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The Jacobi form D_L, theta functions, and Eisenstein series

par Abdelmejid BAYAD et Gilles ROBERT

Résumé. Dans ce travail, consacré à la forme modulaire de Jacobi D_L, nous clarifions sa relation à la fonction thêta de genre un, son écriture possible comme une série de Kronecker, et leur lien avec les séries d'Eisenstein. Nous calculons la dérivée logarithmique de $x \to D_L(x,z)$, et déduisons de ce calcul une représentation intégrale de D_L valable pour x dans le disque épointé, de rayon la distance de z à L. Concernant les coefficients du développement en série de Laurent de D_L, nous établissons une relation de récurrence liant ceux-ci aux valeurs des séries d'Eisenstein $E_k^*(z;L) - e_k^*(L)$, pour $k \geq 1$. Ces coefficients sont les analogues des fonctions de Bernoulli périodiques. Plusieurs identités intéressantes peuvent être décrites par notre étude. Certaines de ces relations ont déjà été étudiées dans le livre de A. Weil [10]. Des identités nouvelles sont obtenues ici.

Abstract. In this paper we study the Jacobi modular form D_L, we clarify its relation to theta function of genus one and its link with Eisenstein series, and also its relationship with Kronecker's series. We compute the logarithmic derivative of $x \to D_L(x,z)$, and derive from it an integral representative of D_L valid for x in the punctured disk of radius the distance of z to L. We consider the Laurent series coefficients of D_L and establish a recurrence formula for them. Those coefficients are elliptic analogue to periodized Bernoulli functions. Several interesting identities can be obtained from our study. These interrelations were already the object of the Weil's book [10]. New relations are here obtained.

1. Introduction and motivation

Let $z \in \mathbb{R}$, we denote by $\{z\}$ its fractional part. We are motivated by elliptic analogues of the periodized Bernoulli functions $B_n(\{z\})$. Let us precise classical and elliptic situations.

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1.1. Classical case: Revisited. For z real, $0 < z < 1$, we have the Cauchy expansion in x

$$2\pi ix \; \frac{e(xz)}{e(x)-1} = 1 + \sum_{n=1}^{\infty} B_n(z) \frac{(2\pi ix)^n}{n!} \quad , \; |x| < 1 \tag{1.1}$$

where $B_n(z)$ are the Bernoulli polynomials restricted to $]0,1[$, and $e(t) := \exp(2\pi it)$. We denote by $B_n := B_n(0)$ the n-th Bernoulli number.

As $B_1(z) = z - \frac{1}{2}$, one can write the left side of the relation (1.1) in the following form

$$e(B_1(z)x) \frac{\pi x}{\sin(\pi x)}. \tag{1.2}$$

For real x, $|x| < 1$, the expression (1.2) can be written as follows

$$\exp\left\{ \int_0^x 2\pi i B_1(z) \; dp - \int_0^x \left(\epsilon_1(p) - \frac{1}{p} \right) dp \right\} \tag{1.3}$$

where $\epsilon_1(x) = \pi \cot(\pi x)$. The exponential term of (1.3) can be written as the series in x

$$2\pi ix B_1(z) - \sum_{m=1}^{\infty} \frac{B_{2m}}{2m} \frac{(2\pi ix)^{2m}}{(2m)!} \; . \tag{1.4}$$

So that we have the formula

$$1 + \sum_{n=1}^{\infty} B_n(z) \frac{(2\pi ix)^n}{n!} = \exp\left\{ 2\pi ix B_1(z) - \sum_{m=1}^{\infty} \frac{B_{2m}}{2m} \frac{(2\pi ix)^{2m}}{(2m)!} \right\} \tag{1.5}$$

of which an elliptic analogue will be investigated below.

On the other hand, writing the Fourier series of the left side of the relation (1.1), we find the expressions

$$\frac{(2\pi i)^n}{n!} \; B_n(z) = (-1)^{n-1} \sideset{}{^*}\sum_{k\in\mathbb{Z}} \frac{e(-kz)}{k^n}, \; 0 < z < 1 \; , \; n \geq 1 \; , \tag{1.6}$$

where the sum $\sideset{}{^*}\sum_{k\in\mathbb{Z}}$ means that $k = 0$ is to be omitted (and, in the non-absolutely convergent case $n = 1$, the sum has to be interpreted as a Cauchy principal value).

1.2. Elliptic case. Let L be a complex lattice and note by $\pi A = a(L)$ the area of the parallelogram period of L , and set $E_L(s,t) := \frac{1}{2\pi i A}(\bar{s}t - s\bar{t})$ for any complex s, t. For the lattice L the analogue of the expressions (1.6) is given by

$$d_n(z; L) = (-1)^{n-1} \sum_{\omega \neq 0} \frac{e(E_L(z,\omega))}{\omega^n} \; , \; z \in \mathbb{C} \backslash L \; , \; n \geq 1 \; , \tag{1.7}$$

periodic in z with period lattice L, we call them *elliptic periodized Bernoulli functions*, see also [4, 6, 5]. In Section 5, for $\mid x \mid< d(L,z)$ these quantities satisfy

$$xD_L(x,z) = 1 + \sum_{n\geq 1} d_n(z;L)x^n \ , \ z \in \mathbb{C}\backslash L \ , \tag{1.8}$$

where $d(L,z)$ is the distance of z to the lattice L. Here D_L is the Jacobi modular form studied below and introduced in [2]. In Section 4, we prove for $xD_L(x,z)$ an analogue to the identity (1.3), given by the equation

$$xD_L(x,z) = \exp\left(\int_{[0,x]} f(p)\ dp\right) \ , \tag{1.9}$$

for $0 < |x| < d(L,z)$, where the function $f(p)$ is defined on the punctured disc $0 < |p| < d(L,z)$, by

$$f(p) := E_1^*(z+p;L) - E_1^*(p;L) + \frac{1}{p} \ ; \tag{1.10}$$

this function is holomorphic, and $E_1^*(x;L)$ is the non-holomorphic Eisenstein series of weight 1, defined as in [10, Chap. VI, §2, p.43] with constant anticomplex derivative. We have

$$\lim_{p\to 0} f(p) = E_1^*(z;L).$$

The Cauchy expansion of $f(x)$, for $0 < |x| < d(L,z)$, can be expressed in terms of the quantities

$$s_k(z;L) := E_k^*(z;L) - e_k^*(L), \tag{1.11}$$

where E_k^* and e_k^* are the Eisenstein series of weight k, see Lemma 5.2, related by

$$e_k^*(L) = \lim_{z\to 0}\left(E_k^*(z;L) - \frac{1}{z^k}\right), \ k\geq 2, \ e_1^*(L) = 0.$$

Applying the indeterminates coefficients method, we obtain in Theorem 5.4, the recurrence formula
(1.12)

$$(-1)^n d_n(z;L) + \frac{1}{n}s_n(z;L) + \frac{1}{n}\sum_{k=1}^{n-1}(-1)^k d_k(z;L)s_{n-k}(z;L) = 0 \ , \ n\geq 1.$$

1.3. Link with the classical case. From the recurrence formula (1.12) we can obtain the well-known Euler's identity

$$\sum_{k=0}^{n}\binom{n}{k}B_k(z)B_{n-k}(0) = n(z-1)B_{n-1}(z) - (n-1)B_n(z). \tag{1.13}$$

In fact, for the complex lattice $L_\tau = \mathbb{Z}+\mathbb{Z}\tau$ with $\mathrm{Im}(\tau) > 0$, and $z = z_1\tau+z_2$ with $z_1, z_2 \in \mathbb{R}$, we have for $\{z_1\} \neq 0$

$$\lim_{\mathrm{Im}(\tau)\to\infty} E_1^*(z_1\tau + z_2; L_\tau) = 2\pi i B_1(\{z_1\})$$

uniformly on z_2. So for a real x, the integral $\int_{[0,x]} f(p)\, dp$ tends to the integral in (1.3) when $\mathrm{Im}(\tau) \to \infty$.

2. Jacobi forms D_L and theta functions

Let L be a complex lattice, and D_L the Jacobi form introduced in [2]. One has the formula

$$D_L(x, z) = \exp\left[-A^{-1}x\bar{z}\right] \frac{\theta_L(x + z)}{\theta_L(x)\theta_L(z)}\ ,\ x \notin L\ , z \notin L\ , \tag{2.1}$$

which is a rewriting of the formula (7) in A. Weil [10, Chap. VIII, p.71], where he used ϕ (see later) depending on the choice of the $\mathbb{Z}$-basis of L.

For the general definition of reduced theta function see [9, Chap.VI,§3]. Here θ_L denotes the reduced theta function associated to the lattice L such that

i) holomorphic with simple zeros at L,
ii) $\theta_L(x) \sim x$

and can be given by

$$x/\theta_L(x) = \exp\left\{\frac{e_2^*(L)}{2}x^2 + \sum_{m\geq 2} \frac{e_{2m}(L)}{2m}x^{2m}\right\}, \quad |\, x\, |<|\, \omega\, |,\ \forall\ \omega \in L\backslash\{0\}\ . \tag{2.2}$$

For fixed $\mathbb{Z}$-basis (u, v) of L , we recall the definitions of the Eisenstein's series

$$e_2^*(L) = e_2(u, v) - \frac{1}{A}\frac{\overline{u}}{u}, \tag{2.3}$$

$$e_2(u, v) = {\sum}'_e\ w^{-2} := \sum_\nu \left(\sum_\mu \frac{1}{(\mu u + \nu v)^2}\right), \tag{2.4}$$

$$e_m(L) = {\sum}'\ w^{-m}\ ,\ m \geq 3, \tag{2.5}$$

the summation being over $w \in L,\ w \neq 0$.

The function θ_L can be rewritten as follows.

$$\theta_L(x) = \exp\left[\frac{1}{2}\mathrm{A}^{-1}\frac{\bar{\mathrm{u}}}{\mathrm{u}}\mathrm{x}^2\right]\phi(\mathrm{x}). \tag{2.6}$$

where $\phi(x) = \phi(x; u, v)$ is borrowed from A. Weil [10, Chap.IV], with

$$\phi(x) = \frac{u}{2\pi i}\frac{X_q(z)}{P(q)^2},$$

$$X_q(z) = \left(z^{1/2} - z^{-1/2}\right)\prod_{m=1}^{\infty}(1 - q^m z)(1 - q^m z^{-1}),$$

$$P(q) = \prod_{m=1}^{\infty}(1 - q^m),$$

and $z^{1/2} = e(x/2u),\ q = e(\tau),\ \tau = \delta v/u$, with $\delta = \pm 1$ so that $\mathrm{Im}(\tau) > 0$.

Theorem 2.1 ([4, 10])**.** *One has*

$$D_L(x, z) = \sum{}^{*}\ \frac{e(E_L(z, \omega))}{x + \omega} = K_1(x, z, 1) \tag{2.7}$$

where the summation is taken on ω such that $x + \omega \notin L$, and $K_1(x, z, 1)$ is as defined below.

Among others, from equality (2.7) and properties of $K_1(x, z, 1)$, we can extend the form $D_L(x, z)$ when x or z belongs to L.

3. Jacobi form D_L, Eisenstein and Kronecker series

3.1. Eisenstein series. We consider the two Eisenstein series

$$E_1(x; u, v) := \sum_{\nu}\left(\sum_{\mu}\frac{1}{x + \mu u + \nu v}\right) \tag{3.1}$$

$$E_1^*(x; L) = E_1(x; u, v) + \frac{1}{A}\left(\frac{\bar{u}}{u}x - \bar{x}\right). \tag{3.2}$$

Note that

i) The serie $E_1(x; u, v)$ is holomorphic on $x \notin L$ and depends on the $\mathbb{Z}$-basis (u, v) of L,

ii) The serie $E_1^*(x; L)$ is periodic with periods L, but non holomorphic on x . In fact,

$$\frac{\partial}{\partial \bar{x}}E_1^*(x; L) = -A^{-1} \quad \text{is constant,}$$

where $\pi\ A = a(L)$ is the area of the parallelogram period of L.

We quote from Weil [10, Chap. IV, p.25, formula (11)] the following lemma.

Lemma 3.1. *We have*

$$\frac{\partial}{\partial x}\log\phi(x) = E_1(x; u, v). \tag{3.3}$$

3.2. Kronecker's series [10, Chap.VIII §§12–14, pp.78-81]**.** The Kronecker series $K_a(x, x_0, s)$ are given by

$$K_a(x, x_0, s) = \sum\nolimits^* (\bar{x} + \bar{w})^a \mid x + w \mid^{-2s} e(E_L(x_0, w)) \tag{3.4}$$

where the $\sum^*$ means the summation is over all $w \in L$ other than $-x$ if $x \in L$,

i) the series (3.4) is absolutely convergent for $\Re(s) > \frac{a}{2} + 1$,
ii) defined by analytical continuation as a meromorphic function to the whole complex s-plane,
iii) with possible poles of order 1 at $s = 0$ (if $a = 0, x \in L$) and at $s = 1$ (if $a = 0, x_0 \in L$),
iv) satisfy functional equation : $s \leftrightarrow a + 1 - s$.
In fact, following C.L. Siegel and putting $s_1 = s - \frac{a}{2}$, we find that

$$G^*(x, z; s_1) := A^{s_1}\Gamma\left(s_1 + \frac{a}{2}\right) K_a\left(x, z, s_1 + \frac{a}{2}\right), \ \mathrm{Re}(s_1) \geq 1, \tag{3.5}$$

is independent of the nonnegative integer a, has an holomorphic continuation to the whole s_1-plane and satisfies the functional equation

$$G^*(x, z; s_1) = e\left(-E_L(z, x)\right) G^*(z, x; 1 - s_1) \tag{3.6}$$

A proof of (3.5) and (3.6) can be obtained from C.L. Siegel [8, p.48] about "*Kronecker's limit formula*". See also H. Ito [3, §2, p.152] and R. Sczech [7].

Directly from A. Weil, formula (7) [10, Chap. VIII, p.71], we obtain the results

$$\begin{aligned}\frac{\partial}{\partial x}\left(\log D_L(x, z)\right) &= \frac{1}{A}\left(\frac{\overline{u}}{u}z - \overline{z}\right) + E_1(x + z; u, v) - E_1(x; u, v) \\ &= E_1^*(x + z; L) - E_1^*(x; L), \ x, z, x + z \notin L\ .\end{aligned}$$

Note that $E_1^*(x + z; L) - E_1^*(x; L)$ is holomorphic in $x \notin L$, as

$$\frac{\partial}{\partial \bar{x}} E_1^*(x; L) = -A^{-1} \ \text{ is constant.}$$

On the other hand we have

$$\frac{\partial}{\partial x} K_1(x, z, 1) = -K_2(x, z, 2),$$

so that it comes

Theorem 3.2. *We have the formula*

$$\begin{aligned} K_2(x, z, 2) &= K_1(x, z, 1)\Big(E_1^*(x; L) + E_1^*(y; L)\Big), \\ &\qquad x, y, z \notin L, \ x + y + z \in L\ . \end{aligned} \tag{3.7}$$

The statement of this theorem seems new. We will use it in Section 5 in order to obtain recurrences formulas. Also from the equality (2.1) we obtain the functional equation

$$D_L(x,z) = \exp\left(\frac{\bar{x}z - x\bar{z}}{A}\right) D_L(z,x),$$

and hence

Corollary 3.3. *We have the following partial differential relations*

$$\frac{\partial}{\partial x}(\log D_L(x,z)) = E_1^*(x+z;L) - E_1^*(x;L) \tag{3.8}$$

$$\frac{\partial}{\partial z}(\log D_L(x,z)) = E_1^*(x+z;L) - E_1^*(z;L) + \frac{\pi}{a(L)}\overline{x} \tag{3.9}$$

$$\frac{\partial}{\partial \bar{x}}(\log D_L(x,z)) = 0, \tag{3.10}$$

$$\frac{\partial}{\partial \bar{z}}(\log D_L(x,z)) = -\frac{\pi}{a(L)}x. \tag{3.11}$$

4. Integral representation of the form D_L

We consider the function

$$f(p) := E_1^*(z+p;L) - E_1^*(p;L) + \frac{1}{p} \tag{4.1}$$

i) is holomorphic for p in the punctured disc $0 < |p| < d(L,z)$, where $d(L,z)$ is the distance of z to the lattice L,

ii) $f(p)$ tends to $E_1^*(z;L)$ when $p \to 0$.

The following theorem outside its own interest, opens a convenient passage for the limit formulas of Section 6.

Theorem 4.1. *Assume that $0 < |x|,\ |x_1| < d(L,z)$, then we have the identities*

$$xD_L(x,z) = \exp\left(\int_{[0,x]} f(p)\ dp\right)\ , \tag{4.2}$$

$$x_1 D_L(x_1,z) = xD_L(x,z)\ \exp\left(\int_{[x,x_1]} f(p)\ dp\right)\ . \tag{4.3}$$

Proof. We observe that the both sides of the equality (4.2) tend to 1, as $x \to 0$, and the right side of the equality (4.3) tend to $xD_L(x,z)$, when $x_1 \to x$. Moreover, from the Corollary 3.3, the logarithmic derivatives $\frac{\partial}{\partial x}\log$, $\frac{\partial}{\partial z}\log$, $\frac{\partial}{\partial \bar{x}}\log$, $\frac{\partial}{\partial \bar{z}}\log$ (resp. $\frac{\partial}{\partial x_1}\log$, $\frac{\partial}{\partial z}\log$, $\frac{\partial}{\partial \bar{x_1}}\log$, $\frac{\partial}{\partial \bar{z}}\log$) of both sides of the equality (4.2)(resp. (4.3)) give the same values. Therefore the left side and right side of the equality (4.2) (resp. (4.3)) are equal. Let us give more details, for example we apply the logarithmic derivative

$\frac{\partial}{\partial z}\log$ for the equality (4.3). We quote from A. Weil [10, Chap.VI, p.43, formula (6)]

$$E_2^*(x;L) = -\frac{\partial}{\partial x}E_1^*(x;L), \quad \text{and} \quad E_2(x;u,v) = -\frac{\partial}{\partial x}E_1(x;u,v). \tag{4.4}$$

As termwise differentiation is allowed see A.Weil [10, Chap.I, p.5], we obtain the following equalities

$$\begin{aligned}\frac{\partial}{\partial z}\int_{[x,x_1]} f(p)dp &= \int_{[x,x_1]} \frac{\partial}{\partial z} f(p)\, dp \\ &= \int_{[x,x_1]} -E_2^*(p+z;L)\, dp \\ &= E_1(x_1+z;u,v) - E_1(x+z;u,v) + (x_1-x)(\pi/a(L))\bar{u}/u \\ &= E_1^*(x_1+z;L) - E_1^*(x+z;L) + (\bar{x_1}-\bar{x})\pi/a(L).\end{aligned}$$

Note that to obtain the third equality, we have written

$$E_2^*(p+z;L) = E_2(p+z;u,v) - (\pi/a(L))\bar{u}/u$$

(take differential operator $-\frac{\partial}{\partial x}$ on the formula (3.2) relating $E_1^*(x;L)$ and $E_1(x;u,v)$) and have used the well-known integral formula

$$E_1(x_1+z;u,v) - E_1(x+z;u,v) = \int_{[x,x_1]} -E_2(p+z;u,v)dp,$$

with $|x|, |x_1| < d(L,z)$, and $z \notin L$. Henceforth, adding by Corollary 3.3 the log derivative $\frac{\partial}{\partial z}\log$ of $D_L(x,z)$ to the above expression we obtain by the same way the log derivative $\frac{\partial}{\partial z}\log$ of $D_L(x_1,z)$, which proves the desired result. □

5. Elliptic periodized Bernoulli functions $d_n(z;L)$

We define the elliptic periodized Bernoulli functions $d_n(z,L)$ as follows. They are elliptic analogues of the classical Bernoulli periodized functions.

Definition 5.1. For $|\, x \,| < d(L,z)$ we have the Cauchy series

$$xD_L(x,z) = 1 + \sum_{n\geq 1} d_n(z;L)x^n.$$

Lemma 5.2. *Under the same hypothesis, we have*

$$f(x) := E_1^*(x+z;L) - E_1^*(x;L) + \frac{1}{x} = \sum_{k\geq 1}(-1)^{k-1}s_k(z;L)x^{k-1}$$

where

$$s_k(z;L) = E_k^*(z;L) - e_k^*(L), \;\; k \geq 1,$$

with $E_k^*(z;L) = \frac{1}{(k-1)!}\left(-\frac{\partial}{\partial z}\right)^{k-1} E_1^*(z;L)$, the Eisenstein series of weight k; for $k \geq 3$ it is given by the absolutely convergent series

$$E_k^*(z;L) := \sum_w \frac{1}{(z+w)^k}, \ z \notin L \tag{5.1}$$

5.1. $d_n(z;L)$ and Eisenstein-Kronecker's series. We link the elliptic periodized Bernoulli functions $d_n(z;L)$ to Eisenstein-Kronecker's series as follows.

Proposition 5.3. *We have*

$$d_n(z;L) = (-1)^{n-1} \sum_{\omega \neq 0} \frac{e(E_L(z,\omega))}{\omega^n} = (-1)^{n-1} K_n(0,z,n). \tag{5.2}$$

Proof. Two ways of proof: write, in case $z \notin L$, the expression $xD_L(x,z)$ in term of Eisenstein summation; that is to say for (u,v) a $\mathbb{Z}$-basis of L, with u choosen so that $E_L(z,u) \notin L$, write

$$xD_L(x,z) = \sum_\nu \left(\sum_\mu \frac{xe\left(E_L(z, \mu u + \nu v)\right)}{x + \mu u + \nu v} \right), \ x \notin L,$$

and develop the double sum in powers of the variable x.
For general z, use the Leibnitz rule to write

$$\frac{1}{n!}\left(\frac{\partial}{\partial x}\right)^n (xD_L(x,z)) \\ = (-1)^n \left(x \left[K_{n+1}(x,z,n+1) - \frac{1}{x^{n+1}} \right] - \left[K_n(x,z,n) - \frac{1}{x^n} \right] \right)$$

Now when x tend to 0, $x \neq 0$, by [10, Chap. VIII, p.81] the two bracketed terms of the right member tend respectively to $K_{n+1}(0,z,n+1)$ (resp. $K_n(0,z,n)$); meanwhile the left member tend to $d_n(z;L)$. Hence the equality

$$d_n(z;L) = (-1)^{n-1} K_n(0,z,n).$$

□

One can see that $d_n(z;L)$ is the elliptic analogue to the periodized Bernoulli functions

$$\frac{(2\pi i)^n}{n!} B_n(\{z\}) = (-1)^{n-1} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{e(-kz)}{k^n} \ , \ z \notin \mathbb{Z} \ . \tag{5.3}$$

Those elliptic analogue are considered by K. Katayama [4] to prove Von Staudt congruences, also T. Machide in [5] obtained *Dedekind reciprocity laws* and in [6] proved *recurrences formulas* of order ≥ 2.

5.2. Recurrences formulas. The elliptic periodized Bernoulli functions $d_n(z;L)$ satisfy the following interesting formulas.

Theorem 5.4. *For any $n \geq 1$, we have*

$$(-1)^n d_n(z;L) + \frac{1}{n} s_n(z;L) + \frac{1}{n}\sum_{k=1}^{n-1}(-1)^k d_k(z;L) s_{n-k}(z;L) = 0\ , \quad z \notin L\ ; \tag{5.4}$$

we have also

$$K_{n+1}(x,z,n+1) + \frac{1}{n}\sum_{k=0}^{n-1} K_{k+1}(x,z,k+1) s_{n-k}(x,z;L) = 0, \quad x,z,x+z \notin L, \tag{5.5}$$

where

$$s_k(x,z;L) = E_k^*(x+z;L) - E_k^*(x;L).$$

Proof. The first formula (5.4) comes from searching (by the method of indeterminates coefficients) the relation existing between the coefficient of $xD_L(x,z)$, for $\mid x \mid < d(L;z)$, and its logarithmic derivative on the variable x, that is the coefficient of the function

$$f(x) = E_1^*(x+z;L) - E_1^*(x;L) + \frac{1}{x}, \text{ for } \mid x \mid < d(L;z).$$

The formula (5.4) is valid for any $z \in \mathbb{C}\backslash L$, and any integer $n \geq 1$.
The second formula (5.5) comes by iterating n times the differential operator $-\frac{\partial}{\partial x}$ on both members of the equality of Theorem 3.2. In fact one uses the formulas

$$-\frac{\partial}{\partial x} K_k(x,z,1) = kK_{k+1}(x,z,k+1),$$

and

$$-\frac{\partial}{\partial x}\left(E_k^*(x;L) - E_k^*(-y;L)\right) = k\left(E_{k+1}^*(x;L) - E_{k+1}^*(-y;L)\right)$$

where $y = -(x+z)$, any integer $k \geq 1$. The formula (5.5) is valid for any $x, z, x+z \notin L$, and any integer $n \geq 1$.

From Theorem 4.1, for $\mid x \mid < d(L,z)$, we get

$$1 + \sum_{n\geq 1} d_n(z;L)x^n = \exp\left(\sum_{k\geq 1}(-1)^{k-1} s_k(z;L)\frac{x^k}{k}\right)\ . \tag{5.6}$$

Thus the desired recurrence relation (5.5) is obtained by comparing the coefficients of x^n of the series (5.6). Moreover, assuming $0 <\mid x \mid, \mid x_1 \mid <$

$d(L,z)$ and $\mid x_1 - x \mid < d(L,x)$, we also get

$$
\begin{aligned}
&K_1(x,z,1) + \sum_{n\geq 1}(-1)^n K_{n+1}(x,z,n+1)(x_1-x)^n \\
&\qquad = K_1(x,z,1)\ \exp\left(\sum_{k\geq 1}(-1)^{k-1} s_k(x,z;L)\frac{(x_1-x)^k}{k}\right) .
\end{aligned}
\tag{5.7}
$$

Therefore we get the second recurrence relation (5.5) by comparing the coefficients of $(x_1 - x)^n$ of the series (5.7). □

5.3. Comments on $d_n(z;L)$, $G^*\left(x,z;\frac{1}{2}+\frac{n}{2}\right)$ and $G^*\left(x,0;\frac{k}{2}\right) - G^*\left(-y,0;\frac{k}{2}\right)$.

Note that the formula (5.4) answers the question of how to construct recursively the $d_n(z;L)$, $n \geq 1$. A first occurence of this question was given in [2]; see also [1] for more explicit formulation. Moreover the formula (5.5) is new: it relies, when $x+y+z \in L$ with $x,y,z \notin L$, the values $K_{n+1}(x,z,n+1), n \geq 0$, of Kronecker's series to the values of Eisenstein series

$$E_k^*(x;L) - E_k^*(-y;L) = K_k(x,0,k) - K_k(-y,0,k), k \geq 1.$$

In fact the above formulas can be rewritten in terms of

$$G^*\left(x,z;\frac{1}{2}+\frac{n}{2}\right) \quad \text{and} \quad G^*\left(x,0;\frac{k}{2}\right) - G^*\left(-y,0;\frac{k}{2}\right).$$

For example, we have:

Corollary 5.5. *Let x,y,z be complex numbers such that $x+y+z \in L$, and $x,y,z \notin L$. Then, for any integer $n \geq 1$, we have the relations*

$$
\begin{aligned}
&G^*\left(x,z;\frac{n+1}{2}\right) \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k} G^*\left(x,z;\frac{k+1}{2}\right)\left[G^*\left(x,0;\frac{n-k}{2}\right) - G^*\left(-y,0;\frac{n-k}{2}\right)\right],
\end{aligned}
\tag{5.8}
$$

where $\binom{n-1}{k}$ are the binomial coefficients that is $\frac{(n-1)!}{k!(n-k-1)!}$.

They can be written together: if $0 < \mid x \mid < d(L,z)$, so that $x \notin L, y \notin L$, we obtain assuming $0 < \mid x_1 \mid < d(L,z)$ and $\mid x_1 - x \mid < d(L,x)$, the relation

(5.9)

$$
\begin{aligned}
&G^*\left(x,z;\frac{1}{2}\right) + \sum_{n\geq 1}(-1)^n G^*\left(x,z;\frac{n+1}{2}\right)\frac{t^n}{n!} \\
&= G^*\left(x,z;\frac{1}{2}\right) \exp\left(\sum_{k\geq 1}^{n-1}(-1)^k\binom{n-1}{k}\left[G^*\left(x,0;\frac{k}{2}\right) - G^*\left(-y,0;\frac{k}{2}\right)\right]\frac{t^k}{k!}\right),
\end{aligned}
$$

with $t = \left(\frac{\pi}{a(L)}\right)^{1/2}(x_1 - x)$.

6. Some notes on the limiting case

We consider the complex lattice $L_\tau = \mathbb{Z} + \mathbb{Z}\tau$ with $\mathrm{Im}(\tau) > 0$. For a complex z we write $z = z_1\tau + z_2$ with $z_1, z_2 \in \mathbb{R}$. Let $\{z_1\}$ be the fractional part of z_1. When $\mathrm{Im}(\tau) \to \infty$ we get classical periodized Bernoulli functions and numbers. More precisely, we have

Proposition 6.1. *Let x be a real number, $x \notin \mathbb{Z}$. Then, for any $z \in \mathbb{C}\backslash L_\tau$, we have*

i) *if $\{z_1\} \neq 0$, it comes*

$$\lim_{\mathrm{Im}(\tau)\to\infty} D_{L_\tau}(x,z) = \frac{2\pi i\ e(x\{z_1\})}{e(x)-1} = e\Big(B_1(\{z_1\})x\Big)\frac{\pi}{\sin(\pi x)}, \tag{6.1}$$

ii) *if $\{z_1\} = 0$, we obtain*

$$\lim_{\mathrm{Im}(\tau)\to\infty} D_{L_\tau}(x,z) = \epsilon_1(x) + \epsilon_1(z_2), \tag{6.2}$$

with $\epsilon_1(t) = \pi\cot(\pi t)$.

We also have :

Proposition 6.2.

i) *For $\{z_1\} \neq 0$, we have*

$$\lim_{\mathrm{Im}(\tau)\to\infty} E_1^*(z_1\tau + z_2; L_\tau) = 2\pi i B_1(\{z_1\})$$

uniformly on z_2.

ii) *For $\{z_1\} = 0$, we have*

$$\lim_{\mathrm{Im}(\tau)\to\infty} E_1^*(z_1\tau + z_2; L_\tau) = \epsilon_1(z_2)$$

uniformly on z_2 in any compact subset of $\mathbb{R}\backslash\mathbb{Z}$.

Proof. When $\{z_1\} \neq 0$, we first use the functional equation

$$E_1^*(z; L_\tau) = K_1(z,0,1) = K_1(0,z,1),$$

and further, by [10, Chap. VIII, p. 81], we write

$$K_1(0,z,1) = \lim_{\substack{x\to 0\\ x\notin L_\tau}} \left(K_1(x,z,1) - \frac{1}{x}\right).$$

To simplify the expression $K_1(x,z,1)$ with $x, z \notin L$, use the Eisenstein summation to let appear the expression

$$\sum_{k\in\mathbb{Z}} \frac{e(-kz_1)}{z_2 + k}$$

the other terms tending uniformly to 0.

When $\{z_1\} = 0$, directly use the Eisenstein summation to let appear the expression

$$\sum_{k\in\mathbb{Z}} \frac{1}{z_2+k}.$$

Note also that Theorem 4.1 may be used to deduce Proposition 6.1 from Proposition 6.2. □

Corollary 6.3. *Let $x, z \in \mathbb{R}$, such that $0 < |x| < |z| < 1$. Then we have*

$$x\left(\epsilon_1(x)+\epsilon_1(z)\right) = \exp\left\{\int_{[0,x]}\left(\epsilon_1(p+z)-\epsilon_1(p)+\frac{1}{p}\right)dp\right\}. \tag{6.3}$$

For the statement of the next result, we write formally

$$\log(1+t) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^n}{n!}\ .$$

Theorem 6.4.

i) *For any complex z we have*

$$\log\left(1+2\pi i x B_1(z)+\sum_{n\geq 2} B_n(z)\frac{(2\pi i x)^n}{n!}\right) = 2\pi i x B_1(z) - \sum_{m\geq 1}\frac{B_{2m}}{2m}\frac{(2\pi i x)^{2m}}{(2m)!} \tag{6.4}$$

ii) *For $z \in \mathbb{C}\backslash\mathbb{Z}$, we have*

$$\log\left(1+\epsilon_1(z)x+\sum_{n\geq 2} B_n\frac{(2\pi i x)^n}{n!}\right) = \epsilon_1(z)x+\sum_{j\geq 2}(-1)^{j-1}\epsilon_j(z)\frac{x^j}{j!} - \sum_{m\geq 1}\frac{B_{2m}}{2m}\frac{(2\pi i x)^{2m}}{(2m)!}, \tag{6.5}$$

where the trigonometric functions

$$\epsilon_j(z) := \sum_{k\in\mathbb{Z}}(z+k)^{-j},$$

with integer $j \geq 1$, satisfy the identity

$$\int_{[0,x]}\epsilon_1(p+z)\ dp = \epsilon_1(z)x+\sum_{j\geq 2}(-1)^{j-1}\epsilon_j(z)\frac{x^j}{j},$$

with, in this relation, $z \in \mathbb{R}\backslash\mathbb{Z}$ and x being real, satisfying $\mid x \mid < d(\mathbb{Z}, z)$.

Proof. It's an immediate consequence of Theorem 4.1, Proposition 6.1 and Proposition 6.2. □

For the second formula (6.5), despite its classical aspect, we did not find any convenient reference.

References

[1] A. BAYAD, "Jacobi forms in two variables: Analytic theory and elliptic Dedekind sums", 2011.

[2] A. BAYAD & G. ROBERT, "Note sur une forme de Jacobi méromorphe", *C. R. Acad. Sci. Paris Sér. I Math.* **325** (1997), no. 5, p. 455-460.

[3] H. ITO, "A function on the upper half space which is analogous to the imaginary part of $\log \eta(z)$", *J. Reine Angew. Math.* **373** (1987), p. 148-165.

[4] K. KATAYAMA, "On the values of Eisenstein series", *Tokyo J. Math.* **1** (1978), no. 1, p. 157-188.

[5] T. MACHIDE, "An elliptic analogue of generalized Dedekind-Rademacher sums", *J. Number Theory* **128** (2008), no. 4, p. 1060-1073.

[6] ———, "Sums of products of Kronecker's double series", *J. Number Theory* **128** (2008), no. 4, p. 820-834.

[7] R. SCZECH, "Dedekindsummen mit elliptischen Funktionen", *Invent. Math.* **76** (1984), no. 3, p. 523-551.

[8] C. L. SIEGEL, *Advanced analytic number theory*, second ed., Tata Institute of Fundamental Research Studies in Mathematics, vol. 9, Tata Institute of Fundamental Research, Bombay, 1980, v+268 pages.

[9] A. WEIL, *Introduction à l'étude des variétés kählériennes*, Publications de l'Institut de Mathématique de l'Université de Nancago, VI. Actualités Sci. Ind. no. 1267, Hermann, Paris, 1958, 175 pages.

[10] ———, *Elliptic functions according to Eisenstein and Kronecker*, Springer-Verlag, Berlin-New York, 1976, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 88, ii+93 pages.

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