := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and β acts trivially on the cusps of Γ, the groups Γ and G have the same cusps. Given a cusp c, its stabilizer in G (modulo $\pm Id$) is of the form $\mathbb{Z}^2 \rtimes \mathbb{Z}/2\mathbb{Z}$. This implies that the connected components of the boundary of Borel-Serre compactification of Y_G are 2-orbifolds whose underlying manifolds are 2-spheres. In turn, the cohomology of the boundary vanishes and we get
$$H^1(G, E_{k,k}) = H^1_{cusp}(G, E_{k,k}).$$
From the inflation-restriction sequence we see that
$$H^1(G, E_{k,k}) = H^1(\Gamma, E_{k,k})^{\langle\beta\rangle}.$$

The involutions σ^1 and τ^1 commute and $\sigma^1\tau^1$ equals the action of β. Hence we get
$$H^1(G, E_{k,k}) = H^1(\Gamma, E_{k,k})^{\sigma^1\tau^1}.$$

Counting the dimensions of the common eigenspaces, we see by comparison that
$$\mathrm{tr}(\tau^1, \Gamma, E_{k,k}) + \mathrm{tr}(\sigma^1, \Gamma, E_{k,k}) = 2 \cdot \mathrm{tr}(\sigma^1, G, E_{k,k}).$$

The matrix β acts on $E_{k,k}$ trivially and acts as $-Id$ on $H^1(\partial X_\Gamma, E_{k,k})$. This implies that

$$\mathrm{tr}(\tau^1_{Eis}, \Gamma, E_{k,k}) = -\mathrm{tr}(\sigma^1_{Eis}, \Gamma, E_{k,k}).$$

Using this last identity, together with the previous facts, we get (dropping $E_{k,k}$ from the notation for convenience)

$$\begin{aligned} &L(\tau,\Gamma) + L(\sigma,\Gamma) \\ &\qquad = -4 \cdot \mathrm{tr}(\sigma^1, G) + \mathrm{tr}(\tau^0,\Gamma) + \mathrm{tr}(\sigma^0,\Gamma) + \mathrm{tr}(\tau^2_{Eis},\Gamma) + \mathrm{tr}(\sigma^2_{Eis},\Gamma). \end{aligned}$$

Using results from previous sections, we get the following simplified formula for the trace of σ on $H^1(\mathrm{GL}_2(\mathcal{O}), E_{k,k})$.

Theorem 5.3. *Let $L(\tau, \mathrm{SL}_2(\mathcal{O}), E_{k,k})$ and $L(\sigma, \mathrm{SL}_2(\mathcal{O}), E_{k,k})$ be as in Theorem 3.2. Then,*

$$\begin{aligned} &\mathrm{tr}(\sigma^1 \mid H^1(\mathrm{GL}_2(\mathcal{O}), E_{k,k})) \\ &\qquad = \frac{-1}{4}\Big(L(\tau, \mathrm{SL}_2(\mathcal{O}), E_{k,k}) + L(\sigma, \mathrm{SL}_2(\mathcal{O}), E_{k,k}) + 2^t - 4\cdot\delta(k,0)\Big) \end{aligned}$$

where t is the number of rational primes which ramify over K and $\delta(k,0)$ is the Kronecker δ–function as defined in Theorem 4.6.

Using Theorem 3.2 and the fact that

$$\dim H^1(\mathrm{GL}_2(\mathcal{O}), E_{k,k}) \geq |\mathrm{tr}(\sigma^1, \mathrm{GL}_2(\mathcal{O}), E_{k,k})|,$$

we get the following asymptotics.

Corollary 5.4. *Let D be the discriminant of $K/\mathbb{Q}$ and $\mathcal{O}_K$ be its ring of integers. As $K/\mathbb{Q}$ is fixed and $k \to \infty$, we have*

$$\dim H^1(\mathrm{GL}_2(\mathcal{O}_K), E_{k,k}) \gg k$$

where the implicit constant depends on the discriminant D. As k is fixed and $|D| \to \infty$, we have

$$\dim H^1(\mathrm{GL}_2(\mathcal{O}_K), E_{k,k}) \gg \varphi(D)$$

where φ is the Euler phi-function and the implicit constant depends on the weight k.

Note that one can write a more precise formula for the lower bounds above. As the formulas for the Lefschetz numbers are complicated, we stated our results in a slightly weaker form for the sake of simplicity.

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