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par Amanda KNECHT

Résumé. Nous abordons la question du degré de paramétrisation unirationnelle de surfaces de del Pezzo de degré quatre et trois. Plus précisément, nous montrons que les surfaces de del Pezzo de degré quatre sur les corps finis admettent des paramétrisations de degré deux, et que les surfaces cubiques minimales admettent des paramétrisations de degré six. Il reste incertain s'il existe des paramétrisations de degré trois ou quatre pour ces dernières surfaces.

Abstract. We address the question of the degree of unirational parameterizations of degree four and degree three del Pezzo surfaces. Specifically we show that degree four del Pezzo surfaces over finite fields admit degree two parameterizations and minimal cubic surfaces admit parameterizations of degree six. It is an open question whether or not minimal cubic surfaces over finite fields can admit degree three or four parameterizations.

1. Introduction

It is a classical result that for every cubic surface S_3 defined over an algebraically closed field there exists a degree one rational map $\mathbb{P}^2 \dashrightarrow S_3$. We say that such a surface is *rational*. But over a non-algebraically closed field there are many examples of non-rational cubic surfaces. A surface S_3 is *unirational* if there exists a finite to one rational map $\mathbb{P}^2 \dashrightarrow S_3$. Such a map is called a *unirational parameterization* and the degree of the map is called the *degree of unirationality*. In 1943 Segre proved that a smooth cubic surface defined over $\mathbb{Q}$ is unirational if and only if it contains a rational point [11]. Manin then showed that the same is true for cubic surfaces over finite fields containing at least 34 elements [8]. More recently Kollár proved that over an arbitrary field a cubic hypersurface with a rational point is always unirational [6]. Cubic surfaces over finite fields always have points [2] so are unirational. The aim of this note is to describe the possible degrees of the unirational maps to non-rational cubic surfaces over finite fields. Let $\mathbb{F}_q$ be

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a finite field of size $q = p^r$. We say that a cubic surface is *minimal* over $\mathbb{F}_q$ if it does not contain a line defined over that field.

Theorem 1.1. *Let S_3 be a non-rational cubic surface defined over a finite field $\mathbb{F}_q$. If there exists a degree two rational map $\mathbb{P}^2 \dashrightarrow S_3$ and $q \neq 2$, then S_3 contains a line defined over $\mathbb{F}_q$. If S_3 does not contain a line defined over $\mathbb{F}_q$, then there exists a degree six unirational parameterization of S_3.*

Fix an algebraic closure $\overline{\mathbb{F}}_q$, let $G = \mathrm{Gal}\,(\overline{\mathbb{F}}_q, \mathbb{F}_q)$, $G_r = \mathrm{Gal}\,(\overline{\mathbb{F}}_q, \mathbb{F}_{q^r})$, and X be a smooth surface defined over $\mathbb{F}_q$. Let $N(X) = \mathrm{Pic}\,(X \otimes \overline{\mathbb{F}}_q)$. In his book *Cubic Forms* [8, Thm 29.4, 30.1], Manin proves that there is always a degree six parameterization of a cubic surface over large enough fields of characteristic different from two. He suggests that his rough lower bound of 35 for the size of the field may not be optimal but gives an example over $\mathbb{F}_4$ where his proof falls apart [8, Rmk 30.1.1]. Manin also proves [8, Thm 29.2] that if there exists a rational map of finite degree $\varphi : \mathbb{P}^2 \dashrightarrow X$ over $\mathbb{F}_q$, then the degree of φ is divisible by the least common multiple of the exponents of the groups $H^1(G_r, N(X))$ where r ranges over all integers.

By Theorem 1.1, we know that a minimal cubic surface never has a rational parameterization of degree two but always has one of degree six. The 27 lines on a cubic surface are acted on by the Weyl group of E_6, $W(E_6)$. There are twenty-five conjugacy classes in this group which Frame denotes $C_1, \ldots, C_{25}$ [3]. For each class C_i, let H_i be the cyclic subgroup in $W(E_6)$ generated by some element of the class. Manin calculates $H^1(H_i, N(S_3))$ for each class and places them in a table on pages 176-177 of the book *Cubic Forms* [8]. There are two mistakes in his table. In Corollary 1.17 Urabe proved that the orders of these groups are all square [14], so $H^1(H_i, N(S_3)) = 0$ for $i = 4$ and $i = 20$. Noting this correction to the chart, we find that the only possible nonzero H^1 groups are $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$. A correct table can be found in a paper by Li [7]. It should be noted that over a perfect field a cubic surface S_3 is minimal if and only the only curves on S_3 fixed by the absolute Galois group G are multiples of the canonical divisor [8, Thm 28.1].

There are two questions left open concerning the minimal degree of the parameterization:

Question. If S_3 is a minimal cubic surface such that

$$H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q, K), N(S_3)) = \mathbb{Z}_2 \times \mathbb{Z}_2$$

for some algebraic extension K of $\mathbb{F}_q$, does there exist a degree four unirational parameterization of S_3?

There is only one Frame conjugacy class C_{10} for which this question applies. The lines on surfaces of type C_{10} are permuted by Frobenius in one group of three and four groups of six. None of the lines are defined over a degree two extension, but over a degree three extension, three of the lines are defined and the other twenty-four are Galois conjugate pairs. For example the equation $X^3 + 3Y^3 + Z^3 + 4W^3 - 2(X + Y + Z + W)^3 = 0$ defines such a surface over $\mathbb{F}_{11}$.

Question. If S_3 is a minimal cubic surface such that

$$H^1(Gal(\overline{\mathbb{F}}_q, K), N(S_3)) = \mathbb{Z}_3 \times \mathbb{Z}_3$$

for some algebraic extension K of $\mathbb{F}_q$, does there exist a degree three unirational parameterization of S_3?

Again there is only one Frame conjugacy class C_{11} for which this question applies. The lines on these surfaces are permuted in nine sets of three, and all twenty-seven lines are defined over a degree three extension. One such surface to consider is defined by the equation $X^3 + Y^3 + Z^3 + \alpha W^3 = 0$ over $\mathbb{F}_4 = \mathbb{F}_2(\alpha)$ where $\alpha^2 + \alpha + 1 = 0$. We refer the reader to [12] for more details on the conjugacy classes of cubic surfaces.

It should be noted that if S is a degree d del Pezzo surface defined over a finite field, then S is rational when $d \geq 5$ [8, Thm 29.3] and unirational of degree two when $d = 4$ and $|k| > 22$ [8, Thm 30.1]. In the following section we extend the result for degree four del Pezzo surfaces to all finite fields. There has also been recent progress in the study of degree two del Pezzo surfaces over finite fields. Salgado, Testa, and Várilly-Alvarado prove that degree two del Pezzo surfaces over finite fields are unirational except for possibly three exceptional cases [10].

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2. Degree 2 Maps for degree 4 del Pezzo Surfaces

Over an algebraically closed field, a degree four del Pezzo surface S_4 is the blow up of the projective plane at five points, no three of which are collinear. Such a surface may also be thought of as the intersection of two quadrics in $\mathbb{P}^4$. Manin showed that a degree four del Pezzo surface is unirational of degree two over a finite field k when $|k| > 22$ [8]. His proof needs the field to contain 22 elements because his rational map uses a k-rational point on the surface that is not on any exceptional curves. Points on degree four del Pezzo surfaces can be contained in at most two exceptional curves. If a line L contained in S_4 contains an $\mathbb{F}_q$-rational point p, then all of the Galois conjugates of L contain p as well. Thus if p is contained in exactly one exceptional curve L, then L has no conjugates, and therefore is $\mathbb{F}_q$-rational. If p is contained in exactly two exceptional curves of a minimal surface, then each of those lines is defined over the quadratic extension $\mathbb{F}_{q^2}$ and their union is defined over $\mathbb{F}_q$.

Below we give a proof that there is a degree two unirational parameterization for every degree four del Pezzo surface over any finite field.

Theorem 2.1. *Suppose S_4 is a degree four del Pezzo surface defined over a finite field k. Then there exists a unirational parameterization of S_4 of degree two defined over k.*

Proof. There are three cases to consider.

(i) There exists a point $x \in S_4(k)$ not contained in an exceptional curve of S_4.
(ii) All the points in $S_4(k)$ are contained in exceptional curves but there is a point contained in exactly one exceptional curve.
(iii) All k-rational points are the intersection of two exceptional curves.

The first case is proven in Manin's book [8, Thm 29.4]. Given a degree four del Pezzo surface S_4 defined over k and a k-rational point that is not contained in any exceptional line of S_4, Manin constructs a unirational parameterization of degree two in part (vi) of the proof of the theorem. He then uses the following rational point count result of Weil to show that when $|k| > 22$ such a k-rational point exists [8, Thm 30.1].

Theorem 2.2 (A.Weil [15])**.** *Let $q = p^r$ and fix an algebraic closure $\overline{\mathbb{F}}_q$, let $G = Gal\ (\overline{\mathbb{F}}_q, \mathbb{F}_q)$ and $F \in G$ the Frobenius automorphism sending an element z to z^q. Let S be a del Pezzo surface, $N(S) = Pic\ (S \otimes \overline{\mathbb{F}}_q)$ and $F^* : N(S) \to N(S)$ be the action of Frobenius on the Picard group. Then:*

$$|S(\mathbb{F}_q)| = q^2 + q\ TrF^* + 1.$$

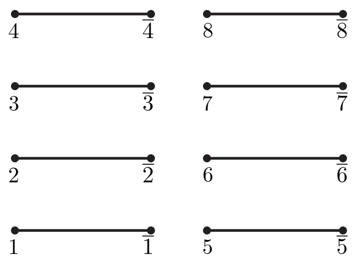
In the second case, suppose we have a k-rational point x contained on exactly one line on the surface. Then this line is defined over k and we can blow it down to produce a birational map to a degree five del Pezzo surface. These are known to be rational.

The third case breaks down into two sub-cases. If S_4 is not minimal, then lines on S_4 can be blown down to produce a birational map to a higher degree del Pezzo surface that is known to be rational. The case where S_4 is minimal never actually occurs as we prove in the following lemma. □

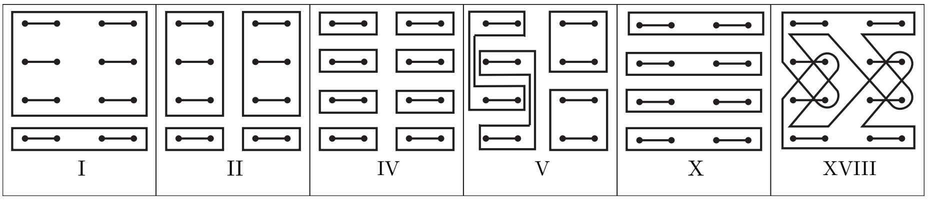
Lemma 2.1. *Let S_4 be a minimal degree four del Pezzo surface defined over a finite field k. Then there exists a k-rational point on S_4 that is not the intersection of two exceptional curves.*

Proof. Suppose by way of contradiction that all of the k-rational points on S_4 are the intersections of two exceptional curves. Let $x \in S_4(k)$ be a point contained in two exceptional curves L_1 and L_2. As noted above, since x is k-rational, the union $L_1 \cup L_2$ is defined over a quadratic extension k' of k and the curves L_1 and L_2 are Galois conjugate. If the surface has $|S_4(k)| = n$ such points defined over k, then S_4 contains at least $2n$ exceptional curves defined over k'. A degree four del Pezzo surface over an algebraically closed field contains exactly 16 exceptional curves, so $n \leq 8$. The above result of Weil applied to the field $k = \mathbb{F}_q$ yields the inequality $q^2 + q\,\mathrm{Tr}F^* + 1 \leq 8$. The trace of the Frobenius automorphism on S_4 is bounded below by -2 [9, Thm 1.1], so q must be 2 or 3.

In order to arrive at our contradiction for surfaces defined over $\mathbb{F}_2$ and $\mathbb{F}_3$, we study the Galois action on the sixteen exceptional curves on the surface. The image below is one way of depicting the exceptional curves on a degree four del Pezzo surface. The vertices are the exceptional curves labeled $1, \overline{1}, \ldots, 8, \overline{8}$. Two curves intersect if their corresponding vertices are joined by a line. The lines connecting the left and right columns are not depicted but are constructed as follows. The left (right) vertex of each pair is joined with the left (right) vertex of the pair in the same row of the other column and with the other four right (left) vertices in the other column. For example, curve 4 intersects curves $\overline{4}, 8, \overline{7}, \overline{6}$ and $\overline{5}$.



Manin classified the Galois conjugacy class decompositions of the lines on minimal degree four del Pezzo surfaces [8, Table 31.2]. He proved that there are six decompositions that occur over finite fields. Using his numbering system these are cases I, II, IV, V, X, and XVIII depicted below. Curves are in the same Galois orbit if they are contained in the same rectangle. For example, in case I curves $1, \overline{1}, 5$ and $\overline{5}$ make up an orbit of four elements while in case II they break into two orbits of two curves.



As mentioned above, a point on S_4 is the intersection of two exceptional curves if and only if those two curves are Galois conjugate over a degree two extension. In the graphic above such a point corresponds to a rectangle containing exactly two vertices. Thus, the only decompositions whose surfaces contain $\mathbb{F}_q$-rational points as the intersection of two exceptional curves are II and IV which contain 2 and 8 of these points, respectively. Over $\mathbb{F}_2$ our surface would have 1,3,5, or 7 points, and over $\mathbb{F}_3$ the surface could only have 4 or 7 points. Hence, such an S_4 does not exist, and we have arrived at the desired contradiction.

□

A corollary of Theorem 2.1 is the following fact known to experts.

Corollary 2.1. *Suppose S_3 is a cubic surface defined over a finite field k. Suppose further that S_3 contains a line defined over k. Then there exists a unirational parameterization of S_3 of degree two defined over k.*

Proof. If S_3 contains a line defined over k, we can blow the line down to get a birational map from S_3 to a degree four del Pezzo surface. Then we apply Theorem 2.1. □

3. Degree 2 Maps for Cubic Surfaces

We saw in the previous section that a cubic surface S_3 defined over a finite field k has a degree two unirational parameterization if the surface contains a line defined over k. In this section we show that having a line defined over k is actually a necessary condition for being unirational of degree two when k is of odd characteristic. This result is a direct consequence of the following theorem of Bayle and Beauville.

Let k be an algebraically closed field of odd characteristic and S a smooth, projective, connected rational surface over k. Also let σ be a non-trivial biregular involution of S. We say the pair (S, σ) is *minimal* if any birational morphism $g : S \to S'$ such that there exists a biregular involution σ' of S' with $g \circ \sigma = \sigma' \circ g$ is an isomorphism.

Theorem 3.1 (Bayle, Beauville [1]). *Let (S, σ) be a minimal pair. One of the following holds:*

(i) *There exists a smooth $\mathbb{P}^1$-fibration $f : S \to \mathbb{P}^1$ and a non-trivial involution τ of $\mathbb{P}^1$ such that $f \circ \sigma = \tau \circ f$.*
(ii) *There exists a fibration $f : S \to \mathbb{P}^1$ such that $f \circ \sigma = f$; the smooth fibers of f are rational curves, on which σ induces a non-trivial involution; any singular fiber is the union of two rational curves exchanged by σ, meeting at one point.*
(iii) *S is isomorphic to $\mathbb{P}^2$ with linear involution σ.*
(iv) *S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with the involution $(x, y) \mapsto (y, x)$.*
(v) *S is a del Pezzo surface of degree two and σ the Geiser involution.*
(vi) *S is a del Pezzo surface of degree one and σ the Bertini involution.*

Suppose S_3 is a smooth, projective, minimal cubic surface that admits a degree two unirational parameterization $\varphi : \mathbb{P}^2 \dashrightarrow S_3$. Then φ induces a birational involution ι on $\mathbb{P}^2$ that exchanges the two pre-images of the points of S_3 and the quotient $\mathbb{P}^2/ < \iota >$ is birational to S_3. By [1, Lemma 2.1] there also exists a birational morphism $f : S \to \mathbb{P}^2$ and a biregular involution σ of S such that $f \circ \sigma = \iota \circ f$ and the pair (S, σ) is minimal. Again the quotient $T = S/ < \sigma >$ is birational to S_3. We argue by contradiction and show that the minimal pair (S, σ) does not satisfy any of the cases above, so no such surface S_3 exists over a field of odd characteristic.

A minimal cubic surface over a perfect field is never birational to a surface with a rational bundle structure [8, Thm 37.1], so cases (i) and (ii) above are ruled out for S. We are left show that cases (iii)-(vi) are also impossible.

Consider the quotient $q : S \to T$ and decompose the fixed locus of σ as a union of isolated points p_i and disjoint curves B_ℓ. We call the points p_i the *exceptional points* of S. Notice that when restricting ourselves to cases (iii)-(vi) there are no curves of negative self intersection on S and $-K_S$ is ample. Thus the pullback of the anticanonical bundle of T is the sum of an ample divisor and nef divisors, $q^*(-K_T) = -K_S + \sum B_\ell$. Hence the surface T is a del Pezzo. Furthermore, since T is birational to S_3 and del Pezzo surfaces of degree at least five are rational over finite fields [8, Thm 29.3], we can assume T is a del Pezzo of degree at most 4. But if T is a degree four del Pezzo surface birational to a cubic surface, the birational map from S_3 to T must be the blow-down of an exceptional curve contradicting the minimality of S_3. Thus $K_T^2 \leq 3$.

We can easily compute

$$2K_T^2 = K_S^2 - 2K_S \cdot \sum B_\ell + \sum B_\ell^2. \tag{3.1}$$

Since $K_T^2 \leq 3$ and each $-K_S \cdot B_\ell$ is strictly positive, S can only be a degree two or one del Pezzo surface, as cases (iii) and (iv) have $K_S^2 = 9$ and 8, respectively. Also note that the degree of each curve B_ℓ as a curve in $\mathbb{P}^3$ is $-K_S \cdot B_\ell$. If $-K_S \cdot B_\ell = 1$ for some B_ℓ fixed by σ, then T and S_3 contain a line contradicting the minimality of S_3. So $-K_S \cdot B_\ell \geq 2$ for each ℓ. Also, a quick glance at equation (3.1) shows us that T cannot be a degree one or two del Pezzo when $K_S^2 = 1$ or 2. We now rule out cases (v) and (vi) of Theorem 3.1 using the fact that T is a cubic surface.

(v) Suppose S is a degree two del Pezzo surface. Equation (3.1) becomes

$$4 = \sum B_\ell^2 - 2K_S \cdot \sum B_\ell, \quad K_S \cdot \sum B_\ell \geq 2.$$

There is only one curve B_1 fixed by σ, and that curve is a conic passing through four of the exceptional points of S. But that means there is a conic on the cubic surfaces T and S_3. We arrive at a contradiction to the minimality of S_3 since a cubic surface containing a conic also contains a line coplanar to that conic.

(vi) Suppose S is a degree one del Pezzo surface. Equation (3.1) becomes

$$5 = \sum B_\ell^2 - 2K_S \cdot \sum B_\ell, \quad K_S \cdot \sum B_\ell \geq 2.$$

The only possibility for $-K_S \cdot \sum B_\ell$ is 2, and as before we arrive at a contradiction. So S is not a degree one del Pezzo, and we have proven the following theorem.

Theorem 3.2. *If S_3 is a minimal cubic surface defined over a field of odd characteristic, then there does not exist a unirational parameterization of S_3 of degree two.*

4. Degree 6 Maps for Cubic Surfaces

We start this section by recalling Manin's theorem on the existence of a degree six unirational parameterization for cubic surfaces over large enough finite fields [8, Thm 29.4].

Theorem 4.1 (Y. Manin)**.** *Let S_3 be a cubic surface defined over a finite field k and suppose that there is a point $p \in S_3(k)$ which is not on an exceptional curve. Then there exists a rational map $\varphi : \mathbb{P}^2 \dashrightarrow S_3$ of degree six.*

Manin goes on to state that a cubic surface defined over a field with at least 34 elements, will contain a point not on an exceptional curve. A cubic surface with all 27 lines defined over the base field k is called *split*. Hirschfeld [5] classified the split cubic surfaces where all the k-rational points are contained in the lines. He proved that $\mathbb{F}_{16}$ is the largest field for which this occurs and the unique surface in this case is the Fermat cubic surface $X^3 + Y^3 + Z^3 + W^3 = 0$. If S_3 is a cubic surface for which at least two of the lines are not defined over the base field $\mathbb{F}_q$, can all the $\mathbb{F}_q$-rational points be contained in the exceptional curves when $16 < q < 34$? We can lower the upper bound from 34 to 19 via the following calculations. Over the field $\mathbb{F}_q$, the number of points on the lines of a cubic surface is no more that $25(q+1) - e$ where e is the number of points contained in three exceptional curves. Such points are called *Eckardt points.* We have the bounds $e \leq 18$ when the field is of odd characteristic and $e \leq 45$ over $\mathbb{F}_{2^r}$ [5]. The trace of Frobenius for a cubic surface containing a line is bounded below by -1 [8, Table 31.1], so the number of points on the surface is at least $q^2 - q + 1$. The largest prime for which $25(q + 1) - 18 \geq q^2 - q + 1$ is 19. The largest power of 2 for which $25(q + 1) - 45 \geq q^2 - q + 1$ is 16.

Question. Over $k = \mathbb{F}_{17}, \mathbb{F}_{19}$ do there exist non-split cubic surfaces whose k-rational points are all contained in the exceptional curves of the surface?

For minimal cubic surfaces, we can lower that bound to five. Swinnerton-Dyer classified cubic surfaces not containing a line and having only Eckardt points as rational points [13]. He proved that these surfaces only exist when $q = 2$ or 4. It should be noted that Hirschfeld [5] proved the Fermat cubic is the unique split cubic surface defined over $\mathbb{F}_4$ with all $\mathbb{F}_4$-rational points Eckardt points.

Theorem 4.2. *Let S_3 be a minimal cubic surface defined over $\mathbb{F}_q$. Either there is a point $x \in S_3(\mathbb{F}_q)$ not contained on an exceptional curve, or $q < 5$ and every $x \in S_3(\mathbb{F}_q)$ is an Eckardt point.*

Proof. Let S_3 be a minimal cubic surface. Suppose all the points defined over $\mathbb{F}_q$ lie on exceptional curves and let $x \in S_3(\mathbb{F}_q)$. If x is contained in exactly one exceptional curve, then that line is defined over $\mathbb{F}_q$. If x is contained in exactly two exceptional curves, they must be Galois conjugates defined over $\mathbb{F}_{q^2}$. Consider the plane P spanned by the two lines. The intersection of the plane and the surface is three lines. The third line must be defined over the ground field. In both cases the surfaces in not minimal and contradicts our hypothesis.

Thus, x must be contained in three exceptional curves. A cubic surface defined over a field of odd characteristic can have 1, 2, 3, 4, 6, 9, 10 or 18 Eckardt points [4]. Over a field of characteristic two the possibilities are 1, 3, 5, 9, 13, and 45 [4]. Recall the theorem of Weil, Theorem 2.2, on the number of $\mathbb{F}_q$ rational points on a cubic surface;

$$|S_3(\mathbb{F}_q)| = q^2 + q\ \mathrm{Tr}F^* + 1.$$

If the surface S_3 is minimal, then the possibilities for $\mathrm{Tr}F^*$ are $-2, -1, 0, 1$ and 2 [8, Table 31.1]. A simple computation shows that the only fields in which all points on the surface can be Eckardt points are $\mathbb{F}_2, \mathbb{F}_3$ and $\mathbb{F}_4$. □

Corollary 4.1. *Let S_3 be a cubic surface defined over a finite field with at least five elements. Then the minimal degree of a unirational parameterization of S_3 is at most six.*

Proof. If S_3 is minimal, there is a point not contained on the exceptional curves, and Theorem 4.1 gives a degree six unirational map. If S_3 is not minimal, then Theorem 2.1 gives a degree two unirational map. □

The following surfaces were found by Swinnerton-Dyer [13]. Over the field $\mathbb{F}_2$ there is a unique cubic surface with one point. The surface is defined by the equation,

$$Y^3 + Y^2Z + Z^3 + W(X^2 + Y^2 + YZ + Z^2) + XW^2 + W^3 = 0. \tag{4.1}$$

The surface contains only one point $[1, 0, 0, 0]$ over $\mathbb{F}_2$, and that point is an Eckardt point. The lines on the surface are all defined over the degree three extension $\mathbb{F}_8$ where all 121 points of the surface are contained on the lines and thirteen of the points are Eckardt points. Also over the field of two elements there is a unique cubic surface with three Eckardt points as its rational points. This surface is given by the equation,

$$XY(X+Y)+Z^3+Z^2W+W^3=0. \tag{4.2}$$

It is in Frame's conjugacy class C_{22} and is a rational surface. The surface splits over $\mathbb{F}_{64}$. Fifteen of the exceptional curves are defined over $\mathbb{F}_8$ and the other twelve over $\mathbb{F}_{64}$.

Swinnerton-Dyer found two inequivalent surfaces over $\mathbb{F}_4$ all of whose $\mathbb{F}_4$-points are Eckardt. If we let the elements of $\mathbb{F}_4$ be $0, 1, \alpha$ and $\alpha+1$, where $\alpha^2+\alpha+1=0$, then the equations of the two surfaces are,

$$X^3+Y^3+Z^3+\theta W^3=0 \text{ where } \theta=\alpha \text{ or } \alpha+1. \tag{4.3}$$

These surfaces each have nine rational points over $\mathbb{F}_4$, and they are all Eckardt points. The exceptional curves on these surfaces are defined over the degree six extension $\mathbb{F}_{64}$.

In order to find a degree six unirational parameterization of these four surfaces, we consider points over degree two extensions instead of just points over the ground field. The following theorem is Kollár's second unirationality construction [6] which yields the desired degree six rational map.

Theorem 4.3 (J. Kollár)**.** *Let S_3 be a cubic surface defined over a finite field $\mathbb{F}_q$. Fix a point $x \in S_3(\mathbb{F}_q)$ and a line $L \in \mathbb{P}^3$ containing x. If L is not contained in the surface and not tangent to the surface, then $L \cap S_3 = \{x, s, s'\}$ where $s, s' \in S_3(\mathbb{F}_{q^2})$ are Galois conjugate points. If there exists an $x \in S_3(\mathbb{F}_q)$ such that s and s' do not lie on the exceptional curves of S_3, then there exists a dominant rational map $\varphi : \mathbb{P}^2 \dashrightarrow S_3$.*

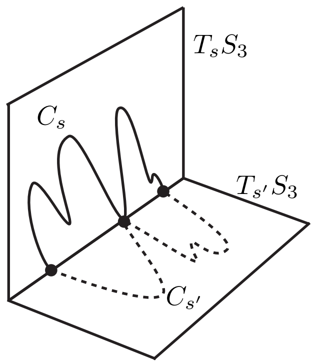
We will assume that the surface is minimal, since we have already seen how to produce a degree two parameterization for non-minimal surfaces.

Suppose there exist an $x \in S_3(\mathbb{F}_q)$ and a line L such that $L \cap S_3 = \{x, s, s'\}$ where s and s' are Galois conjugate points over a quadratic extension and do not lie on the exceptional curves of S_3. Let C_s be the singular cubic curve given as the intersection of the surface S_3 and the tangent plane at s, T_sS_3. We similarly define $C_{s'}$. Since C_s and $C_{s'}$ are Galois conjugate rational cubic curves, there exists a birational map $\psi : \mathbb{P}^2 \dashrightarrow C_s \times C_{s'}$ defined over $\mathbb{F}_q$. We define the *third point map* $\tau : C_s \times C_{s'} \dashrightarrow S_3$ by making $\tau(p, p')$ the third intersection point on the surface S_3 when we draw a line $L_{p,p'}$ between p and p'.

We will now show that $T_sS_3 \neq T_{s'}S_3$, which implies that this map is dominant because C_s and $C_{s'}$ do not lie in the same plane. If we suppose these planes are the same then we are saying that the intersection of a plane with our cubic surface is a cubic curve with two nodes at s and s'. This cubic curve is contained in the minimal surface S_3 so is irreducible, since if it were reducible the surface S_3 would contain a line. It is not possible for an irreducible cubic curve to have two nodes because a cubic with two nodes has negative genus. Thus the map τ is dominant.

Kollár's second unirationality construction [6] defines the map φ as $\tau \circ \psi : \mathbb{P}^2 \dashrightarrow S_3$. We can see that the degree of the third point map is six as follows.

Pick another point $z \in S_3$ such that $z \notin C_s \cup C_{s'}$. In order to compute the degree of φ, we want to know how many pairs (p, p') in $C_s \times C_{s'}$ satisfy $L_{p,p'} \cap S_3 = \{z, p, p'\}$. Let $\pi_z : \mathbb{P}^3 \backslash \{z\} \dashrightarrow T_{s'}S_3$ be the projection from z onto the tangent plane of S_3 at s'. Since $\pi_z(C_s)$ is a cubic curve in $T_{s'}S_3$, it will intersect $C_{s'}$ in nine points. So it appears at first that there are nine pairs of points that are mapped to z via τ. Let $\mathcal{L} = T_s S_3 \cap T_{s'} S_3$. Since $\mathcal{L}$ is defined over $\mathbb{F}_q$ and S_3 is minimal, $\mathcal{L}$ does not lie on S_3 and intersects S_3 at three points. These three points are also in the intersection of C_s and $C_{s'}$. Since the tangent lines to these points lie in different planes, we know that C_s and $C_{s'}$ intersect transversely here. These three points are in the indeterminacy locus of the map φ since they are on both C_s and $C_{s'}$, so we do not count them when counting the degree of φ. For a general point z, the other six points of intersection will be distinct points. Any special points $z \in S_3$ that result in points of multiple intersection do not affect the degree of this map. Thus the degree of φ is six, and we have proven the following proposition.



Proposition 4.1. *Let S_3 be a minimal cubic surface over a finite field k. If over a quadratic extension of k, S_3 contains two Galois conjugate points not contained on any of the 27 exceptional curves, then there exists a degree six unirational parameterization of S_3.*

In order to finish the proof of Theorem 1.1, we must show that the exceptional cases defined by equations (4.1) and (4.3) satisfy the assumptions of Proposition 4.1, recalling that the surface defined by equation (4.2) is rational. These surfaces are in the conjugacy class C_{11} of Frame [3], which means the 27 lines are permuted as nine triples of coplanar lines and $\mathrm{Tr} F^* = -2$.

They split over a degree three extension and contain no lines over $\mathbb{F}_q$ or $\mathbb{F}_{q^2}$. Over a degree two extension the surfaces are still in the conjugacy class C_{11}, so they contain q^4-2q^2+1 points. The only $\mathbb{F}_{q^2}$-rational points lying on exceptional curves are the original q^2-2q+1 Eckardt points. Thus there are $q(q+2)(q-1)^2$ $\mathbb{F}_{q^2}$-points on the surface away from the exceptional locus where $q=2$ or $q=4$. So the assumptions of the proposition are met, and there exists a degree six unirational parameterization.

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