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A local large sieve inequality for cusp forms

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Résumé. Nous démontrons une inégalité du type grand crible pour les formes de Maass et les formes cuspidales holomorphes de niveau au moins un et de poids entier ou demi-entier dans un petit intervalle.

Abstract. We prove a large sieve type inequality for Maass forms and holomorphic cusp forms with level greater or equal to one and of integral or half-integral weight in short interval.

1. Introduction

Large sieve type inequality involving Fourier coefficients of either Maass cusp forms or holomorphic cusp forms has been a subject of intense study since the pioneering work of Deshouillers, Iwaniec and their collaborators (see for example [6], [2] and [3]). In this paper we will prove such inequality for Maass forms and holomorphic cusp forms of integral or half-integral weight when the weight varies in a short interval (refered to as local large sieve inequality) and the level is not necessarily one.

We introduce our notations and state our main results in section 2; the proof of the local large sieve inequality for the integral weight holomorphic cusp forms will be given in section 3; the half-integral weight will be dealt with in section 4 and the Maass forms case in section 5.

In forthcoming papers, we will establish large sieve type inequality for cusp forms over totally real number fields and totally imaginary quadratic number fields.

2. Definitions and statement of result

We first start with the integral wight case.

Let $S_k(\Gamma_0(q))$ be the space of holomorphic cusp forms of even weight $k \geq 2$ and level q; $B_{k,q}$ be an orthonormal basis of $S_k(\Gamma_0(q))$ (for Theorem 2.1, we may choose any such basis, but we prefer to take a Hecke basis $H_{k,q}$ consisting of common eigenfunctions for all Hecke operators T_n with

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$(n,q)=1$) with respect to the Petersson inner product: For $f,g\in S_k(\Gamma_0(q))$

$$< f,g >= \frac{1}{\mathrm{vol}(X_0(q))}\int_{X_0(q)} y^k f(z)\overline{g(z)}\frac{dxdy}{y^2}$$

where $X_0(q)=\Gamma_0(q)\backslash\mathbb{H}$.

For $f=\sum_{n\geq 1} a_f(n)e(nz)\in B_{k,q}$, define

$$\rho_f(n)=\sqrt{\frac{(k-1)!}{(4\pi n)^{k-1}}}a_f(n).$$

Let $\{a_n\}_{N\leq n\leq 2N}$be any complex sequence.

Theorem 2.1. *With the notations as above, for* $1\leq G\leq K$. *For any* $\epsilon>0$, *we have*

$$\sum_{\substack{K\leq k\leq K+G\\ k\ even}}\sum_{f\in H_{k,M}}\left|\sum_{n=N}^{2N} a_n\rho_f(n)\right|^2 \ll (MKN)^\epsilon(MKG+N)\sum_{n=N}^{2N}|a_n|^2.$$

For the sake of convenience we assume q is a square-free number and $H^*_{k,q}$ be the subset of all new forms in $H_{k,q}$. For $f\in H^*_{k,q}$, we have

$$a_f(n)=a_f(1)\lambda_f(n)n^{\frac{k-1}{2}},$$

where $\lambda_f(n)$ is the normalized n-th Hecke eigenvalue of f satisfying $|\lambda_f(n)|\leq d(n)$; and

$$|a_f(1)|^2=\frac{(4\pi)^{k-1}}{\Gamma(k)}\frac{2\pi^2}{qL(1,\mathrm{sym}^2 f)}\mathrm{vol}(X_0(q)).$$

Thus, (see P.74 and 83 in [8])

$$\begin{aligned}\rho_f(n) &= \sqrt{\frac{(k-1)!}{(4\pi n)^{k-1}}}a_f(n)\\ &= \sqrt{\frac{(k-1)!}{(4\pi)^{k-1}}}a_f(1)\lambda_f(n)\\ &= \frac{\sqrt{2}\pi}{\sqrt{L(1,\mathrm{sym}^2 f)}}\sqrt{\frac{\pi}{3}\prod_{p|q}\left(1+\frac{1}{p}\right)}\lambda_f(n)\\ &= \rho_f(1)\lambda_f(n)\end{aligned}$$

where $(kq)^{-\epsilon}\ll\rho_f(1)\ll(kq)^\epsilon$ (see P.6 of [6] and Lemma 4.2 of [1]).

As an immediate corollary of Theorem 2.1, we have

Theorem 2.2. *For square-free* M *and with the same notation as above.*

$$\sum_{\substack{K\leq k\leq K+G\\ k even}}\sum_{f\in H^*_{k,M}}\left|\sum_{n=N}^{2N} a_n\lambda_f(n)\right|^2 \ll_\epsilon (MKN)^\epsilon(MKG+N)\sum_{n=N}^{2N}|a_n|^2$$

For $G = 1$, it was proved by Duke, Frielander and Iwaniec (see [3])and for $G = K$ by Deshouillers and Iwaniec (see [2]). We learnt later that Jutila and Motohashi had derived the same result in [10] when $M = 1$ (See Lemma 8 in[10]).Our proof is however much simpler.

For the half-integral weight case. Let $k = \frac{1}{2} + l$ with $l \in \mathbb{N}$; $S_k(\Gamma_0(q))$ be the space of holomorphic cusp forms of weight k and level q with $4|q$; $B_{k,q}$ be a basis of $S_k(\Gamma_0(q))$, orthonormal with respect to the Petersson inner product: For $f, g \in S_k(\Gamma_0(q))$

$$< f, g >= \frac{1}{\mathrm{vol}(X_0(q))} \int_{X_0(q)} y^k f(z)\overline{g(z)} \frac{dxdy}{y^2}$$

where $X_0(q) = \Gamma_0(q)\backslash\mathbb{H}$.

For $f(z) = \sum_{n\geq 1} a_f(n)e(nz) \in B_{k,q}$, define

$$\rho_f(n) = \sqrt{\frac{(k-1)!}{(4\pi n)^{k-1}}} a_f(n).$$

Let $\{a_n\}_{N\leq n\leq 2N}$ be any complex sequence and $1 \leq G \leq K^{1-\epsilon}$ for any $\epsilon > 0$.

Theorem 2.3. *With the notations as above. For any natural number M divisible by 4 and any $\epsilon > 0$, we have*

$$\sum_{\substack{K\leq k\leq K+G \\ k-\frac{1}{2} \ even}} \sum_{f\in B_{k,M}} \left| \sum_{n=N}^{2N} a_n \rho_f(n) \right|^2 \ll (MKN)^\epsilon (MKG + N) \sum_{n=N}^{2N} |a_n|^2.$$

We would like to remark that following P.224-225 of [3], one can prove

Theorem 2.4. *With the notations as above, for all natural number M divisible by 4 and any $\epsilon > 0$,*

$$\sum_{f\in B_{K,M}} \left| \sum_{n=N}^{2N} a_n \rho_f(n) \right|^2 \ll (MKN)^\epsilon (MK + N) \sum_{n=N}^{2N} |a_n|^2.$$

We now turn to the Maass form case. Let $S(\Gamma_0(q))$ be the space of Maass csup forms of level q; $\mathfrak{P}_q$ be the set of inequivalent cusps; B_q be an orthonormal basis for $S(\Gamma_0(q))$ (for the main Theorem below, we may choose any such basis, but we prefer to take a Hecke basis H_q consisting of common eigenfunctions for all Hecke operators T_n with $(n, q) = 1$) with respect to the Petersson inner product: For $f, g \in S(\Gamma_0(q))$

$$< f, g >= \frac{1}{\mathrm{vol}(X_0(q))} \int_{X_0(q)} f(z)\overline{g(z)} \frac{dxdy}{y^2}$$

where $X_0(q) = \Gamma_0(q)/\mathbb{H}$.

For each $f \in B_q$, we have the Fourier expansion,

$$f(z) = \sqrt{y} \sum_{m \neq 0} \rho_{t_f}(m) K_{it_f}(2\pi |m| y) e(mx).$$

Let $\{a_n\}_{N \leq n \leq 2N}$ be any complex sequence.

Theorem 2.5. *With the notions as above, for* $1 \leq G \leq K$, $M \geq 1$ *and any* $\epsilon > 0$, *we have*

$$\sum_{\substack{f \in H_M \\ K \leq t_f \leq K+G}} \frac{1}{\cosh \pi t_f} \left| \sum_{n=N}^{2N} a_n \rho_{t_f}(n) \right|^2$$
$$\ll \quad (MKN)^\epsilon (MKG + N) \sum_{n=N}^{2N} |a_n|^2.$$

For $G = K$, it was proved by Deshouillers and Iwaniec in [2]; G=1 by Luo in [12] and for $1 \leq G \leq K$ by Jutila in [9] and Motohashi in [14]. Independently, Zhang ([17]) proved the above large sieve inequality (including the contribution from the Fourier coefficients of the Eisenstein Series) by extending the argument in [12].

Let q be a square-free number and H_q^* be the set of all new forms in H_q.

For $f \in H_q^*$ whenever $(n, q) = 1$, we have

$$\rho_{t_f}(n) = \rho_{t_f}(1) \lambda_{t_f}(n)$$

where $T_n f(z) = \lambda_{t_f}(n) f(z)$ for $(n, q) = 1$ and $\lambda_{t_f}(1) = 1$.

As stated in P.119, 120 of [5],

$$(t_f q)^{-\epsilon} \ll |\rho_{t_f}(1)|^2 \ll (t_f q)^\epsilon.$$

As an immediate corollary of Theorem 2.5, we have

Theorem 2.6. *For square-free* M *and with the same notations as above*

$$\sum_{\substack{f \in H_M^* \\ K \leq t_f \leq K+G}} \left| \sum_{n=N}^{2N} a_n \lambda_{t_f}(n) \right|^2 \ll (MKN)^\epsilon (MKG + N) \sum_{n=N}^{2N} |a_n|^2.$$

3. Proof of Theorem 2.1

3.1. Strategy of the Proof. The starting point is the Petersson formula

$$\frac{1}{\operatorname{vol}(X_0(M))} \sum_{f \in H_{k,M}} \rho_f(m) \overline{\rho_f(n)} = (k-1)\delta_{mn}$$
$$+ \quad 2\pi i^{-k} (k-1) \sum_{c \equiv 0 (M)} \frac{S(m, n; c)}{c} J_{k-1} \left(\frac{4\pi \sqrt{mn}}{c} \right).$$

We will employ an embedding idea of Duke, Frielander and Iwaniec (see for example P.225 of [3]) in a quantitative manner. We take a prime p such that $pMKG > N(MKN)^{\epsilon}$. Note that any function f in $H_{k,M}$ is naturally a holomorphic cusp form of weight k and level pM with L^2-norm 1. Hence we can embed $H_{k,M}$ into $H_{k,pM}$ and by positivity study the sum

$$\sum_{\substack{K\leq k\leq K+G\\ k \text{ even}}} \sum_{f\in H_{k,pM}} \left|\sum_{n=N}^{2N} a_n \rho_f(n)\right|^2 \tag{3.1}$$

instead. In which case the corresponding Petersson formula is

$$\frac{1}{\mathrm{vol}(X_0(pM))} \sum_{f\in H_{k,pM}} \rho_f(m)\overline{\rho_f(n)} = (k-1)\delta_{mn}$$
$$+ \quad 2\pi i^{-k}(k-1) \sum_{c\equiv 0(pM)} \frac{S(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

notice that $\frac{4\pi\sqrt{mn}}{c}$ is no more than $\frac{N}{pM} < (MNK)^{-\epsilon}KG$ this will be crucial in subsequent steps. Such improvement does not come for free as one can see that $\mathrm{vol}(X_0(pM)) \ll p\ \mathrm{vol}(X_0(M))$ hence we has increased in particular the size of the diagonal term by p. Also, note that the terms with c large can be dealt with via well-known bound on Bessel functions and Kloosterman sum, so we can consider the sum up to certain large C.

The sum

$$\sum_{\substack{K\leq k\leq K+G\\ k \text{ even}}} \sum_{f\in H_{k,pM}} \left|\sum_{n=N}^{2N} a_n \rho_f(n)\right|^2$$

is bounded by

$$\sum_{\substack{k=1\\ k \text{ even}}}^{\infty} g(k) \sum_{f\in H_{k,pM}} \left|\sum_{n=N}^{2N} a_n \rho_f(n)\right|^2$$

where g is a suitable smooth function with compact support such that it dominates the characteristic function of the interval $[K, K+G]$. The corresponding sum on the Kloosterman sum side is

$$\sum_{\substack{c\equiv 0(pM)\\ c\leq C}} \frac{S(m,n;c)}{c} \sum_{\substack{k=1\\ k \text{ even}}}^{\infty} g(k)(k-1)i^{-k} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \tag{3.2}$$

Applying Neumann's theory, (3.2) is then transformed into a sum exponentially small due to $\frac{4\pi\sqrt{mn}}{cKG} \ll \frac{N}{pMKG} \ll (MNK)^{-\epsilon}$. The remainder of the proof is then a simple exercise of integration by parts.

3.2. First Manipulations. Assume $K^{1-\epsilon} \geq G \geq (KMN)^\epsilon$ for all $\epsilon > 0$, otherwise the result is known from [3] and Theorem 2 of [6]. We assume $M < K^D$ for some $D > 0$ otherwise the result is trivial. We also assume $N \ll K^E$ for some $E > 0$ (we think of K, N as varying) otherwise the result follows easily from direct estimation using Deligne's bound. We will now choose the test function that will smooth our sum. Let $g(x) \in C_c^\infty(0,\infty)$ such that $\mathrm{supp}(g(x)) \subseteq \left[\frac{1}{2}, \frac{5}{2}\right]$, $g^{(j)} \ll 1$ for all $j \geq 0$, and $g(x) = 1$ for $x \in [1,2]$. To prove Theorem 2.1, it suffices to bound, for sufficiently large K, the sum (we shift the sum for convenience but the adjustment to the original is simple)

$$S_M = \sum_{\substack{k\geq 1\\ 2|k}} g\left(\frac{k-K}{G}\right) \sum_{f\in H_{k,M}} \left|\sum_{n=N}^{2N} a_n\rho_f(n)\right|^2 .$$

Choose prime p such that

$$p \in \left[(KMN)^\epsilon \max\left\{\frac{N}{MKG}, 1\right\}, 2(KMN)^\epsilon \max\left\{\frac{N}{MKG}, 1\right\}\right].$$

As indicated in Section 4,

$$S_M \leq \sum_{\substack{k\geq 1\\ 2|k}} g\left(\frac{k-K}{G}\right) \sum_{f\in H_{k,pM}} \left|\sum_{n=N}^{2N} a_n\rho_f(n)\right|^2 .$$

Applying Petersson formula, we obtain

$$\sum_{f\in H_{k,pM}} \left|\sum_{n=N}^{2N} a_n\rho_f(n)\right|^2 = \sum_{N\leq n_1,n_2\leq 2N} a_{n_1}\overline{a_{n_2}} \sum_{f\in H_{k,pM}} \rho_f(n_1)\overline{\rho_f(n_2)}$$

$$= \sum_{N\leq n_1,n_2\leq 2N} a_{n_1}\overline{a_{n_2}}(k-1)\mathrm{vol}(X_0(pM)) \times$$

$$\left\{\delta_{n_1n_2} + 2\pi i^{-k} \sum_{c\equiv 0(pM)} \frac{S(n_1,n_2,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{n_1n_2}}{c}\right)\right\}.$$

Now summing over even k, weighted by $g\left(\frac{k-K}{G}\right)$,

$$\sum_{k \text{ even}} g\left(\frac{k-K}{G}\right) \sum_{f\in H_{k,pM}} \left|\sum_{n=N}^{2N} a_n\rho_f(n)\right|^2 =$$

$$\mathrm{vol}(X_0(pM)) \sum_{k \text{ even}} g\left(\frac{k-K}{G}\right)(k-1) \sum_{n=N}^{2N} |a_n|^2 \tag{3.3}$$

$$+\sum_{N\le n_1,n_2\le 2N} a_{n_1}\overline{a_{n_2}}\mathrm{vol}(X_0(pM))\sum_{c\equiv 0(pM)}\frac{S(n_1,n_2;c)}{c}\times \tag{3.4}$$

$$2\pi\sum_{k\text{ even}} i^{-k}g\left(\frac{k-K}{G}\right)(k-1)J_{k-1}\left(\frac{4\pi\sqrt{n_1n_2}}{c}\right).$$

3.3. The diagonal contribution. The diagonal term (3.3) contributes at most $\sum_{m=N}^{2N}|a_m|^2$ times

$$KG(pM)^{1+\epsilon}\ll (KM)^\epsilon pKGM \ll (KNM)^{2\epsilon}\max\{KGM,N\}.$$

3.4. The off-diagonal contribution. Since $J_{k-1}(x)\ll\left(\frac{ex}{2k}\right)^{k-1}$ (see (5.10.2) in [11]) and $\frac{4\pi\sqrt{n_1n_2}}{c}\le\frac{8\pi N}{c}$, so the contribution from those c with $c>\frac{24\pi N}{K}$ is exponentially small. Thus it suffices to consider those c less than or equal N.

We now come to our second ingredient.

Lemma 3.1. *For $h\in C_0^\infty(0,\infty)$, we have*

$$\sum_{\substack{k\ge 1\\ 2|k}}2\pi(-1)^{\frac{k}{2}}J_{k-1}(x)h(k-1)=-2\pi\int_{-\infty}^{\infty}\hat{h}(t)\sin(x\cos(2\pi t))dt$$

where $\hat{h}(t)=\int_{-\infty}^{\infty}h(y)e(ty)dy$ is the Fourier transform of $h(t)$.

Proof. See Lemma 4.1 of [13]. □

Let $h(y)=yg\left(\frac{y-(K-1)}{G}\right)$, then

$$\begin{aligned}\hat{h}(t) &= \int_{-\infty}^{\infty}yg\left(\frac{y-(K-1)}{G}\right)e(yt)dy\\ &= \int_{-\infty}^{\infty}[y-(K-1)]\,g\left(\frac{y-(K-1)}{G}\right)e(yt)dy\\ &+ (K-1)\int_{-\infty}^{\infty}g\left(\frac{y-(K-1)}{G}\right)e(ty)dy\\ &= e((K-1)t)\int_{-\infty}^{\infty}xg\left(\frac{x}{G}\right)e(xt)dx\\ &+ (K-1)e((K-1)t)\int_{-\infty}^{\infty}g\left(\frac{x}{G}\right)e(xt)dx\\ &= e((K-1)t)\hat{h}_1(t)+(K-1)e((K-1)t)\hat{h}_2(t)\end{aligned}$$

where $h_1(x)=xg\left(\frac{x}{G}\right)$ and $h_2(x)=g\left(\frac{x}{G}\right)$.

Hence,

$$\begin{aligned}&\sum_{\substack{k\geq 1\\ 2|k}}(-1)^{\frac{k}{2}}J_{k-1}(x)h(k-1)\\ = \quad & -\sum_{j=1}^{2}(K-1)^{j-1}\int_{-\infty}^{\infty}\hat{h}_j(t)e((K-1)t)\sin(x\cos(2\pi t))dt.\end{aligned}$$

We will show in the next subsection that assuming $\frac{x}{KG} \ll (KMN)^{-\epsilon}$ and $G \leq K^{1-\epsilon}$ for any $\epsilon > 0$ then for $j = 1, 2$

Lemma 3.2.

$$\int_{-\infty}^{\infty}\hat{h}_j(t)e((K-1)t)\sin(x\cos(2\pi t))dt \ll_{B,\epsilon} (KMN)^{-B}$$

for all $B > 0$.

Assuming Lemma 3.2 and takes $x = \frac{2\sqrt{n_1n_2}}{c}$, $N \leq n_1, n_2 \leq 2N$, the off-diagonal term (3.4)

$$\begin{aligned}\ll \quad & \sum_{N\leq n_1,n_2\leq 2N}|a_{n_1}||a_{n_2}|(pM)^{1+\epsilon}\sum_{\substack{c\equiv 0(pM)\\ c\leq N}}\frac{|S(n_1,n_2;c)|}{c}(KMN)^{-B}\\ \ll \quad & (PM)^{1+\epsilon}N^{2+\epsilon}(KMN)^{-B}\sum_{n=N}^{2N}|a_n|^2\\ \ll \quad & (MNK)^{\epsilon}(MKG+N)\sum_{n=N}^{2N}|a_n|^2\end{aligned}$$

upon taking, say, $B = 100$. This establishes Theorem 2.1 when $G \leq K^{1-\epsilon}$ for all small $\epsilon > 0$.

3.5. Proof of Lemma 3.2. We first prove the following basic Lemma in Fourier analysis.

Lemma 3.3. *Let* $h \in C_c^{\infty}(0,\infty)$ *be such that supp* $h \subset [K, K+G]$ *and* $h^{(j)} \ll G^{-j+1}$ *for all* $j > 0$. *Then*

$$\hat{h}(t) \ll |t|^{-m}G^{-m+1}K$$

for all $m > 0$.

Proof. For each $m \in \mathbb{N}$, integrating by parts m times

$$\hat{h}(t) = \left(-\frac{1}{2\pi it}\right)^m\int_{-\infty}^{\infty}h^{(m)}(x)e^{-2\pi ixt}dx.$$

Hence,

$$|\hat{h}|(t) \leq \frac{1}{(2\pi|t|)^m}\int_{-\infty}^{\infty}|h^{(m)}(x)|dx \ll |t|^{-m}G^{-m+1}K.$$

□

As a corollary (for more details, we refer to the proof of Proposition 4.8 below), we can consider instead

$$\int_{-K^{\epsilon/2}/G}^{K^{\epsilon/2}/G} \hat{h}_j(t)e((K-1)t)\sin(x\cos(2\pi t))dt,$$

and form now on we work with the more general integral

$$\int_{-K^{\epsilon/2}/G}^{K^{\epsilon/2}/G} \hat{h}_j(t)e((K-1)t + x\cos(2\pi t))dt. \tag{3.5}$$

Let $\omega(t) = 2\pi(K-1)t + x\cos(2\pi t)$. We claim that for $|t| \leq \frac{K^{\epsilon/2}}{G}$, $x \leq (MNK)^{-\epsilon}KG$, then

$$|\omega'(t)| > \frac{K}{2}.$$

To see this, note that

$$\omega'(t) = 2\pi(K-1) - x\sin(2\pi t) = 2\pi(K-1) - (2\pi)xt\frac{\sin(2\pi t)}{2\pi t}.$$

Since $\lim_{u\to 0}\frac{\sin u}{u} = 1$ and $G \gg K^\epsilon$, we have for $|t| \leq \frac{K^{\epsilon/2}}{G}$

$$\left|\frac{\sin(2\pi t)}{2\pi t}\right| \leq \frac{3}{2}.$$

From which we have

$$\left|(2\pi)xt\frac{\sin(2\pi t)}{2\pi t}\right| \leq (2\pi)\frac{3}{2}|xt| \leq (2\pi)\frac{3}{2}(KNM)^{-\epsilon}KG\frac{K^{\epsilon/2}}{G} < \frac{K}{2}.$$

Hence our claim.

Note that

$$\begin{aligned} \hat{h}_1^{(\nu)}(t) &= (2\pi i)^\nu \int_{-\infty}^{\infty} x^{\nu+1} g\left(\frac{x}{G}\right) e(xt)dx \\ &= (2\pi i G)^\nu G \int_{-\infty}^{\infty} \left(\frac{x}{G}\right)^{\nu+1} g\left(\frac{x}{G}\right) e(xt)dx \\ &\ll G^{\nu+2}. \end{aligned}$$

Similarly,

$$\hat{h}_2^{(\nu)}(t) \ll G^{\nu+1}.$$

Define the differential operator $\mathfrak{D}$ by, for any smooth function f,

$$(\mathfrak{D}f)(r) = -\frac{1}{2i}\frac{d}{dr}\left(\frac{f(r)}{\omega'(r)}\right).$$

By mathematical induction,

$$\left(\mathfrak{D}^A f\right)(r) = \sideset{}{'}\sum_{\substack{\nu+\sum \nu_j\mu_j=2A \\ \nu+\sum(\nu_j-1)\mu_j=A}} c_{\nu,\nu_j} f^{(\nu)}(r) \frac{\prod \left(\omega^{(\nu_j)}(r)\right)^{\mu_j}}{(\omega'(r))^{\nu+\sum \mu_j\nu_j}}$$

for some constants $\{c_{\nu,\nu_j}\}$ and $\sum'$ is a sum over a subset of all $\{\nu,\nu_j\}$ satisfying the stated conditions. Integrating by parts and let $\omega(t) = 2\pi(K-1)t + x\cos(2\pi t)$, $\int_{-\frac{K^{\epsilon/2}}{G}}^{\frac{K^{\epsilon/2}}{G}} \hat{h}_j(t) e^{i\omega(t)} dt$

$$= \sideset{}{'}\sum_{\substack{\nu+\sum \nu_j\mu_j=2A \\ \nu+\sum(\nu_j-1)\mu_j=A}} c_{\nu,\nu_j} \int_{-\frac{K^{\epsilon/2}}{G}}^{\frac{K^{\epsilon/2}}{G}} \hat{h}_j^{(\nu)}(t) \frac{\prod \left(\omega^{(\nu_j)}(t)\right)^{\mu_j}}{(\omega'(t))^{\nu+\sum \mu_j\nu_j}} e^{i\omega(t)} dt$$

For each summand, assume that $\nu_1 = 1$, then

$$\begin{aligned} \int_{-\frac{K^{\epsilon/2}}{G}}^{\frac{K^{\epsilon/2}}{G}} \hat{h}_j^{(\nu)}(t) \frac{\prod \left(\omega^{(\nu_j)}(t)\right)^{\mu_j}}{(\omega'(t))^{\nu+\sum \mu_j\nu_j}} e^{i\omega(t)} dt &\ll G^{2-j} K^{\epsilon/2} G^{\nu} \frac{K^{\mu_1} x^{\sum_{j>1}\mu_j}}{K^{\nu+\sum \nu_j\mu_j}} \\ = \quad G^{2-j} K^{\epsilon} \left(\frac{G}{K}\right)^{\nu+\sum_{j>1}\mu_j} \left(\frac{x}{KG}\right)^{\sum_{j>1}\mu_j} \left(\frac{1}{K}\right)^{\sum_{j>1}(\nu_j-2)\mu_j}. \end{aligned}$$

Note that the sum of exponents is $(\nu + \sum_{j>1}\mu_j) + (\sum_{j>1}\mu_j) + (\sum_{j>1}(\nu_j - 2)\mu_j) = \nu + \sum_{j>1}\nu_j\mu_j$. Since $\nu_j \le A$ and $\nu + \sum \nu_j\mu_j = 2A$, $\nu + \sum_{j>1}\nu_j\mu_j \ge A$. Recall that

$$\frac{x}{KG} \ll (KMN)^{-\epsilon} \text{ and } \frac{G}{K} \ll K^{-\epsilon},$$

so for each $B > 0$, choose $A = \lfloor \frac{100E+100D+100}{\epsilon} \rfloor + 10B$, we have (3.5) =

$$\sideset{}{'}\sum_{\substack{\nu+\sum \nu_j\mu_j=2A \\ \nu+\sum(\nu_j-1)\mu_j=A}} c_{\nu,\nu_j} \int_{-\frac{K^{\epsilon/2}}{G}}^{\frac{K^{\epsilon/2}}{G}} \hat{h}_j^{(\nu)}(t) \frac{\prod \left(\omega^{(\nu_j)}(t)\right)^{\mu_j}}{(\omega'(t))^{\nu+\sum \mu_j\nu_j}} e^{i\omega(t)} dt \ll (KNM)^{-B}.$$

4. Proof of Theorem 2.3

4.1. Strategy of the proof. The starting point is the Petersson formula (see P.389 of [4])

$$\frac{1}{\mathrm{vol}(X_0(M))}\sum_{f\in H_{k,M}}\rho_f(m)\overline{\rho_f(n)} = (k-1)\delta_{mn}$$
$$+ \quad 2\pi i^{-k}\sum_{c\equiv 0(M)}\frac{K_k(m,n;c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

where $K_k(m,n,;c)$ is the generalized Kloosterman sum,

$$K_k(m,n;c) = \sum_{d \mod c}\epsilon_d^{-2k}\left(\frac{c}{d}\right)e\left(\frac{m\bar{d}+nd}{c}\right)$$

with $d\bar{d}\equiv 1(c)$, $\epsilon_d = \begin{cases}1 \text{ if } d\equiv 1(4)\\ i \text{ if } d\equiv -1(4)\end{cases}$ and $\left(\frac{c}{d}\right)$ is the extended Kronecker's symbol (see P.388 of [4]).

We will employ an embedding idea of Duke, Frielander and Iwaniec (see for example P.225 of [3]) in a quantitative manner. We take a prime p such that $pMKG > N(MKN)^\epsilon$. Note that any function f in $B_{k,M}$ is naturally a holomorphic cusp form of weight k, level pM and L^2- norm 1. Hence we can embed $B_{k,M}$ into $B_{k,pM}$ and by positivity study the sum

$$\sum_{\substack{K\le k\le K+G\\ k-\frac{1}{2}\text{ even}}}\ \sum_{f\in B_{k,pM}}\left|\sum_{n=N}^{2N}a_n\rho_f(n)\right|^2$$

instead. Furthermore, the above sum is bounded by

$$\sum_{\substack{k\ge 1\\ k-\frac{1}{2}\text{ even}}} g(k)\sum_{f\in H_{k,pM}}\left|\sum_{n=N}^{2N}a_n\rho_f(n)\right|^2$$

where g is a suitable smooth function with compact support such that it dominates the characteristic function of the interval $[K, K+G]$. Contrary to the integral case, the half-integral weight J-Bessel function has two terms: For $l\in\mathbb{N}\cup\{0\}$, (see P.231 of [16])

$$J_{l+\frac{1}{2}}(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l+\frac{1}{2}\right)t\right)e^{-ix\sin 2\pi t}dt \tag{4.1}$$

$$+ \quad \frac{(-1)^l}{\pi} \int_0^\infty e^{-(l+\frac{1}{2})t - x\sinh t} dt.$$

The first term (or more precisely, the sum of Kloosterman sums involving the first term above) can be dealt with just as in the integral case after developing the appropriate Neumann theory while the second term is a new feature and require the full force of the hybrid large sieve inequality.

4.2. First manipulations. Assume $N \leq K^D$ for some $D > 0$ otherwise the result is trivial. Let $g(x) \in C_c^\infty(0,\infty)$ such that $\mathrm{supp}(g) \subset [\frac{1}{2}, \frac{5}{2}]$, $g^{(j)} \ll 1$ for all $j \geq 0$, and $g(x) = 1$ for $x \in [1,2]$. To prove Theorem 3.1, it suffices to bound, for sufficiently large K, the sum (we shift the sum for convenience but the adjustment to the original is simple)

$$S_M = \sum_{\substack{k \geq 1 \\ k-\frac{1}{2} \text{ even}}} g\left(\frac{k-K}{G}\right) \sum_{f \in H_{k,M}} \left| \sum_{n=N}^{2N} a_n \rho_f(n) \right|^2.$$

Choose a prime p such that

$$p \in \left[(KMN)^\epsilon \max\left\{ \frac{N}{MKG}, 1 \right\}, 2(KMN)^\epsilon \max\left\{ \frac{N}{MKG}, 1 \right\} \right].$$

As indicated in section 4.1,

$$S_M \leq \sum_{\substack{k \geq 1 \\ k-\frac{1}{2} \text{ even}}} g\left(\frac{k-K}{G}\right) \sum_{f \in H_{k,pM}} \left| \sum_{n=N}^{2N} a_n \rho_f(n) \right|^2.$$

By Petersson formula,

$$\begin{aligned} &\sum_{\substack{k \geq 1 \\ k-\frac{1}{2} \text{ even}}} g\left(\frac{k-K}{G}\right) \sum_{f \in B_{k,pM}} \left| \sum_{n=N}^{2N} a_n \rho_f(n) \right|^2 \\ = \quad & \mathrm{vol}(X_0(pM)) \sum_{\substack{k \geq 1 \\ k-\frac{1}{2} \text{ even}}} g\left(\frac{k-K}{G}\right) \sum_{N \leq n_1, n_2 \leq 2N} a_{n_1} \overline{a_{n_2}} \left\{ (k-1)\delta_{n_1 n_2} \right. \\ + \quad & \left. 2\pi i^{-k}(k-1) \sum_{c \equiv 0 (pM)} \frac{K_k(n_1, n_2; c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{n_1 n_2}}{c} \right) \right\} \end{aligned}$$

where we define for each complex number $z \neq 0$ and real number ν,

$$z^\nu = |z|^\nu \exp(i\nu \arg z) \text{ with } -\pi < \arg z \leq \pi.$$

4.3. The diagonal contribution. The diagonal term

$$\mathrm{vol}(X_0(pM)) \sum_{\substack{k\geq 1\\ k-\frac{1}{2}\text{ even}}} (k-1)g\left(\frac{k-K}{G}\right)\sum_{m=N}^{2N}|a_m|^2$$

contributes at most $\sum_{m=N}^{2N}|a_m|^2$ times

$$KG(pM)^{1+\epsilon} \ll (KM)^{\epsilon}pKGM \ll (KNM)^{2\epsilon}\max\{KGM, N\}.$$

4.4. The off-diagonal contribution. Since $J_{k-1}(x) \ll \left(\frac{ex}{2k}\right)^{k-1}$ (see (5.10.2) in [11]) and $\frac{4\pi\sqrt{n_1n_2}}{c} \leq \frac{8\pi N}{c}$, so the contribution from those c with $c > \frac{24\pi N}{K}$ is exponentially small. Thus it suffices to consider those c less than or equal to $\frac{24\pi N}{K}$, i.e. we can consider instead

$$\mathrm{vol}(X_0(pM)) \sum_{\substack{k\geq 1\\ k-\frac{1}{2}\text{ even}}} g\left(\frac{k-K}{G}\right)\sum_{N\leq n_1,n_2\leq 2N} a_{n_1}\overline{a_{n_2}}\times \tag{4.2}$$

$$\left\{2\pi i^{-k}(k-1)\sum_{\substack{c\equiv 0(pM)\\ c\leq\frac{24\pi N}{K}}}\frac{K_k(n_1,n_2;c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{n_1n_2}}{c}\right)\right\}.$$

By (4.1), (4.2) =

$$2\mathrm{vol}(X_0(pM))\sum_{\alpha=1,2}\sum_{\substack{c\equiv 0(pM)\\ c\leq\frac{24\pi N}{K}}}\sum_{N\leq n_1,n_2\leq 2N}\sum_{\substack{l\text{ even}\\ l\geq 2}}\frac{K_{l+\frac{1}{2}}(n_1,n_2;c)}{c}a_{n_1}\overline{a_{n_2}}\times$$

$$i^{-(l+\frac{1}{2})}\left(l-\frac{1}{2}\right)g\left(\frac{l+\frac{1}{2}-K}{G}\right)I_{l,\alpha}\left(\frac{4\pi\sqrt{n_1n_2}}{c}\right)$$

where

$$I_{l,1}(x) = \pi\int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l-\frac{1}{2}\right)t\right)e^{-ix\sin 2\pi t}dt \text{ and}$$

$$I_{l,2}(x) = \int_0^{\infty} e^{-\left(l-\frac{1}{2}\right)t-x\sinh t}dt.$$

Opening the twisted Kloosterman sum (see section 4.1 for the equation of $K_k(n_1,n_2;c)$) and by our definition of z^{ν} (see P.12), (4.2) =

(4.3)

$$\sum_{m=0,1} \sum_{\substack{c\equiv 0(pM)\\ c\le \frac{24\pi N}{K}}} \frac{2\mathrm{vol}(X_0(pM))}{c} \sum_{N\le n_1,n_2\le 2N} \sum_{\substack{d(c)\\ d\equiv(-1)^m(4)}} \left(\frac{c}{d}\right)$$

$$\times \quad e\left(\frac{n_1 d + n_2\bar{d}}{c}\right) a_{n_1}\overline{a_{n_2}} \sum_{j=\pm 1} i^{-1+\frac{j}{2}} \sum_{\substack{l \text{ even}\\ l\ge 2}} \left(l-\frac{1}{2}\right) i^{l(j-2)} \times$$

$$g\left(\frac{l+\frac{1}{2}-K}{G}\right) I_{l,1}\left(\frac{4\pi\sqrt{n_1 n_2}}{c}\right) +$$

(4.4)

$$\sum_{m=0,1} \sum_{\substack{c\equiv 0(pM)\\ c\le \frac{24\pi N}{K}}} \frac{2\mathrm{vol}(X_0(pM))}{c} \sum_{N\le n_1,n_2\le 2N} \sum_{\substack{d(c)\\ d\equiv(-1)^m(4)}} \left(\frac{c}{d}\right)$$

$$\times \quad e\left(\frac{n_1 d + n_2\bar{d}}{c}\right) a_{n_1}\overline{a_{n_2}} \sum_{j=\pm 1} i^{-1+\frac{j}{2}} \sum_{\substack{l \text{ even}\\ l\ge 2}} \left(l-\frac{1}{2}\right) i^{l(j-2)}$$

$$\times \quad g\left(\frac{l+\frac{1}{2}-K}{G}\right) I_{l,2}\left(\frac{4\pi\sqrt{n_1 n_2}}{c}\right)$$

The complicated integral $I_{l,2}(x)$ can be evaluated with an arbitrarily small error term.

Lemma 4.1. *For $A \ge 0$, $l \ge 2$*

$$I_{l,2}(x) = \sum_{n=0}^{A-1}(-1)^n \sideset{}{'}\sum_{\substack{\sum \nu_j\mu_j = 2n\\ \sum(\nu_j-1)\mu_j = n}} d_{n,\nu_j} \frac{\prod\left(\omega^{(\nu_j)}(0)\right)^{\mu_j}}{\left(l-\frac{1}{2}+x\right)^{2n+1}} + O\left(x^{-A}\right)$$

where $\omega(t) = -\left(l-\frac{1}{2}\right)t - x\sinh t$, $\{d_{n,\nu_j}\}$ constants and $\sum'$ is a sum over a subset of all $\{\nu_j\}$ satisfying the stated conditions.

Proof. Define a differential operator $\mathfrak{D}$ by: For each smooth h

$$(\mathfrak{D}h)(t) = \frac{d}{dt}\left(\frac{h(t)}{\omega'(t)}\right)$$

and $\mathfrak{D}^0 h = h$.

By induction,

$$(\mathfrak{D}^n 1)(t) = \sum_{\substack{\nu+\sum \nu_j\mu_j=2n \\ \nu+\sum(\nu_j-1)\mu_j=n}}{}^{'} c_{n,\nu,\nu_j} \frac{d^\nu 1}{dt^\nu} \frac{\prod(\omega^{(\nu_j)}(t))^{\mu_j}}{(\omega'(t))^{\nu+\sum \nu_j\mu_j}}$$

$$= \sum_{\substack{\sum \nu_j\mu_j=2n \\ \sum(\nu_j-1)\mu_j=n}}{}^{'} d_{n,\nu_j} \frac{\prod(\omega^{(\nu_j)}(t))^{\mu_j}}{(\omega'(t))^{2n}}$$

for some constants $\{d_{n,\nu_j}\}$ and $\sum'$ is a sum over a subset of all $\{\nu_j\}$ satisfying the stated conditions. Integrating by parts,

$$\int_0^\infty e^{\omega(t)} dt = \sum_{n=0}^{A-1} (-1)^{n+1} \frac{(\mathfrak{D}^n 1)(0)}{\omega'(0)}$$

$$+ \quad (-1)^A \sum_{\substack{\sum \nu_j\mu_j=2A \\ \sum(\nu_j-1)\mu_j=A}}{}^{'} d_{A,\nu_j} \int_0^\infty \frac{\prod(\omega^{(\nu_j)}(t))^{\mu_j}}{(l-\frac{1}{2}+x\cosh t)^{2A}} dt.$$

Since for $n \geq 2$

$$\omega^{(n)}(t) = \begin{cases} -x\sinh t \text{ if } n \text{ even} \\ -x\cosh t \text{ if } n \text{ odd} \end{cases}$$

and

$$\int_0^\infty \frac{\sinh^A t}{\left(\frac{l}{x}+\cosh t\right)^{2A}} dt, \int_0^\infty \frac{\cosh^A t}{\left(\frac{l}{x}+\cosh t\right)^{2A}} dt < \infty.$$

We conclude that

$$\int_0^\infty e^{\omega(t)} dt = \sum_{n=0}^{A-1} (-1)^{n+1} \frac{(\mathfrak{D}^n 1)(0)}{\omega'(0)} + O(x^{-A}).$$

Hence,

$$\int_0^\infty e^{\omega(t)} dt = \sum_{n=0}^{A-1} (-1)^n \frac{(\mathfrak{D}^n 1)(0)}{l-\frac{1}{2}+x} + O(x^{-A})$$

$$= \sum_{n=0}^{A-1} (-1)^n \sum_{\substack{\sum \nu_j\mu_j=2n \\ \sum(\nu_j-1)\mu_j=n}}{}^{'} d_{n,\nu_j} \frac{\prod(\omega^{(\nu_j)}(0))^{\mu_j}}{(l-\frac{1}{2}+x)^{2n+1}} + O(x^{-A})$$

□

The contribution from the error term $O(x^{-A})$ in (4.4) is

$$\ll \quad pMKG \sum_{\substack{c\equiv 0(pM)\\ c\leq \frac{24\pi N}{K}}} c^A \left(\sum_{n=N}^{2N} \frac{|a_n|}{n^{A/2}}\right)^2 \ll \frac{pMKG(pM)^A}{N^{A-1}} \sum_{t=1}^{\frac{24\pi N}{KpM}} t^A \sum_{n=N}^{2N} |a_n|^2$$

$$\ll \quad \frac{N^2}{K^{A-1}} \sum_{n=N}^{2N} |a_n|^2.$$

Since we assume a prior that $N \ll K^B$ for some $B > 0$. By taking $A = B+1$, we have the term above $\ll N \sum |a_n|^2$. Hence to show (4.4) $\ll (MKN)^\epsilon (KGM + N) \sum_{n=N}^{2N} |a_n|^2$ (thanks to Lemma 4.1), it suffices to show

Lemma 4.2.

(4.5)

$$\begin{aligned}
&vol(X_0(pM)) \sum_{m=0,1} \sum_{\substack{c\equiv 0(pM)\\ c\leq \frac{24\pi N}{K}}} \frac{1}{c} \sum_{\substack{d(c)\\ d\equiv(-1)^m(4)}} \left(\frac{c}{d}\right) \sum_{j=\pm 1} i^{-1+\frac{j}{2}} \\
&\sum_{\substack{l \text{ even}\\ l\geq 2}} \left(l-\frac{1}{2}\right) i^{l(j-2)} g\left(\frac{l+\frac{1}{2}-K}{G}\right) \\
\times \quad &\sum_{N\leq n_1,n_2\leq 2N} e\left(\frac{n_1\bar{d}+n_2 d}{c}\right) \frac{a_{n_1}\overline{a_{n_2}} \left(\frac{4\pi\sqrt{n_1n_2}}{c}\right)^a}{\left(l-\frac{1}{2}+\frac{4\pi\sqrt{n_1n_2}}{c}\right)^{b+1+a}} \\
&\ll (MKN)^\epsilon (KGM+N) \sum_{n=N}^{2N} |a_n|^2
\end{aligned}$$

for all natural number b, $0 \leq a \leq b$.

To proof Lemma 4.2 we need the following hybrid large sieve inequality.

Theorem 4.1. *For any complex numbers $\{a_n\}$, $M < n \leq M+N$, $x_1 \cdots, x_R$ be real numbers which are distinct mod 1. Let $\delta = \min_{\substack{r,s\\ r\neq s}} ||x_r - x_s||$, where if $R \geq 2$, $||x|| := \min_{k\in\mathbb{Z}} |x-k|$; $\delta := \infty$, if $R = 1$. Then*

$$\sum_{r=1}^{R} \left|\sum_{n=M+1}^{M+N} a_n e(nx_r)\right|^2 \leq (\pi N + \delta^{-1}) \sum_{n=N+1}^{M+N} |a_n|^2.$$

Proof. See Theorem 2.1 in [15] □

Corollary 4.1. *Notations as before. Let Q, s be natural numbers. Then*

$$\sum_{\substack{q\le Q\\ q\equiv 0(s)}}\sum_{\substack{(l,q)=1\\ 0<l<q}}\left|\sum_{n=M+1}^{M+N} a_n e\left(\frac{nl}{q}\right)\right|^2 \ll \left(N+\frac{Q^2}{s}\right)\sum_{n=M+1}^{M+N}|a_n|^2.$$

Proof. For $(l,q)=1$, $0<l<q$, $q\le Q$, $q\equiv 0(s)$. Let $x_{l,q}=\frac{l}{q}$. Then for each $q\neq q'$,

$$||x_{l,q}-x_{l',q'}|| = ||\frac{l}{q}-\frac{l'}{q'}|| = ||\frac{l}{sv}-\frac{l'}{sv'}||$$

for some $v, v'\in\mathbb{N}$. Hence,

$$||x_{l,q}-x_{l',q'}|| \ge \frac{1}{svv'} \ge \frac{1}{\frac{Q^2}{s}}.$$

Hence taking $\delta=\frac{Q^2}{s}$ in Theorem 7.2,

$$\sum_{\substack{q\le Q\\ q\equiv 0(s)}}\sum_{\substack{(l,q)=1\\ 0<l<q}}\left|\sum_{n=M+1}^{M+N} a_n e\left(\frac{nl}{q}\right)\right|^2 \ll \left(N+\frac{Q^2}{s}\right)\sum_{n=M+1}^{M+N}|a_n|^2.$$

□

Proposition 4.1. *Notations as before. For $a_n\in\mathbb{C}$, let*

$$\sum_{n=1}^{\infty}|a_n|<\infty.$$

Then, for $T\ge 1$,

$$\sum_{\substack{q\le Q\\ q\equiv 0(s)}}\sum_{\substack{(l,q)=1\\ 0<l<q}}\int_{-T}^{T}\left|\sum_{n=1}^{\infty} a_n e\left(\frac{nl}{q}\right)n^{-it}\right|^2 dt \ll \sum_{n=1}^{\infty}\left(\frac{TQ^2}{s}+n\right)|a_n|^2.$$

Proof. See the proof of Theorem 5.1 in [15]. □

We define, for any complex sequence $\{b_n\}_{N\le n\le 2N}$,

$$B(\{b_n\},c,N,t)=\sum_{m=0,1}\sum_{\substack{d(c)\\ d\equiv(-1)^m(4)}}\left|\sum_{n=N}^{2N} b_n e\left(\frac{nd}{c}\right)n^{-it}\right|^2.$$

We need to show that for $0\le a\le b$,

Lemma 4.3.

$$vol(X_0(pM))\sum_{m=0,1}\sum_{\substack{c\equiv 0(pM)\\ c\le\frac{24\pi N}{K}}}\frac{1}{c}\sum_{\substack{d(c)\\ d\equiv(-1)^m(4)}}\left(\frac{c}{d}\right)\times \tag{4.6}$$

$$\sum_{N\le n_1,n_2\le 2N} e\left(\frac{n_1\overline{d}+n_2d}{c}\right) a_{n_1}\overline{a_{n_2}}\frac{\left(\frac{4\pi\sqrt{n_1n_2}}{c}\right)^a}{\left(l-\frac{1}{2}+\frac{4\pi\sqrt{n_1n_2}}{c}\right)^{b+1+a}}$$
$$\ll \quad \left(\frac{N}{K^{b+2}}+\frac{pM}{K^b}\right)\sum |a_n|^2$$

Proof. We follow the argument on P.256 of [2]. The above sum remains unchanged if a smooth weight $h\left(\frac{\sqrt{n_1n_2}}{N}\right)$ is attached to each term provided that

$$h(x)=\begin{cases}1 \text{ if } x\in(1,2]\\ 0 \text{ if } x\notin(\frac{1}{2},3].\end{cases}$$

In what follows we demand $h(x)$ to be of C^∞ class. Then

$$\frac{h(x)}{\left(l-\frac{1}{2}+\frac{4\pi}{c}xN\right)^{b+1+a}}=\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} H(s)x^{-s}ds$$

with

$$H(s)=\int_0^\infty h(x)\frac{1}{\left(l-\frac{1}{2}+\frac{4\pi}{c}xN\right)^{b+1+a}}x^{it}dx.$$

Hence, (4.6) =

$$\sum_{m=0,1}\sum_{\substack{c\equiv 0(pM)\\ c\le\frac{24\pi N}{K}}}\frac{1}{2\pi ic}\sum_{\substack{d(c)\\ d\equiv(-1)^m(4)}}\left(\frac{c}{d}\right)\sum_{N\le n_1,n_2\le 2N} e\left(\frac{n_1\overline{d}+n_2d}{c}\right)$$
$$\times \quad \mathrm{vol}(X_0(pM))a_{n_1}\overline{a_{n_2}}\left(\frac{4\pi\sqrt{n_1n_2}}{c}\right)^a\int_{1-i\infty}^{1+i\infty} H(s)\left(\frac{\sqrt{n_1n_2}}{N}\right)^{-s}ds$$

$$= \quad \frac{\mathrm{vol}(X_0(pM))}{2\pi i}\sum_{m=0,1}\sum_{\substack{c\equiv 0(pM)\\ c\le\frac{24\pi N}{K}}}\frac{(4\pi)^a}{c^{a+1}}\sum_{\substack{d(c)\\ d\equiv(-1)^m(4)}}\left(\frac{c}{d}\right)\int_{1-i\infty}^{1+i\infty} H(s)\times$$
$$\left(\sum_{n_1=N}^{2N} a_{n_1}n_1^{\frac{a}{2}}\left(\frac{n_1}{N}\right)^{-\frac{s}{2}} e\left(\frac{n_1\overline{d}}{c}\right)\right)$$
$$\times \quad \left(\sum_{n_2=N}^{2N} \overline{a_{n_2}}n_2^{\frac{a}{2}}\left(\frac{n_2}{N}\right)^{-\frac{s}{2}} e\left(\frac{n_2d}{c}\right)\right)ds$$

$$\ll \sum_{\substack{c\equiv 0(pM)\\ c\leq \frac{24\pi N}{K}}} \frac{pMN}{c^{a+1}} \int_{-\infty}^{\infty} B(\{a_n n^{\frac{a-1}{2}}\}, c, N, t)\, |H(1+2it)|\, dt$$

$$\leq \sum_{\substack{c\equiv 0(pM)\\ c\leq \frac{24\pi N}{K}}} \frac{pMN}{c^{a+1}} \int_{-1}^{1} B(\{a_n n^{\frac{a-1}{2}}\}, c, N, t)\, |H(1+2it)|\, dt$$

$$+ \sum_{\substack{c\equiv 0(pM)\\ c\leq \frac{24\pi N}{K}}} \frac{pMN}{c^{a+1}} \sum_{\nu\geq 0} \int_{J_\nu} B(\{a_n n^{\frac{a-1}{2}}\}, c, N, t)\, |H(1+2it)|\, dt$$

where $J_\nu = [-2^{\nu+1}, -2^\nu] \cup [2^\nu, 2^{\nu+1}]$. Integrating by parts,

$$H(1+it) \ll \begin{cases} (1+|t|)^{-\frac{1}{2}} \frac{1}{\left(l-\frac{1}{2}+\frac{4\pi N}{c}\right)^{b+1+a}} \\ t^{-2} \frac{1}{\left(l-\frac{1}{2}+\frac{4\pi N}{c}\right)^{b+1+a}} \text{ for } |t| > \frac{16\pi N}{c}. \end{cases}$$

Then, (4.6)

$$\ll \sum_{\substack{c\equiv 0(pM)\\ c\leq \frac{24\pi N}{K}}} \frac{pMN}{c^{a+1}} \int_{-1}^{1} B(\{a_n n^{\frac{a-1}{2}}\}, c, N, t) \left(\frac{c}{N}\right)^{b+1+a} dt + \sum_{v=0}^{\infty} \sum_{\substack{c\equiv 0(pM)\\ c\leq \frac{24\pi N}{K}}}$$

$$\times \frac{pMN}{c^{a+1}} \int_{J_\nu} B(\{a_n n^{\frac{a-1}{2}}\}, c, N, t) \min\left\{\frac{1}{\sqrt{1+2^v}}, 2^{-2v}\right\} \left(\frac{c}{N}\right)^{b+1+a} dt$$

$$= \frac{pM}{N^{b+a}} \sum_{\substack{c\equiv 0(pM)\\ c\leq \frac{24\pi N}{K}}} c^b \int_{-1}^{1} B(\{a_n n^{\frac{a-1}{2}}\}, c, N, t) dt + \frac{pM}{N^{b+a}} \sum_{v=0}^{\infty} \sum_{\substack{c\equiv 0(pM)\\ c\leq \frac{24\pi N}{K}}} c^b$$

$$\times \quad B(2^\nu, c) \min\left\{\frac{1}{\sqrt{1+2^v}}, 2^{-2v}\right\}$$

where

$$B(T,c) = \int_{[T,2T]\cup[-2T,-T]} B(\{a_n n^{\frac{a-1}{2}}\}, c, N, t) dt.$$

By Abel summation formula,

$$\frac{pM}{N^{b+a}}\sum_{\substack{c\equiv 0(pM)\\ c\le \frac{24\pi N}{K}}} c^b B(T,c) = \frac{(pM)^{b+1}}{N^{a+b}}\sum_{u=1}^{\frac{24\pi N}{KpM}} u^b B(T,pMu)$$

$$\ll \frac{(pM)^{n+1}}{N^{a+b}}\left(\sum_{u=1}^{\frac{24\pi N}{KpM}} B(T,pMu)\right)\left(\frac{24\pi N}{KpM}\right)^b$$

$$+ \quad b\int_1^{\frac{24\pi N}{KpM}} x^{b-1}\left(\sum_{u=1}^{x} B(T,pMu)\right)dx\frac{(pM)^{b+1}}{N^{a+b}}$$

$$\ll \frac{pM}{N^aK^b}\sum_{\substack{c\equiv 0(pM)\\ c\le \frac{24\pi N}{K}}} B(T,c) + \frac{(pM)^{b+1}}{N^{a+b}}\int_1^{\frac{24\pi N}{KpM}} x^{b-1}\left(\sum_{\substack{c\equiv 0(pM)\\ c\le xpM}} B(T,c)\right)dx$$

By Proposition 4.5,

$$\sum_{\substack{c\equiv 0(pM)\\ c\le xpM}} B(T,c) \ll \sum_{n=N}^{2N}(Tx^2pM+n)|a_n|^2 n^{a-1}$$

$$\ll \quad (Tx^2pMN^{a-1}+N^a)\sum_{n=N}^{2N}|a_n|^2.$$

Hence,

$$\frac{pM}{N^{b+a}}\sum_{\substack{c\equiv 0(pM)\\ c\le \frac{24\pi N}{K}}} c^b B(T,c) \ll \frac{pM}{N^aK^b}\left(T\frac{N^2}{K^2pM}N^{a-1}+N^a\right)\sum_{n=N}^{2N}|a_n|^2$$

$$+ \quad \frac{(pM)^{b+1}}{N^{a+b}}\int_1^{\frac{24\pi N}{KpM}} x^{b-1}(Tx^2pMN^{a-1}+N^a)dx\sum_{n=N}^{2N}|a_n|^2$$

$$\ll \quad \left(\frac{TN}{K^{b+2}}+\frac{pM}{K^b}\right)\sum_{n=N}^{2N}|a_n|^2$$

$$+ \quad \frac{(pM)^{b+1}}{N^{a+b}}\left[TpMN^{a-1}\left(\frac{N}{KpM}\right)^{b+2} + N^a\left(\frac{N}{KpM}\right)^{b}\right]\sum_{n=N}^{2N}|a_n|^2$$

$$\ll \quad \left[\frac{TN}{K^{b+2}} + \frac{pM}{K^b}\right]\sum_{n=N}^{2N}|a_n|^2.$$

Hence, (4.6)

$$\ll \quad \left[\frac{N}{K^{b+2}} + \frac{pM}{K^b}\right]\sum_{n=N}^{2N}|a_n|^2.$$

□

Proof. (of Lemma 4.2)

By Lemma 4.6, (4.5)

$$\ll \quad \sum_{k\geq 1} g\left(\frac{2k-K+\frac{1}{2}}{G}\right)\left(2k-\frac{1}{2}\right)\left(\frac{N}{K^2}+pM\right)\sum_{m=N}^{2N}|a_m|^2$$

$$\ll \quad KG\left(\frac{N}{K^2}+pM\right)\sum_{m=N}^{2N}|a_m|^2 \ll (MKN)^{\epsilon}(KG+N)\sum_{m=N}^{2N}|a_m|^2.$$

□

We now develop the appropriate Neumann theory for (4.3). We start with

Lemma 4.4. *For $h \in C_0^{\infty}(0,\infty)$, $\alpha \in \mathbb{R}$,*

$$\sum_l h(l)e(\alpha l)\int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l+\frac{1}{2}\right)t\right)e^{-ix\sin 2\pi t}dt$$

$$= \int_{-\infty}^{\infty}\hat{h}(t)e\left(\frac{\lfloor \alpha - t\rfloor + \alpha - t}{2}\right)e^{-ix\sin 2\pi(\alpha-t)}dt$$

Proof. Let $g_x(t) = e\left(\frac{t}{2}\right)e^{-ix\sin 2\pi t}$ and

$$f(t) = \begin{cases} g_x(t) & t \in \left(-\frac{1}{2},\frac{1}{2}\right] \\ 0 & \text{otherwise}\end{cases}$$

then, $\int_{-\frac{1}{2}}^{\frac{1}{2}} e(lt)g_x(t)dt = \int_{-\infty}^{\infty} e(lt)f(t)dt = \hat{f}(l)$, the Fourier transform of f at l. By Poisson summation formula, denoting the Fourier transform by $\mathfrak{F}$,

$$\sum_l h(l)\int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l+\frac{1}{2}\right)t\right)e^{-ix\sin 2\pi t}dt$$

$$= \sum_l h(l)\hat{f}(l) = \sum_l \mathfrak{F}(\hat{h}(-\cdot) * f)(l)$$

$$= \sum_l (\hat{h}(-\cdot) * f)(l) = \sum_l \int_{-\infty}^{\infty} \hat{h}(-t) f(l-t)dt$$

$$= \int_{-\infty}^{\infty} \hat{h}(t) \sum_l f(l+t)dt.$$

We now show have to show

$$\sum_l f(l+t) = e\left(\frac{\lfloor t \rfloor + t}{2}\right) e^{-ix\sin 2\pi t}.$$

For each $t \in \mathbb{R}$, if $t = 2n + y$ for some $y \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ and $n \in \mathbb{Z}$, then

$$\sum_l f(l+t) = f(y) = g_x(y) = e\left(\frac{y}{2}\right) e^{-ix\sin 2\pi y}$$

$$= e\left(\frac{2n+y}{2}\right) e^{-ix\sin 2\pi(2n+y)} = e\left(\frac{\lfloor t \rfloor + t}{2}\right) e^{-ix\sin 2\pi t};$$

if $t = 2n + 1 + y$ for some $y \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ and $n \in \mathbb{Z}$, then

$$\sum_l f(l+t) = f(y) = g_x(y) = e\left(\frac{y}{2}\right) e^{-ix\sin 2\pi y}$$

$$= -e\left(\frac{2n+1+y}{2}\right) e^{-ix\sin 2\pi(2n+1+y)} = e\left(\frac{\lfloor t \rfloor + t}{2}\right) e^{-ix\sin 2\pi t}.$$

Hence $\sum_l f(l+t) = e\left(\frac{\lfloor t \rfloor + t}{2}\right) e^{-ix\sin 2\pi t}$.

Applying Lemma 4.7 to $h_\alpha(t) = h(t)e(\alpha t)$, then

$$\sum_l h(l)e(\alpha l) \int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l + \frac{1}{2}\right) t\right) e^{-ix\sin 2\pi t} dt$$

$$= \sum_l h_\alpha(l) \int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l + \frac{1}{2}\right) t\right) e^{-ix\sin 2\pi t} dt$$

$$= \int_{-\infty}^{\infty} \widehat{h_\alpha}(t) e\left(\frac{\lfloor t \rfloor + t}{2}\right) e^{-ix\sin 2\pi t} dt.$$

Notice that

$$\widehat{h_\alpha}(t) = \int_{-\infty}^{\infty} h(y)e(\alpha y)e(-yt)dy$$

$$= \int_{-\infty}^{\infty} h(y)e((\alpha - t)y)dy = \hat{h}(\alpha - t).$$

Hence,

$$\begin{aligned}
&\sum_l h(l)e(\alpha l)\int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l+\frac{1}{2}\right)t\right)e^{-ix\sin 2\pi t}dt \\
=\ & \int_{-\infty}^{\infty}\hat{h}(\alpha-t)e\left(\frac{\lfloor t\rfloor+t}{2}\right)e^{-ix\sin 2\pi t}dt \\
=\ & \int_{-\infty}^{\infty}\hat{h}(t)e\left(\frac{\lfloor \alpha-t\rfloor+(\alpha-t)}{2}\right)e^{-ix\sin 2\pi(\alpha-t)}dt.
\end{aligned}$$

□

In particular, take $\alpha=\frac{1}{4}$

$$\sum_l h(l)e\left(\frac{l}{4}\right)\int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l+\frac{1}{2}\right)t\right)e^{-ix\sin 2\pi t}dt$$

$$\begin{aligned}
=\ & \sum_{l\text{ even}}(-1)^{\frac{l}{2}}h(l)\int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l+\frac{1}{2}\right)t\right)e^{-ix\sin 2\pi t}dt \\
+\ & i\sum_{l\text{ odd}}(-1)^{\frac{l-1}{2}}h(l)\int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l+\frac{1}{2}\right)t\right)e^{-ix\sin 2\pi t}dt.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{l\text{ odd}}(-1)^{\frac{l-1}{2}}h(l)\int_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l+\frac{1}{2}\right)t\right)e^{-ix\sin 2\pi t}dt \\
=\ & \operatorname{Im}\left(\int_{-\infty}^{\infty}\hat{h}(t)e\left(\frac{\lfloor\frac{1}{4}-t\rfloor+\frac{1}{4}-t}{2}\right)e^{-ix\cos 2\pi t}dt\right).
\end{aligned}$$

To get rid of the annoying floor function $\lfloor\frac{1}{4}-t\rfloor$, note that $\hat{h}$ is essentially supported in $\left[-\frac{1}{G},\frac{1}{G}\right]$ whenever h is supported in $[K,K+G]$ whose j-derivative is majored by G^{-j+1}, i.e. $h(t)=tg\left(\frac{t-(K-1)}{G}\right)$ satisfies the conditions in Lemma 3.3. Thus,

Proposition 4.2. *notation as above. For all $A > 0$,*

$$\int_{-\infty}^{\infty} \hat{h}(t) e\left(\frac{\lfloor \frac{1}{4} - t \rfloor + \frac{1}{4} - t}{2}\right) e^{-ix\cos 2\pi t} dt$$
$$= \int_{-\infty}^{\infty} \hat{h}(t) e\left(\frac{1}{8} - \frac{t}{2}\right) e^{-ix\cos 2\pi t} dt + O(K^{-A}).$$

Proof. Fix $0 < \delta < 1$ such that $\frac{G^\delta}{G} < \frac{1}{4}$,

$$\int_{-\infty}^{\infty} \hat{h}(t) e\left(\frac{\lfloor \frac{1}{4} - t \rfloor + \frac{1}{4} - t}{2}\right) e^{-ix\cos 2\pi t} dt$$
$$= \int\limits_{|t| \leq \frac{G^\delta}{G}} \hat{h}(t) e\left(\frac{\lfloor \frac{1}{4} - t \rfloor + \frac{1}{4} - t}{2}\right) e^{-ix\cos 2\pi t} dt$$
$$+ \int\limits_{|t| > \frac{G^\delta}{G}} \hat{h}(t) e\left(\frac{\lfloor \frac{1}{4} - t \rfloor + \frac{1}{4} - t}{2}\right) e^{-ix\cos 2\pi t} dt.$$

By Lemma 7.4,

$$\int\limits_{|t| > \frac{G^\delta}{G}} \hat{h}(t) e\left(\frac{\lfloor \frac{1}{4} - t \rfloor + \frac{1}{4} - t}{2}\right) e^{-ix\cos 2\pi t} dt$$
$$\ll KG^{-m+1}\left(\frac{G^\delta}{G}\right)^{-m+1} = KG^{\delta(-m+1)}.$$

Taking $m = \left\lfloor 3\frac{A+1}{\delta} \right\rfloor + 2$,

$$\int\limits_{|t| > \frac{G^\delta}{G}} \hat{h}(t) e\left(\frac{\lfloor \frac{1}{4} - t \rfloor + \frac{1}{4} - t}{2}\right) e^{-ix\cos 2\pi t} dt \ll K^{-A}.$$

On the other hand when $|t| \leq \frac{G^\delta}{G} \leq \frac{1}{4}$, $\left\lfloor \frac{1}{4} - t \right\rfloor = 0$. Hence,

$$\int_{-\infty}^{\infty} \hat{h}(t) e\left(\frac{\lfloor \frac{1}{4} - t \rfloor + \frac{1}{4} - t}{2}\right) e^{-ix\cos 2\pi t} dt$$
$$= \int\limits_{|t| \leq \frac{G^\delta}{G}} \hat{h}(t) e\left(\frac{1}{8} - \frac{t}{2}\right) e^{-ix\cos 2\pi t} dt + O(K^{-A})$$
$$= \int_{-\infty}^{\infty} \hat{h}(t) e\left(\frac{1}{8} - \frac{t}{2}\right) e^{-ix\cos 2\pi t} dt + O(K^{-A})$$

by using $\int\limits_{|t|>G^{\delta-1}} \hat{h}(t)e\left(\frac{1}{8}-\frac{t}{2}\right)e^{-ix\cos 2\pi t}dt = O(K^{-A})$. □

From this we conclude that

$$\sum_{l \text{ even}} (-1)^{\frac{l}{2}} h(l-1) \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l+\frac{1}{2}\right)t\right) e^{-ix\sin 2\pi t}dt$$
$$= \operatorname{Im}\left(\int_{-\infty}^{\infty} \hat{h}(t)e\left(\frac{1}{8}-\frac{t}{2}\right)e^{-ix\cos 2\pi t}dt\right) + O(K^{-A}).$$

As before (see the discussion proceeding Lemma 3.1),

$$\hat{h}(t) = e((K-1)t)\hat{h}_1(t) + (K-1)e((K-1)t)\hat{h}_2(t)$$

where $h_1(x) = xg\left(\frac{x}{G}\right)$ and $h_2(x) = g\left(\frac{x}{G}\right)$. Hence

$$\sum_{l\text{even}} (-1)^{\frac{l}{2}} h(l-1) \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} e\left(\left(l+\frac{1}{2}\right)t\right) e^{-ix\sin 2\pi t}dt$$
$$= \operatorname{Im}\left(\int_{-\infty}^{\infty} \hat{h}_1(t)e\left(\frac{1}{8}+\left(K-\frac{3}{2}\right)t-\frac{x}{2\pi}\cos 2\pi t\right)dt\right.$$
$$+ \left.(K-1)\int_{-\infty}^{\infty} \hat{h}_2(t)e\left(\frac{1}{8}+\left(K-\frac{3}{2}\right)t-\frac{x}{2\pi}\cos 2\pi t\right)dt\right).$$

The rest of the proof is then completely analogous to that of Theorem 2.1.

5. Proof of Theorem 2.5

The proof is similar to that of the holomorphic case, but in lieu of Neumann's theory we will appeal to the asymptotic expansion of the Bessel function. We assume $N \leq K^E$ for some $E > 0$ otherwise we can appeal to the duality principle (see P.137 of [14]).

For each cusp $\mathfrak{a} \in \mathfrak{P}_q$, there exists $\sigma_{\mathfrak{a}}$ in $PSL_2(\mathbb{R})$ such that $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$ and $\sigma_{\mathfrak{a}}^{-1}\Gamma_0(q)_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty$ where $\Gamma_0(q)_{\mathfrak{a}}$, $\Gamma_0(q)_\infty$ are the group of stabilizers of $\mathfrak{a}$ and ∞ in $\Gamma_0(q)$ respectively. We now define the Eisenstein series for $\Gamma_0(q)$,

$$E_{\mathfrak{a},\Gamma_0(q)}(z,s) = \sum_{\gamma\in\Gamma_0(q)_{\mathfrak{a}}/\Gamma_0(q)} \left(\operatorname{Im}\,\sigma_{\mathfrak{a}}^{-1}\gamma z\right)^s$$

which has the Fourier expansion at $\mathfrak{b} \in \mathfrak{P}_q$ (see P.388 of [7])

$$E_{\mathfrak{a},\Gamma_0(q)}(\sigma_{\mathfrak{b}}z,s) = \delta_{\mathfrak{a}\mathfrak{b}}y^s + \phi_{\mathfrak{a}\mathfrak{b}}^{\Gamma_0(q)}y^{1-s} +$$
$$\sum_{n\neq 0} \phi_{\mathfrak{a}\mathfrak{b}}^{\Gamma_0(q)}(n,s)y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|y)e(nx).$$

Let $g(x) \in C_0^\infty(0,\infty)$such that $\mathrm{supp}(g(x)) \subset \left[\frac{1}{2},\frac{5}{2}\right]$, $g^{(j)} \ll 1$ for all $j \geq 1$ and $g(x) = 1$ for all $x \in [1,2]$. To prove Theorem A.1, it suffices to bound, for sufficiently large K, the sum (we shift the sum for convenience but the adjustment to the original is simple)

$$S_M = \sum_{f\in H_M} \frac{1}{\cosh \pi t_f} g\left(\frac{t_f - K}{G}\right)\left|\sum_{n=N}^{2N} a_n \rho_{t_f}(n)\right|^2 .$$

Choose a prime p such that

$$p \in \left[(KMN)^\epsilon \max\left\{\frac{N}{MKG},1\right\}, 2(KMN)^\epsilon \max\left\{\frac{N}{MKG},1\right\}\right].$$

We embed H_M into H_{pM} and by positivity

$$\begin{aligned} S_M &\leq \sum_{f\in H_{pM}} \frac{1}{\cosh \pi t_f} g\left(\frac{t_f - K}{G}\right)\left|\sum_{n=N}^{2N} a_n \rho_{t_f}(n)\right|^2 \\ &\leq \sum_{f\in H_{pM}} \frac{1}{\cosh \pi t_f} g\left(\frac{t_f - K}{G}\right)\left|\sum_{n=N}^{2N} a_n \rho_{t_f}(n)\right|^2 \\ &+ \sum_{\mathfrak{a}\in\mathfrak{P}_{pM}} \frac{1}{4\pi}\int_{-\infty}^{\infty}\left|\sum_{n=N}^{2N} a_n \phi_{\mathfrak{a}\infty}^{\Gamma_0(pM)}\left(n,\frac{1}{2}+ir\right)\right|^2 \times \\ &\quad g\left(\frac{r-K}{G}\right)\frac{dr}{\cosh \pi r}. \end{aligned}$$

Applying Kuznetsov formula (see P.409 of [7])

$$\begin{aligned} &\sum_{f\in H_{pM}} \frac{1}{\cosh \pi t_f} g\left(\frac{t_f - K}{G}\right)\left|\sum_{n=N}^{2N} a_n \rho_{t_f}(n)\right|^2 \\ + &\sum_{\mathfrak{a}\in\mathfrak{P}_{pM}} \frac{1}{4\pi}\int_{-\infty}^{\infty}\left|\sum_{n=N}^{2N} a_n \phi_{\mathfrak{a}\infty}^{\Gamma_0(pM)}\left(n,\frac{1}{2}+ir\right)\right|^2 \times \\ &g\left(\frac{r-K}{G}\right)\frac{dr}{\cosh \pi r} \\ = &\sum_{N\leq n_1,n_2\leq 2N} a_{n_1}\overline{a_{n_2}}\mathrm{vol}(X_0(pM)) \times \\ &\left\{\frac{\delta_{n_1 n_2}}{\pi^2}\int_{-\infty}^{\infty} rg\left(\frac{r-K}{G}\right)\tanh \pi r dr + \frac{2i}{\pi}\right. \\ &\left.\sum_{c\equiv 0(pM)} \frac{S(n_1,n_2;c)}{c}\int_{-\infty}^{\infty} J_{2ir}\left(\frac{4\pi\sqrt{n_1 n_2}}{c}\right) rg\left(\frac{r-K}{G}\right)\frac{dr}{\cosh \pi r}\right\}. \end{aligned}$$

The diagonal terms contribute at most $\sum_{n=N}^{2N}|a_n|^2$ times

$$KG(pM)^{1+\epsilon} \ll (KM)^{\epsilon}pKGM \ll (KNM)^{2\epsilon}\max\{KGM, N\}.$$

Thus it remains to treat the non-diagonal terms.

Let $C = NK^{\frac{1}{4}\epsilon-1}$, for $c > C$, the integral

$$\frac{2i}{\pi}\int_{-\infty}^{\infty} J_{2ir}\left(\frac{4\pi\sqrt{n_1n_2}}{c}\right)\frac{rh(r)}{\cosh \pi r}dr$$

can be made as small as we wish (see P. 631 of [12]). Hence, it suffices to consider those c less than $NK^{\frac{1}{4}\epsilon-1}$. We have the following asymptotic expansion for $J_{2ir}(x)$ (P.627 in [12])

$$\begin{aligned} J_{2ir}(x) = & \frac{e^{i\omega_x(2r)+\pi r-i\frac{\pi}{4}}}{\pi\sqrt{2}}\left(\sum_{m=0}^{B-1} t_m(4r^2+x^2)^{-\frac{m}{2}-\frac{1}{4}}\right. \\ + & \left. O\left((4r^2+x^2)^{-\frac{B}{2}}\right)\right) \end{aligned}$$

where $\omega_x(r) = \sqrt{r^2+x^2} - r\log\left(\frac{r}{x}+\sqrt{\left(\frac{r}{x}\right)^2+1}\right)$ and $\sum_{n=0}^{\infty}|t_n|S^{-n} < \infty$ for some $S > 0$. We assme $K \geq S$ from now on.

Using the above asymptotic expansion we will show for $K^{1-\frac{\epsilon}{4}} \ll x \ll \frac{N}{pM}$,

$$\int_{-\infty}^{\infty} J_{2ir}(x)rg\left(\frac{r-K}{G}\right)\frac{dr}{\cosh \pi r} \ll (MKN)^{-10}.$$

Indeed,

$$\begin{aligned} & \int_{-\infty}^{\infty} J_{2ir}(x)rg\left(\frac{r-K}{G}\right)\frac{dr}{\cosh \pi r} \\ = & \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}\pi}\sum_{m=0}^{B-1} t_m\int_{-\infty}^{\infty} rg\left(\frac{r-K}{G}\right)\frac{e^{\pi r}}{\cosh \pi r}(4r^2+x^2)^{-\frac{1}{4}-\frac{m}{2}}e^{i\omega_x(2r)}dr \\ + & O\left(\int_{-\infty}^{\infty} rg\left(\frac{r-K}{G}\right)\frac{e^{\pi r}}{\cosh \pi r}(4r^2+x^2)^{-\frac{B}{2}}dr\right). \end{aligned}$$

Taking $B = \left\lfloor \frac{\log(MNK)}{\log K}100\right\rfloor$ and bounding trivially

$$\begin{aligned} & \int_{-\infty}^{\infty} rg\left(\frac{r-K}{G}\right)\frac{e^{\pi r}}{\cosh \pi r}(4r^2+x^2)^{-\frac{B}{2}}dr \ll KK^{-B}G \\ \ll & (MKN)^{-10}. \end{aligned}$$

For each $0 \leq m \leq B-1$. Let

$$h_{m,x}(r) = rg\left(\frac{r-K}{G}\right)\frac{e^{\pi r}}{\cosh \pi r}(4r^2+x^2)^{-\frac{1}{4}-\frac{m}{2}}.$$

Then by Leibniz's rule,

$$h_{m,x}^{(l)}(r) = \sum_{j=0}^{l} \binom{l}{j} \left[\frac{d^{l-j}}{dr^{l-j}} \left(rg\left(\frac{r-K}{G}\right) \frac{e^{\pi r}}{\cosh \pi r} \right) \right]$$
$$\times \quad \left(\frac{d^j}{dr^j} (4r^2 + x^2)^{-\frac{1}{4}-\frac{m}{2}} \right).$$

And by induction, for $K \le r \le K+G$,

$$\frac{d^j}{dr^j}(4r^2+x^2)^{-\frac{1}{4}-\frac{m}{2}} \ll m^j K^{-\frac{1}{2}-m-j}$$

and

$$\frac{d^{l-j}}{dr^{l-j}} \left(rg\left(\frac{r-K}{G}\right) \frac{e^{\pi r}}{\cosh \pi r} \right) \ll G^{-(l-j)} K.$$

Hence,

$$\begin{aligned} h_{m,x}^{(l)}(r) &\ll \sum_{j=0}^{l} \binom{l}{j} m^j G^{-l+j} K^{\frac{1}{2}-m-j} \\ &= G^{-l} K^{\frac{1}{2}-m} \sum_{j=0}^{l} \binom{l}{j} \left(\frac{Gm}{K}\right)^j \\ &= G^{-l} K^{\frac{1}{2}-m} \left(\frac{Gm}{K}+1\right)^l \\ &\le G^{-l} K^{\frac{1}{2}-m} B^l. \end{aligned}$$

Integrating by parts,

$$\int_{-\infty}^{\infty} h_{m,x}(r) e^{i\omega_x(2r)} dr = \int_{-\infty}^{\infty} \left(\mathfrak{D}^A h_{m,x}\right)(r) e^{i\omega_x(2r)} dr$$

where for any smooth function f,

$$(\mathfrak{D}f)(r) = -\frac{1}{2i}\frac{d}{dr}\left(\frac{f(r)}{\omega_x'(2r)}\right).$$

By induction,

$$\text{(5.1)} \quad \left(\mathfrak{D}^A h_{m,x}\right)(r) = \sideset{}{'}\sum_{\substack{\nu+\sum \nu_j\mu_j = 2A \\ \nu+\sum(\nu_j-1)\mu_j = A}} c_{\nu,\nu_j} h_{m,x}^{(\nu)}(r) \frac{\prod \left(\omega_x^{(\nu_j)}(2r)\right)^{\mu_j}}{[\omega_x'(2r)]^{\nu+\sum \nu_j\mu_j}}$$

for some constants $\{c_{\nu,\nu_j}\}$, and $\sum'$ is a sum over a subset of all $\{\nu, \nu_j\}$ such that $\nu + \sum \nu_j\mu_j = 2A$ and $\nu + \sum(\nu_j - 1)\mu_j = A$.

We first have to estimate the size of $\omega_x'(2r)$ in the range $K \le r \le K+G$ and $K^{1-\frac{\epsilon}{4}} \ll x := x_{n_1,n_2} = \frac{4\pi\sqrt{n_1n_2}}{c} \ll \frac{N}{pM}$.

Let $u = \frac{2r}{x}$, then $\frac{pKM}{N} \ll u \ll K^{\frac{\epsilon}{4}}$ and

$$\omega_x'(2r) = -\log(u + \sqrt{u^2+1}) = -u\frac{\log(u+\sqrt{u^2+1})}{u} < 0.$$

We will find a lower bound of $f(u) := \frac{\log(u+\sqrt{u^2+1})}{u}$ in the given range range.

The derivative

$$f'(u) = \frac{u - \sqrt{u^2+1}\log(\sqrt{u^2+1}+u)}{u^2\sqrt{u^2+1}} := \frac{h(u)}{u^2\sqrt{u^2+1}}$$

has exactly one zero u_0 in $(0,\infty)$ and $f'(u) > 0$ for all $0 < u < u_0$; $f'(u) < 0$ for all $u_0 < u$. To see this notice that $h'(u) = \frac{u\log(\sqrt{u^2+1}+u)}{\sqrt{u^2+1}} < 0$ for all $u > 0$ (hence $h(u)$ is decreasing in $(0,\infty)$), $\lim_{u\to 0^+} h(u) > 0$ and $\lim_{u\to\infty} h(u) = -\infty$. Thus for $\frac{pMK}{N} \ll u \ll K^{\frac{\epsilon}{4}}$,

$$f(u) \gg \min\{\lim_{u\to 0^+} f(u), f(K^{\frac{\epsilon}{4}})\} \geq K^{-\frac{\epsilon}{4}}.$$

Hence,

$$\omega_x'(2r)^{-1} \ll \frac{x}{r}K^{\frac{\epsilon}{4}}$$

and by induction,

$$\omega_x^{(s)}(2r) \ll r^{-s+1}.$$

Thus, using (A.1),

$$\int_{-\infty}^{\infty} h_{m,x}(r)e^{i\omega_x(2r)}dr = \int_{-\infty}^{\infty}\left(\mathfrak{D}^A h_{m,x}\right)(r)e^{i\omega_x(2r)}dr$$

$$= \sideset{}{'}\sum_{\substack{\nu+\sum\nu_j\mu_j=2A \\ \nu+\sum(\nu_j-1)\mu_j=A}} c_{\nu,\nu_j}\int_{-\infty}^{\infty} h_{m,x}^{(\nu)}(r)\frac{\prod\left(\omega_x^{(\nu_j)}(2r)\right)^{\mu_j}}{[\omega_x'(2r)]^{\nu+\sum\nu_j\mu_j}}dr$$

$$\ll \sideset{}{'}\sum_{\substack{\nu+\sum\nu_j\mu_j=2A \\ \nu+\sum(\nu_j-1)\mu_j=A}} GG^{-\nu}K^{\frac{1}{2}-m}K^{-\sum(\nu_j-1)\mu_j}B^{\nu}\left(\frac{x}{K}K^{\frac{\epsilon}{4}}\right)^{\nu+\sum(\nu_j-1)\mu_j}$$

$$
\begin{aligned}
&\ll \sideset{}{'}\sum_{\substack{\nu+\sum \nu_j\mu_j=2A\\ \nu+\sum(\nu_j-1)\mu_j=A}} G^{-\nu+1}K^{\frac{1}{2}-m-A+\nu}B^{\nu}\left(\frac{x}{K}K^{\frac{\epsilon}{4}}\right)^A\\
&\ll GK^{\frac{1}{2}-m-A}B^A\left(\frac{K}{G}\right)^A\left(\frac{x}{K}\right)^A K^{\frac{\epsilon}{4}A}\\
&= GK^{\frac{1}{2}-m}B^A\left(\frac{x}{KG}\right)^A K^{\frac{\epsilon}{4}A}\\
&\ll GK^{\frac{1}{2}-m}(KMN)^{-\epsilon A}K^{\frac{\epsilon}{4}A}B^A\\
&\ll GK^{\frac{1}{2}-m}(KMN)^{-\epsilon A}K^{\frac{\epsilon}{4}A}\left(\log(MNK)\right)^A\\
&< K^{-m}(KMN)^{-10}\frac{(\log(MNK))^A}{MNK}
\end{aligned}
$$

by taking $A = \left\lfloor \frac{100}{\epsilon} \right\rfloor + 1$.

Let $w(x) = \frac{(\log x)^A}{x}$, then $w'(x) = \frac{(\log x)^{A-1}}{x^2}(A - \log x)$. Hence $w(x)$ has maximum $\left(\frac{A}{e}\right)^A$.

Hence,

$$
\begin{aligned}
&\sum_{m=1}^{B-1} t_m \int_{-\infty}^{\infty} rg\left(\frac{r-K}{G}\right)\frac{e^{\pi r}}{\cosh \pi r}(4r^2+x^2)^{-\frac{1}{4}-\frac{m}{2}}e^{i\omega_x(2r)}dr\\
\ll\ &\sum_{m=0}^{B-1}|t_m|K^{-m}(MKN)^{-10} \le (KMN)^{-10}\sum_{m=0}^{\infty}|t_m|S^{-m}\\
\ll\ &(KMN)^{-10}.
\end{aligned}
$$

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