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Refined class number formulas for \mathbb{G}_m

par BARRY MAZUR et KARL RUBIN

RÉSUMÉ. Nous formulons une généralisation d’une “formule du nombre de classes raffinée” de Darmon. Notre conjecture concerne des éléments de type Stickelberger formés à partir d’unités de Stark généralisées. En utilisant la théorie des systèmes de Kolyvagin, nous démontrons une grande partie de cette conjecture lorsque l’ordre d’annulation de la fonction L complexe correspondante est 1.

ABSTRACT. We formulate a generalization of a “refined class number formula” of Darmon. Our conjecture deals with Stickelberger-type elements formed from generalized Stark units, and has two parts: the “order of vanishing” and the “leading term”. Using the theory of Kolyvagin systems we prove a large part of this conjecture when the order of vanishing of the corresponding complex L -function is 1.

1. Introduction

In [3], Darmon conjectured a “refined class number formula” for real quadratic fields, inspired by work of Gross [5], of the first author and Tate [7], and of Hayes [6]. The common setting for these conjectures included a finite abelian extension L/K and a Stickelberger-type element $\theta \in \mathbb{Z}[\text{Gal}(L/K)]$. In analogy with the Birch and Swinnerton-Dyer conjecture, these conjectures predicted the “order of vanishing” (a nonnegative integer r such that θ lies in the r -th power of the augmentation ideal \mathcal{A} of $\mathbb{Z}[\text{Gal}(L/K)]$) and the “leading term” (the image of θ in $\mathcal{A}^r/\mathcal{A}^{r+1}$) of θ .

In [9], we proved most (the “non-2-part”) of Darmon’s conjecture, using the theory of Kolyvagin systems [8]. The key idea is that in nice situations, the space of Kolyvagin systems is a free \mathbb{Z}_p -module of rank one, and hence two Kolyvagin systems that agree “at $n = 1$ ” must be equal. Darmon’s conjecture for $n = 1$ follows from the classical evaluation of $L'(0, \chi)$ for a real quadratic Dirichlet character χ .

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Mots-clefs. Class number formulas, Euler systems, Kolyvagin systems, Stark conjectures, L -functions.

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In this paper we attempt to generalize both the statement and proof of Darmon’s conjecture. To generalize the statement we rely on a suitable version of Stark’s conjectures. Namely, given a finite abelian tower of number fields $L/K/k$, our proposed Conjecture 5.2 relates the so-called “Rubin-Stark” elements ϵ_{L,S_L} attached to L/k (see §3) with an “algebraic regulator” (see Definition 4.5) constructed from Rubin-Stark elements ϵ_{K,S_L} attached to K/k and L . Similar generalizations of Darmon’s conjecture have recently been proposed independently by Sano [16, Conjecture 4] and Popescu [12].

Our conjecture has two parts, the “order of vanishing” and the “leading term”. We prove a large portion of the order of vanishing part of the conjecture in Theorem 6.3. We prove a large part of the leading term statement in Theorem 10.7 following the method of [9], but only under the rather strong assumption that the order of vanishing (the “core rank”, in the language of [10]) is one. As L varies, the elements ϵ_{L,S_L} form an Euler system, and the elements ϵ_{K,S_L} form what we call a Stark system. When the order of vanishing is one we can relate these systems and prove the leading term formula. In the final section we prove a weakened version of the leading term statement for general r , under some additional hypotheses.

Notation. Suppose throughout this paper that \mathcal{O} is an integral domain with field of fractions F , and let $R = \mathcal{O}[\Gamma]$ with a finite abelian group Γ . We are mainly interested in the case where $\mathcal{O} = \mathbb{Z}$ or \mathbb{Z}_p for some prime p .

If M is an R -module, we let $M^* := \text{Hom}_R(M, R)$. If $\rho \in R$, then $M[\rho]$ will denote the kernel of multiplication by ρ in M .

If $r \geq 0$, then $\wedge^r M$ (or $\wedge_R^r M$, if we need to emphasize the ring R) will denote the r -th exterior power of M in the category of R -modules, with the convention that $\wedge^0 M = R$. See Appendix A for more on the exterior algebra that we use. In particular, in Definition A.3 we define an R -lattice $\wedge^{r,0} M \subset \wedge^r M \otimes F$, containing the image of $\wedge^r M$, that will play an important role.

2. Unit groups

Suppose that K/k is a finite abelian extension of number fields, and let $\Gamma = \text{Gal}(K/k)$ and $R = \mathbb{Z}[\Gamma]$. Fix a finite set S of places of k containing all infinite places and all places ramified in K/k , and a second finite set T of places of k , disjoint from S . Define:

$$\begin{aligned} S_K &= \{\text{places of } K \text{ lying above places in } S\}, \\ T_K &= \{\text{places of } K \text{ lying above places in } T\}, \\ U_{K,S,T} &= \{x \in K^\times : |x|_w = 1 \text{ for all } w \notin S_K, \\ &\quad \text{and } x \equiv 1 \pmod{w} \text{ for all } w \in T_K\}. \end{aligned}$$

We assume further that K has no roots of unity congruent to 1 modulo all places in T_K , so that $U_{K,S,T}$ is a free \mathbb{Z} -module. When there is no fear of confusion, we will suppress the S and T and write $U_K := U_{K,S,T}$.

Suppose now that L is a finite abelian extension of k containing K . Let $G := \text{Gal}(L/k)$ and $H := \text{Gal}(L/K)$, so $G/H = \Gamma$. Let $\mathcal{A}_H \subset \mathbb{Z}[H]$ be the augmentation ideal, the ideal generated by $\{h - 1 : h \in H\}$.

Proposition 2.1. *For every $s \leq r$ and every $\rho \in \mathbb{Q}[\Gamma]$, Proposition A.6 gives a canonical pairing*

$$\begin{aligned} (\wedge^{r,0}U_K)[\rho] \times \wedge^{r-s}\text{Hom}_\Gamma(U_K, \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \mathcal{A}_H/\mathcal{A}_H^2) \\ \longrightarrow (\wedge^{s,0}U_K)[\rho] \otimes_{\mathbb{Z}} \mathcal{A}_H^{r-s}/\mathcal{A}_H^{r-s+1}. \end{aligned}$$

Proof. Apply Proposition A.6 with $B := \bigoplus_{i \geq 0} \mathcal{A}_H^i/\mathcal{A}_H^{i+1}$ and $n = 1$. □

3. A Stark conjecture over \mathbb{Z}

In this section we recall the so-called Rubin-Stark conjecture over \mathbb{Z} for arbitrary order of vanishing from [14]. When the order of vanishing (the integer r below) is one, this is essentially the “classical” Stark conjecture over \mathbb{Z} (see for example [18, §IV.2] and [14, Proposition 2.5]).

Keep the finite abelian extension K/k of number fields from §2, with $\Gamma = \text{Gal}(K/k)$, and the sets S, T of places of K . We define the Stickelberger function attached to K/k (and S and T) to be the meromorphic $\mathbb{C}[\Gamma]$ -valued function

$$\theta_{K/k}(s) = \theta_{K/k,S,T}(s) = \prod_{\mathfrak{p} \notin S} (1 - \text{Fr}_{\mathfrak{p}}^{-1} \mathbf{N}\mathfrak{p}^{-s})^{-1} \prod_{\mathfrak{p} \in T} (1 - \text{Fr}_{\mathfrak{p}}^{-1} \mathbf{N}\mathfrak{p}^{1-s})$$

where $\text{Fr}_{\mathfrak{p}} \in \Gamma$ is the Frobenius of the (unramified) prime \mathfrak{p} . If χ is a character in $\hat{\Gamma} := \text{Hom}(\Gamma, \mathbb{C}^\times)$, then applying χ to the Stickelberger function yields the (modified at S and T) Artin L -function

$$\chi(\theta_{K/k}(s)) = L_{S,T}(K/k; \bar{\chi}, s).$$

Definition 3.1. If w is a place of K we write K_w for the completion of K at w and $|\cdot|_w : K_w \rightarrow \mathbb{R}^+ \cup \{0\}$ for the absolute value normalized so that

$$|x|_w = \begin{cases} \pm x \text{ (the usual absolute value)} & \text{if } K_w = \mathbb{R}, \\ x\bar{x} & \text{if } K_w = \mathbb{C}, \\ \mathbf{N}w^{-\text{ord}_w(x)} & \text{if } K_w \text{ is nonarchimedean} \end{cases}$$

where $\mathbf{N}w$ is the cardinality of the residue field of the finite place w .

Definition 3.2. Suppose now that $S' \subset S$ is a subset such that every $v \in S'$ splits completely in K/k . Let $r = |S'| \geq 0$. Let S'_K denote the set of primes of K above S' , and let $W_{K,S'}$ denote the free abelian group on S'_K , so $W_{K,S'}$ is a free $\mathbb{Z}[\Gamma]$ -module of rank r .

Define a $\mathbb{Z}[\Gamma]$ -homomorphism $\eta_K^{\log} : U_K \rightarrow W_{K,S'} \otimes \mathbb{R}$ by

$$\eta_K^{\log}(u) = \sum_{w \in S'_K} w \otimes \log |u|_w.$$

If L is an abelian extension of K with Galois group $H := \text{Gal}(L/K)$, and $\mathcal{A}_H \subset \mathbb{Z}[H]$ is the augmentation ideal, let $[\cdot, L_w/K_w] : K_w^\times \rightarrow H$ denote the local Artin symbol (this is independent of the choice of place of L above w) and define a $\mathbb{Z}[\Gamma]$ -homomorphism $\eta_{L/K}^{\text{Art}} : U_K \rightarrow W_{K,S'} \otimes_{\mathbb{Z}} \mathcal{A}_H / \mathcal{A}_H^2$ by

$$\eta_{L/K}^{\text{Art}}(u) := \sum_{w \in S'_K} w \otimes ([u, L_w/K_w] - 1).$$

Definition 3.3. Let

$$\mathcal{R}^\infty = \mathcal{R}_{K,S,T,S'}^\infty : \wedge^r U_K \otimes_\Gamma \wedge^r W_{K,S'}^* \longrightarrow \mathbb{R}[\Gamma]$$

be the classical regulator map induced by $\eta_K^{\log} : \wedge^r U_K \rightarrow \wedge^r W_{K,S'} \otimes \mathbb{R}$ and the natural isomorphism $\wedge^r W_{K,S'} \otimes \wedge^r W_{K,S'}^* \rightarrow \mathbb{Z}[\Gamma]$.

Concretely, the map \mathcal{R}^∞ is given as follows. If $w \in S'_K$, let $w^* \in W_{K,S'}^*$, be the map

$$w^* \left(\sum_{z \in S'_K} a_z z \right) := \sum_{\gamma \in \Gamma} a_{\gamma w} \gamma.$$

If v_1, \dots, v_r is an ordering of the places in S' , and for each i we choose a place w_i of K above v_i , then $w_1 \wedge \dots \wedge w_r$ is a $\mathbb{Z}[\Gamma]$ -basis of $\wedge^r W_{K,S'}$, and $w_1^* \wedge \dots \wedge w_r^*$ is the dual basis of $\wedge^r W_{K,S'}^*$. Then

$$\mathcal{R}^\infty((u_1 \wedge \dots \wedge u_r) \otimes (w_1^* \wedge \dots \wedge w_r^*)) = \det \left(\sum_{\gamma \in \Gamma} \log |u_i^\gamma|_{w_j} \gamma^{-1} \right).$$

Definition 3.4. Write $\mathbf{1}$ for the trivial character of Γ . For every $\chi \in \hat{\Gamma}$ there is an idempotent

$$e_\chi = |\Gamma|^{-1} \sum_{\gamma \in \Gamma} \chi(\gamma) \gamma^{-1} \in \mathbb{C}[\Gamma],$$

and we define a nonnegative integer $r(\chi) = r(\chi, S)$ by

(3.5)

$$r(\chi) = \text{ord}_{s=0} L_S(s, \bar{\chi}) = \dim_{\mathbb{C}} e_\chi \mathbb{C}U_K = \begin{cases} |\{v \in S : \chi(\Gamma_v) = 1\}| & \text{if } \chi \neq \mathbf{1} \\ |S| - 1 & \text{if } \chi = \mathbf{1} \end{cases}$$

where Γ_v is the decomposition group of v in Γ (see for example [18, Proposition I.3.4]). If $r \geq 0$ is such that S contains r places that split completely in K/k , and $|S| \geq r + 1$, then $r(\chi) \geq r$ for every $\chi \in \hat{\Gamma}$, and we let

$$\rho_{K,r} := \sum_{\chi \in \hat{\Gamma}, r(\chi) \neq r} e_\chi \in \mathbb{Q}[\Gamma].$$

The following is the ‘‘Stark conjecture over \mathbb{Z} ’’ that we will use.

Conjecture $\text{St}(K/k, S, T, S')$ (= Conjecture B' of [14]). *Suppose that:*

- (i) S is a finite set of places of k containing all archimedean places and all places ramifying in K/k ,
- (ii) T is a finite set of places of K , disjoint from S , such that $U_{K,S,T}$ contains no roots of unity,
- (iii) $S' \subsetneq S$ contains only places that split completely in K .

Let $r = |S'|$. Then there is a unique element

$$\epsilon_K = \epsilon_{K,S,T,S'} \in (\wedge^{r,0}U_{K,S,T})[\rho_{K,r}] \otimes_{\Gamma} \wedge^r W_{K,S'}^*$$

such that

$$\mathcal{R}^\infty(\epsilon_K) = \lim_{s \rightarrow 0} s^{-r} \theta_{K/k}(s).$$

By Conjecture $\text{St}(K/k)$ we mean the conjecture that $\text{St}(K/k, S, T, S')$ holds for all choices of S, T , and S' satisfying the hypotheses above.

Recall that $\wedge^r W_{K,S'}^*$ is free of rank one over $\mathbb{Z}[\Gamma]$. The uniqueness of $\epsilon_{K,S,T,S'}$ is automatic because the regulator map \mathcal{R}^∞ is injective on $(\wedge^{r,0}U_{K,S,T})[\rho_{K,r}] \otimes_{\Gamma} \wedge^r W_{K,S'}^*$ (see for example [14, Lemma 2.7]).

Conjecture $\text{St}(K/k, S, T, S')$ is known to be true in the following cases:

- $r = 0$ (in which case $\epsilon_K := \theta_{K/k}(0) \in \mathbb{Z}[\Gamma]$, which was proved independently by Deligne and Ribet, Cassou-Noguès, and Barsky),
- K/k is quadratic ([14, Theorem 3.5]),
- $k = \mathbb{Q}$ (proved by Burns in [1, Theorem A]),
- $S - S'$ contains a prime that splits completely in K/k ([14, Proposition 3.1]).

Lemma 3.6. *Suppose that $S - S'$ contains a place that splits completely in K/k , and $|S - S'| \geq 2$. Then $\epsilon_K = 0$ satisfies Conjecture $\text{St}(K/k, S, T, S')$.*

Proof. In this case we have $r(\chi, S) > r = |S'|$ for every $\chi \in \hat{\Gamma}$, so

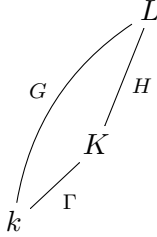
$$\lim_{s \rightarrow 0} s^{-r} \theta_{K/k}(s) = 0$$

and $\rho_{K,r} = 1$ in Definition 3.4. The lemma follows. □

4. The Artin regulator

Fix now a finite abelian extension L/k of number fields, and an intermediate field K , $k \subset K \subset L$. Let $G := \text{Gal}(L/k)$, $H := \text{Gal}(L/K)$ and

$\Gamma := \text{Gal}(K/k) = G/H.$



Fix a finite set S of places of k containing all archimedean places and all primes ramifying in L/k . Fix a second finite set of primes T of k , disjoint from S , such that $U_L = U_{L,S,T}$ contains no roots of unity.

Suppose that we have a filtration $S' \subset S'' \subsetneq S$, where every $v \in S''$ splits completely in K/k , and every $v \in S'$ splits completely in L/k . Let $r = |S'|$ and $s = |S''| - |S'|$.

For the rest of this section, we keep S, S', S'' and T fixed, and we suppress them from the notation when possible.

For every subset $\Sigma \subset S''$, let $W_{K,\Sigma}$ denote the free abelian group on the set of primes of K above Σ , and similarly with L in place of K . Then $W_{K,\Sigma}$ is a free $\mathbb{Z}[\Gamma]$ -module of rank $|\Sigma|$, we have

$$(4.1) \quad W_{K,S''} = W_{K,S'} \oplus W_{K,S''-S'}, \quad \wedge^{r+s} W_{K,S''} = \wedge^r W_{K,S'} \otimes_{\Gamma} \wedge^s W_{K,S''-S'},$$

and the natural map $S_L \rightarrow S_K$, that takes a place of L to its restriction to K , induces an isomorphism of free modules

$$(4.2) \quad W_{L,S'} \otimes_G \mathbb{Z}[\Gamma] \xrightarrow{\sim} W_{K,S'}.$$

Let $\eta_{L/K}^{\text{Art}} \in \text{Hom}_{\Gamma}(U_K, W_{K,S''-S'} \otimes_{\mathbb{Z}} \mathcal{A}_H/\mathcal{A}_H^2)$ be the map of Definition 3.2, with the augmentation ideal \mathcal{A}_H as in §2. Composition with $\eta_{L/K}^{\text{Art}}$ gives a $\mathbb{Z}[\Gamma]$ -homomorphism

$$(4.3) \quad W_{K,S''-S'}^* \longrightarrow \text{Hom}_{\Gamma}(U_K, \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \mathcal{A}_H/\mathcal{A}_H^2).$$

Proposition 2.1 gives a canonical pairing

$$(\wedge^{r+s,0} U_K) \times \wedge^s \text{Hom}_{\Gamma}(U_K, \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \mathcal{A}_H/\mathcal{A}_H^2) \longrightarrow (\wedge^{r,0} U_K) \otimes_{\mathbb{Z}} \mathcal{A}_H^s/\mathcal{A}_H^{s+1},$$

and using (4.3) we can pull this back to a pairing

$$(4.4) \quad (\wedge^{r+s,0} U_K) \times \wedge^s W_{K,S''-S'}^* \longrightarrow (\wedge^{r,0} U_K) \otimes_{\mathbb{Z}} \mathcal{A}_H^s/\mathcal{A}_H^{s+1}.$$

Definition 4.5. Tensoring both sides of (4.4) with $\wedge^r W_{K,S'}^*$ and using (4.1), we define an algebraic regulator map $\mathcal{R}_{L/K}^{\text{Art}} = \mathcal{R}_{L/K,S,S',S''}^{\text{Art}}$

$$\mathcal{R}_{L/K}^{\text{Art}} : (\wedge^{r+s,0} U_K) \otimes_{\Gamma} \wedge^{r+s} W_{K,S''}^* \longrightarrow (\wedge^{r,0} U_K) \otimes_{\Gamma} \wedge^r W_{K,S'}^* \otimes_{\mathbb{Z}} \mathcal{A}_H^s/\mathcal{A}_H^{s+1}.$$

Definition 4.6. Let $\iota_{L/K} : \mathbb{Z}[\Gamma] \hookrightarrow \mathbb{Z}[G]$ denote the natural $\mathbb{Z}[G]$ -module homomorphism that sends $\gamma \in \Gamma$ to $\sum_{g \in \gamma} g$, viewing γ as an H -coset. Then $\iota_{L/K}$ is not a ring homomorphism, but rather

$$(4.7) \quad \iota_{L/K}(\alpha)\iota_{L/K}(\beta) = [L : K]\iota_{L/K}(\alpha\beta).$$

Note that $\iota_{L/K}$ is a $\mathbb{Z}[G]$ -module isomorphism $\mathbb{Z}[\Gamma] \xrightarrow{\sim} \mathbb{Z}[G]^H$.

As in §2, let

$$U_K^* := \text{Hom}_\Gamma(U_K, \mathbb{Z}[\Gamma]), \quad U_L^* := \text{Hom}_G(U_L, \mathbb{Z}[G]).$$

If $\varphi \in U_L^*$, then $\varphi(U_K) \subset \mathbb{Z}[G]^H$, and we define $\varphi^K = \iota_{L/K}^{-1} \circ \varphi|_{U_K} \in U_K^*$.

Let $j_{L/K} : U_K \hookrightarrow U_L$ denote the natural inclusion, and

$$\wedge^s j_{L/K} : \wedge^s U_K \longrightarrow \wedge^s U_L$$

the induced map (if $s = 0$, we let $\wedge^0 j_{L/K} = \iota_{L/K} : \mathbb{Z}[\Gamma] \rightarrow \mathbb{Z}[G]$).

Lemma 4.8. $\wedge^r j_{L/K}(\wedge^{r,0} U_K) \subset [L : K]^{\max\{0, r-1\}} \wedge^{r,0} U_L$.

Proof. If $r = 0$ there is nothing to prove, so assume $r \geq 1$. Suppose $\varphi_1, \dots, \varphi_r \in U_L^*$. Let

$$\varphi := \varphi_1 \wedge \dots \wedge \varphi_r \in \wedge^r U_L^*, \quad \varphi^K := \varphi_1^K \wedge \dots \wedge \varphi_r^K \in \wedge^r U_K^*.$$

Using (4.7) and the evaluation (A.2) of φ and φ^K as determinants, we have a commutative diagram

$$\begin{array}{ccccc} \wedge^{r,0} U_L & \hookrightarrow & \wedge^r U_L \otimes \mathbb{Q} & \xrightarrow{\varphi} & \mathbb{Q}[G] \\ & & \uparrow \wedge^r j_{L/K} & & \uparrow [L:K]^{r-1} \iota_{L/K} \\ \wedge^{r,0} U_K & \hookrightarrow & \wedge^r U_K \otimes \mathbb{Q} & \xrightarrow{\varphi^K} & \mathbb{Q}[\Gamma] \end{array}$$

By definition $\varphi^K(\wedge^{r,0} U_K) \subset \mathbb{Z}[\Gamma]$, so $\varphi(\wedge^r j_{L/K}(\wedge^{r,0} U_K)) \subset [L : K]^{r-1} \mathbb{Z}[G]$. Since these φ generate $\wedge^r U_L^*$, this proves the lemma. \square

Lemma 4.9. The map $[L : K]^{-\max\{0, r-1\}} \wedge^r j_{L/K} : \wedge^{r,0} U_K \rightarrow \wedge^{r,0} U_L$ and the inverse of the isomorphism (4.2) induce a map

$$\mathbf{j}_{L/K} : (\wedge^{r,0} U_K) \otimes_\Gamma \wedge^r W_{K,S'}^* \longrightarrow (\wedge^{r,0} U_L) \otimes_G \wedge^r W_{L,S'}^*.$$

Proof. Using (4.2) for the second equality, we have

$$\begin{aligned} (\wedge^{r,0} U_K) \otimes_\Gamma \wedge^r W_{K,S'}^* &= (\wedge^{r,0} U_K) \otimes_G \wedge^r W_{K,S'}^* \\ &= (\wedge^{r,0} U_K) \otimes_G (\wedge^r W_{L,S'}^* \otimes_G \mathbb{Z}[\Gamma]) \\ &= ((\wedge^{r,0} U_K) \otimes_G \mathbb{Z}[\Gamma]) \otimes_G \wedge^r W_{L,S'}^* \\ &= (\wedge^{r,0} U_K) \otimes_G \wedge^r W_{L,S'}^*. \end{aligned}$$

Now the lemma follows from Lemma 4.8. \square

5. The conjecture

Let $L/K/k, G, H, \Gamma, S, T, S', S''$ be as in §4. The hypotheses of Conjectures $\text{St}(L/k, S, T, S')$ and $\text{St}(K/k, S, T, S'')$ are all satisfied, and if those conjectures both hold they provide us with elements

$$\begin{aligned} \epsilon_L &:= \epsilon_{L,S,T,S'} \in (\wedge^{r,0}U_L)[\rho_{L,r}] \otimes_G \wedge^r W_{L,S'}^* \subset \wedge^r U_L \otimes_G \wedge^r W_{L,S'}^* \otimes_{\mathbb{Z}} \mathbb{Q}, \\ \epsilon_K &:= \epsilon_{K,S,T,S''} \in (\wedge^{r+s,0}U_K)[\rho_{K,r+s}] \otimes_{\Gamma} \wedge^{r+s} W_{K,S''}^* \\ &\subset \wedge^{r+s} U_K \otimes_{\Gamma} \wedge^{r+s} W_{K,S''}^* \otimes_{\mathbb{Z}} \mathbb{Q}. \end{aligned}$$

Definition 5.1. If M is a $\mathbb{Z}[H]$ -module, define the *twisted trace*

$$\text{Tw}_{L/K} : M \longrightarrow M \otimes_{\mathbb{Z}} \mathbb{Z}[H]$$

by $\text{Tw}_{L/K}(m) := \sum_{h \in H} m^h \otimes h^{-1} \in M \otimes_{\mathbb{Z}} \mathbb{Z}[H]$.

Note that for every m , $\text{Tw}_{L/K}(m)$ is invariant under the natural anti-diagonal action of H . We will think of $\text{Tw}_{L/K}(\epsilon_L)$ as a generalized Stickelberger element. The following conjecture is inspired by conjectures in [7, 5, 4, 3].

Conjecture 5.2. *With $(L/K/k, S, T, S', S'')$ as in §4, suppose that Conjectures $\text{St}(L/k, S, T, S')$ and $\text{St}(K/k, S, T, S'')$ both hold.*

(i) “Order of vanishing”:

$$\text{Tw}_{L/K}(\epsilon_L) \in (\wedge^{r,0}U_L) \otimes_G \wedge^r W_{L,S'}^* \otimes_{\mathbb{Z}} \mathcal{A}_H^s.$$

(ii) “Leading term”: with the maps $\mathbf{j}_{L/K}$ of Lemma 4.9 and $\mathcal{R}_{L/K}^{\text{Art}}$ of Definition 4.5, we have

$$\begin{aligned} \text{Tw}_{L/K}(\epsilon_L) &\equiv (\mathbf{j}_{L/K} \otimes 1)(\mathcal{R}_{L/K}^{\text{Art}}(\epsilon_K)) \\ &\text{in } (\wedge^{r,0}U_L) \otimes_G \wedge^r W_{L,S'}^* \otimes_{\mathbb{Z}} \mathcal{A}_H^s / \mathcal{A}_H^{s+1}. \end{aligned}$$

Remark 5.3. Suppose $k = \mathbb{Q}$, K is a real quadratic field, n is an integer prime to the conductor of K/\mathbb{Q} , and let $L = K(\mu_n)^+$ (the real subfield of the extension of K generated by the n -th roots of unity). Let $S' := \{\infty\}$ and $S'' := \{\infty\} \cup \{\ell : \ell \mid n\}$ (so $r = 1$). In this case $\text{St}(L/k, S, T, S')$ and $\text{St}(K/k, S, T, S'')$ are known to hold, and Conjecture 5.2 is essentially the same as Darmon’s conjecture in [3, §4]. This case was studied in detail in [9].

See §10 for more about the case $r = 1$.

Proposition 5.4. *If $r = 0$ then Conjecture 5.2 is equivalent to the conjecture of Gross in [5, Conjecture 7.6] and [4] (for the statement see also Conjecture $\tilde{A}_{\mathbb{Z}}(L/K/k, S, T, s)$ of [13]).*

Before proving Proposition 5.4, we have the following two lemmas. Let $J_H := \mathbb{Z}[G]\mathcal{A}_H$ be the kernel of the natural projection $\mathbb{Z}[G] \rightarrow \mathbb{Z}[\Gamma]$.

Lemma 5.5. *There are natural isomorphisms*

- (i) $H \xrightarrow{\sim} \mathcal{A}_H/\mathcal{A}_H^2$, given by $h \mapsto (h - 1)$,
- (ii) $\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \mathcal{A}_H^r/\mathcal{A}_H^{r+1} \xrightarrow{\sim} J_H^r/J_H^{r+1}$ for every $r \geq 0$, given by $\gamma \otimes \alpha \mapsto \alpha\bar{\gamma}$, where $\bar{\gamma}$ is any lift of γ to $\mathbb{Z}[G]$.

Proof. This is a standard exercise. □

Lemma 5.6. *Define $\psi : \mathbb{Z}[G] \rightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[H]$ by $\psi(\rho) = \sum_{h \in H} h\rho \otimes h^{-1}$. Then:*

- (i) ψ is an injective $\mathbb{Z}[G]$ -module homomorphism (with G acting on the left on $\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[H]$),
- (ii) $\psi(h\rho) = \psi(\rho)h$ for every $h \in H$,
- (iii) $\psi(J_H^t) \subset \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathcal{A}_H^t$ for every $t \geq 0$,
- (iv) for every $t \geq 0$ there is a commutative diagram

$$\begin{array}{ccc}
 J_H^t/J_H^{t+1} & & \\
 \cong \uparrow & \searrow \psi & \\
 \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \mathcal{A}_H^t/\mathcal{A}_H^{t+1} & \xrightarrow{\iota_{L/K} \otimes 1} & \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathcal{A}_H^t/\mathcal{A}_H^{t+1}
 \end{array}$$

where the vertical map is the isomorphism of Lemma 5.5(ii).

Proof. The first two assertions are clear, and (iii) follows from (ii).

To check the commutativity of the diagram in (iv), take $\gamma \in \Gamma$ and $\alpha \in \mathcal{A}_H^t$. Using (ii), the image of $\gamma \otimes \alpha$ in $\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathcal{A}_H^t/\mathcal{A}_H^{t+1}$ by the upper path is

$$\psi(\alpha\bar{\gamma}) = \sum_{h \in H} h\alpha\bar{\gamma} \otimes h^{-1} = \sum_{h \in H} \bar{\gamma}h \otimes \alpha h^{-1},$$

where $\bar{\gamma}$ is any lift of γ to G . The image of $\gamma \otimes \alpha$ by the lower path is $\sum_{h \in H} \bar{\gamma}h \otimes \alpha$. Since $\alpha(h^{-1} - 1) \in \mathcal{A}_H^{t+1}$ for every h , these are equal in $\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathcal{A}_H^t/\mathcal{A}_H^{t+1}$. This shows that the diagram in (iv) commutes, and the injectivity of the map induced by ψ now follows from the injectivity of $\iota_{L/K}$. □

Proof of Proposition 5.4. Let ψ be as in Lemma 5.6. If $r = 0$, then $\epsilon_L = \theta_{L/k}(0)$, so $\text{Tw}_{L/k}(\epsilon_L) = \psi(\theta_{L/k}(0))$. Thus by Lemma 5.6(iii,iv), Conjecture 5.2 is equivalent in this case to the assertions

- (i) $\theta_{L/k}(0) \in J_H^s$,
- (ii) $\theta_{L/k}(0) \equiv \mathcal{R}_{L/K}^{\text{Art}}(\epsilon_K) \pmod{J_H^{s+1}}$,

where we view $\mathcal{R}_{L/K}^{\text{Art}}(\epsilon_K) \in J_H^{s+1}$ via the isomorphism of Lemma 5.5(ii). This is Gross' conjecture [13, Conjecture $\tilde{A}_{\mathbb{Z}}(L/K/k, S, T, s)$]. □

Proposition 5.7. *If $s = 0$ (i.e., if $S'' = S'$), then Conjecture 5.2 is true.*

Proof. Conjecture 5.2(i) is vacuous when $r = 0$, since by definition

$$\mathrm{Tw}_{L/K}(\epsilon_L) \in (\wedge^{r,0}U_L) \otimes_{\Gamma} \wedge^r W_{S'}^* \otimes_{\mathbb{Z}} \mathbb{Z}[H] = (\wedge^{r,0}U_L) \otimes_{\Gamma} \wedge^r W_{S'}^* \otimes_{\mathbb{Z}} \mathcal{A}_H^0.$$

Let $\mathbf{N}_H = \sum_{h \in H} h$. In $(\wedge^{r,0}U_L) \otimes_{\Gamma} \wedge^r W_{S'}^* \otimes_{\mathbb{Z}} \mathbb{Z}[H]/\mathcal{A}_H$ we have

$$(5.8) \quad \mathrm{Tw}_{L/K}(\epsilon_L) = \sum_{h \in H} \epsilon_L^h \otimes h \equiv \sum_{h \in H} \epsilon_L^h \otimes 1 = (\mathbf{N}_H \epsilon_L) \otimes 1.$$

If $r = 0$, then since the image of the Stickelberger element $\theta_{L/k}(0)$ under the restriction map $\mathbb{Z}[G] \rightarrow \mathbb{Z}[\Gamma]$ is $\theta_{K/k}(0)$, we have

$$\mathbf{N}_H \epsilon_L = \mathbf{N}_H \theta_{L/k}(0) = \iota_{L/K} \theta_{K/k}(0) = \mathbf{j}_{L/K}(\epsilon_K).$$

By (5.8), this proves Conjecture 5.2(ii) when $r = 0$.

Suppose now that $r > 0$. Fix generators

$$\mathbf{w}_L = w_1 \wedge \cdots \wedge w_r \text{ of } \wedge^r W_{L,S'}, \quad \mathbf{w}_L^* = w_1^* \wedge \cdots \wedge w_r^* \text{ of } \wedge^r W_{L,S'}^*.$$

Let \mathbf{w}_K and \mathbf{w}_K^* be the corresponding generators of $\wedge^r W_{K,S'}$ and $\wedge^r W_{K,S'}^*$ obtained by restricting the w_i to K (note that since $s = 0$, we have $S' = S''$). Choose $\mathbf{u}_L \in \wedge^{r,0}U_L$ and $\mathbf{u}_K \in \wedge^{r,0}U_K$ such that $\epsilon_L = \mathbf{u}_L \otimes \mathbf{w}_L^*$ and $\epsilon_K = \mathbf{u}_K \otimes \mathbf{w}_K^*$. Then [14, Proposition 6.1] shows that $(\mathbf{N}_H)^r \mathbf{u}_L = (\wedge^r j)(\mathbf{u}_K)$, and so we also have

$$(5.9) \quad \begin{aligned} \mathbf{N}_H \epsilon_L &= [L : K]^{1-r} (\mathbf{N}_H)^r \epsilon_L = ([L : K]^{1-r} (\mathbf{N}_H)^r \mathbf{u}_L) \otimes \mathbf{w}_L^* \\ &= ([L : K]^{1-r} (\wedge^r j)(\mathbf{u}_K)) \otimes \mathbf{w}_L^* = \mathbf{j}_{L/K}(\epsilon_K). \end{aligned}$$

Since $s = 0$, the map (4.4) is just the map $\wedge^{r,0}U_K \rightarrow (\wedge^{r,0}U_K) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/\mathcal{A}_H$ that sends \mathbf{u} to $\mathbf{u} \otimes 1$, so $\mathcal{R}_{L/K}^{\mathrm{Art}}$ is the map

$$(\wedge^{r,0}U_K) \otimes_{\Gamma} \wedge^r W_{K,S'}^* \longrightarrow (\wedge^{r,0}U_K) \otimes_{\Gamma} \wedge^r W_{K,S'}^* \otimes_{\mathbb{Z}} \mathbb{Z}[H]/\mathcal{A}_H$$

that sends $\mathbf{u} \otimes \mathbf{w}$ to $\mathbf{u} \otimes \mathbf{w} \otimes 1$. Hence by (5.8) and (5.9) we have

$$\mathrm{Tw}_{L/K}(\epsilon_L) \equiv \mathbf{j}_{L/K}(\epsilon_K) \otimes 1 = (\mathbf{j}_{L/K} \otimes 1)(\mathcal{R}_{L/K}^{\mathrm{Art}}(\epsilon_K))$$

in $(\wedge^{r,0}U_K) \otimes_{\Gamma} \wedge^r W_{K,S'}^* \otimes_{\mathbb{Z}} \mathbb{Z}[H]/\mathcal{A}_H$, which is Conjecture 5.2(ii). \square

Proposition 5.10. *If $L = K$, then Conjecture 5.2 is true.*

Proof. If $S'' = S'$, then this follows from Proposition 5.7. If $S'' \neq S'$ then for every character χ of Γ , we have $r(\chi, S) \geq |S''| > |S'| = r$, so $\rho_{L,r} = 1$ in Definition 3.4 and by definition $\epsilon_L = \epsilon_{L,S,T,S'} = 0$. Further, we have $\mathcal{A}_H = 0$ in this case, so $\mathcal{R}_{K/K}^{\mathrm{Art}} = 0$ and Conjecture 5.2 holds. \square

6. Order of vanishing

Fix a number field k , and a set S' of archimedean places of k . Let $r := |S'|$. Let T be a finite set of primes of k , containing at least one prime not dividing 2, and containing primes of at least two different residue characteristics if S' contains no real places. (This ensures that an extension of k in which all places in S' split completely has no roots of unity congruent to one modulo all primes in T .)

For example (perhaps the most interesting example), k could be a totally real field and S' the set of all archimedean places, in which case $r = [k : \mathbb{Q}]$.

Fix a finite abelian extension K of k such that all places in S' split completely in K/k , and all places in T are unramified in K/k . Fix a finite set S of places of K disjoint from T , containing all archimedean places, all primes ramifying in K/k , and at least one place not in S' . Let \mathcal{P} be the set of all primes of k not in $S \cup T$ that split completely in K/k , and let \mathcal{N} be the set of all squarefree products of primes in \mathcal{P} .

For every $\mathfrak{q} \in \mathcal{P}$ suppose that $K(\mathfrak{q})$ is a finite abelian extension of k containing K , such that $K(\mathfrak{q})/K$ is totally tamely ramified above \mathfrak{q} and unramified everywhere else, and all places above S' split completely in $K(\mathfrak{q})/K$. (For example, if K contains the Hilbert class field of k then we could take $K(\mathfrak{q})$ to be the compositum of K with the ray class field of k modulo \mathfrak{q} .) If $\mathfrak{n} \in \mathcal{N}$ define $K(\mathfrak{n})$ to be the compositum of the fields $K(\mathfrak{q})$ for \mathfrak{q} dividing \mathfrak{n} . Ramification considerations show that all the $K(\mathfrak{q})$ are linearly disjoint over K , so if we define $H(\mathfrak{n}) := \text{Gal}(K(\mathfrak{n})/K)$ then

$$H(\mathfrak{n}) = \prod_{\mathfrak{q}|\mathfrak{n}} H(\mathfrak{q})$$

and if $\mathfrak{m} \mid \mathfrak{n}$ we can view $H(\mathfrak{m})$ both as a quotient and a subgroup of $H(\mathfrak{n})$. Let

$$\pi_{\mathfrak{m}} : H(\mathfrak{n}) \twoheadrightarrow H(\mathfrak{m}) \hookrightarrow H(\mathfrak{n}),$$

denote the projection map.

Let $S(\mathfrak{n}) := S \cup \{\mathfrak{q} : \mathfrak{q} \mid \mathfrak{n}\}$ and $S'(\mathfrak{n}) := S' \cup \{\mathfrak{q} : \mathfrak{q} \mid \mathfrak{n}\}$. Assume for the rest of this section that the generalized Stark conjecture $\text{St}(K(\mathfrak{n})/k, S(\mathfrak{n}), T, S')$ holds for every $\mathfrak{n} \in \mathcal{N}$, with an element

$$\epsilon_{\mathfrak{n}} := \epsilon_{K(\mathfrak{n}), S(\mathfrak{n}), T, S'} \in \wedge^{r,0} U_{K(\mathfrak{n}), S(\mathfrak{n})} \otimes \wedge^r W_{K(\mathfrak{n}), S'}^*$$

Lemma 6.1. *If $\mathfrak{d} \mid \mathfrak{n}$ then*

$$\sum_{\gamma \in H(\mathfrak{n}/\mathfrak{d})} \gamma \epsilon_{\mathfrak{n}} = \left(\prod_{\mathfrak{q}|\mathfrak{n}/\mathfrak{d}} (1 - \text{Fr}_{\mathfrak{q}}^{-1}) \right) \mathbf{j}_{K(\mathfrak{n})/K(\mathfrak{d})}(\epsilon_{\mathfrak{d}}).$$

Proof. This follows from [14, Proposition 6.1] and the definition of $\mathbf{j}_{K(\mathfrak{n})/K(\mathfrak{d})}$ from Lemma 4.9. □

Let $\nu(\mathfrak{n})$ denote the number of prime factors of \mathfrak{n} .

Lemma 6.2. *We have*

$$\begin{aligned} \sum_{\gamma \in H(\mathfrak{n})} \gamma \epsilon_{\mathfrak{n}} \otimes \prod_{\mathfrak{q}|\mathfrak{n}} (\pi_{\mathfrak{q}}(\gamma) - 1) \\ = \sum_{\mathfrak{d}|\mathfrak{n}} \sum_{\gamma \in H(\mathfrak{d})} \gamma \mathbf{j}_{K(\mathfrak{n})/K(\mathfrak{d})}(\epsilon_{\mathfrak{d}}) \otimes \gamma \prod_{\mathfrak{q}|\mathfrak{n}/\mathfrak{d}} (\pi_{\mathfrak{d}}(\text{Fr}_{\mathfrak{q}}) - 1) \end{aligned}$$

in $\wedge^{r,0} U_{K(\mathfrak{n}),S(\mathfrak{n})} \otimes \wedge^r W_{K(\mathfrak{n}),S'}^* \otimes \mathbb{Z}[H(\mathfrak{n})]$.

Proof. Expanding gives

$$\sum_{\gamma \in H(\mathfrak{n})} \gamma \epsilon_{\mathfrak{n}} \otimes \prod_{\mathfrak{q}|\mathfrak{n}} (\pi_{\mathfrak{q}}(\gamma) - 1) = \sum_{\gamma \in H(\mathfrak{n})} \sum_{\mathfrak{d}|\mathfrak{n}} (-1)^{\nu(\mathfrak{n}/\mathfrak{d})} \gamma \epsilon_{\mathfrak{n}} \otimes \pi_{\mathfrak{d}}(\gamma).$$

For every \mathfrak{d} dividing \mathfrak{n} , using Lemma 6.1 we have

$$\begin{aligned} \sum_{\gamma \in H(\mathfrak{n})} \gamma \epsilon_{\mathfrak{n}} \otimes \pi_{\mathfrak{d}}(\gamma) &= \sum_{\gamma \in H(\mathfrak{d})} (\gamma \sum_{h \in H(\mathfrak{n}/\mathfrak{d})} h \epsilon_{\mathfrak{n}}) \otimes \gamma \\ &= \sum_{\gamma \in H(\mathfrak{d})} \gamma \prod_{\mathfrak{q}|\mathfrak{n}/\mathfrak{d}} (1 - \text{Fr}_{\mathfrak{q}}^{-1}) \mathbf{j}_{K(\mathfrak{n})/K(\mathfrak{d})}(\epsilon_{\mathfrak{d}}) \otimes \gamma \\ &= \sum_{\gamma \in H(\mathfrak{d})} \gamma \mathbf{j}_{K(\mathfrak{n})/K(\mathfrak{d})}(\epsilon_{\mathfrak{d}}) \otimes \gamma \prod_{\mathfrak{q}|\mathfrak{n}/\mathfrak{d}} (1 - \pi_{\mathfrak{d}}(\text{Fr}_{\mathfrak{q}})). \end{aligned}$$

Combining these identities proves the lemma. □

Theorem 6.3. *Suppose that the Stark conjecture $\text{St}(K(\mathfrak{n})/k, S(\mathfrak{n}), T, S')$ holds for every $\mathfrak{n} \in \mathcal{N}$. Then for every $\mathfrak{n} \in \mathcal{N}$, we have*

$$\text{Tw}_{K(\mathfrak{n})/K}(\epsilon_{\mathfrak{n}}) \in \wedge^{r,0} U_{K(\mathfrak{n}),S(\mathfrak{n})} \otimes \wedge^r W_{K(\mathfrak{n}),S'}^* \otimes \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})}.$$

In other words, Conjecture 5.2(i) holds for $(K(\mathfrak{n})/K/k, S(\mathfrak{n}), T, S', S'(\mathfrak{n}))$.

Proof. The proof is by induction on $\nu(\mathfrak{n})$, and is essentially the same as the proof of [3, Lemma 8.1]. In the equality of Lemma 6.2, every term except possibly $\text{Tw}_{K(\mathfrak{n})/K}(\epsilon_{\mathfrak{n}})$ (the summand on the right with $\mathfrak{d} = \mathfrak{n}$) lies in $\wedge^{r,0} U_{K(\mathfrak{n}),S(\mathfrak{n})} \otimes \wedge^r W_{K(\mathfrak{n}),S'}^* \otimes \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})}$ by our induction hypothesis. Therefore $\text{Tw}_{K(\mathfrak{n})/K}(\epsilon_{\mathfrak{n}})$ does as well. □

7. The case $K = k$

In this section we consider the case $K = k$. Let $S', S, T, \mathcal{N}, k(\mathfrak{q}), k(\mathfrak{n}), H(\mathfrak{n}), S(\mathfrak{n}), S'(\mathfrak{n})$ be as in §6, and recall that $r := |S'|$. We will show under mild hypotheses that Conjecture 5.2 holds in this case (with both sides of Conjecture 5.2(ii) equal to zero). This is needed for the proof of Theorem 10.7 below, because our general techniques only work for nontrivial characters of K/k .

Lemma 7.1. *Suppose that S' does not contain all archimedean places of k . Then $\epsilon_{k(\mathfrak{n}),S(\mathfrak{n}),T,S'} = 0$ for every $\mathfrak{n} \neq 1$.*

Proof. Let w be an archimedean place not in S' . By definition $k(\mathfrak{n})/k$ is unramified outside of \mathfrak{n} , so w splits completely in $k(\mathfrak{n})/k$. Hence if $\mathfrak{n} \neq 1$ then $\epsilon_{k(\mathfrak{n}),S(\mathfrak{n}),T,S'} = 0$ by Lemma 3.6. \square

Theorem 7.2. *Suppose $\mathfrak{n} \in \mathcal{N}$ and Conjecture $\text{St}(k(\mathfrak{n})/k)$ holds. If S' does not contain all archimedean places of k , or if $|S - S'| \geq 2$, then Conjecture 5.2 holds for $(k(\mathfrak{n})/k, S(\mathfrak{n}), T, S', S'(\mathfrak{n}))$.*

Proof. Conjecture 5.2(i) holds by Theorem 6.3, and Conjecture 5.2(ii) holds when $\mathfrak{n} = 1$ by Proposition 5.10. To prove the theorem we will show that for every $\mathfrak{n} \neq 1$,

$$(7.3) \quad \text{Tw}_{k(\mathfrak{n})/k}(\epsilon_{k(\mathfrak{n}),S(\mathfrak{n}),T,S'}) \in \wedge^{r,0} U_{k(\mathfrak{n}),S_0(\mathfrak{n})} \otimes \wedge^r W_{k(\mathfrak{n}),S'}^* \otimes \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})+1},$$

$$(7.4) \quad \mathcal{R}_{k(\mathfrak{n})/k}^{\text{Art}}(\epsilon_{k,S(\mathfrak{n}),T,S'(\mathfrak{n})}) \in \wedge^{r,0} U_{k(\mathfrak{n}),S_0(\mathfrak{n})} \otimes \wedge^r W_{k(\mathfrak{n}),S'}^* \otimes \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})+1}.$$

Suppose first that $|S - S'| \geq 2$. Then $\epsilon_{k,S(\mathfrak{n}),T,S'(\mathfrak{n})} = 0$ by Lemma 3.6, so (7.4) holds. If k has an archimedean place not in S' , then $\epsilon_{k(\mathfrak{n}),S(\mathfrak{n}),T,S'(\mathfrak{n})} = 0$ for $\mathfrak{n} \neq 1$ by Lemma 7.1, so (7.3) holds. If not, then S contains two nonarchimedean primes; call one of them v and let $S_0 := S - \{v\}$. Since v does not divide \mathfrak{n} and S_0 is still strictly larger than S' , all the hypotheses of Conjecture $\text{St}(k(\mathfrak{n})/k, S_0(\mathfrak{n}), T, S')$ are satisfied, so by Theorem 6.3 we have

$$(7.5) \quad \text{Tw}_{k(\mathfrak{n})/k}(\epsilon_{k(\mathfrak{n}),S_0(\mathfrak{n}),T,S'}) \in \wedge^{r,0} U_{k(\mathfrak{n}),S_0(\mathfrak{n})} \otimes \wedge^r W_{k(\mathfrak{n}),S'}^* \otimes \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})}.$$

It follows directly from the defining properties (see for example [14, Proposition 3.6]) that $\epsilon_{k(\mathfrak{n}),S(\mathfrak{n}),T,S'} = (1 - \text{Fr}_v^{-1})\epsilon_{k(\mathfrak{n}),S_0(\mathfrak{n}),T,S'}$, so using (7.5)

$$\begin{aligned} \text{Tw}_{k(\mathfrak{n})/k}(\epsilon_{k(\mathfrak{n}),S(\mathfrak{n}),T,S'}) &= \text{Tw}_{k(\mathfrak{n})/k}(\epsilon_{k(\mathfrak{n}),S_0(\mathfrak{n}),T,S'})(1 - \text{Fr}_v^{-1}) \\ &\in \wedge^{r,0} U_{k(\mathfrak{n}),S_0(\mathfrak{n})} \otimes \wedge^r W_{k(\mathfrak{n}),S'}^* \otimes \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})+1}. \end{aligned}$$

This is (7.3).

Now suppose that S' does not contain all archimedean places of k . By Lemma 7.1 we have $\epsilon_{k(\mathfrak{n}),S(\mathfrak{n}),T,S'} = 0$ for every $\mathfrak{n} \neq 1$, so (7.3) holds. If S contains a nonarchimedean place then $|S - S'| \geq 2$, and we are in the case treated above. So we may assume that S is the set of all archimedean places. Let $S' = \{v_1, \dots, v_r\}$ and $\mathfrak{n} = \prod_{i=1}^s \mathfrak{q}_i$. For $1 \leq i \leq s$ define $\eta_i : U_{k,S(\mathfrak{n})} \rightarrow \mathcal{A}_{H(\mathfrak{n})}/\mathcal{A}_{H(\mathfrak{n})}^2$ to be the map given by the local Artin symbol

$$\eta_i(u) := [u, k(\mathfrak{n})_{\mathfrak{q}_i}/k_{\mathfrak{q}_i}] - 1$$

where $k(\mathfrak{n})_{\mathfrak{q}_i}$ is the completion of $k(\mathfrak{n})$ at a prime above \mathfrak{q}_i . Fix an expression

$$\epsilon_{k,S(\mathfrak{n}),T,S'(\mathfrak{n})} = (u_1 \wedge \dots \wedge u_{r+s}) \otimes (v_1^* \wedge \dots \wedge v_r^* \wedge \mathfrak{q}_1^* \wedge \dots \wedge \mathfrak{q}_s^*)$$

with $u_i \in U_{k,S(\mathfrak{n})}$ (we have $\wedge^{r+s,0}U_{k,S(\mathfrak{n})} = \wedge^{r+s}U_{k,S(\mathfrak{n})}$ since $\mathbb{Z}[\Gamma] = \mathbb{Z}$). Then concretely (ignoring the sign, which will not be important)

$$(7.6) \quad \mathcal{R}_{k(\mathfrak{n})/k}^{\text{Art}}(\epsilon_{k,S(\mathfrak{n}),T,S'(\mathfrak{n})}) = \pm(\eta_1 \wedge \cdots \wedge \eta_s)(u_1 \wedge \cdots \wedge u_{r+s}) \otimes (v_1^* \wedge \cdots \wedge v_r^*).$$

In $\mathcal{A}_{H(\mathfrak{n})}/\mathcal{A}_{H(\mathfrak{n})}^2$, using the reciprocity law of global class field theory, we have for every $u \in U_{k,S(\mathfrak{n})}$

$$\sum_{i=1}^s \eta_i(u) = \left(\prod_{\mathfrak{q}|\mathfrak{n}} [u, k(\mathfrak{n})_{\mathfrak{q}}/k_{\mathfrak{q}}] \right) - 1 = \prod_{w|\mathfrak{n}} [u, k(\mathfrak{n})_w/k_w]^{-1} - 1.$$

If w is nonarchimedean and does not divide \mathfrak{n} , then u is a unit at w and w is unramified in $k(\mathfrak{n})/k$, so $[u, k(\mathfrak{n})_w/k_w] = 1$. If w is archimedean, then w splits completely in $k(\mathfrak{n})/k$, so again $[u, k(\mathfrak{n})_w/k_w] = 1$. Therefore $\sum_{i=1}^s \eta_i : U_{k,S(\mathfrak{n})} \rightarrow \mathcal{A}_{H(\mathfrak{n})}/\mathcal{A}_{H(\mathfrak{n})}^2$ is the zero map, and we conclude using (7.6) that

$$\begin{aligned} \mathcal{R}_{k(\mathfrak{n})/k}^{\text{Art}}(\epsilon_{k,S(\mathfrak{n}),T,S'(\mathfrak{n})}) &= \pm(\eta_1 \wedge \cdots \wedge \eta_s)(u_1 \wedge \cdots \wedge u_{r+s}) \otimes (v_1^* \wedge \cdots \wedge v_r^*) \\ &= \pm(\eta_1 \wedge \cdots \wedge \eta_{s-1} \wedge (\sum_i \eta_i))(u_1 \wedge \cdots \wedge u_{r+s}) \otimes (v_1^* \wedge \cdots \wedge v_r^*) = 0. \end{aligned}$$

Thus (7.4) holds in this case as well, and the theorem follows. □

8. Connection with Euler systems

Let $K/k, S', S, T, \mathcal{P}, \mathcal{N}, K(\mathfrak{q}), K(\mathfrak{n}), S(\mathfrak{n}), S'(\mathfrak{n})$ be as in §6, and let $\Gamma = \text{Gal}(K/k)$. Recall that $r := |S'|$.

We assume further (by shrinking $K(\mathfrak{q})$ if necessary) that $[K(\mathfrak{q}) : K]$ is prime to $[K : k]$ for every $\mathfrak{q} \in \mathcal{P}$. It follows that for every \mathfrak{q} there is a unique extension $k(\mathfrak{q})/k$, totally ramified at \mathfrak{q} and unramified elsewhere, such that $K(\mathfrak{q}) = Kk(\mathfrak{q})$. Then if $k(\mathfrak{n})$ denotes the compositum of the $k(\mathfrak{q})$ for \mathfrak{q} dividing \mathfrak{n} , we have $K(\mathfrak{n}) = Kk(\mathfrak{n})$ for every $\mathfrak{n} \in \mathcal{N}$, and

$$(8.1) \quad \text{Gal}(K(\mathfrak{n})/k) \cong \Gamma \times H(\mathfrak{n}).$$

Since all archimedean places split completely in $k(\mathfrak{q})/k$ for every \mathfrak{q} , every $v \in S'$ splits completely in $K(\mathfrak{n})/k$ for every \mathfrak{n} . Hence all hypotheses of Conjecture $\text{St}(K(\mathfrak{n})/k, S(\mathfrak{n}), T, S')$ are satisfied.

Fix an ordering v_1, \dots, v_r of the places in S' , and for each i choose a place w_i of the algebraic closure \bar{k} above v_i . Then for every \mathfrak{n} , the element

$$\mathbf{w}_{\mathfrak{n}}^* := (w_1|_{K(\mathfrak{n})})^* \wedge \cdots \wedge (w_r|_{K(\mathfrak{n})})^*$$

is a generator of the free, rank-one $\mathbb{Z}[\text{Gal}(K(\mathfrak{n})/k)]$ -module $\wedge^r W_{K(\mathfrak{n}),S'}^*$. When $\mathfrak{n} = 1$ we will write \mathbf{w}_K^* instead of \mathbf{w}_1^* .

Definition 8.2. As in §6, for every $\mathfrak{n} \in \mathcal{N}$ we define

$$\epsilon_{\mathfrak{n}} := \epsilon_{K(\mathfrak{n}),S(\mathfrak{n}),T,S'} \in (\wedge^{r,0}U_{K(\mathfrak{n}),S(\mathfrak{n})}) \otimes \wedge^r W_{K(\mathfrak{n}),S'}^*$$

to be the element predicted by Conjecture $\text{St}(K(\mathfrak{n})/k, S(\mathfrak{n}), T, S')$, and we define

$$\xi_{\mathfrak{n}} \in \wedge^{r,0} U_{K(\mathfrak{n}),S(\mathfrak{n})} \subset \wedge^r U_{K(\mathfrak{n}),S(\mathfrak{n})} \otimes \mathbb{Q}$$

to be the unique element satisfying

$$\xi_{\mathfrak{n}} \otimes \mathbf{w}_{\mathfrak{n}}^* = \epsilon_{\mathfrak{n}}.$$

Proposition 8.3. *If $\mathfrak{m}, \mathfrak{n} \in \mathcal{N}$, and $\mathfrak{m} \mid \mathfrak{n}$, then*

$$\mathbf{N}_{K(\mathfrak{n})/K(\mathfrak{m})}^r \xi_{\mathfrak{n}} = \prod_{\mathfrak{q} \mid (\mathfrak{n}/\mathfrak{m})} (1 - \text{Fr}_{\mathfrak{q}}^{-1}) \xi_{\mathfrak{m}}.$$

Proof. This is [14, Proposition 6.1]. □

By (8.1), for every $\mathfrak{n} \in \mathcal{N}$ we can view any $\text{Gal}(K(\mathfrak{n})/k)$ -module as a Γ -module.

Fix a rational prime p , not lying below any prime in T , and not dividing $[K : k]$. Fix also a character $\chi : \Gamma \rightarrow \overline{\mathbb{Q}}_p^\times$. Let $\mathcal{O} := \mathbb{Z}_p[\chi]$, the extension of \mathbb{Z}_p generated by the values of χ . Since $p \nmid [K : k]$, the order of χ is prime to p so \mathcal{O} is unramified over \mathbb{Z}_p . If M is a $\mathbb{Z}[\Gamma]$ -module, we let M^χ be the submodule of $M \otimes_{\mathbb{Z}} \mathcal{O}$ on which Γ acts via χ . If $m \in M$, then

$$(8.4) \quad m^\chi := \frac{1}{[K : k]} \sum_{\gamma \in \Gamma} m^\gamma \otimes \chi^{-1}(\gamma) \in M^\chi$$

is the projection of m into M^χ .

Let $\mathcal{M}_\chi := \mathbb{Z}_p(1) \otimes \chi^{-1}$ denote a free \mathcal{O} -module of rank one on which G_k acts via χ^{-1} times the cyclotomic character.

Proposition 8.5. *For every $\mathfrak{n} \in \mathcal{N}$, Kummer theory gives Galois-equivariant isomorphisms*

$$(K(\mathfrak{n})^\times)^\chi \cong H^1(k(\mathfrak{n}), \mathcal{M}_\chi),$$

and if \mathfrak{q} is a prime of k

$$((K \otimes_k k_{\mathfrak{q}})^\times)^\chi \cong H^1(k_{\mathfrak{q}}, \mathcal{M}_\chi).$$

Proof. This is a standard calculation; see for example [8, §6.1] or [15, §1.6.C]. □

Theorem 8.6. *Suppose that $r = 1$, and Conjecture $\text{St}(K(\mathfrak{n})/k, S(\mathfrak{n}), T, S')$ holds for every $\mathfrak{n} \in \mathcal{N}$. Let $c_{\mathfrak{n}} \in H^1(k(\mathfrak{n}), \mathcal{M}_\chi)$ denote the image of $\xi_{\mathfrak{n}}^\chi$ under the Kummer map of Proposition 8.5. Then the collection*

$$\{c_{\mathfrak{n}} : \mathfrak{n} \in \mathcal{N}\}$$

is an Euler system for the G_k -representation \mathcal{M}_χ in the sense of [8, Definition 3.2.2] or [15, §9.1].

Proof. It follows from Proposition 8.3 and (8.4) that if $\mathfrak{m}, \mathfrak{n} \in \mathcal{N}$ and $\mathfrak{m} \mid \mathfrak{n}$, then

$$\mathbf{N}_{K(\mathfrak{n})/K(\mathfrak{m})}\xi_{\mathfrak{n}}^{\chi} = \prod_{\mathfrak{q} \mid (\mathfrak{n}/\mathfrak{m})} (1 - \text{Fr}_{\mathfrak{q}}^{-1})\xi_{\mathfrak{m}}^{\chi}.$$

Translated to the elements $c_{\mathfrak{n}}$ and $c_{\mathfrak{m}}$, this is the defining property of an Euler system for \mathcal{M}_{χ} . (Note that by the definition of \mathcal{N} in §6, we have $\chi(\mathfrak{q}) = 1$ if $\mathfrak{q} \mid \mathfrak{n}$.) \square

Remark 8.7. For general $r \geq 1$, the collection $\{c_{\mathfrak{n}} : \mathfrak{n} \in \mathcal{N}\}$ is not necessarily an Euler system in the sense of [11, Definition 1.2.2], because the elements $c_{\mathfrak{n}}$ lie in $\wedge^{r,0}H^1(k(\mathfrak{n}), \mathcal{M}_{\chi})$ rather than $\wedge^r H^1(k(\mathfrak{n}), \mathcal{M}_{\chi})$. This suggests that one might want to relax the definition of Euler system to allow elements to lie in the larger lattice.

9. Connection with Stark systems

Let $K(\mathfrak{n})/K/k$, Γ , S' , r , S , T , \mathcal{P} , \mathcal{N} , $S(\mathfrak{n})$, $S'(\mathfrak{n})$, χ and \mathcal{M}_{χ} be as in §6 and §8. For $\mathfrak{n} \in \mathcal{N}$ let $\nu(\mathfrak{n})$ denote the number of primes dividing \mathfrak{n} . We continue to suppose that $[K(\mathfrak{q}) : K]$ is prime to $[K : k]$ for every $\mathfrak{q} \in \mathcal{P}$, and we now suppose in addition that

$$(9.1) \quad p \nmid [K : k] \prod_{\lambda \in T_K} (\mathbf{N}\lambda - 1)$$

Let A denote the ring of integers of K , and for every $\mathfrak{n} \in \mathcal{N}$ let $A_{S(\mathfrak{n})}$ denote the $S(\mathfrak{n})$ -integers of K

$$A_{S(\mathfrak{n})} := \{x \in K : \text{ord}_{\lambda}(x) \geq 0 \text{ for every } \lambda \notin S(\mathfrak{n})_K\}.$$

Then $U_{K,S(\mathfrak{n})} = \{u \in A_{S(\mathfrak{n})}^{\times} : u \equiv 1 \pmod{\lambda} \text{ for every } \lambda \in T_K\}$.

Lemma 9.2. *For every $\mathfrak{n} \in \mathcal{N}$ we have $p \nmid [A_{S(\mathfrak{n})}^{\times} : U_{K,S(\mathfrak{n})}]$*

Proof. Reduction gives an injection $A_{S(\mathfrak{n})}^{\times}/U_{K,S(\mathfrak{n})} \hookrightarrow \bigoplus_{\lambda \in T_K} (A/\lambda)^{\times}$, so the lemma follows from our assumption (9.1). \square

Lemma 9.3. *For all $\mathfrak{n} \in \mathcal{N}$ we have $(\wedge^{r+\nu(\mathfrak{n}),0}U_{K,S(\mathfrak{n})})^{\chi} = \wedge^{r+\nu(\mathfrak{n})}U_{K,S(\mathfrak{n})}^{\chi}$.*

Proof. By our choice of T , the group $U_{K,S(\mathfrak{n})}$ is torsion-free. Since p does not divide $[K : k]$, we have $[K : k] \in \mathcal{O}^{\times}$, so $U_{K,S(\mathfrak{n})} \otimes \mathcal{O}$ is a projective $\mathcal{O}[\Gamma]$ -module. It now follows from Lemma A.4 that

$$\wedge^{r+\nu(\mathfrak{n}),0}U_{K,S(\mathfrak{n})} \otimes \mathcal{O} = \wedge^{r+\nu(\mathfrak{n})}U_{K,S(\mathfrak{n})} \otimes \mathcal{O}.$$

Taking χ -components proves the lemma. \square

Define

$$\mathcal{N}_p = \{\mathfrak{n} \in \mathcal{N} : \mathfrak{n} \text{ is prime to } p\}.$$

For $\mathfrak{n} \in \mathcal{N}_p$ recall that $H(\mathfrak{n}) := \text{Gal}(K(\mathfrak{n})/K)$, and $\mathcal{A}_{H(\mathfrak{n})} \subset \mathcal{O}[H(\mathfrak{n})]$ is the augmentation ideal. Define an ideal $I_{\mathfrak{n}} \subset \mathcal{O}$ by

$$I_{\mathfrak{n}} := \sum_{\mathfrak{q}|\mathfrak{n}} ([k(\mathfrak{q}) : k]\mathcal{O})$$

(with the convention $I_1 = 0$). Let $W_{K,\mathfrak{n}}$ denote the free abelian group on the set of primes of K dividing \mathfrak{n} , so $W_{K,S'(\mathfrak{n})} = W_{K,S'} \oplus W_{K,\mathfrak{n}}$ and

$$(9.4) \quad \wedge^{r+\nu(\mathfrak{n})} W_{K,S'(\mathfrak{n})}^* = \wedge^{\nu(\mathfrak{n})} W_{K,\mathfrak{n}}^* \otimes \wedge^r W_{K,S'}^*.$$

Definition 9.5. For every $\mathfrak{n} \in \mathcal{N}_p$, define

$$Y_{\mathfrak{n}} := \wedge^{r+\nu(\mathfrak{n})} U_{K,S(\mathfrak{n})}^{\chi} \otimes \wedge^{\nu(\mathfrak{n})} (W_{K,S'(\mathfrak{n})}^*)^{\chi} \otimes (\mathcal{O}/I_{\mathfrak{n}}).$$

If $\mathfrak{m} \mid \mathfrak{n}$, we define a map

$$\Psi_{\mathfrak{n},\mathfrak{m}} : Y_{\mathfrak{n}} \longrightarrow Y_{\mathfrak{m}} \otimes (\mathcal{O}/I_{\mathfrak{n}})$$

as follows. Fix a prime factorization $\mathfrak{n}/\mathfrak{m} = \mathfrak{q}_1 \cdots \mathfrak{q}_t$ and for each i fix a prime \mathfrak{Q}_i of K above \mathfrak{q}_i . Define $\psi_i \in U_{K,S(\mathfrak{n})}^*$ by

$$\psi_i(u) := \sum_{\gamma \in \Gamma} \text{ord}_{\mathfrak{Q}_i}(u^\gamma) \gamma^{-1}.$$

By Definition A.1 we get a map

$$\psi_1 \wedge \cdots \wedge \psi_t : \wedge^{r+\nu(\mathfrak{n})} U_{K,S(\mathfrak{n})}^{\chi} \longrightarrow \wedge^{r+\nu(\mathfrak{m})} U_{K,S(\mathfrak{n})}^{\chi}$$

and by [14, Lemma 5.1] or [10, Proposition A.1] the image of this map is contained in $\wedge^{r+\nu(\mathfrak{m})} U_{K,S(\mathfrak{m})}^{\chi}$. Further, viewing $\mathfrak{Q}_1 \wedge \cdots \wedge \mathfrak{Q}_t$ as a generator of $\wedge^t W_{\mathfrak{n}/\mathfrak{m}}$ the map

$$(9.6) \quad (\psi_1 \wedge \cdots \wedge \psi_t) \otimes (\mathfrak{Q}_1 \wedge \cdots \wedge \mathfrak{Q}_t) : \wedge^{r+\nu(\mathfrak{n})} U_{K,S(\mathfrak{n})}^{\chi} \otimes \wedge^t (W_{\mathfrak{n}/\mathfrak{m}}^*)^{\chi} \longrightarrow \wedge^{r+\nu(\mathfrak{m})} U_{K,S(\mathfrak{m})}^{\chi}$$

is independent of the choice of the \mathfrak{Q}_i and the order of the \mathfrak{q}_i . Now we define $\Psi_{\mathfrak{n},\mathfrak{m}}$ to be the composition

$$\begin{aligned} Y_{\mathfrak{n}} &= \wedge^{r+\nu(\mathfrak{n})} U_{K,S(\mathfrak{n})}^{\chi} \otimes \wedge^{\nu(\mathfrak{n})} (W_{K,S'(\mathfrak{n})}^*)^{\chi} \otimes (\mathcal{O}/I_{\mathfrak{n}}) \\ &\xrightarrow{\sim} \wedge^{r+\nu(\mathfrak{n})} U_{K,S(\mathfrak{n})}^{\chi} \otimes \wedge^{\nu(\mathfrak{m})} (W_{K,S'(\mathfrak{m})}^*)^{\chi} \otimes \wedge^{\nu(\mathfrak{n}/\mathfrak{m})} (W_{\mathfrak{n}/\mathfrak{m}}^*)^{\chi} \otimes (\mathcal{O}/I_{\mathfrak{n}}) \\ &\longrightarrow \wedge^{r+\nu(\mathfrak{m})} U_{K,S(\mathfrak{m})}^{\chi} \otimes \wedge^{\nu(\mathfrak{m})} (W_{K,S'(\mathfrak{m})}^*)^{\chi} \otimes (\mathcal{O}/I_{\mathfrak{n}}) = Y_{\mathfrak{m}} \otimes (\mathcal{O}/I_{\mathfrak{n}}), \end{aligned}$$

where the last map is induced by (9.6). Note that $\Psi_{\mathfrak{n},\mathfrak{m}}$ is the map Φ of [14, §5].

Using Lemma 9.3 we can view $\epsilon_{\mathfrak{n}}^{\chi} \in Y_{\mathfrak{n}}$, where $\epsilon_{\mathfrak{n}}$ is the element of Definition 8.2 predicted by Conjecture $\text{St}(K(\mathfrak{n})/k, S(\mathfrak{n}), T, S')$. The following lemma allows us to apply the results of [10] to the family of $Y_{\mathfrak{n}}$.

Lemma 9.7. *The modules $Y_{\mathfrak{n}}$ and the maps $\Psi_{\mathfrak{n},\mathfrak{m}}$ defined above are the same as the $Y_{\mathfrak{n}}$ and $\Psi_{\mathfrak{n},\mathfrak{m}}$ of [10, Definition 7.1] for the Galois representation \mathcal{M}_{χ} .*

Proof. The proof is an exercise, using the natural Kummer theory isomorphisms $(K^{\times})^{\times} \cong H^1(k, \mathcal{M}_{\chi})$ and $((K \otimes k_v)^{\times})^{\times} \cong H^1(k_v, \mathcal{M}_{\chi})$ for places v of k (Proposition 8.5), along with Lemma 9.2. \square

Definition 9.8. As in [10, Definition 7.1] we say that a collection

$$\{\sigma_{\mathfrak{n}} \in Y_{\mathfrak{n}} : \mathfrak{n} \in \mathcal{N}_p\}$$

is a *Stark system of rank r* if

$$\Psi_{\mathfrak{n},\mathfrak{m}}(\sigma_{\mathfrak{n}}) = \sigma_{\mathfrak{m}} \otimes 1 \in Y_{\mathfrak{m}} \otimes (\mathcal{O}/I_{\mathfrak{n}}) \quad \text{whenever } \mathfrak{m} \mid \mathfrak{n} \in \mathcal{N}_p.$$

Let $\mathbf{SS}_r(\mathcal{M}_{\chi})$ denote the \mathcal{O} -module of Stark systems of rank r .

Suppose for the rest of this section that Conjecture $\text{St}(K/k, S(\mathfrak{n}), T, S'(\mathfrak{n}))$ holds for every $\mathfrak{n} \in \mathcal{N}$. Recall that \mathbf{w}_K^* is the generator of $\wedge^r W_{K,S'}^*$ fixed at the beginning of §8, and $\mathbf{1}$ denotes the trivial character of Γ .

Definition 9.9. For $\mathfrak{n} \in \mathcal{N}$ let $\delta_{\mathfrak{n}} \in (\wedge^{r+\nu(\mathfrak{n})} U_{K,S(\mathfrak{n})}) \otimes \wedge^{\nu(\mathfrak{n})} (W_{K,\mathfrak{n}}^*)$ be the unique element such that

$$\delta_{\mathfrak{n}} \otimes \mathbf{w}_K^* := \epsilon_{K,S(\mathfrak{n}),S'(\mathfrak{n})} \in (\wedge^{r+\nu(\mathfrak{n})} U_{K,S(\mathfrak{n})}) \otimes \wedge^{r+\nu(\mathfrak{n})} W_{K,S'(\mathfrak{n})}^*$$

is the element predicted by Conjecture $\text{St}(K/k, S(\mathfrak{n}), T, S'(\mathfrak{n}))$, using the identifications of Lemma 9.3 and (9.4). Then

$$\delta_{\mathfrak{n}}^{\chi} \otimes 1 \in \wedge^{r+\nu(\mathfrak{n})} U_{K,S(\mathfrak{n})}^{\chi} \otimes \wedge^{\nu(\mathfrak{n})} (W_{K,S'(\mathfrak{n})}^*)^{\chi} \otimes (\mathcal{O}/I_{\mathfrak{n}}) = Y_{\mathfrak{n}},$$

and we denote by δ^{χ} the collection $\{\delta_{\mathfrak{n}}^{\chi} \otimes 1 \in Y_{\mathfrak{n}} : \mathfrak{n} \in \mathcal{N}_p\}$.

Proposition 9.10. *We have $\delta^{\chi} \in \mathbf{SS}_r(\mathcal{M}_{\chi})$, i.e., δ^{χ} is a Stark system of rank r .*

Proof. If $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{m} \mid \mathfrak{n}$, then $\Psi_{\mathfrak{n},\mathfrak{m}}(\delta_{\mathfrak{n}}^{\chi} \otimes 1) = \delta_{\mathfrak{m}}^{\chi} \otimes 1$ by [14, Proposition 5.2]. \square

Let $r(\chi, S)$ be as in Definition 3.4.

Lemma 9.11.

- (i) *If $r(\chi, S) > r$, then $\delta_{\mathfrak{n}}^{\chi} = 0$ for every $\mathfrak{n} \in \mathcal{N}$.*
- (ii) *If $r(\chi, S) = r$, then $\delta_{\mathfrak{n}}^{\chi}$ is a nonzero element of the free, rank-one \mathcal{O} -module $\wedge^{r+\nu(\mathfrak{n})} U_{K,S(\mathfrak{n})}^{\chi} \otimes \wedge^{\nu(\mathfrak{n})} (W_{K,\mathfrak{n}}^*)^{\chi}$.*

Proof. The $\mathbb{Z}[\Gamma]$ -module $W_{K,\mathfrak{n}}^*$ is free of rank $\nu(\mathfrak{n})$. By the basic properties of Conjecture $\text{St}(K/k, S(\mathfrak{n}), T, S'(\mathfrak{n}))$ we have

$$\delta_{\mathfrak{n}}^{\chi} \neq 0 \iff r(\chi, S(\mathfrak{n})) = r + \nu(\mathfrak{n}) \iff r(\chi, S) = r,$$

and if these equivalent conditions hold then $U_{K,S(\mathfrak{n})}^{\chi}$ is free of rank $r + \nu(\mathfrak{n})$ over \mathcal{O} . The lemma follows. \square

10. The case $r = 1$

Keep the setting and notation of the previous two sections. In this section we will prove (Theorem 10.7) a part of Conjecture 5.2(ii) when $r = 1$. The idea of the proof is as follows.

The Stark system δ^χ of §9 gives rise (via an explicit construction) to a Kolyvagin system for \mathcal{M}_χ . When $r = 1$, the Euler system of Stark elements of Theorem 8.6 also gives rise (via an explicit construction) to a Kolyvagin system for \mathcal{M}_χ . The \mathcal{O} -module of Kolyvagin systems for \mathcal{M}_χ is free of rank one, and the two Kolyvagin systems agree when $\mathfrak{n} = 1$ by construction. Hence the two Kolyvagin systems agree for every \mathfrak{n} , and unwinding the two explicit constructions shows that the agreement for \mathfrak{n} is equivalent to the “ (p, χ) -part” of Conjecture 5.2(ii) for $(K(\mathfrak{n})/K/k, S(\mathfrak{n}), T, S')$.

As in §6, if $\mathfrak{m} \mid \mathfrak{n}$ we can view $H(\mathfrak{m})$ as both a subgroup and a quotient of $H(\mathfrak{n})$, and $\pi_{\mathfrak{m}} : H(\mathfrak{n}) \rightarrow H(\mathfrak{m}) \hookrightarrow H(\mathfrak{n})$ is the projection map.

Definition 10.1. If $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{d} = \prod_{i=1}^t \mathfrak{q}_i$ divides \mathfrak{n} , let $M_{\mathfrak{n}, \mathfrak{d}} = (m_{ij})$ be the $t \times t$ matrix with entries in $\mathcal{A}_{H(\mathfrak{n})} / \mathcal{A}_{H(\mathfrak{n})}^2$

$$m_{ij} = \begin{cases} \pi_{\mathfrak{n}/\mathfrak{d}}(\text{Fr}_{\mathfrak{q}_i} - 1) & \text{if } i = j, \\ \pi_{\mathfrak{q}_j}(\text{Fr}_{\mathfrak{q}_i} - 1) & \text{if } i \neq j, \end{cases}$$

and define

$$\mathcal{D}_{\mathfrak{n}, \mathfrak{d}} := \det(M_{\mathfrak{n}, \mathfrak{d}}) \in \mathcal{A}_{H(\mathfrak{n})}^t / \mathcal{A}_{H(\mathfrak{n})}^{t+1}$$

(this is independent of the ordering of the prime factors of \mathfrak{d}). By convention we let $\mathcal{D}_{\mathfrak{n}, 1} = 1$. For $\mathfrak{n} \in \mathcal{N}$, let $\mathcal{B}_{\mathfrak{n}}$ denote the cyclic group

$$\mathcal{B}_{\mathfrak{n}} := \{ \prod_{\mathfrak{q} \mid \mathfrak{n}} (\gamma_{\mathfrak{q}} - 1) : \gamma_{\mathfrak{q}} \in H(\mathfrak{q}) \} \subset \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})} / \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})+1}.$$

By [9, Proposition 4.2], $\mathcal{B}_{\mathfrak{n}}$ is a direct summand of $\mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})} / \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})+1}$.

Let $\mathbf{KS}_r(\mathcal{M}_\chi)$ denote the \mathcal{O} -module of Kolyvagin systems of rank r for \mathcal{M}_χ (with the natural Selmer structure of [10, §5.2]) as defined in [10, §10] (see also [10, §5.2] and [8, §3.1 and §6.1]). A Kolyvagin system of rank r for \mathcal{M}_χ is a collection

$$\{ \kappa_{\mathfrak{n}} \in \wedge^r U_{K, S(\mathfrak{n})}^\chi \otimes \mathcal{B}_{\mathfrak{n}} : \mathfrak{n} \in \mathcal{N}_p \}$$

satisfying properties that we do not need to review here. We are identifying $\otimes_{\mathfrak{q} \mid \mathfrak{n}} H(\mathfrak{q})$ with $\mathcal{B}_{\mathfrak{n}}$ via $\otimes_{\mathfrak{q}} \gamma_{\mathfrak{q}} \mapsto \prod_{\mathfrak{q}} (\gamma_{\mathfrak{q}} - 1)$.

Definition 10.2. For $\mathfrak{n} \in \mathcal{N}$ let $\delta_{\mathfrak{n}}^\chi \in \wedge^{r+\nu(\mathfrak{n})} U_{K, S(\mathfrak{n})}^\chi \otimes \wedge^{\nu(\mathfrak{n})} (W_{K, \mathfrak{n}}^*)^\chi$ be as in Definition 9.9, and define

$$\beta_{\mathfrak{n}}^{\text{St}} := \sum_{\mathfrak{d} \mid \mathfrak{n}} \mathcal{R}_{K(\mathfrak{d})/K}^{\text{Art}}(\delta_{\mathfrak{d}}^\chi) \cdot \mathcal{D}_{\mathfrak{n}, \mathfrak{n}/\mathfrak{d}} \in \wedge^r U_{K, S(\mathfrak{n})}^\chi \otimes \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})} / \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})+1}.$$

Proposition 10.3. For $\mathfrak{n} \in \mathcal{N}$ we have $\beta_{\mathfrak{n}}^{\text{St}} \in \wedge^r U_{K(\mathfrak{n}), S(\mathfrak{n})}^{\chi} \otimes \mathcal{B}_{\mathfrak{n}}$, and the collection

$$\beta^{\text{St}} := \{\beta_{\mathfrak{n}}^{\text{St}} : \mathfrak{n} \in \mathcal{N}_p\}$$

is a Kolyvagin system of rank r for \mathcal{M}_{χ} .

Proof. In the special case where $k = \mathbb{Q}$, $S' = \{\infty\}$, and χ is an even quadratic character, this is [9, Theorem 8.7 and Proposition 6.5]. The proof in general is similar. The general case is also proved by Sano in [17, §4] (what we call a Stark system is called a unit system in [17]). \square

For the rest of this section we assume that $r = 1$, i.e., S' consists of a single archimedean place. Since $r = 1$, the Stark unit Euler system of Theorem 8.6 gives rise, via the map of [8, Theorem 3.2.4], to a Kolyvagin system of rank one

$$\kappa^{\text{St}} = \{\kappa_{\mathfrak{n}}^{\text{St}} : \mathfrak{n} \in \mathcal{N}_p\} \in \mathbf{KS}_1(\mathcal{M}_{\chi}).$$

(The results of [8] are stated only for $k = \mathbb{Q}$, but the proofs in the general case are the same; see [10].)

Proposition 10.4. Suppose $\mathfrak{n} \in \mathcal{N}_p$. Under restriction $K^{\times} \rightarrow K(\mathfrak{n})^{\times}$ and inclusion $\mathcal{B}_{\mathfrak{n}} \subset \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})} / \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})+1}$, with $\xi_{\mathfrak{n}}$ as in Definition 8.2 we have

$$\kappa_{\mathfrak{n}}^{\text{St}} \mapsto \sum_{\mathfrak{d}|\mathfrak{n}} \text{Tw}_{K(\mathfrak{d})/K}(\xi_{\mathfrak{d}}^{\chi}) \cdot \mathcal{D}_{\mathfrak{n}, \mathfrak{n}/\mathfrak{d}} \in U_{K(\mathfrak{n}), S(\mathfrak{n})}^{\chi} \otimes \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})} / \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})+1}.$$

Proof. Note that $\text{Tw}_{K(\mathfrak{d})/K}(\xi_{\mathfrak{d}}^{\chi})$ lies in $U_{K(\mathfrak{n}), S(\mathfrak{n})}^{\chi} \otimes \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{d})} / \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{d})+1}$ by Theorem 6.3 and Lemma 9.3, and $\mathcal{D}_{\mathfrak{n}, \mathfrak{n}/\mathfrak{d}}$ lies in $\mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n}/\mathfrak{d})} / \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n}/\mathfrak{d})+1}$ by definition.

In the special case where $k = \mathbb{Q}$ and χ is a real quadratic character, this is [9, Theorem 7.2 and Proposition 6.5]. The proof in general is the same. The general case also follows from calculations of Sano [17, §3]. \square

Theorem 10.5. If $\chi \neq \mathbf{1}$ then for every $\mathfrak{n} \in \mathcal{N}$ we have $\kappa_{\mathfrak{n}}^{\text{St}} = \beta_{\mathfrak{n}}^{\text{St}}$.

Proof. Let $r(\chi, S)$ be as in Definition 3.4, and suppose first that $r(\chi, S) = 1$. We have $\kappa^{\text{St}}, \beta^{\text{St}} \in \mathbf{KS}_1(\mathcal{M}_{\chi})$. Since $\chi \neq \mathbf{1}$ and K contains no nontrivial p -th roots of unity by Lemma 9.2, all the hypotheses of [8, §3.5] hold, so $\mathbf{KS}_1(\mathcal{M}_{\chi})$ is a free \mathcal{O} -module of rank one by [8, Theorem 5.2.10]. We have $\beta_1^{\text{St}} = \delta_1^{\chi} = \xi_1^{\chi} = \kappa_1^{\text{St}}$ by definition, and by Lemma 9.11(ii) this is a nonzero element of the free, rank-one \mathcal{O} -module $U_{K, S}^{\chi}$. Hence $\beta^{\text{St}} = \kappa^{\text{St}}$, i.e., $\kappa_{\mathfrak{n}}^{\text{St}} = \beta_{\mathfrak{n}}^{\text{St}}$ for every $\mathfrak{n} \in \mathcal{N}_p$.

Now suppose $r(\chi, S) > 1$. By Lemma 9.11(i), we have $\delta_{\mathfrak{n}}^{\chi} = 0$ for every \mathfrak{n} , so $\beta_{\mathfrak{n}}^{\text{St}} = 0$ for every \mathfrak{n} . Since $\kappa_1^{\text{St}} = 0$, the finiteness of the ideal class group together with [10, Theorem 13.4(iv) and Proposition 5.7] (see also [8, Theorem 5.2.12]) shows that $\kappa^{\text{St}} = 0$, i.e., $\kappa_{\mathfrak{n}}^{\text{St}} = 0$ for every $\mathfrak{n} \in \mathcal{N}_p$.

It remains to show that $\kappa_n^{\text{St}} = \beta_n^{\text{St}} \in U_{K,S(n)}^\chi \otimes \mathcal{B}_n$ when $n \in \mathcal{N} - \mathcal{N}_p$. But the exponent of the cyclic group \mathcal{B}_n is the greatest common divisor of the $|H(\mathfrak{q})|$ for \mathfrak{q} dividing n . If $\mathfrak{q} \mid p$ then (since $K(\mathfrak{q})$ is tamely ramified by definition) $H(\mathfrak{q})$ has order prime to p . Hence \mathcal{B}_n has order prime to p if $n \in \mathcal{N} - \mathcal{N}_p$, so $\mathcal{B}_n \otimes \mathcal{O} = 0$ and $\kappa_n^{\text{St}} = \beta_n^{\text{St}} = 0$. This completes the proof. \square

Theorem 10.6. *Suppose that $|S'| = 1$, that Conjectures $\text{St}(K/k)$ and $\text{St}(K(\mathbf{n})/k)$ hold for every \mathbf{n} , and that at least one of the following holds:*

- (a) $\chi \neq \mathbf{1}$,
- (b) $\chi = \mathbf{1}$ and $|S - S'| \geq 2$,
- (c) $\chi = \mathbf{1}$ and k has more than one archimedean place,

Then for every $n \in \mathcal{N}$,

$$\text{Tw}_{K(\mathbf{n})/K}(\epsilon_{K(\mathbf{n}),S(\mathbf{n}),T,S'}^\chi) = \mathcal{R}_{K(\mathbf{n})/K}^{\text{Art}}(\epsilon_{K,S(\mathbf{n}),T,S'}^\chi)$$

in $U_{K(\mathbf{n}),S(\mathbf{n})} \otimes_{\text{Gal}(K(\mathbf{n})/k)} W_{K(\mathbf{n}),S'}^* \otimes \mathcal{A}_{H(\mathbf{n})}^{\nu(\mathbf{n})} / \mathcal{A}_{H(\mathbf{n})}^{\nu(\mathbf{n})+1}$. In other words, the (p, χ) part of Conjecture 5.2(ii) holds for $(K(\mathbf{n})/K/k, S(\mathbf{n}), T, S')$.

Proof. If $\chi \neq \mathbf{1}$, then this follows directly from Theorem 10.5 by induction on \mathbf{n} , using Proposition 10.4 and Definition 10.2 for the induction. If $\chi = \mathbf{1}$, then this is Theorem 7.2. \square

Let $\Sigma = \Sigma(K/k, T)$ be the set of primes dividing $[K : k] \prod_{\lambda \in T_K} (\mathbf{N}\lambda - 1)$.

Theorem 10.7. *Suppose that $|S'| = 1$, that Conjectures $\text{St}(K/k)$ and $\text{St}(K(\mathbf{n})/k)$ hold for every \mathbf{n} , and that either k has more than one infinite place or $|S| \geq 3$. Then Conjecture 5.2(ii) holds for $(K(\mathbf{n})/K/k, S(\mathbf{n}), T, S')$ away from Σ , i.e., for every $p \notin \Sigma$ the leading term formula holds if we tensor with \mathbb{Z}_p .*

Proof. We can apply Theorem 10.6 for every prime $p \notin \Sigma$, and every character χ of Γ . Summing the conclusion of Theorem 10.6 over all χ gives the equality of Conjecture 5.2(ii) tensored with \mathcal{O} . \square

11. Evidence in the case of general r

Keep the notation of the previous sections. When $r > 1$, the proof of §10 breaks down. Namely, the elements ξ_n of Definition 8.2 naturally form an Euler system of rank r , but when $r > 1$ we do not know how to use this Euler system to produce a Kolyvagin system of rank r . However, using ideas of [14, §6] and [2] we define a family of “projectors” Φ , each of which maps the collection $\{\xi_n^\chi\}$ to an Euler system ξ_Φ^{St} of rank one, and maps the rank- r Kolyvagin system β^{St} to a rank-one Kolyvagin system β_Φ^{St} . We can associate to ξ_Φ^{St} a Kolyvagin system κ_Φ^{St} of rank one, and the arguments of

§10 will show that $\beta_{\Phi}^{\text{St}} = \kappa_{\Phi}^{\text{St}}$. Unwinding the definitions, this shows that the Φ -projection of the leading term formula of Conjecture 5.2 holds.

For this section we make the extra assumptions that

- S contains no primes above p ,
- k is totally real of degree r and S' is the set of its archimedean places,
- Leopoldt’s conjecture holds for K .

In particular K is totally real and K/k is unramified above p .

Definition 11.1. For every $\mathfrak{n} \in \mathcal{N}_p$ let $V_{K(\mathfrak{n})}$ denote the p -adic completion of the local units of $K(\mathfrak{n}) \otimes \mathbb{Q}_p$, and

$$V_{K(\mathfrak{n})}^* := \text{Hom}_{\text{Gal}(K(\mathfrak{n})/k)}(V_{K(\mathfrak{n})}, \mathbb{Z}_p[\text{Gal}(K(\mathfrak{n})/k)]).$$

If $\phi \in V_{K(\mathfrak{n})}^*$, then $\tilde{\phi}$ will denote the composition

$$\tilde{\phi} : U_{K(\mathfrak{n}), S(\mathfrak{n})} \longrightarrow V_{K(\mathfrak{n})} \longrightarrow \mathbb{Z}_p[\text{Gal}(K(\mathfrak{n})/k)].$$

Define $V_{\infty}^* := \varprojlim V_{K(\mathfrak{n})}^*$, where the inverse limit is taken with respect to the maps $V_{K(\mathfrak{n}\mathfrak{q})}^* \rightarrow V_{K(\mathfrak{n})}^*$ induced by

$$V_{K(\mathfrak{n})} \subset V_{K(\mathfrak{n}\mathfrak{q})}, \quad \mathbb{Z}_p[\text{Gal}(K(\mathfrak{n}\mathfrak{q})/k)]^{\text{Gal}(K(\mathfrak{n}\mathfrak{q})/K(\mathfrak{n}))} = \mathbb{Z}_p[\text{Gal}(K(\mathfrak{n})/k)].$$

If $\Phi := \phi_1 \wedge \dots \wedge \phi_{r-1} \in \wedge^{r-1} V_{\infty}^*$, with $\phi_i \in V_{\infty}^*$, let

$$\phi_{i, K(\mathfrak{n})} : V_{K(\mathfrak{n})} \longrightarrow \mathbb{Z}_p[\text{Gal}(K(\mathfrak{n})/k)]$$

denote the projection of ϕ_i to $V_{K(\mathfrak{n})}^*$, let

$$\tilde{\Phi}_{K(\mathfrak{n})} := \tilde{\phi}_{1, K(\mathfrak{n})} \wedge \dots \wedge \tilde{\phi}_{r-1, K(\mathfrak{n})} : \wedge^{r, 0} U_{K(\mathfrak{n}), S(\mathfrak{n})} \longrightarrow U_{K(\mathfrak{n}), S(\mathfrak{n})}$$

be the map of Definition A.1 (combined with Lemmas A.4 and A.5), and let

$$\mathcal{L}_{\Phi} := \cap_i \ker(\phi_{i, K}) \subset V_K.$$

Using the identification $V_K^{\chi} \subset \oplus_{\mathfrak{p}|p} H^1(k_{\mathfrak{p}}, \mathcal{M}_{\chi})$ of Proposition 8.5, we define a Selmer structure (see [10, Definition 2.1] or [8, Definition 2.1.1]) \mathcal{F}_{Φ} on \mathcal{M}_{χ} by modifying the natural Selmer structure \mathcal{F}_{ur} of [10, §5.2] at primes above p , namely we set

$$\oplus_{\mathfrak{p}|p} H_{\mathcal{F}_{\Phi}}^1(k_{\mathfrak{p}}, \mathcal{M}_{\chi}) := \mathcal{L}_{\Phi}^{\chi} \subset V_K^{\chi} \subset \oplus_{\mathfrak{p}|p} H^1(k_{\mathfrak{p}}, \mathcal{M}_{\chi}).$$

Let $\xi_{\mathfrak{n}} \in \wedge^{r, 0} U_{K(\mathfrak{n}), S(\mathfrak{n})}$ be as in Definition 8.2, and recall the Kolyvagin system $\beta^{\text{St}} = \{\beta_{\mathfrak{n}}^{\text{St}} : \mathfrak{n} \in \mathcal{N}_p\} \in \mathbf{KS}_r(\mathcal{M}_{\chi})$ of Definition 10.2 and Proposition 10.3.

Proposition 11.2. *Suppose $\Phi := \phi_1 \wedge \dots \wedge \phi_{r-1} \in \wedge^{r-1} V_{\infty}^*$.*

- (i) *The collection $\{\tilde{\Phi}_{K(\mathfrak{n})}(\xi_{\mathfrak{n}}^{\chi}) \in U_{K(\mathfrak{n}), S(\mathfrak{n})}^{\chi} : \mathfrak{n} \in \mathcal{N}_p\}$ is an Euler system of rank one for the representation \mathcal{M}_{χ} .*

- (ii) Let $\kappa_{\Phi}^{\text{St}} = \{\kappa_{\Phi, \mathfrak{n}}^{\text{St}} : \mathfrak{n} \in \mathcal{N}_p\} \in \mathbf{KS}_1(\mathcal{M}_\chi)$ be the Kolyvagin system of rank one attached to the Euler system of (i) by [8, Theorem 3.2.4]. Then $\kappa_{\Phi}^{\text{St}} \in \mathbf{KS}_1(\mathcal{M}_\chi, \mathcal{F}_\Phi)$, where \mathcal{F}_Φ is the Selmer structure of Definition 11.1.
- (iii) The collection $\beta_{\Phi}^{\text{St}} := \{\tilde{\Phi}_K(\beta_{\mathfrak{n}}^{\text{St}}) : \mathfrak{n} \in \mathcal{N}_p\}$ is a Kolyvagin system of rank one for $(\mathcal{M}_\chi, \mathcal{F}_\Phi)$.

Proof. The first assertion is proved in [14, Proposition 6.6]. Both (i) and (ii) are proved in [2, Proposition 2.2 and Theorem 2.19]. Assertion (iii) follows from Proposition 10.3 by direct calculation. \square

Proposition 11.3. *Suppose that $\Phi := \phi_1 \wedge \dots \wedge \phi_{r-1} \in \wedge^{r-1} V_\infty^*$ and that $\mathfrak{n} \in \mathcal{N}_p$. Under the restriction map $K^\times \rightarrow K(\mathfrak{n})^\times$ and the inclusion $\mathcal{B}_{\mathfrak{n}} \subset \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})} / \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})+1}$, we have*

$$\kappa_{\Phi, \mathfrak{n}}^{\text{St}} \mapsto \sum_{\mathfrak{d}|\mathfrak{n}} \text{Tw}_{K(\mathfrak{d})/K}(\tilde{\Phi}_{K(\mathfrak{d})}(\xi_{\mathfrak{d}}^\chi)) \cdot \mathcal{D}_{\mathfrak{n}, \mathfrak{n}/\mathfrak{d}} \in U_{K(\mathfrak{n}), S(\mathfrak{n})}^\chi \otimes \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})} / \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})+1}.$$

Proof. The proof is similar to Proposition 10.4, or see [17, §3]. \square

Theorem 11.4. *Suppose $\Phi := \phi_1 \wedge \dots \wedge \phi_{r-1} \in \wedge^{r-1} V_\infty^*$. If $\chi \neq \mathbf{1}$ then for every $\mathfrak{n} \in \mathcal{N}$ we have $\kappa_{\Phi, \mathfrak{n}}^{\text{St}} = \tilde{\Phi}_K(\beta_{\mathfrak{n}}^{\text{St}})$.*

Proof. The proof is similar to that of Theorem 10.5.

Let $r(\chi, S)$ be as in Definition 3.4. Suppose first that $r(\chi, S) = r$ and $\phi_{1,K}, \dots, \phi_{r-1,K}$ are \mathbb{Z}_p -linearly independent. By Proposition 11.2 we have that $\kappa_{\Phi}^{\text{St}}, \beta_{\Phi}^{\text{St}} \in \mathbf{KS}_1(\mathcal{M}_\chi, \mathcal{F}_\Phi)$. The core rank of $(\mathcal{M}_\chi, \mathcal{F}_\Phi)$ is one by [2, Proposition 1.8]. All the hypotheses of [8, §3.5] hold, so $\mathbf{KS}_1(\mathcal{M}_\chi, \mathcal{F}_\Phi)$ is a free \mathcal{O} -module of rank one by [8, Theorem 5.2.10]. We have $\tilde{\Phi}_K(\beta_1^{\text{St}}) = \tilde{\Phi}_K(\delta_1^\chi) = \tilde{\Phi}_K(\xi_1^\chi) = \kappa_{\Phi, 1}^{\text{St}}$ by definition, and it follows from Lemma 9.11(ii), our assumption on the independence of the $\phi_{i,K}$, and Leopoldt’s conjecture that this has infinite order in \mathcal{L}_Φ^χ . Hence $\beta_{\Phi}^{\text{St}} = \kappa_{\Phi}^{\text{St}}$, i.e., $\kappa_{\Phi, \mathfrak{n}}^{\text{St}} = \tilde{\Phi}_K(\beta_{\mathfrak{n}}^{\text{St}})$ for every $\mathfrak{n} \in \mathcal{N}_p$.

Now suppose that either $r(\chi, S) > r$ or the $\phi_{i,K}$ are linearly dependent. In the former case Lemma 9.11(i) shows that $\delta_{\mathfrak{n}}^\chi = 0$ for every \mathfrak{n} , and in the latter case $\tilde{\Phi}_K = 0$, so in either case $\tilde{\Phi}_K(\beta_{\mathfrak{n}}^{\text{St}}) = 0$ for every \mathfrak{n} . Since $\kappa_{\Phi, 1}^{\text{St}} = 0$, the finiteness of the ideal class group together with [10, Theorem 13.4(iv) and Proposition 5.7] (see also [8, Theorem 5.2.12]) and Leopoldt’s conjecture (see [2, Remark 1.7]) shows that $\kappa_{\Phi}^{\text{St}} = 0$, i.e., $\kappa_{\Phi, \mathfrak{n}}^{\text{St}} = 0$ for every $\mathfrak{n} \in \mathcal{N}_p$.

It remains to show that $\kappa_{\mathfrak{n}}^{\text{St}} = \beta_{\mathfrak{n}}^{\text{St}} \in U_{K, S(\mathfrak{n})}^\chi \otimes \mathcal{B}_{\mathfrak{n}}$ when $\mathfrak{n} \in \mathcal{N} - \mathcal{N}_p$. This follows exactly as in the proof of Theorem 10.5, since $\mathcal{B}_{\mathfrak{n}}$ has order prime to p if $\mathfrak{n} \in \mathcal{N} - \mathcal{N}_p$. This completes the proof. \square

Theorem 11.5. *Suppose Conjecture $\text{St}(K(\mathfrak{n})/k)$ holds for every \mathfrak{n} , and either $\chi \neq \mathbf{1}$ or $|S - S'| \geq 2$. Then for every $\Phi := \phi_1 \wedge \dots \wedge \phi_{r-1} \in \wedge^{r-1} V_\infty^*$ and every $\mathfrak{n} \in \mathcal{N}$,*

$$\tilde{\Phi}_{K(\mathfrak{n})}(\text{Tw}_{K(\mathfrak{n})/K}(\epsilon_{K(\mathfrak{n}),S(\mathfrak{n}),T,S'}^\chi)) = \tilde{\Phi}_{K(\mathfrak{n})}(\mathcal{R}_{K(\mathfrak{n})/K}^{\text{Art}}(\epsilon_{K,S(\mathfrak{n}),T,S'(\mathfrak{n})}^\chi))$$

in $U_{K(\mathfrak{n}),S(\mathfrak{n})} \otimes_{\text{Gal}(K(\mathfrak{n})/k)} W_{K(\mathfrak{n}),S'}^ \otimes \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})} / \mathcal{A}_{H(\mathfrak{n})}^{\nu(\mathfrak{n})+1}$. In other words, the $\tilde{\Phi}_{K(\mathfrak{n})}$ -projection of the χ part of Conjecture 5.2(ii) holds for the data $(K(\mathfrak{n})/K/k, S(\mathfrak{n}), T, S')$.*

Proof. If $\chi \neq \mathbf{1}$, then this follows directly from Theorem 11.4 by induction on \mathfrak{n} , using Proposition 11.3 and Definition 10.2 for the induction. If $\chi = \mathbf{1}$, then this is Theorem 7.2. □

Let $\Sigma = \Sigma(K/k, S, T)$ be the set of rational primes dividing

$$[K : k] \prod_{\lambda \in S - S'} \mathbf{N}\lambda \prod_{\lambda \in T_K} (\mathbf{N}\lambda - 1).$$

Theorem 11.6. *Suppose that k is totally real, S' is the set of all archimedean places of k , Conjectures $\text{St}(K/k)$ and $\text{St}(K(\mathfrak{n})/k)$ hold for every \mathfrak{n} , Leopoldt's conjecture holds for K , and $|S - S'| \geq 2$. Then for every $p \notin \Sigma$, every $\Phi \in \wedge^{r-1} V_\infty^*$, and every $\mathfrak{n} \in \mathcal{N}$, we have*

$$\tilde{\Phi}_{K(\mathfrak{n})}(\text{Tw}_{K(\mathfrak{n})/K}(\epsilon_{K(\mathfrak{n}),S(\mathfrak{n}),T,S'})) = \tilde{\Phi}_{K(\mathfrak{n})}(\mathcal{R}_{K(\mathfrak{n})/K}^{\text{Art}}(\epsilon_{K,S(\mathfrak{n}),T,S'(\mathfrak{n})})).$$

In other words, for every $\Phi \in \wedge^{r-1} V_\infty^$ the leading term formula of Conjecture 5.2(ii) holds for $(K(\mathfrak{n})/K/k, S(\mathfrak{n}), T, S')$ after applying $\tilde{\Phi}_{K(\mathfrak{n})}$.*

Proof. We can apply Theorem 11.5 for every prime $p \notin \Sigma$, and every character χ of Γ . Summing the conclusion of Theorem 11.5 over all χ gives the $\tilde{\Phi}_{K(\mathfrak{n})}$ -projection of Conjecture 5.2(ii) tensored with \mathcal{O} . □

Appendix A. Exterior algebras and determinants

Let \mathcal{O} be an integral domain with field of fractions F , and let $R = \mathcal{O}[\Gamma]$ with a finite abelian group Γ .

If M is an R -module, we let $M^* := \text{Hom}_R(M, R)$, and \overline{M} will denote the image of M in $M \otimes_{\mathcal{O}} F$. If $\rho \in R$, then $M[\rho]$ denotes the kernel of multiplication by ρ in M .

Fix for this appendix an R -module M of finite type.

Definition A.1. If $r \geq 0$, then $\wedge^r M$ (or $\wedge_R^r M$, if we need to emphasize the ring R) will denote the r -th exterior power of M in the category of R -modules, with the convention that $\wedge^0 M = R$. If $\psi \in M^*$ and $r \geq 1$, we view $\psi \in \text{Hom}(\wedge^r M, \wedge^{r-1} M)$ by

$$\psi(m_1 \wedge \dots \wedge m_r) := \sum_{i=1}^r (-1)^{i+1} \psi(m_i)(m_1 \wedge \dots \wedge m_{i-1} \wedge m_{i+1} \wedge \dots \wedge m_r).$$

If $\psi \in \wedge^s M^*$ with $s \leq r$, we view $\psi \in \text{Hom}(\wedge^r M, \wedge^{r-s} M)$ by

$$(\psi_1 \wedge \cdots \wedge \psi_s)(\mathbf{m}) := \psi_s \circ \psi_{s-1} \circ \cdots \circ \psi_1(\mathbf{m}).$$

In particular

$$(A.2) \quad (\phi_1 \wedge \cdots \wedge \phi_r)(m_1 \wedge \cdots \wedge m_r) = \det(\phi_i(m_j)).$$

Definition A.3. For every $r \geq 0$, define

$$\wedge^{r,0} M := \{\mathbf{m} \in \overline{\wedge^r M} : \psi(\mathbf{m}) \in R \text{ for every } \psi \in \wedge^r M^*\}.$$

In other words, $\wedge^{r,0} M$ is the dual lattice to $\overline{\wedge^r M^*}$ in $\wedge^r M \otimes F$.

Lemma A.4. We have $\overline{\wedge^r M} \subset \wedge^{r,0} M$, with equality if $|\Gamma| \in \mathcal{O}^\times$ or if $r = 1$.

Proof. The inclusion follows directly from the definition, and for the rest see [14, Proposition 1.2]. (If $|\Gamma| \in \mathcal{O}^\times$ the equality holds because \overline{M} is a projective R -module.) \square

Lemma A.5. If $\mathbf{m} \in \wedge^{r,0} M$ and $\psi \in \wedge^s M^*$ with $s \leq r$, then

$$\psi(\mathbf{m}) \in \wedge^{r-s,0} M.$$

Proof. If $\psi' \in \wedge^{r-s} M^*$ then we have $\psi'(\psi(\mathbf{m})) = (\psi \wedge \psi')(\mathbf{m}) \in R$ because $\mathbf{m} \in \wedge^{r,0} M$, so $\psi \wedge \mathbf{m} \in \wedge^{r-s,0} M$ by definition. \square

Proposition A.6. Suppose M is an R -module that is projective as an \mathcal{O} -module, and B is an \mathcal{O} -module. For every $s \leq r$ and $\rho \in F[\Gamma]$, the construction of Definition A.1 induces a canonical pairing

$$(\wedge^{r,0} M)[\rho] \times \wedge^{r-s} \text{Hom}_R(M, R \otimes_{\mathcal{O}} B) \longrightarrow (\wedge^{s,0} M)[\rho] \otimes_{\mathcal{O}} B^{\otimes(r-s)}.$$

In particular, when $s = 0$ this pairing takes values in $R[\rho] \otimes_{\mathbb{Z}} B^{\otimes r}$.

Proof. There are natural isomorphisms

$$\text{Hom}_R(M, R) \cong \text{Hom}_{\mathcal{O}}(M, \mathcal{O}), \quad \text{Hom}_R(M, R \otimes_{\mathcal{O}} B) \cong \text{Hom}_{\mathcal{O}}(M, B).$$

Since M is a projective \mathcal{O} -module, the natural map

$$M^* \otimes_{\mathcal{O}} B \longrightarrow \text{Hom}_R(M, R \otimes_{\mathcal{O}} B)$$

is an isomorphism. This isomorphism gives the first map of

$$\begin{aligned} (\wedge^{r,0} M)[\rho] \times \wedge^{r-s} \text{Hom}_R(M, R \otimes_{\mathcal{O}} B) &\xrightarrow{\sim} (\wedge^{r,0} M)[\rho] \times \wedge^{r-s} (M^* \otimes_{\mathcal{O}} B) \\ &\longrightarrow (\wedge^{r,0} M)[\rho] \times (\wedge^{r-s} M^*) \otimes_{\mathcal{O}} B^{\otimes(r-s)} \\ &\longrightarrow (\wedge^{s,0} M)[\rho] \otimes_{\mathcal{O}} B^{\otimes(r-s)} \end{aligned}$$

and the last map comes from Definition A.1, using Lemma A.5.

If $s = 0$, then $\wedge^{0,0} M = R$ by definition. \square

Remark A.7. If $m_1, \dots, m_r \in M$, $\phi_1, \dots, \phi_r \in \text{Hom}_R(M, R \otimes_{\mathcal{O}} B)$, and $s = 0$, then the pairing of Proposition A.6 is given by

$$(m_1 \wedge \cdots \wedge m_r, \phi_1 \wedge \cdots \wedge \phi_r) \mapsto \det(\phi_i(m_j)).$$

The content of Proposition A.6 is that this pairing is defined on all of $\wedge^{r,0} M$, not just on $\overline{\wedge^r M}$.

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