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Generalized jacobians and Pellian polynomials

par Daniel BERTRAND

On the occasion of A. Thue's 150th birthday

Résumé. Bien qu'elles aient une infinité de solutions, on peut voir les équations de Pell-Fermat comme des ancêtres des équations de Thue. L'analogie se resserre lorsqu'on les étudie sur les anneaux de polynômes en caractéristique nulle. Nous poursuivons l'étude entreprise par D. Masser et U. Zannier dans ce cadre, en considérant le cas de discriminants admettant une racine double.

Abstract. Pell equations over the ring of integers are the forerunners of Thue equations. In fact, they too often have only finitely many solutions, when set over polynomial rings in characteristic zero. How often this happens has been the theme of recent work of D. Masser and U. Zannier. We pursue this study by considering Pell equations with non square-free discriminants over such rings.

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1. Introduction

Let k be a field, and let V be an absolutely irreducible variety over k. Let further R be a finitely generated k-subalgebra of the field $F = k(V)$ of rational functions on V, and let R^* be the group of units of R. Then, R^*/k^* is a finitely generated group (see [11], 2.7). Combined with Dirichlet's theorem on units in number fields, this implies Samuel's classical theorem that the group of units of a reduced ring, finitely generated over $\mathbb{Z}$, is itself finitely generated. See [9] for extensions of this result in the setting of schemes.

Computing the rank of R^*/k^* is another matter, which strongly depends on the base field k. For instance, if k is a finite field, and if V/k is an affine curve, with a given number, say $s+1$, of k-rational places at infinity, then, the group of units of the subalgebra $R = k[V]$ of $k(V)$ has rank s, in perfect analogy with Dirichlet's theorem. But when k has characteristic 0 and $k(V)$ has positive genus, the rank of R^*/k^*, although always bounded from above by s, can be much smaller : it is governed in a (theoretically) computable way by elements in the class-group of R, and it is an interesting question to try and determine it explicitely on any concretely given curve V.

As a corollary of their work on the relative Manin-Mumford conjecture (see in particular [15], [18], [19]), D. Masser and U. Zannier have recently addressed this problem when

$$k = \overline{\mathbb{Q}}$$

is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and V runs through an algebraic family of algebraic curves defined over k. Their work encompasses the study of a one-parameter family $\{V_\lambda, \lambda \in S(k)\}$ of plane curves

$$(V_\lambda) : v^2 = D_\lambda(t), \quad D_\lambda(t) = D(\lambda, t),$$

where $D \in k[\eta, t]$ is a polynomial in two variables of positive degree d in t, the parameter space S is a cofinite subset of the affine line, while at the generic point η of S, with

$$K = k(\eta) = k(S),$$

the following conditions are satisfied :

(i) the curve V_η/K is irreducible over $\overline{K}$ and has $s+1 = 2$ places at infinity, defined over K, i.e. the polynomial $D_\eta(t) \in K[t]$ is not a square, its degree d is even, and a K-rational homothety on t turns it into a monic polynomial;

(ii) V_η is non-singular, i.e. $D_\eta \in K[t]$ *has no square factor*, so the genus of the field $F = K(V_\eta)$ is $g = \frac{1}{2}d - 1$.

Restricting to a non-empty open subset of S if necessary, we can assume that these conditions are satisfied at all closed points $\lambda \in S(k)$. For each

such λ, the field $k(V_\lambda)$ is a quadratic extension of $k(t)$, the ring $R_\lambda := k[V_\lambda]$ is $k[t, \sqrt{D_\lambda(t)}]$ and up to a constant factor, its units $X(t)+Y(t)v$ are given by the solutions of Pell's equation in polynomial unknowns

$$X^2 - D_\lambda(t)Y^2 = 1, \text{ with } X, Y \in k[t].$$

From our introductory remark, the rank of R_λ^*/k^* is at most $s = 1$. It is equal to 1 if and only if this Pell equation has a solution (X, Y) with $Y \neq 0$, in which case, following [19], we will say that $D_\lambda \in \overline{\mathbb{Q}}[t]$ is a *Pellian polynomial*. As in the classical case, there then exists a solution $(X_1^\lambda, Y_1^\lambda)$ with minimal degrees such that on setting $(X_1^\lambda + \sqrt{D_\lambda}Y_1^\lambda)^n = X_n^\lambda + \sqrt{D_\lambda}Y_n^\lambda, n \in \mathbb{Z}_{\geq 1}$, all non trivial solutions take the shape $(\pm X_n^\lambda, \pm Y_n^\lambda)$.

Let $\infty_+(\lambda), \infty_-(\lambda)$ be the two places at infinity of V_λ. By a standard criterion, recalled in Proposition 1 below, D_λ is Pellian if and only if the point

$$p(\lambda) = \{\text{class of the divisor } (\infty_+(\lambda)) - (\infty_-(\lambda))\} \in \mathrm{Pic}^0(\hat{V}_\lambda)$$

in the jacobian $\mathrm{Pic}^0(\hat{V}_\lambda/k)$ of the smooth completion $\hat{V}_\lambda$ of the curve V_λ is a torsion point. The relative Manin-Mumford conjecture predicts how often this occurs, and Masser and Zannier deduce from their work [15] on abelian surface schemes that for $g = 2$, i.e. in *the sextic case* $d = 6$, *the set of parameters*

$$S_D := \{\lambda \in S(\overline{\mathbb{Q}}), D_\lambda(t) \text{ is Pellian}\}$$

is finite, unless we are in one of the exceptional cases deduced from the list of Theorem 1 below. On the other hand, all D_λ's are Pellian if $d = 2$. As for *the quartic case* $d = 4$, it usually leads to an *infinite, but sparse* set S_D, in the sense that for any integer δ, the set S_D contains only finitely many algebraic numbers λ of degree $\leq \delta$ over $\mathbb{Q}$. In fact, more is true (cf. [18], 3.2.3, [4], 3.1) : all the sparse sets we will meet in the present paper have bounded height. See [19] for a general survey on these problems, and [10], [8], for connections with the arithmetic case and with Zolotarev polynomials.

Now, what happens if the curve under study has a singularity? Not surprisingly, generalized jacobians provide the required answer. For the sake of clarity, let us call W such a singular curve, with normalisation $V \to W$ and completions $\hat{V} \to \hat{W}$, and let us assume that W has a unique singularity which is nodal over k, so defines two k-rational points $\mathfrak{q}^+, \mathfrak{q}^-$ on the normalized curve V. Then, the generalized jacobian $\mathrm{Pic}^0(\hat{W}/k)$ is a semi-abelian variety, which is an extension

$$1 \to \mathbb{G}_m \to \mathrm{Pic}^0(\hat{W}) \to \mathrm{Pic}^0(\hat{V}) \to 0$$

of the standard jacobian $\mathrm{Pic}^0(\hat{V}/k)$ by $\mathbb{G}_m$, and the isomorphism class of this extension is given by the class q of the divisor $(\mathfrak{q}^+) - (\mathfrak{q}^-)$ in (the dual of) $\mathrm{Pic}^0(\hat{V})$.

In the above setting with hyperelliptic curves and with $k = \overline{\mathbb{Q}}$, this brings us to the study of a family of Pell equations with discriminants $D_\lambda(t)$ of the form

$$D_\lambda(t) = (t - \rho(\lambda))^2 \Delta_\lambda(t),\ \lambda \in S(k),$$

where $\rho \in k(S) = K$ is a rational (or more generally, $\rho \in \overline{K}$ is an algebraic) function of the parameter, and Δ_λ is a separable polynomial which does not vanish at $\rho(\lambda)$ (see §4.1 for a more precise formulation). The corresponding affine curves are

$$(W_\lambda) : u^2 = D_\lambda(t)\ ;\ (V_\lambda) : v^2 = \Delta_\lambda(t),$$

and $\mathfrak{q}_\pm(\lambda)$ are the two distinct points on $V_\lambda(k)$ with abscissa $t = \rho(\lambda)$. Clearly, for any value of $\lambda \in S(k)$, the polynomial $D_\lambda(t)$ is Pellian if and only if $\Delta_\lambda(t)$ is Pellian *and* one of the non-trivial solutions $(X = X_n^\lambda, Y = Y_n^\lambda) \in k[t] \times k[t]$ of $X^2(t) - \Delta_\lambda(t)Y^2(t) = 1$ satisfies : $Y(\rho(\lambda)) = 0$, i.e. defines a regular function $X(t) + \frac{Y(t)}{(t-\rho(\lambda))}u$ on the affine singular curve W_λ.

The above criterion then still holds for D_λ to be Pellian, at the cost of replacing $p(\lambda)$ by the point

$$\tilde{p}(\lambda) = \ [\text{class of the divisor } (\infty_+(\lambda)) - (\infty_-(\lambda))] \in \mathrm{Pic}^0(\hat{W}_\lambda)$$

in the generalized jacobian $\mathrm{Pic}^0(\hat{W}_\lambda/k)$. From my recent joint work [4] with D. Masser, A. Pillay and U. Zannier on the relative Manin-Mumford conjecture for semi-abelian surface schemes, finiteness statements for the set S_D of non-separable Pellian sextic D_λ's can again be obtained (for a study of quartics and over $\mathbb{Q}$, see [16], §3). After recalling these results in Theorem 2 of §3.2, we will sharpen them in §4 by showing that one of the families of possibly exceptional cases alluded to in [4], Appendix II, is actually a finite set, and by constructing, on the converse side, an "extremely exceptional" family of non-separable Pellian sextics : see Examples 1 and 2 just below.

For the classical Pell equations over $\mathbb{Z}$, such problems correspond to non-square free numbers $D = f^2\Delta$, boiling down to the study of units of a non-maximal order R of the quadratic field $F = \mathbb{Q}(\sqrt{\Delta})$. Now, the ranks of the unit groups do not change when we go from the maximal order $\mathcal{O}$ of a number field F to a non-maximal one R : indeed, R^* contains the kernel of the projection of $\mathcal{O}^*$ to the finite group $(\mathcal{O}/n\mathcal{O})^*$, where n denotes the (finite) index of R in $\mathcal{O}$. This finiteness has no counterpart when R is replaced $k[W]$, and $\mathcal{O}$ by its integral closure $k[V]$ in $k(W)$, with $k = \overline{\mathbb{Q}}$. So, one can guess that even if Δ_λ is Pellian, the non-separable discriminant D_λ will rarely be so. Or said in geometric terms : even if the projection $p(\lambda)$

of $\tilde{p}(\lambda)$ to $\mathrm{Pic}^0(\hat{V}_\lambda)(\overline{\mathbb{Q}})$ is a torsion point, the point $\tilde{p}(\lambda)$ will usually have infinite order in $\mathrm{Pic}^0(\hat{W}_\lambda)(\overline{\mathbb{Q}})$. The case studies of §4 illustrate the scope of this expression, resp. its limits, as follows :

Example 1 (see Corollary 2). - *Let S_Δ be the set of algebraic numbers λ such that*

$$\Delta_\lambda(t) = t^4 + t + \lambda \in \overline{\mathbb{Q}}[t]$$

is a Pellian polynomial. Then, S_Δ is infinite, but for any algebraic function ρ outside a finite (and conjecturally empty) subset of $\overline{K}$, there are only finitely many λ's in S_Δ such that the polynomial $(t-\rho(\lambda))^2\Delta_\lambda(t)$ is Pellian. For instance, there are only finitely many λ's in S_Δ such that the polynomial $(t+\frac{1}{2})^2\Delta_\lambda(t)$ is Pellian.

Example 2 (see Corollary 3). - *Let $S_{\Delta'}$ be the set of algebraic numbers $\lambda \neq 0, 1$ such that*

$$\Delta'_\lambda(t) = t^4 + (2\lambda+1)t^3 + 3\lambda t^2 + \lambda t \in \overline{\mathbb{Q}}[t]$$

is a Pellian polynomial. Then, $S_{\Delta'}$ is infinite. Moreover, for all λ's in $S_{\Delta'}$, the polynomial $(t+\frac{1}{2})^2\Delta'_\lambda(t)$ too is Pellian. However, there exists an infinite sequence of points λ_ℓ in $S_{\Delta'}$ such that among the solutions of the polynomial equation $X^2 - \Delta'_{\lambda_\ell}Y^2 = 1$, the proportion of those satisfying $Y(-\frac{1}{2}) = 0$ tends to 0 when ℓ tends to ∞.

In both cases, the infinite sets $S_\Delta, S_{\Delta'}$ are sparse in the sense given above. As will become apparent in §3.2, it is its last conclusion which then makes Example 2 so exceptional. We will express this in a more precise way as a comparison between the degrees of the fundamental solutions of the corresponding Pell equations.

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2. Generalized jacobians

Here k denotes a field of characteristic 0. Following [17] (see also [6] §10.3, [5] §9.2.8, [12] §7.5, [3]), we begin by recalling the construction of the generalized jacobian of the semi-stable curve $\hat{W}$. Instead of blowing up the singularites of $\hat{W}$, we start with the normal curve $\hat{V}$. Collapsing pairs of points and removing other points from $\hat{V}$ then provides us with the affine

curves W, V we want to study (and with a description of the one-motive attached to the affine singular curve W).

2.1. Affine singular curves. So, let $\hat{V}$ be a complete smooth geometrically irreducible curve of positive genus g over k, with field of rational functions $F = k(\hat{V})$, and let $\mathrm{Pic}^0(\hat{V}/k)$ be its jacobian variety. This abelian variety represents the set of equivalence classes $\{D\}$ of divisors D of degree 0 on $\hat{V}$, for the standard ("weak") equivalence relation

$$\{D\} = 0 \Leftrightarrow \exists f \in F^*, \mathrm{div}(f) = D.$$

Let further $\mathfrak{p}_0, ..., \mathfrak{p}_s$ be $s+1$ points in $\hat{V}(k)$, let V be the complement in $\hat{V}$ of this set of points, and for $i = 1, ..., s$, let p_i, be the class in $\mathrm{Pic}^0(\hat{V})$ of the degree 0 divisor $(\mathfrak{p}_i) - (\mathfrak{p}_0)$.

Let finally $\mathcal{Q} = \{(\mathfrak{q}_j^+, \mathfrak{q}_j^-), j = 1, ..., r\}$ be a set of r pairs of distinct points of $V(k)$, and let $\hat{W}$ be the curve obtained by pinching $\hat{V}$ at these pairs. Then $\hat{W}$ and the complement W of $\mathfrak{p}_0, ..., \mathfrak{p}_s$ in $\hat{W}$ are singular curves, all of whose r singular points are double points with k-rational tangents, cf. [17], IV, §1, and V, resp. $\hat{V}$, are the normalizations of W, resp. $\hat{W}$. In particular, the ring $R := k[V]$ of regular functions on the smooth affine curve V is the integral closure in $F = k(V) = k(W) = k(\hat{V}) = k(\hat{W})$ of the ring $S := k[W]$ of regular functions on W, and the arithmetic genus of $\hat{W}$ is equal to $g + r$.

Let $\mathrm{Pic}^0(\hat{W})$ be the generalized jacobian of the complete singular curve $\hat{W}$. This algebraic group represents the set of equivalence classes $[D]$ of divisors D of degree 0 on $\hat{V}$, with support disjoint from $\mathcal{Q}$, for the "strict" equivalence relation

$$[D] = 0 \Leftrightarrow \exists f \in F^*_{\mathcal{Q}}, \mathrm{div}(f) = D, \text{ and } \forall j = 1, ..., r, \frac{f(\mathfrak{q}_j^+)}{f(\mathfrak{q}_j^-)} = 1,$$

where $F^*_{\mathcal{Q}} \subset F^*$ is the subgroup of functions f on $\hat{V}$ whose divisors avoid $\mathcal{Q}$. We point out that since $(\mathfrak{q}_j^+) - (\mathfrak{q}_j^-)$ has degree 0, the latter quotients depend only on $\mathrm{div}(f)$.

A moving lemma (see [12], 9.1.1) ensures that the natural map $[D] \to \{D\}$ induces a surjection $\mathrm{Pic}^0(\hat{W}) \to \mathrm{Pic}^0(\hat{V})$, which is a morphism of algebraic groups, whose kernel T consists of the strict classes $[\mathrm{div}(f)]$ of divisors of rational functions in $F^*_{\mathcal{Q}}$. The map $(f \in F^*_{\mathcal{Q}}) \mapsto ((\frac{f(\mathfrak{q}_1^+)}{f(\mathfrak{q}_1^-)}, ..., \frac{f(\mathfrak{q}_r^+)}{f(\mathfrak{q}_r^-)}) \in (k^*)^r)$ identifies T with the torus $(\mathbb{G}_m)^r$, and

$$\{1\} \to (\mathbb{G}_m)^r \to \mathrm{Pic}^0(\hat{W}) \to \mathrm{Pic}^0(\hat{V}) \to \{0\}$$

is a semi-abelian variety, extension of the abelian variety $\mathrm{Pic}^0(\hat{V})$ by $(\mathbb{G}_m)^r$.

The group of isomorphism classes of extensions of an abelian variety A by $\mathbb{G}_m$ is represented by the abelian variety dual to A : for any rational section σ of an extension G, the equivalence class of the divisor of σ in $\mathrm{Pic}^0(A)$ does not depend on the choice of σ, and represents G. Using the description of jacobians and generalized jacobians via symmetric products (see [17], V.9), and the self-duality $\mathrm{Pic}^0(\mathrm{Pic}^0(\hat{V})) \simeq \mathrm{Pic}^0(\hat{V})$, one deduces that up to signs, the semi-abelian variety $\mathrm{Pic}^0(\hat{W})$ is the extension of $\mathrm{Pic}^0(\hat{V})$ by $(\mathbb{G}_m)^r$ represented by the r-tuple of (weak) equivalence classes

$$q_j = \{(\mathfrak{q}_j^+) - (\mathfrak{q}_j^-)\} \in \mathrm{Pic}^0(\hat{V}) \ ; \ j = 1, ..., r.$$

For each $i = 1, ..., s$, let

$$\tilde{p}_i = [(\mathfrak{p}_i) - (\mathfrak{p}_0)] \in \mathrm{Pic}^0(\hat{W})$$

be the strict equivalence class of the divisor $(\mathfrak{p}_i) - (\mathfrak{p}_0)$, which is prime to $\mathcal{Q}$ by construction. The projection

$$p_i = \{(\mathfrak{p}_i) - (\mathfrak{p}_0)\} \in \mathrm{Pic}^0(\hat{V})$$

of $\tilde{p}_i$ to $\mathrm{Pic}^0(\hat{V})$ is the previously introduced point p_i. Since we will not need this notion, we merely mention in passing that the one-motive attached to the curve W is the complex $[\mathbb{Z}^r \to \mathrm{Pic}^0(\hat{W}/k)]$, sending the i-th basis element of $\mathbb{Z}^r$ to $\tilde{p}_i$; see [6], 10.3, and more generally, [1].

2.2. Ranks of units. We can now state the well-known

Proposition 1. *i) The rank of $(k[V])^*/k^*$ is $\leq s$, and attains the value s if and only all the weak divisor classes $p_1, ..., p_s$ in the jacobian $\mathrm{Pic}^0(\hat{V})$ are torsion points.*

ii) The rank of $(k[W])^*/k^*$ is $\leq s$, and attains the value s if and only all the strict divisor classes $\tilde{p}_1, ..., \tilde{p}_s$ in the generalized jacobian $\mathrm{Pic}^0(\hat{W})$ are torsion points.

iii) More generally, these ranks are equal (i.e. the group $(k[V])^*/(k[W])^*$ is finite) if and only for every rational function $f \in F^*$ with $\mathrm{div}(f)$ supported by the points $\mathfrak{p}_0, ..., \mathfrak{p}_s$, the numbers $\frac{f(\mathfrak{q}_j^+)}{f(\mathfrak{q}_j^-)}, i = 1, ..., r$, are all roots of unity.

iv) Restricting for simplicity to the case $s = 1$, assume that $(k[W])^*/k^*$, hence $(k[V])^*/k^*$, have rank 1, and let n_V, resp. n_W be the order of the torsion point p_1, resp. $\tilde{p}_1$. Then, the index $\nu := [(k[V])^* : (k[W])^*]$ is equal to $\frac{n_W}{n_V}$.

Proof : i, ii) let $\mathcal{D}^0_\infty = \{D \in \oplus_{i=0,...,s}\mathbb{Z}.\mathfrak{p}_i, \deg(D) = 0\} \subset \mathrm{Div}^0(\hat{V})$ be the group of degree 0 divisors supported by the points at infinity of $\hat{V}$ (hence disjoint from $\mathcal{Q}$). We then have exact sequences :

$$\{1\} \to k[W]^*/k^* \to \mathcal{D}^0_\infty \to \mathrm{Pic}^0(\hat{W})$$

$$\{1\} \to k[V]^*/k^* \to \mathcal{D}^0_\infty \to \mathrm{Pic}^0(\hat{V}),$$

where the second arrows are the restrictions to the units of the maps $f \mapsto \mathrm{div}(f)$, and the third ones consist in taking the strict and weak equivalence classes $[D], \{D\}$, of the divisor D. Since $\mathcal{D}^0_\infty$ is a $\mathbb{Z}$-module of rank s, the first two conclusions immediately follow.

iii) Let

$$[\mathcal{D}^0_\infty] := \mathbb{Z}\tilde{p}_1 + ... + \mathbb{Z}\tilde{p}_s \subset \mathrm{Pic}^0(\hat{W}), \text{ resp. } \{\mathcal{D}^0_\infty\} := \mathbb{Z}p_1 + ... + \mathbb{Z}p_s \subset \mathrm{Pic}^0(\hat{V})$$

be the image of the third upper, resp. lower, maps, and let $\Delta_{V/W}$ be the kernel of the restriction to $[\mathcal{D}^0_\infty]$ of the natural surjection $\mathrm{Pic}^0(\hat{W}) \to \mathrm{Pic}^0(\hat{V})$ (cf. [13], §9). By, say, the snake lemma, the cokernel $(k[V])^*/(k[W])^*$ identifies with $\Delta_{V/W}$. In particular,

$$rk((k[V])^*/(k[W])^*) = rk((\mathbb{Z}\tilde{p}_1 + ... + \mathbb{Z}\tilde{p}_s) \cap (\mathbb{G}_m)^r).$$

So, the corank of $(k[W])^*$ in $(k[V])^*$ vanishes if and only if any point $[D] \in [\mathcal{D}^0_\infty]$ which projects to a torsion point $\{D\} \in \mathrm{Pic}^0(\hat{V})$ is itself a torsion point in $\mathrm{Pic}^0(\hat{W})$. This is exactly what conclusion (iii) asserts.

iv) In general, the exponent of the group $(k[V])^*/(k[W])^* \simeq \Delta_{V/W}$ is the l.c.m. of the orders of the roots of unity appearing in conclusion (iii). Here, the two unit groups are cyclic modulo k^*, so the index ν is the order of $\Delta_{V/W}$, which is indeed equal to $\frac{n_W}{n_V}$.

Remark 1 (*Heights of fundamental units*) : when the hypotheses $1 = s = rk((k[V])^*/k^*) = rk((k[W])^*/k^*)$ are fulfilled, one can speak of the fundamental units u_V, u_W of $k[V], k[W]$ (well defined up to a constant factor, and an inversion). The index $\nu = [(k[V])^* : (k[W])^*]$ satisfies $u_W \sim (u_V)^{\pm\nu}$. Since the order n_V of the torsion point p_1 of $\mathrm{Pic}^0(V/k)$ is the smallest positive integer n such that $n.(\mathfrak{p}_1) - n(\mathfrak{p}_0) = \mathrm{div}(u_V)$, it is equal to the degree of u_V, viewed as a rational function on $\hat{V}$. The index ν is then the l.c.m. of the orders of the roots of unity $\frac{u_V(\mathfrak{q}_j^+)}{u_V(\mathfrak{q}_j^-)}$, and the order $n_W = \nu n_V$ of $\tilde{p}_1$ is the degree of the rational function u_W. These degrees are functional heights. In the setting of §1, n_{V_λ} is the (polynomial) degree

$$\deg_t(X_1^\lambda) = \frac{d}{2} + \deg_t(Y_1^\lambda) = \frac{1}{2}\deg(u_{V_\lambda} + u_{V_\lambda}^{-1}) = \deg(u_{V_\lambda})$$

of the minimal solution of the Pell equation attached to Δ_λ (see [15], Lemma 10.1), and similarly with n_{W_λ} for $D_\lambda = (t - \rho(\lambda))^2\Delta_\lambda$. In the exceptional cases where the set S_D is infinite, the variation in $\lambda \in S_D$ of the relative complexity

$$\nu(\lambda) := [(k[V_\lambda])^* : (k[W_\lambda])^*]$$

will help us measure how special the family of non-separable polynomials D_λ truly is.

2.3. Isosplit extensions. Proposition 1 shows that the ranks of units are not affected by the description of the (generalized) jacobian itself as a (semi-)abelian variety. However, if we know in advance that the points p_i ($\tilde{p}_i$) lie in a strict algebraic subgroup (meaning : of positive dimension and codimension) of the (generalized) jacobian, the condition that they be torsion becomes easier to fulfill. In the relative situations studied in the next Section, this observation will lead to the exceptional cases mentioned in the introduction.

An abelian variety such as $A = \mathrm{Pic}^0(\hat{V}/k)$ contains a strict abelian subvariety (and is then isogenous to a non trivial product) if and only there is a surjection $\varpi : \hat{V} \to \hat{V}'$ to a curve of genus $g' \in [1, g-1]$; criteria for the points p_i to lie in torsion translates of $\varpi^*(\mathrm{Pic}^0(\hat{V}'/k))$ are then known (see [15], p. 17, for an example). As for the semi-abelian variety $G = \mathrm{Pic}^0(\hat{W}/k)$, it always contains subgroups of the torus T as strict algebraic subgroups (and the criterion for $\tilde{p}_i$ to lie in a torsion translate of T is just that p_i be torsion), while for general values of $q_1, ..., q_r$, no further strict algebraic subgroups appear.

The proposition below describes the extreme case when on the contrary, $q_1, ..., q_r$ are all torsion points, which means that up to an isogeny, $G = \mathrm{Pic}^0(\hat{W}/k)$ is split as an extension of $A = \mathrm{Pic}^0(\hat{V}/k)$ by T, so contains an abelian variety A' of dimension g among its strict algebraic subgroups. Under this condition and recalling the notations of §2.1, we have the following criterion for the points $\tilde{p}_i$ to all lie in torsion translates of A'.

Proposition 2. *let N be a common multiple of the orders of the torsion points $q_1, ..., q_r$, and for $j = 1, ..., r$, let φ_j be a rational function on $\hat{V}$ with divisor $N.(\mathfrak{q}_j^+) - N.(\mathfrak{q}_j^-)$. Then, the pushout $G' = [N]_*G$ of the extension $G = \mathrm{Pic}^0(\hat{W}/k)$ of $A = \mathrm{Pic}^0(\hat{V}/k)$ by $T = (\mathbb{G}_m)^r$ is isomorphic to $T \times A$, and*

i) for each $i = 1, ..., s$, the image of the point $[N]_*(\tilde{p}_i) \in G'$ under the corresponding projection $G' \to T$ is given by $(\frac{\varphi_j(\mathfrak{p}_i)}{\varphi_j(\mathfrak{p}_0)})_{j=1,...,r}^N$.

ii) in particular, the points $\tilde{p}_1, ..., \tilde{p}_s$ all lie in torsion translates of an abelian variety contained in $\mathrm{Pic}^0(\hat{W}/k)$ if and only if for any divisor D of degree 0 supported by the points $\mathfrak{p}_0, ..., \mathfrak{p}_s$, the numbers $\varphi_j(D), j = 1, ..., r$, are all roots of unity.

Proof. - The standard notation $\varphi_j(D)$ in (ii) is recalled below. We also recall that the push-out $G' = [N]_*G$ is the quotient $G/(\mu_N)^r$ of G by the kernel $(\mu_N)^r$ of the multiplication by N on T. The corresponding extension

is parametrized by $(Nq_j = 0, j = 1, ..., r)$, so is indeed isomorphic to $T \times A$. Any abelian variety contained in G (and implicitely assumed to pass through 0) then maps under $[N]_*$ into the factor A of this product, so, (ii) clearly follows from (i).

The splitting $[N]_*G \simeq \mathbb{G}_m \times A$ is given by a regular section $\sigma' : A \to G'$ of the natural projection, which we are going to describe explicitely. First, consider a degree 0 divisor D on $\hat{V}$ prime to $\mathcal{Q}$, and associate to D the class

$$\tilde{\sigma}(D) := [D] -_{(G)} \boldsymbol{\varphi}(D) \in \mathrm{Pic}_0(\hat{W});$$

here, $-_{(G)}$ represents the group law on G, while $[D]$ denotes as before the class of D for the strict equivalence, $\boldsymbol{\varphi} = (\varphi_1, ..., \varphi_r)$ and $\boldsymbol{\varphi}(D) \in (k^*)^r$ is defined componentwise for $\varphi \in k(\hat{V})^*$ and a degree 0 divisor $D = \Sigma_{i=1,...,m} n_i \mathfrak{r}_i$, prime to $\mathrm{div}(\varphi)$, by the formula $\varphi(D) = \Pi_{i=1,...,m} \varphi(\mathfrak{r}_i)^{n_i}$.

We now show that $[N]_* \circ \tilde{\sigma}$ factors through a well defined section of algebraic groups $A = \mathrm{Pic}_0(\hat{V}) \to \mathrm{Pic}_0(\hat{W})/(\mu_N)^r = G'$, which must be the searched for section σ', since $Hom(A, T) = 0$. For this (and appealing in the end to a moving lemma), it suffices to show that $[N]_*\tilde{\sigma}(D)$ depends only on the weak equivalence class $\{D\}$ of D. So, let $D' \in \mathrm{Div}_0(\hat{V})$, still prime to $\mathcal{Q}$, be linearly equivalent to D, and let g be a rational function on $\hat{V}$ such that $D' = D + \mathrm{div}(g)$. Then, $\tilde{\sigma}(D') - \tilde{\sigma}(D) = \tilde{\sigma}(\mathrm{div}(g))$. But as recalled in §2.1, the strict class $[\mathrm{div}(g)] \in T \subset \mathrm{Pic}_0(\hat{W})$ of a principal divisor is given by the element of $(\mathbb{G}_m)^r(k)$ with j-th component $g(\mathfrak{q}_j^+)/g(\mathfrak{q}_j^-)$. Now, appealing to Weil's law of reciprocity for the last equality, we have :

$$(g(\mathfrak{q}_j^+)/g(\mathfrak{q}_j^-))^N = g(N.(\mathfrak{q}_j^+) - N.(\mathfrak{q}_j^-)) = g(\mathrm{div}(\varphi_j)) = \varphi_j(\mathrm{div}(g)).$$

Therefore, the j-th component of the point $[N]_*(\tilde{\sigma}(\mathrm{div}(g)) = \tilde{\sigma}(\mathrm{div}(g^N)) = [\mathrm{div}(g^N)] -_{(G)} \boldsymbol{\varphi}(\mathrm{div}(g)) \in T/(\mu_N)^r \simeq T$, which reads in multiplicative notations as $(\frac{g(\mathfrak{q}_j^+)}{g(\mathfrak{q}_j^-)})^N \times (\varphi_j(\mathrm{div}(g)))^{-1}$, is equal to $1_{\mathbb{G}_m}$, i.e. vanishes. Consequently, $[N]_*(\tilde{\sigma}(D)) = [N]_*(\tilde{\sigma}(D'))$ depends only $\{D\}$, and $\sigma' := [N]_* \circ \tilde{\sigma}$ does define a regular section of $G' \to A$.

Finally, for each $i = 1, ..., s$, let $\tilde{p}_i'$ be the image under $[N]_*$ of the strict class $\tilde{p}_i \in G$ of the divisor $(\mathfrak{p}_i) - (\mathfrak{p}_0)$. Both $\tilde{p}_i'$ and $\tilde{p}_i$ project to the same point $p_i = \{(\mathfrak{p}_i) - (\mathfrak{p}_0)\} \in A$, and by definition, $\tilde{\sigma}((\mathfrak{p}_i) - (\mathfrak{p}_0)) = \tilde{p}_i -_{(G)} \boldsymbol{\varphi}((\mathfrak{p}_i) - (\mathfrak{p}_0))$. Therefore, the retraction $\gamma : G' \to T$ attached to the section σ' satisfies

$$\gamma(\tilde{p}_i') = \tilde{p}_i' -_{(G')} \sigma'(p_i) = [N]_*\big(\tilde{p}_i -_{(G)} \tilde{\sigma}((\mathfrak{p}_i) - (\mathfrak{p}_0))\big) =$$
$$= [N]_* \frac{\boldsymbol{\varphi}(\mathfrak{p}_i)}{\boldsymbol{\varphi}(\mathfrak{p}_0)} = \big(\frac{\boldsymbol{\varphi}(\mathfrak{p}_i)}{\boldsymbol{\varphi}(\mathfrak{p}_0)}\big)^N$$

(equalities in $T(k)$), as asserted in (i).

Remark 2 : assume that in addition to the points $q_1, ..., q_r$, all the points $p_1, ..., p_s$ too are torsion points, of order dividing M, and for $i = 1, ..., s$, let f_i be a rational function such that $M.(\mathfrak{p}_i) - M.(\mathfrak{p}_0) = \mathrm{div}(f_i)$. By Proposition 1.(iii), the points $\tilde{p}_1, ..., \tilde{p}_s$ are then all torsion if and only if the rs numbers $\frac{f_i(\mathfrak{q}_j^+)}{f_i(\mathfrak{q}_j^-)}$ are roots of unity. But by Proposition 2.(ii), since they project to the torsion points $p_1, ..., p_s$ in A, they are all torsion if and only if the sr numbers $\frac{\varphi_j(\mathfrak{p}_i)}{\varphi_j(\mathfrak{p}_0)}$ are roots of unity. That these numerical criteria are equivalent can be checked directly from another appeal to Weil's law of reciprocity. Indeed, for any $i = 1, ..., s, j = 1, ..., r$,

$$(\frac{f_i(\mathfrak{q}_j^+)}{f_i(\mathfrak{q}_j^-)})^N = f_i(\mathrm{div}(\varphi_j)) = \varphi_j(\mathrm{div}(f_i)) = (\frac{\varphi_j(\mathfrak{p}_i)}{\varphi_j(\mathfrak{p}_0)})^M.$$

More generally, Cartier duality on one-motives explains the resemblance between Proposition 1.(iii) and Proposition 2.(ii). See [1], Lemma 3.3, for a more precise statement.

3. Relative Manin-Mumford

From now on, S denotes an affine irreducible curve over the field $k = \overline{\mathbb{Q}}$, with generic point η and field of rational functions

$$K = \overline{\mathbb{Q}}(S) = \overline{\mathbb{Q}}(\eta).$$

For any group scheme G/S and any point $\lambda \in S(\overline{\mathbb{Q}})$, we denote by G_λ the fiber of G above λ, and by G_λ^{tor} the torsion subgroup of $G_\lambda(\overline{\mathbb{Q}})$. If G comes after a finite base extension from an algebraic group G^0 defined over $\overline{\mathbb{Q}}$, we will tacitly make this base extension, say that G is isoconstant, and call constant sections those sections of G/S which come from points of $G^0(\overline{\mathbb{Q}})$. By a multiple of a section, we mean a multiple by a non-zero integer.

Let $p \in G(S)$ be a section of G/S. Among other things (see [14], [18]), the relative Manin-Mumford conjecture studies the set

$$S_p := \{\lambda \in S(\overline{\mathbb{Q}}), p(\lambda) \in G_\lambda^{tor}\}.$$

It is not difficult to include in this study points $\lambda \in S(\mathbb{C})$, but we will leave this aside. One expects that the set S_p will be finite if the relative dimension $dim(G/S)$ is strictly larger than that of the base $dim(S) = 1$. On the other hand, it is an easy, although non trivial, observation that *for $dim(G/S) = 1$, the set S_p is infinite, and sparse*, unless p is a torsion or a constant section (or if $G = \mathbb{G}_{a/S}$). We refer to [18], p. 92, for several proofs of this fact, which will be used repeatedly in what follows under the denomination "*relative dimension 1*". We now turn to the two types of group schemes G/S of relative dimension 2 relevent to our study (for additive extensions, see [19], Remark 1.7 and §4).

Remark 3 (*about the base* S). - i) Our results depend only on the study of the generic fiber G_η/K of G/S, and of $G_\eta \otimes_K \overline{K}$. They can then be expressed in terms of finite base extensions $S' \to S$, including restriction to a non-empty, but unspecified, open subset. In what follow, we allow such extensions, usually without mentioning it, and still call S the resulting base. In other words, although we are now a in relative setting, all the constructions of §2, with K playing the role of k, may be used, at the cost of modifying S. We refer to [5], §8.1, [12], §8.3 and [9] for finer presentations of this relative situation.

ii) An affine curve V, resp. W, over $\overline{\mathbb{Q}}$ as in §2 defines a point in the moduli space $\mathcal{M}_{g,s+1}$, resp. $\overline{\mathcal{M}_{g,s+1}}$, of complete smooth, resp. stable, curves of arithmetic genus g with $s+1$ marked (then deleted) points. Ranks of unit groups modulo $\overline{\mathbb{Q}}^*$ are generically 0, but reach the value s on a subset Σ which is dense for the complex topology. In this perspective, our study can be described as follows : we consider a curve S inside $\overline{\mathcal{M}_{2,2}}$ along which the generic rank is still 0, and we give criteria for the finiteness of $S \cap \Sigma$. It would be desirable to extend these results to higher dimensional subvarieties $\mathcal{S}$ of $\overline{\mathcal{M}_{2,2}}$ as criteria for the non Zariski denseness of $\mathcal{S} \cap \Sigma$. This is out of reach of the present techniques. See however [7] for a related result, and [19], §2.2 for the description of an another moduli space relevent to Pell equations.

3.1. Abelian surfaces. The results of this subsection are due to Masser and Zannier, and are discussed in greater generality in [14], [15], [18], [19], and in their forthcoming work on elementary integration (cf. [19], §2.1 and §4). We here restrict to separable sextics (and quartics) over $\overline{\mathbb{Q}}$.

Theorem 1. ([15]) *Let A/S be an abelian scheme of relative dimension 2, over the curve $S/\overline{\mathbb{Q}}$, and let $p \in A(S)$ be a section. Then the set*

$$S_p := \{\lambda \in S(\overline{\mathbb{Q}}), p(\lambda) \in A_\lambda^{tor}\}$$

is infinite if and only if one of the following conditions holds :

a) *p is a torsion section;*

b) *there exists an elliptic subscheme E/S of A/S such that a multiple of p factors through E, and is not a constant section if E/S is isoconstant.*

In view of §2, this translates in terms of unit groups as follows. Let $\hat{V}/S$ be a family of genus 2 smooth proper curves, whose generic fiber $\hat{V}_\eta/K$ is geometrically irreducible (cf. [12], 8.3.6). Deleting finitely many points from S if necessary (in the spirit of Remark 3.(i)), we will henceforth identify a section $\mathfrak{p}$ in $\hat{V}(S)$ with the horizontal divisor its image $\mathfrak{p}(S)$ defines in $\hat{V}$, and with its generic value $\mathfrak{p}(\eta) \in V_\eta(K)$. Let then $\mathfrak{p}_0, \mathfrak{p}_1 \in \hat{V}(S)$ be two disjoint sections, and let V be the complement of their images in $\hat{V}$. Set

$$S_V = \{\lambda \in S(\overline{\mathbb{Q}}), rk((\overline{\mathbb{Q}}[V_\lambda])^*/\overline{\mathbb{Q}}^*) = 1\},$$

and for $\lambda \in S_V$, denote by $n_V(\lambda) = n_{V_\lambda}$ the height of the fundamental unit of $(\overline{\mathbb{Q}}[V_\lambda])^*$ as in Remark 1. Then S_V *is finite (usual case), unless*

(**a**) (trivial case) there exists a rational function on $\hat{V}$ whose relative divisor is supported by $\mathfrak{p}_1 \cup \mathfrak{p}_0$, in which case $p = \{(\mathfrak{p}_1) - (\mathfrak{p}_0)\}$ is a torsion section of the relative jacobian $\mathrm{Pic}^0(\hat{V}/S)$; the group of generic units $(K[V_\eta])^*/K^*$ then has rank 1, and for all λ's in $S(\overline{\mathbb{Q}})$, $(\overline{\mathbb{Q}}[V_\lambda])^*/\overline{\mathbb{Q}}^*$ too has rank 1, with $n_V(\lambda)$ at most equal to the order of $p(\eta)$ (and in fact equal, since by assumption, $\hat{V}/S$ has good reduction everywhere). So,

$$S_V = S(\overline{\mathbb{Q}}), \text{ and } n_V(\lambda) \text{ is bounded as } \lambda \text{ runs through } S_V.$$

(**b**) (exceptional case) there is a surjection ϖ from $\hat{V}$ to a genus 1 relative curve $\hat{V}'$ over S such that a multiple of p factors through $\varpi^* : \mathrm{Pic}^0(\hat{V}'/S) \to \mathrm{Pic}^0(\hat{V}/S)$. Then, we are in fact in a "relative dimension 1" case, so leaving aside the constant cases, infinitely, but sparsely, many groups $(\overline{\mathbb{Q}}[V_\lambda])^*/\overline{\mathbb{Q}}^*$ have rank 1. Now, the orders of the torsion points $p(\lambda), \lambda \in S_p$, are unbounded, so

$$S_V \text{ is infinite but sparse, and } n_V(\lambda) \text{ is unbounded as } \lambda \text{ runs through } S_V.$$

The latter case (resp. the usual case) is illustrated in Pellian terms by Statement (ii) (resp. (i)) below. We have included the easier Statements (iii) and (iv), as they will be our starting points in the study of §4. Recalling the notation S_D, similar to S_V and S_p, from §1, we have :

Corollary 1. ([15], [19]) *i) For the sextic family $D_\lambda(t) = t^6 + t + \lambda$, the set S_D is finite;*

ii) For the sextic family $D'_\lambda(t) = t^6 + t^2 + \lambda$, the set $S_{D'}$ is infinite, but sparse;

iii) For the quartic family $\Delta_\lambda(t) = t^4 + t + \lambda$, the set S_Δ is infinite, but sparse;

iv) For the quartic family $\Delta'_\lambda(t) = t^4 + (2\lambda + 1)t^3 + 3\lambda t^2 + \lambda t$, the set $S_{\Delta'}$ is infinite, but sparse.

By Theorem 1 and the discussion in §2, with $\mathfrak{p}_0(\lambda) = \infty_+(\lambda), \mathfrak{p}_1(\lambda) = \infty_-(\lambda)$, the proof of this corollary reduces to the following observations, for which we refer to [14], [15], [19] : in all four cases, $p = \{(\infty_+) - (\infty_-)\}$ is not a torsion section of the relative jacobian $\mathrm{Pic}^0(\hat{V}/S)$, and

• in (i), the jacobian of the corresponding curve V_η over K is simple over $\overline{K}$, so we are in the usual case (see [15], §10);

• in (ii), the curve $\hat{V}$ covers two genus 1 curves $\hat{V}', \hat{V}''$ over S, with non-isoconstant (and non-isogeneous) jacobians, and it just occurs that the non torsion section p factors through one of their jacobians, so we are in the exceptional Case (**b**) of Theorem 1 (see [15], p. 17, [19], §3, Obstruction (b));

• in (iii), the relative curve $\hat{V}/S$ has genus one. We are in relative dimension 1, with a non-torsion section p of the jacobian, whose j-invariant is not constant since its Weierstrass model is given by $y^2 = x^3 - 4\lambda x + 1$ (see [14], [15], §1).

• in (iv), we are again in genus 1. As we will check in §4.2, the jacobian is the isoconstant curve $y^2 = x^3 + \lambda(1-\lambda)x$, but the section p is not constant, so leads to the same conclusion.

3.2. Semi-abelian surfaces. We now consider a semi-abelian scheme G/S of relative dimension 2, and toric rank 1, over the algebraic curve $S/\overline{\mathbb{Q}}$. So, G is an an S-extension of an elliptic scheme E/S by $\mathbb{G}_m$, whose isomorphism class is represented by a section q of (the dual of) E/S. If $\tilde{p} \in G(S)$, we denote by $p \in E(S)$ its projection to E. Again, a multiple of a section means a multiple by a non-zero integer, and we allow for finite base change of S whenever necessary.

Theorem 2. ([4]) *Let $S/\overline{\mathbb{Q}}$, G/S, $q \in E(S)$ be as above, and let $\tilde{p} \in G(S)$ be a section, with projection $p \in E(S)$. Then, the set*

$$S_{\tilde{p}} := \{\lambda \in S(\overline{\mathbb{Q}}), \tilde{p}(\lambda) \in G_\lambda^{tor}\}$$

is infinite if and only if one of the following conditions is satisfied :

a) $\tilde{p}$ is a torsion section;

b) there exists an elliptic subscheme E'/S of G/S (equivalently, q is a torsion section) and a multiple of $\tilde{p}$ which factors through E', and is not constant if E'/S (equivalently E/S) is isoconstant.

b') a multiple of $\tilde{p}$ factors though $\mathbb{G}_{m/S}$ (equivalently, p is torsion), and is not constant;

c) $\tilde{p}$ is a Ribet section (see below).

Consequently, if $\hat{V}/S$ is a family of smooth proper curves of genus 1, with four disjoint sections $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{q}^+, \mathfrak{q}^- \in \hat{V}(S)$, and if W/S is the relative curve obtained by pinching $\hat{V}/S$ along the images of $\mathfrak{q}^+, \mathfrak{q}^-$, and removing those of $\mathfrak{p}_0, \mathfrak{p}_1$, then the set

$$S_W = \{\lambda \in S(\overline{\mathbb{Q}}), rk(\overline{\mathbb{Q}}[W_\lambda]^*/\overline{\mathbb{Q}}^*) = 1\}$$

is finite (usual case), unless we are in one of the following situations, where we recall the notation $n_V(\lambda) = n_{V_\lambda}, n_W(\lambda) = n_{W_\lambda}$,

$$\nu(\lambda) = \frac{n_W(\lambda)}{n_V(\lambda)} = [(\overline{\mathbb{Q}}[V_\lambda])^* : (\overline{\mathbb{Q}}[W_\lambda])^*]$$

of Remark 1 for the heights and relative complexity of the fundamental units :

(**a**) (trivial case) there exists a rational function f on $\hat{V}$ whose relative divisor is supported by $\mathfrak{p}_1 \cup \mathfrak{p}_0$, so $p = \{(\mathfrak{p}_1) - (\mathfrak{p}_0)\}$ is a torsion section of the

relative jacobian $\mathrm{Pic}^0(\hat{V}/S)$, hence $S_V = S(\overline{\mathbb{Q}})$, and furthermore $\frac{f(\mathfrak{q}^+)}{f(\mathfrak{q}^-)} \in \mathbb{G}_m(S) \subset K^*$ is a (constant) root of unity, so $\tilde{p} = [(\mathfrak{p}_1) - (\mathfrak{p}_0)] \in \mathrm{Pic}^0(\hat{W}/S)$ too is a torsion section. Then, the group of generic units $(K[W_\eta])^*/K^*$ has rank 1, so

$$S_W = S_V = S(\overline{\mathbb{Q}}),\ \textit{and}$$
$$n_W(\lambda) \sim \nu(\lambda)\ \textit{are both bounded as}\ \lambda\ \textit{runs through}\ S_W.$$

(**b'**) (an exceptional, but easy, case, reducing to $G = \mathbb{G}_m$) : same hypothesis on the existence of f as above, but we assume that $\frac{f(\mathfrak{q}^+)}{f(\mathfrak{q}^-)} \in K^*$ is not constant. In this case, $S_V = S(\overline{\mathbb{Q}})$, $n_V(\lambda)$ is constant, and we are reduced to a "relative dimension 1" case, so

$$S_V = S(\overline{\mathbb{Q}}),\ S_W\ \textit{is infinite but sparse, and}$$
$$n_W(\lambda) \sim \nu(\lambda)\ \textit{are both unbounded as}\ \lambda\ \textit{runs through}\ S_W.$$

[NB : we take the opportunity of this case to mention that if $\frac{f(\mathfrak{q}^+)}{f(\mathfrak{q}^-)} \in \overline{\mathbb{Q}}^*$ is constant, but not a root of unity, then all the groups $(\overline{\mathbb{Q}}[W_\lambda])^*/\overline{\mathbb{Q}}^*$ have rank 0, i.e. $S_W = \emptyset$.]

(**b**) (an exceptional, and more delicate, case, reducing to $G = E$) : there exists a rational function φ on $\hat{V}$ whose relative divisor is supported by $\mathfrak{q}^+ \cup \mathfrak{q}^-$, so $q = \{(\mathfrak{q}^+) - (\mathfrak{q}^-)\}$ is a torsion section, say of order N, of the relative jacobian $E = \mathrm{Pic}^0(\hat{V}/S)$, and furthermore $\frac{\varphi(\mathfrak{p}_1)}{\varphi(\mathfrak{p}_0)} \in \mathbb{G}_m(S) \subset K^*$ is a root of unity, say of order M. By Proposition 2, $\tilde{p}' = NM\tilde{p}$ is then contained in an elliptic scheme $E' \subset G$ isogenous to E, and we are reduced to relative dimension 1. In this case, $S_V = S_W$, and for λ in this set, the torsion points $p(\lambda)$ and $\tilde{p}(\lambda)$ have the same order, up to a factor dividing NM. Leaving aside the cases where the section $\tilde{p}'$ is torsion (see Case (**a**)) or constant (see NB above), we have :

$$S_W = S_V\ \textit{is infinite but sparse, and}$$
$$n_W(\lambda)\ \textit{is unbounded,}\ \nu(\lambda)\ \textit{is bounded as}\ \lambda\ \textit{runs through}\ S_W.$$

(**c**) (Ribet sections) : this case can occur only in a very specific situation, namely when $E \simeq E_0 \times_{\overline{\mathbb{Q}}} S$ is isoconstant, where the elliptic curve $E_0/\overline{\mathbb{Q}}$ has CM, the section p and q are non-constant and related by a totally imaginary endomorphism, and $\tilde{p}$ is a well chosen lift of p, as described in [2]. Then, $\tilde{p}$ lifts all torsion values $p(\lambda), \lambda \in S_V$, to torsion points $\tilde{p}(\lambda)$, so $S_V = S_W$. By relative dimension 1, and since p is not constant, S_V is infinite but unlike in Case (**b'**), sparse. Now, what makes this case differ distinctively from Case (**b**) is that the order of $\tilde{p}(\lambda)$ is often much bigger than that of $p(\lambda)$. We will illustrate this case by a specific example in §4.2, and here merely record that if $\tilde{p} = [(\mathfrak{p}_1) - (\mathfrak{p}_0)]$ is a Ribet section of G/S, then

$$S_W = S_V\ \textit{is infinite but sparse, and}$$
$$n_W(\lambda)\ \textbf{and}\ \nu(\lambda)\ \textit{are both unbounded as}\ \lambda\ \textit{runs through}\ S_W.$$

4. Non separable Pellians : two case studies

4.1. Illustrating the usual case. We now come back to the setting of §1 and consider sextic discriminants $D_\lambda, \lambda \in S(\overline{\mathbb{Q}})$, with one double root $\rho(\lambda)$. The "usual case" of §3.2 can then be illustrated as follows. We first choose, more or less at random in terms of $\mathcal{M}_{1,1}$ (though a priori not in terms of the 2-dimensional moduli space $\mathcal{M}_{1,2}$), the family of affine quartic curves

$$(V_\lambda) : v^2 = \Delta_\lambda(t), \text{ where } \Delta_\lambda(t) = t^4 + t + \lambda.$$

As mentioned after Corollary 1.(iii) of the previous section (cf. [14], [15], §1), the jacobian of the smooth completion $\hat{V}_\lambda$ of V_λ is the non isoconstant elliptic curve

$$(E_\lambda) : y^2 = x^3 - 4\lambda x + 1,$$

which defines an elliptic scheme E over $S = \mathbb{P}_1 \setminus \{\lambda^3 = 2(\frac{3}{8})^3, \infty\}$. The two places at infinity $\mathfrak{p}_0(\lambda), \mathfrak{p}_1(\lambda)$ of $\hat{V}_\lambda$ are sent under the birational isomorphism $((t, v) \mapsto (x = 2(t^2 + v), y = 4t(t^2 + v) + 1)$ to the points $\mathfrak{p}'_0(\lambda)$ at infinity and $\mathfrak{p}'_1(\lambda) = (0, -1)$ of E_λ. So, the weak class of the divisor $(\mathfrak{p}_1) - (\mathfrak{p}_0)$ is the section $p = (0, -1)$ of E/S. Computing $2p(\lambda) = (4\lambda^2, -8\lambda^3 + 1)$ and evaluating at $\lambda = \frac{1}{4}$, we deduce as in [14] that p is not a torsion section. As announced in Corollary 1.(iii), this justifies the initial assertion of Corollary 2 below.

Let now $\rho \in \overline{K} = \overline{\mathbb{Q}(S)}$ be an arbitrarily fixed algebraic function, which is not a root of the generic polynomial $\Delta_\eta \in K[t]$. Replacing S by a finite cover, and withdrawing a finite number of its closed points, we may view ρ as a regular function on S (cf. Remark 3), and we can assume that $\Delta_\lambda(\rho(\lambda))$ never vanishes on $S(\overline{\mathbb{Q}})$. As in [4], Appendix II, let then

$$(W_\lambda) : u^2 = D_\lambda(t), \text{ where } D_\lambda(t) = (t - \rho(\lambda))^2(t^4 + t + \lambda),$$

be the singular curve obtained by pinching V_λ at the points $\mathfrak{q}^\pm(\lambda) = (\rho(\lambda), \pm\delta_\rho(\lambda))$, where $\delta_\rho(\lambda) = \sqrt{\Delta_\lambda(\rho(\lambda))}$. Their images in E_λ define sections $\mathfrak{q}'^\pm = \big(2(\rho^2 \pm \delta_\rho), 4\rho(\rho^2 \pm \delta_\rho) + 1\big)$, of E/S, whose difference $q = \mathfrak{q}'^+ -_{(E)} \mathfrak{q}'^-$ is the weak equivalence class of the horizontal divisor $(\mathfrak{q}^+) - (\mathfrak{q}^-)$. Since p, hence $\tilde{p}$, is not a torsion section, we deduce from the analysis of §3.2 that

• if the algebraic function ρ is such that q is not a torsion section, then we are the "usual case", and S_W is finite. So, for such $\rho \in \overline{K}$, we get finiteness of the set S_D of $\lambda \in S(\overline{\mathbb{Q}})$ such that D_λ is Pellian. The (random) choice $\rho \equiv -\frac{1}{2}$ gives an example of this case (see [4], Appendix II, for a similar discussion with $\rho \equiv 0$).

• otherwise, ρ leads to a torsion section q, say of order N (and since $x(q) = -4\rho^2 + \frac{(4\rho^3+1)^2}{\Delta_\eta(\rho)}$, we see that examples of such $\rho \in \overline{K}$ do exist for essentially any order of torsion N). Letting φ denote a rational function in $\overline{K}(V_\eta)$ such

that $N.(\mathfrak{q}^+) - N.(\mathfrak{q}^-) = \mathrm{div}(\varphi)$ at the generic fiber $V_\eta \otimes_K \overline{K}$, we then know that S_D can be infinite only in the exceptional case (**b**) of §3.2 where

$$\frac{\varphi(\mathfrak{p}_1)}{\varphi(\mathfrak{p}_0)} \in \boldsymbol{\mu}_\infty \subset \overline{\mathbb{Q}}^* \subset \overline{K}^*$$

is a root of unity : indeed, Case (**b'**) cannot occur since p is not a torsion section, while Case (**c**) would require E/S isoconstant.

In what follows, we will show that as soon as N is sufficiently large, the latter condition on $\frac{\varphi(\mathfrak{p}_1)}{\varphi(\mathfrak{p}_0)}$ never happens, and more precisely, that $\frac{\varphi(\mathfrak{p}_1)}{\varphi(\mathfrak{p}_0)} \notin \overline{\mathbb{Q}}^*$ is not even constant. Consequently, only finitely many algebraic functions $\rho \in \overline{K}$ can lead to an infinite set S_D. We actually believe that no ρ does so. In other words :

Corollary 2. *Let S_Δ be the set of algebraic numbers λ such that*

$$\Delta_\lambda(t) = t^4 + t + \lambda \in \overline{\mathbb{Q}}[t]$$

is a Pellian polynomial. Then, S_Δ is infinite but sparse, and

i) there are only finitely many points λ's in S_Δ such that the polynomial $(t + \frac{1}{2})^2 \Delta_\lambda(t)$ is Pellian. Moreover,

ii) for any algebraic function ρ outside a finite (and conjecturally empty) subset of $\overline{K}$, there are only finitely many λ's in S_Δ such that the polynomial $(t - \rho(\lambda))^2 \Delta_\lambda(t)$ is Pellian.

To treat (i), it suffices to show that for $\rho \equiv -\frac{1}{2}$, so, $\delta_\rho(\lambda) = \sqrt{\lambda - \frac{7}{16}}$, the section q of E/S is not torsion. Now, the sections $\mathfrak{q}^+$ and $\mathfrak{q}^-$ (over the quadratic cover of the initial base S defined by δ_ρ) meet at $\lambda = \frac{7}{16}$, so $q(\frac{7}{16}) = 0$, while q is not the zero section since it represents a divisor with only one, simple, pole. But E has good reduction at the place above $\lambda = \frac{7}{16}$, so specialization at such a place is injective on torsion subgroups, and q cannot be a torsion section. This argument of course extends to many other choices of ρ in $\overline{K}$, but by definition, not to those, infinite in numbers, which yield a torsion section q.

To treat those ρ's, leading to (ii), we now *fix a field of definition $K = \overline{\mathbb{Q}}(S)$ for $\hat{V}_\eta$ and its points $\mathfrak{p}_0(\eta), \mathfrak{p}_1(\eta)$*, and we drop the variable η from the notations. Since the K-rational points $\mathfrak{p}_0, \mathfrak{p}_1$, resp. the $\overline{K}$-rational points $\mathfrak{q}^+, \mathfrak{q}_-$, of $\hat{V}_\eta := \mathcal{E}$ are interchanged under the involution $\tau : (t, v) \mapsto (t, -v)$, we are left with the task of proving the Corollary below of the following Proposition, whose proof is due to U. Zannier. The elements of $\overline{K}$ will be called scalars, those of $\overline{\mathbb{Q}}$ constants.

Proposition 3. *Given a function field $K = \overline{\mathbb{Q}}(S)$ with algebraic closure $\overline{K}$, let E/K be an elliptic curve with a non-constant j-invariant, and let $\mathfrak{p}_1, \mathfrak{p}_0$ be two non-zero distinct points in $E(K)$. There exists an integer N_0*

depending only on E and K with the following property. For any positive integer N and any point $u \in E(\overline{K})$, of order dividing N, denote by $\Phi_u \in \overline{K}(E)$ a rational function on $E/\overline{K}$ such that $\mathrm{div}(\Phi_u) = N.(u) - N.(0)$. *Then, for any integer $N > N_0$, and any point $q \in E(\overline{K})$ of precise order N (with $q \neq \mathfrak{p}_1, \mathfrak{p}_0$), the scalar $\frac{\Phi_q(\mathfrak{p}_1)}{\Phi_q(\mathfrak{p}_0)} \in \overline{K}^*$ does not lie in K^*. In particular, it is not constant.*

[NB : this Proposition still holds if we replace $(\mathfrak{p}_1) - (\mathfrak{p}_0)$ by any non zero divisor D of degree 0, provided that D is defined over K and its polar degree is bounded. The conclusion then reads : $\Phi_q(D) \notin K^*$. It suffices to replace in the proof the expressions $\mathfrak{p}_i - v$ by $t_v^*(D)$, where t_v denotes translation by v.]

Corollary. - *Let $\mathcal{E}/K$ be a curve of genus 1 with a non isoconstant jacobian, let τ be a K-rational involution on $\mathcal{E}$, and let $\mathfrak{p}_1, \mathfrak{p}_0 = \tau(\mathfrak{p}_1) \in \mathcal{E}(K)$, $\mathfrak{q}^+, \mathfrak{q}^- = \tau(\mathfrak{q}^+) \in \mathcal{E}(\overline{K})$ be four distinct points. There exists an integer $\mathcal{N}_0$ depending only on $\mathcal{E}, K, \tau$, with the following property. Suppose that the class q of the divisor $(\mathfrak{q}^+) - (\mathfrak{q}^-)$ has finite order $N > \mathcal{N}_0$ in* $\mathrm{Pic}^0(\mathcal{E}/\overline{K})$, *and let φ_q be a rational function in $\overline{K}(\mathcal{E})$ such that* $\mathrm{div}(\varphi_q) = N.(\mathfrak{q}^+) - N.(\mathfrak{q}^-)$. *Then, the scalar $\frac{\varphi_q(\mathfrak{p}_1)}{\varphi_q(\mathfrak{p}_0)} \in \overline{K}^*$ is not constant (i.e. it does not lie in $\overline{\mathbb{Q}}^*$).*

Proof of Proposition 3 (following U. Zannier) : we claim that there exists a positive number $c_0 = c_0(E, K) < 1$ with the following property. Let $q \neq \mathfrak{p}_1, \mathfrak{p}_0$, be a torsion point on $E(\overline{K})$, and let N be its order. Assume that $\frac{\Phi_q(\mathfrak{p}_1)}{\Phi_q(\mathfrak{p}_0)}$ lies in K^*. Then, there exists a non zero torsion point u of order dividing N such that for there exist at least $c_0 N^2/(logN)^2$ torsion points v of order N satisfying $\mathfrak{p}_i - v \neq 0, u$ $(i = 0, 1)$, and

$$\frac{\Phi_u(\mathfrak{p}_1 - v)}{\Phi_u(\mathfrak{p}_0 - v)} = 1.$$

So, the rational function $\Phi_u(\mathfrak{p}_1 - x) - \Phi_u(\mathfrak{p}_0 - x) \in \overline{K}(E)$ has at least $c_0 N^2/(logN)^2$ zeroes. But its degree, if defined, is twice the order of u, so $\leq 2N$. Consequenty, the function vanishes identically as soon as $N > N_0 = (c_0/2)^{-4}$. This contradicts the hypothesis $\mathfrak{p}_1 \neq \mathfrak{p}_0$, and implies that $\frac{\Phi_q(\mathfrak{p}_1)}{\Phi_q(\mathfrak{p}_0)} \notin K^*$ for $N > N_0$.

To prove the claim, we consider the Galois group Γ_N of the extension $K(E[N])/K$, and denote by $c_1, \ldots$ positive constants depending only on E and K. Since E/K is not isoconstant, Γ_N is a subgroup of $SL_2(\mathbb{Z}/N\mathbb{Z})$ of index bounded by c_1. Let $\mathcal{V}$ be the orbit $\Gamma_N.q$ of the point q of order N. Since the stabilizer of q in $SL_2(\mathbb{Z}/N\mathbb{Z})$ has order N, this orbit $\mathcal{V}$ has at least $c_1^{-1} N^{3-1}/4logN$ elements, all of order exactly N. By the box principle, there then exists at least one non-zero point $u \in E[N]$ (so, of order dividing

N) such that for at least $c_2(N^2/logN)^2/N^2 = c_2N^2/(logN)^2$ couples of distinct points $(v, v') \in \mathcal{V} \times \mathcal{V}$, with $v, v' \neq \mathfrak{p}_1, \mathfrak{p}_0$, we have $u = v' - v$. We henceforth fix such a point u, and denote by $\mathcal{V}_u$ a set of $[c_2N^2/(logN)^2]$ points v in the orbit $\mathcal{V}$ such that $v' = v+u$ too lies in $\mathcal{V}$ (and $v, v' \neq \mathfrak{p}_1, \mathfrak{p}_0$).

Denoting by x a general element of E, we now appeal to the functional identity :

$$\Phi_{v'-v}(x - v) = c_{v,v'}\frac{\Phi_{v'}(x)}{\Phi_v(x)},$$

where $c_{v,v'} \in \overline{K}^*$ is a normalizing scalar, and which is readily checked, in fact for any points v, v' in $E(\overline{K})$, by comparing the divisors of these functions. Evaluating at $x = \mathfrak{p}_1, \mathfrak{p}_0$, we deduce that for $v' - v = u$,

$$\frac{\Phi_u(\mathfrak{p}_1 - v)}{\Phi_u(\mathfrak{p}_0 - v)} = \frac{\Phi_{v'}(\mathfrak{p}_1)}{\Phi_{v'}(\mathfrak{p}_0)} \times \frac{\Phi_v(\mathfrak{p}_0)}{\Phi_v(\mathfrak{p}_1)}.$$

We finally come back to our choice of $u \in E[N]$ and let v run through $\mathcal{V}_u \subset \Gamma_N.q$, with $v' = v + u$ also in $\Gamma_N.q$. Denote by σ, σ' elements of Γ_N such that $\sigma(q) = v, \sigma'(q) = v'$, and let Γ_N act on $\overline{K}(E)$ by conjugating the coefficients. In particular, the rational functions $\sigma(\Phi_q)$ and $\Phi_{\sigma(q)} = \Phi_v$ have same divisor $N.(\sigma(q)) - N.(0)$, so are equal up to a scalar factor $c_\sigma \in \overline{K}^*$. Similarly, $\sigma'(\Phi_q) = c_{\sigma'}\Phi_{v'}$. Since $\mathfrak{p}_1, \mathfrak{p}_0$ are defined over the fixed field K of Γ_N, we deduce :

$$\sigma\Big(\frac{\Phi_q(\mathfrak{p}_1)}{\Phi_q(\mathfrak{p}_0)}\Big) = \frac{\Phi_v(\mathfrak{p}_1)}{\Phi_v(\mathfrak{p}_0)} \ , \ \sigma'\Big(\frac{\Phi_q(\mathfrak{p}_1)}{\Phi_q(\mathfrak{p}_0)}\Big) = \frac{\Phi_{v'}(\mathfrak{p}_1)}{\Phi_{v'}(\mathfrak{p}_0)}.$$

Assuming now that $\frac{\Phi_q(\mathfrak{p}_1)}{\Phi_q(\mathfrak{p}_0)}$ lies in K^*, we also have $\sigma(\frac{\Phi_q(\mathfrak{p}_1)}{\Phi_q(\mathfrak{p}_0)}) = \frac{\Phi_q(\mathfrak{p}_1)}{\Phi_q(\mathfrak{p}_0)} = \sigma'(\frac{\Phi_q(\mathfrak{p}_1)}{\Phi_q(\mathfrak{p}_0)})$. So, the right-hand side of the last but one formula is

$$\frac{\Phi_{v'}(\mathfrak{p}_1)}{\Phi_{v'}(\mathfrak{p}_0)} \times \frac{\Phi_v(\mathfrak{p}_0)}{\Phi_v(\mathfrak{p}_1)} = \frac{\Phi_q(\mathfrak{p}_1)}{\Phi_q(\mathfrak{p}_0)} \times \frac{\Phi_q(\mathfrak{p}_0)}{\Phi_q(\mathfrak{p}_1)} = 1,$$

and the desired equality $\frac{\Phi_u(\mathfrak{p}_1-v)}{\Phi_u(\mathfrak{p}_0-v)} = 1$ does hold for at least the specified $[c_2N^2/(logN)^2]$ points v of $\mathcal{V}_u$. This proves the claim, with $c_0 = c_2/2$.

Proof of the Corollary : we may assume that the 4 fixed points of τ are defined over K, and choose one as an origin 0, allowing to identify $\mathcal{E}$ with its jacobian E. The involution τ is then represented on E by the standard symmetry, and reading on E, we have $\mathfrak{p}_0 = -\mathfrak{p}_1, \mathfrak{q}^- = -\mathfrak{q}_+$. So, $q = \{(\mathfrak{q}^+) - (\mathfrak{q}^-)\} = [2]_E\mathfrak{q}^+ = [-2]_E\mathfrak{q}^- \in E$ has order N if and only if $\mathfrak{q}^+ \in E(\overline{K})$ has order N or $N/2$. In particular, recalling the notation Φ_u of the Proposition, we see that the rational function φ_q of the corollary satisfies : $\mathrm{div}(\varphi_q) = N.(\mathfrak{q}^+) - N.(\mathfrak{q}^-) = N.((\mathfrak{q}^+) - (0)) - N.((\mathfrak{q}^-) - (0)) = \mathrm{div}(\Phi_{\mathfrak{q}^+}/\Phi_{\mathfrak{q}^-})$. Furthermore, up to scalar factors, $\Phi_{\mathfrak{q}^-}(x) = \Phi_{\mathfrak{q}^+}(-x)$ since

$[-1]_E$ interchanges the divisors $(\mathfrak{q}^+) - (0)$, $(\mathfrak{q}^-) - (0)$, hence $\varphi_q(x) = \frac{\Phi_{\mathfrak{q}^+}(x)}{\Phi_{\mathfrak{q}^+}(-x)}$. Finally, since $\mathfrak{p}_0 = -\mathfrak{p}_1$:

$$\frac{\varphi_q(\mathfrak{p}_1)}{\varphi_q(\mathfrak{p}_0)} = \frac{\Phi_{\mathfrak{q}^+}(\mathfrak{p}_1)}{\Phi_{\mathfrak{q}^+}(-\mathfrak{p}_1)} \times \frac{\Phi_{\mathfrak{q}^+}(-\mathfrak{p}_0)}{\Phi_{\mathfrak{q}^+}(\mathfrak{p}_0)} = \big(\frac{\Phi_{\mathfrak{q}^+}(\mathfrak{p}_1)}{\Phi_{\mathfrak{q}^+}(\mathfrak{p}_0)}\big)^2.$$

On choosing $\mathcal{N}_0 = N_0$ (which now depends on $\mathcal{E}, K$ and τ), the proposition implies that for $N > \mathcal{N}_0$, this algebraic scalar cannot be constant.

4.2. Illustrating Ribet sections. This will be done through the following example, where neither Δ' *nor* $\rho \equiv -\frac{1}{2}$ are chosen at random. Here, the base curve is $S = \mathbb{P}_1 \setminus \{0, 1, \infty\}$, and we still write $K = \overline{\mathbb{Q}}(S) = \overline{\mathbb{Q}}(\eta)$.

Corollary 3. *Let $S_{\Delta'}$ be the set of algebraic numbers $\lambda \neq 0, 1$ such that*

$$\Delta'_\lambda(t) = t^4 + (2\lambda + 1)t^3 + 3\lambda t^2 + \lambda t \in \overline{\mathbb{Q}}[t]$$

is a Pellian polynomial. For each such λ, let $(X_1^\lambda, Y_1^\lambda)$ be a fundamental solution of the corresponding Pell equation, and for any positive integer n, set $(X_1^\lambda + \sqrt{\Delta'_\lambda} Y_1^\lambda)^n = X_n^\lambda + \sqrt{\Delta'_\lambda} Y_n^\lambda$. Then, $S_{\Delta'}$ is infinite but sparse, and

i) for all λ's in $S_{\Delta'}$, the polynomial $(t + \frac{1}{2})^2 \Delta'_\lambda(t)$ too is Pellian. We can therefore define $\nu(\lambda)$ as the smallest integer $n \geq 1$ such that $Y_n^\lambda(-\frac{1}{2}) = 0$;

ii) for every prime number $\ell \geq 3$, there exists an element λ_ℓ in $S_{\Delta'}$ such that $\deg_t(X_1^{\lambda_\ell}) = \ell$ and $\nu(\lambda_\ell) = \ell$ or 2ℓ.

Proof. - A first issue is that the jacobian E_η/K of the normalisation $\hat{V}_\eta$ of the quartic curve $(V_\eta) : v^2 = \Delta'_\eta(t)$ is isomorphic over $\overline{K}$ to the CM elliptic curve $(E^0/\overline{\mathbb{Q}}) : y^2 = x^3 + x$. Let us check this, and then, reinterpet in terms of E^0 the two points $\mathfrak{p}_1(\lambda), \mathfrak{p}_0(\lambda)$ at infinity of the quartic V_λ, and the points $\mathfrak{q}^\pm(\lambda)$ with abscissa $t = -\frac{1}{2}$, whose pinching gives the singular curve W_λ with equation $u^2 = (t + \frac{1}{2})^2 \Delta'_\lambda(t) := D'_\lambda(t)$. The main issue will then be that the degree zero divisors these points define lead to Ribet sections.

Consider the birational isomorphism $(t, v) \mapsto (X = \frac{\lambda}{t}, Y = \lambda \frac{v}{t^2})$. Multiplying the quartic equation by $\frac{\lambda^2}{t^4}$, we see that its image is the curve with equation :

$$Y^2 = X^3 + 3\lambda X^2 + (2\lambda + 1)\lambda X + \lambda^2.$$

Consider now the birational isomorphism $(X, Y) \mapsto (x = X + \lambda, y = Y)$. Developing the powers of $x - \lambda$, we see that its image is the elliptic curve

$$(E_\lambda) : y^2 = x^3 + \lambda(1 - \lambda)x,$$

which, on choosing its point at infinity as origin, can therefore be identified to the jacobian of $\hat{V}_\lambda$, and defines an elliptic scheme E over $S = \mathbb{P}_1 \setminus \{0, 1, \infty\}$.

Let us now follow how the points are mapped under these isomorphisms. The first one sends the places at infinity $\mathfrak{p}_1(\lambda), \mathfrak{p}_0(\lambda)$ of the quartic (V_λ) to the points $(0, \pm\lambda)$ of the first cubic, since $v/t^2 \sim \pm 1$, and the points $\mathfrak{q}^{\pm}(\lambda)$ to its two points with abscissa $X = -2\lambda$. The second isomorphism sends the former points to the points $\mathfrak{p}'_1(\lambda), \mathfrak{p}'_0(\lambda) = (\lambda, \pm\lambda)$ of (E_λ), and the latter points to the points with abscissa $x = -\lambda$, i.e. to the points $\mathfrak{q}'^{\pm}(\lambda) = (-\lambda, \pm i\lambda)$ of (E_λ). Transporting divisors and pinching from the quartic curve to E_λ, we deduce that Δ'_λ (resp. D'_λ) is Pellian if and only if the value at λ of the weak, (resp. strict) equivalence class $p = \{(\mathfrak{p}'_1) - (\mathfrak{p}'_0)\}$ (resp. $\tilde{p} = [(\mathfrak{p}'_1) - (\mathfrak{p}'_0)]$) of the divisor $(\mathfrak{p}'_1) - (\mathfrak{p}'_0)$ is torsion on $\mathrm{Pic}^0(E_\lambda) = E_\lambda$, resp. on the fiber G_λ of the generalized jacobian G/S of the singular curve, with equation $y^2 = (x+\eta)^2(x^3 + \eta(1-\eta)x)$, obtained by pinching E/S along the sections $\mathfrak{q}'^{\pm}$. The latter is the S-extension of E by $\mathbb{G}_m$ parametrized by the weak equivalence class $q = \{(\mathfrak{q}'^{+}) - (\mathfrak{q}'^{-})\} \in \mathrm{Pic}^0(E/S) = E/S$.

As was pointed out, E/S is isoconstant (see E^0 above). Now, on the one hand, the points $\mathfrak{p}'_1(\lambda) = (\lambda, \lambda), \mathfrak{p}'_0(\lambda) = (\lambda, -\lambda) = [-1]\mathfrak{p}'_1(\lambda)$ of E_λ are such that the weak equivalence class $p = \{(\mathfrak{p}'_1) - (\mathfrak{p}'_0)\} = \mathfrak{p}'_1 -_{(E)} \mathfrak{p}'_0 = [2]\mathfrak{p}'_1 \in E(S)$ is not a constant section : indeed, on the constant model E^0, the abscissa of $\mathfrak{p}'_1(\lambda)$ reads $\frac{\lambda}{\sqrt{\lambda(1-\lambda)}} = \left(\frac{\lambda}{1-\lambda}\right)^{\frac{1}{2}}$. So, by "relative dimension 1", $S_{\Delta'}$ is infinite and sparse (as announced in Corollary 1.iv); in fact, the Möbius transform in the formula above directly shows that $\mathfrak{p}'_1(S)$ meets *any* torsion section of E/S of order $\neq 2$. On the other hand, E^0 admits complex multiplications by $\mathbb{Z}[i]$, and the points $\mathfrak{q}'^{+}(\lambda) = (-\lambda, i\lambda) = [i]\mathfrak{p}'_1(\lambda), \mathfrak{q}'^{-}(\lambda) = (-\lambda, -i\lambda) = [i]\mathfrak{p}'_0(\lambda)$ lead to $q = \mathfrak{q}'^{+} -_{(E)} \mathfrak{q}'^{-} = [i]p \in E(S)$. According to [2], §3 (see also [3]), the strict equivalence class $\tilde{p} = [(\mathfrak{p}'_1) - (\mathfrak{p}'_0)] = [(i^{-1}\mathfrak{q}'^{+}) - (i^{-1}\mathfrak{q}'^{-})]$ is then a Ribet section of the extension G/S of E/S by $\mathbb{G}_m$ parametrized by q.

For the convenience of the reader, we recall below how to check that for any λ such that $p(\lambda)$ is a torsion point on E_λ, say of order $n(\lambda)$, then the value $\tilde{p}(\lambda)$ of the Ribet section is a torsion point on G_λ of order $\tilde{n}(\lambda)$ dividing $2n^2(\lambda)$. Hence, $S_{D'} = S_{\Delta'}$, and this concludes the proof of (i). Moreover, this proof implies that for any prime number $\ell \geq 3$, there exists $\lambda_\ell \in S_{\Delta'}$ such that $n(\lambda_\ell) = \ell$, while $\tilde{n}(\lambda_\ell)$ equals ℓ^2 or $2\ell^2$. So, the index

$$\nu(\lambda) := \frac{\tilde{n}(\lambda)}{n(\lambda)}$$

grows as fast as ℓ or $2\ell = 2n(\lambda) = 2\deg_t(X_1^\lambda)$ as λ runs through the infinite sequence $\{\lambda_\ell\}$ of $S_{\Delta'}$, and this concludes the proof of (ii).

Finally, following B. Edixhoven's argument in [2], §3, here is the promised proof that $\tilde{p}$ lifts torsion values $p(\lambda), \lambda \in S_{\Delta'}$, of p to torsion points $\tilde{p}(\lambda)$ of

G_λ, in the quantitative way announced above. First, notice that the four points $\mathfrak{p}'_{0,1}(\lambda), \mathfrak{q}'^{\pm}(\lambda)$ are distinct as soon as $2\mathfrak{p}'_1(\lambda) \neq 0$ and $(1 \pm i)\mathfrak{p}'_1(\lambda) \neq 0$ on E_λ, so $\mathfrak{p}'_1(\lambda) \notin E_\lambda[2]$. And this condition holds for any $\lambda \in S(\overline{\mathbb{Q}})$, since back to E^0, it means that $\frac{\lambda}{1-\lambda} \neq 0, -1$ on S. Let now $n = n(\lambda)$ be the order of $p(\lambda)$, so, dropping the dashes and some of the λ's in the notation, there exists a rational function $f_1 = f$ on E_λ such that $n.(\mathfrak{p}_1) - n.(\mathfrak{p}_0) = \mathrm{div}(f)$. By the description of §2.1, the point $n\tilde{p}$ of G_λ, which lies in $\mathbb{G}_m(\overline{\mathbb{Q}})$, is given by

$$n\tilde{p} = \frac{f(\mathfrak{q}^+)}{f(\mathfrak{q}^-)}.$$

Furthermore, $q = ip$ too has order n, so there is a rational function φ on E_λ such that $n.(\mathfrak{q}^+) - n.(\mathfrak{q}^-) = \mathrm{div}(\varphi)$. In fact $\varphi = f \circ [i]^{-1}$ since $\mathfrak{q}^+ = i\mathfrak{p}_1, \mathfrak{q}^- = i\mathfrak{p}_0$.

Recall now that for two points $p, q \in E_\lambda[n]$ of order dividing n, the Weil pairing $e_n(p, q) \in \boldsymbol{\mu}_n$ can be computed as follows : if D_p, Δ_q are disjoint divisors in the (weak) equivalence classes p, q, and if $nD_p = \mathrm{div}(f), n\Delta_q = \mathrm{div}(\varphi)$, then

$$e_n(q, p) = \frac{f(\Delta_q)}{\varphi(D_p)},$$

which by yet another appeal to Weil's law of reciprocity, is indeed a n-th root of unity. By our previous observations, we may here choose the disjoint divisors $D_p = (\mathfrak{p}_1) - (\mathfrak{p}_0), \Delta_q = (\mathfrak{q}^+) - (\mathfrak{q}^-)$, and recalling that $\mathfrak{p}_0 = -\mathfrak{p}_1$, compute :

$$2n\tilde{p} = \Big(\frac{f(\mathfrak{q}^+)}{f(\mathfrak{q}^-)}\Big)^2 = \frac{f(\mathfrak{q}^+)}{f(\mathfrak{q}^-)} \times \frac{f(i^{-1}\mathfrak{p}_0)}{f(i^{-1}\mathfrak{p}_1)} = \frac{f(\mathfrak{q}^+)}{f(\mathfrak{q}^-)} \times \frac{\varphi(\mathfrak{p}_0)}{\varphi(\mathfrak{p}_1)} = e_n(q, p),$$

Therefore, $\tilde{p}(\lambda)$ is a torsion point of G_λ, of order $\tilde{n}(\lambda)$ dividing $2n^2(\lambda)$.

To check that up to the factor 2, this upper bound is reached infinitely often, we fix any odd prime ℓ, and consider the Weil pairing e^0_ℓ on the constant curve E^0. Since $[i]$ induces an automorphism of $E^0[\ell]$ which is not a homothety, $e^0_\ell([i]\xi, \eta)$ is a perfect symmetric pairing on $E^0[\ell]$, and any point ξ_ℓ of $E^0[\ell]$ outside of its isotropy cone gives a root of unity $\zeta_\ell = e^0_\ell([i]\xi_\ell, \xi_\ell)$ of precise order ℓ. From the explicit formula quoted above for $x(\mathfrak{p}'_1(\lambda))$, we deduce that there exists an element $\lambda_\ell \in S(\overline{\mathbb{Q}})$ such that the point $p(\lambda_\ell)$ is represented by ξ_ℓ in the constant model E^0. Then, $n(\lambda_\ell) = \ell$ by definition, and since

$$2\ell\tilde{p}(\lambda_\ell) = e_\ell(q(\lambda_\ell), p(\lambda_\ell)) = e^0_\ell([i]\xi_\ell, \xi_\ell) = \zeta_\ell,$$

$\tilde{n}(\lambda_\ell)$ is indeed equal to ℓ^2 or $2\ell^2$.

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