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Principalization algorithm via class group structure

par DANIEL C. MAYER

RÉSUMÉ. Pour un corps de nombres algébriques K avec 3-groupe de classes $\text{Cl}_3(K)$ du type $(3, 3)$, la structure des 3-groupes de classes $\text{Cl}_3(N_i)$ des quatre extensions cubiques cycliques non-ramifiées N_i , $1 \leq i \leq 4$, de K est calculée à l'aide de présentations du groupe de Galois métabelien $G_3^2(K) = \text{Gal}(F_3^2(K)|K)$ du deuxième 3-corps de classes de Hilbert $F_3^2(K)$ de K . Dans le cas d'un corps de base quadratique $K = \mathbb{Q}(\sqrt{D})$, il est montré que la structure des 3-groupes de classes des quatre S_3 -corps N_1, \dots, N_4 détermine fréquemment le type de principalisation de 3-groupe de classes de K dans N_1, \dots, N_4 . Ceci offre une alternative à l'algorithme de principalisation classique par Scholz et Taussky. Le nouvel algorithme qui est facilement automatisable et s'exécute très brièvement est implémenté en PARI/GP et est appliqué à tous les 4 596 corps quadratiques K de 3-groupe de classes du type $(3, 3)$ et de discriminant $-10^6 < D < 10^7$ pour obtenir des statistiques détaillées sur leurs types de principalisation et sur la distribution de leurs deuxièmes 3-groupes de classes $G_3^2(K)$ sur des arbres divers de coclasses des graphes de coclasses $\mathcal{G}(3, r)$, $1 \leq r \leq 6$, dans le sens de Eick, Leedham-Green et Newman.

ABSTRACT. For an algebraic number field K with 3-class group $\text{Cl}_3(K)$ of type $(3, 3)$, the structure of the 3-class groups $\text{Cl}_3(N_i)$ of the four unramified cyclic cubic extension fields N_i , $1 \leq i \leq 4$, of K is calculated with the aid of presentations for the metabelian Galois group $G_3^2(K) = \text{Gal}(F_3^2(K)|K)$ of the second Hilbert 3-class field $F_3^2(K)$ of K . In the case of a quadratic base field $K = \mathbb{Q}(\sqrt{D})$ it is shown that the structure of the 3-class groups of the four S_3 -fields N_1, \dots, N_4 frequently determines the type of principalization of the 3-class group of K in N_1, \dots, N_4 . This provides an alternative to the classical principalization algorithm by Scholz and Taussky. The new algorithm, which is easily automatizable

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and executes very quickly, is implemented in PARI/GP and is applied to all 4596 quadratic fields K with 3-class group of type $(3, 3)$ and discriminant $-10^6 < D < 10^7$ to obtain extensive statistics of their principalization types and the distribution of their second 3-class groups $G_3^2(K)$ on various coclass trees of the coclass graphs $\mathcal{G}(3, r)$, $1 \leq r \leq 6$, in the sense of Eick, Leedham-Green, and Newman.

1. Introduction

The principal ideal theorem, which has been conjectured by Hilbert in 1898 [19, p. 14], states that each ideal of a number field K becomes principal when it is extended to the Hilbert class field $F^1(K)$ of K , that is the maximal abelian unramified extension field of K . Inspired by the Artin-Furtwängler proof [2, 19] of the principal ideal theorem, Scholz and Taussky investigated the principalization in intermediate fields $K < N < F_3^1(K)$ between a base field K with 3-class group of type $(3, 3)$ and its Hilbert 3-class field $F_3^1(K)$. They developed an algorithm for computing the principalization of K in its four unramified cyclic cubic extension fields N_1, \dots, N_4 for the case of a complex quadratic base field K [41]. This algorithm is probabilistic, since it decides whether an ideal \mathfrak{a} of K becomes principal in N_i , for some $1 \leq i \leq 4$, by testing local cubic residue characters of a principal ideal cube $(\alpha) = \mathfrak{a}^3$, associated with the ideal \mathfrak{a} , and of a fundamental unit ε_i of the non-Galois cubic subfield L_i of the complex S_3 -field N_i with respect to a series of rational test primes $(p_\ell)_{\ell \geq 1}$ and terminating when a critical test prime occurs [29, Algorithm, Step 8, p. 81]. An upper bound for the minimal critical test prime p_{ℓ_0} cannot be given effectively. p_{ℓ_0} can only be estimated by means of Chebotarëv's density theorem [41, 23], and thus causes uncertainty.

An entirely different approach to the principalization problem will be presented in this article. It is based on a purely group theoretical connection between the structure of the abelianizations M_i/M_i' of the four maximal normal subgroups M_i of an arbitrary metabelian 3-group G with abelianization G/G' of type $(3, 3)$ and the kernels $\ker(T_i)$ of the transfers $T_i : G/G' \rightarrow M_i/M_i'$, $1 \leq i \leq 4$. By the Artin reciprocity law of class field theory [1], a corresponding number theoretical connection is established between the structure of the 3-class groups $\text{Cl}_3(N_i)$ of the four unramified cyclic cubic extension fields N_i of an arbitrary algebraic number field K with 3-class group $\text{Cl}_3(K)$ of type $(3, 3)$ and the principalization kernels $\ker(j_{N_i|K})$ of the class extension homomorphisms $j_{N_i|K} : \text{Cl}_3(K) \rightarrow \text{Cl}_3(N_i)$, $1 \leq i \leq 4$. The correspondence is obtained by applying the group theoretical statements about G and M_1, \dots, M_4 to the second 3-class group $G_3^2(K) = \text{Gal}(F_3^2(K)|K)$ of K [31], that is the Galois group of the second

Hilbert 3-class field $F_3^2(K) = F_3^1(F_3^1(K))$ of K , and its maximal subgroups $\text{Gal}(F_3^2(K)|N_i)$, $1 \leq i \leq 4$. We call the family $\tau(G) = (M_i/M'_i)_{1 \leq i \leq 4}$ the *transfer target type* (TTT) of G and the family $\varkappa(G) = (\ker(T_i))_{1 \leq i \leq 4}$ the *transfer kernel type* (TKT) of G (briefly called transfer type in [31, § 2.2, p. 476]).

We begin by comparing the four little two-stage towers $K < F_3^1(K) < F_3^1(N_i)$, $1 \leq i \leq 4$, and the big two-stage tower $K < F_3^1(K) < F_3^2(K)$ of 3-class fields in § 2. Based on these relationships, § 3 is devoted to proving that the 3-class groups $\text{Cl}_3(N_i)$, $1 \leq i \leq 4$, have the structure of nearly homocyclic abelian 3-groups of 3-rank two, provided the index of nilpotency of the second 3-class group $G = G_3^2(K) = \text{Gal}(F_3^2(K)|K)$ is not too small. The structure of the remaining 3-class groups, which are partially of 3-rank three as predicted at the beginning of § 4, is determined in § 4.1, if G is of coclass 1. The central results of this paper concern groups G of coclass $\text{cc}(G) \geq 2$, based on Nebelung's Thesis [37, 38]. They are developed successively, beginning in § 4.2 on the general method of proof, for sporadic groups of coclass 2 with bicyclic centre in § 4.3 and with cyclic centre in § 4.4, for groups of coclass 2 on coclass trees [16] in § 4.5, and finally, for all groups of coclass $\text{cc}(G) \geq 3$ in § 4.6. These results establish theoretical background for our new principalization algorithm in § 5 which is based on the invariant $\varepsilon = \varepsilon(K)$, the number of elementary abelian 3-groups of type $(3, 3, 3)$ among the 3-class groups $\text{Cl}_3(N_i)$, $1 \leq i \leq 4$. In view of future generalization [35] to base fields K with $\text{Cl}_3(K)$ of type $(9, 3)$, we can also define ε as the number of 3-class groups of elevated 3-rank at least three. As opposed to Scholz and Taussky's classical algorithm for determining the 3-principalization over complex quadratic fields [41, § 1, pp. 20–31], which is also described in [29, Algorithm, pp. 80–83] and in [11, § 3, pp. 9–29, and Appendix A, pp. 96–113], and is extended to the p -principalization with an odd prime p over arbitrary quadratic fields in [23, § 1, pp. 3–6, and § 3–4, pp. 12–24], the new 3-principalization algorithm for arbitrary quadratic fields is easily automatizable and executes very quickly. In § 6 an extensive application of our principalization algorithm via class group structure is presented.

2. Little and big two-stage towers of 3-class fields

Let K be an algebraic number field with 3-class group $\text{Cl}_3(K)$ of type $(3, 3)$. By class field theory [33, Cor. 3.1, p. 838], there exist four unramified cyclic cubic extension fields N_1, \dots, N_4 of K within the first Hilbert 3-class field $F_3^1(K)$ of K . Consequently, the first Hilbert 3-class field $F_3^1(N_i)$ of N_i is an intermediate field between $F_3^1(K)$ and the second Hilbert 3-class field $F_3^2(K)$ of K , for each $1 \leq i \leq 4$.

By $\Gamma_i = \text{Gal}(\mathbb{F}_3^1(N_i)|K)$ we denote the Galois groups of the four *little two-stage towers* of K , $K < \mathbb{F}_3^1(K) \leq \mathbb{F}_3^1(N_i)$, where $1 \leq i \leq 4$, and by $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ the Galois group of the *big two-stage tower* of K , $K < \mathbb{F}_3^1(K) \leq \mathbb{F}_3^2(K)$.

Proposition 2.1. *G and Γ_i are metabelian 3-groups with abelianizations G/G' and Γ_i/Γ'_i of type $(3, 3)$. They are non-abelian, except for a single-stage tower $\mathbb{F}_3^2(K) = \mathbb{F}_3^1(N_i) = \mathbb{F}_3^1(K)$.*

G contains four maximal normal subgroups M_1, \dots, M_4 and each Γ_i contains an abelian maximal subgroup $A_i = \text{Gal}(\mathbb{F}_3^1(N_i)|N_i)$ isomorphic to the 3-class group $\text{Cl}_3(N_i)$ of N_i , for $1 \leq i \leq 4$.

The connection between Γ_i and G , resp. between A_i and M_i , for $1 \leq i \leq 4$, is given by

$$(2.1) \quad \begin{aligned} \Gamma_i &\simeq G/M'_i, \\ A_i &\simeq M_i/M'_i. \end{aligned}$$

Proof. Since $\mathbb{F}_3^1(K)$ is the maximal abelian unramified extension field of K with a power of 3 as relative degree, the Galois groups $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ and $\Gamma_i = \text{Gal}(\mathbb{F}_3^1(N_i)|K)$, $1 \leq i \leq 4$, are non-abelian, except in the degenerate case $\mathbb{F}_3^2(K) = \mathbb{F}_3^1(N_i) = \mathbb{F}_3^1(K)$. That Γ_i is also non-abelian when G is non-abelian follows from the relation $\Gamma_i \simeq G/M'_i$, which is shown at the end of this proof, together with Cor. 3.1 and Cor. 3.2 in [31, pp. 476 and 480], where it is proved that M'_i is contained in the third member of the lower central series of G and thus strictly smaller than G' , for each $1 \leq i \leq 4$. The abelian commutator subgroups of G and Γ_i are given by

$$G' = \text{Gal}(\mathbb{F}_3^2(K)|\mathbb{F}_3^1(K)) \simeq \text{Cl}_3(\mathbb{F}_3^1(K))$$

and

$$\Gamma'_i = \text{Gal}(\mathbb{F}_3^1(N_i)|\mathbb{F}_3^1(K)) < A_i = \text{Gal}(\mathbb{F}_3^1(N_i)|N_i) \simeq \text{Cl}_3(N_i),$$

by the Artin reciprocity law [1]. Hence, G and Γ_i are metabelian 3-groups with abelianizations G/G' and Γ_i/Γ'_i isomorphic to $\text{Gal}(\mathbb{F}_3^1(K)|K) \simeq \text{Cl}_3(K)$ and thus of type $(3, 3)$.

Since the four maximal normal subgroups M_1, \dots, M_4 of G are associated with the extensions N_1, \dots, N_4 by $M_i = \text{Gal}(\mathbb{F}_3^2(K)|N_i)$, their commutator subgroups are given by

$$M'_i = \text{Gal}(\mathbb{F}_3^2(K)|\mathbb{F}_3^1(N_i))$$

and their abelianizations by

$$\begin{aligned} M_i/M'_i &= \text{Gal}(\mathbb{F}_3^2(K)|N_i)/\text{Gal}(\mathbb{F}_3^2(K)|\mathbb{F}_3^1(N_i)) \\ &\simeq A_i = \text{Gal}(\mathbb{F}_3^1(N_i)|N_i) \simeq \text{Cl}_3(N_i). \end{aligned}$$

Since M'_i is a characteristic subgroup of M_i , it is a normal subgroup of G and we have the relation

$$G/M'_i = \text{Gal}(\mathbb{F}_3^2(K)|K)/\text{Gal}(\mathbb{F}_3^2(K)|\mathbb{F}_3^1(N_i)) \simeq \text{Gal}(\mathbb{F}_3^1(N_i)|K) = \Gamma_i.$$

□

In the following § 3, we determine the standard structure of the abelian maximal normal subgroups $A_i \simeq \text{Cl}_3(N_i)$ of Γ_i , $1 \leq i \leq 4$, for a given second 3-class group G of K .

3. Nearly homocyclic 3-class groups of 3-rank two

The concept of a nearly homocyclic abelian p -group with an arbitrary prime $p \geq 2$ appears in [9, Thm. 3.4, p. 68] (see our Appendix) and is treated systematically in [37, § 2.4]. For our purpose, it suffices to consider the special case $p = 3$.

Definition. By the *nearly homocyclic abelian 3-group* $A(3, n)$ of order 3^n , for an integer $n \geq 2$, we understand the abelian group of type $(3^{q+r}, 3^q)$, where $n = 2q + r$ with integers $q \geq 1$ and $0 \leq r < 2$. Additionally, including two degenerate cases, we define that $A(3, 1)$ denotes the cyclic group C_3 of order 3 and $A(3, 0)$ the trivial group 1.

The application of Blackburn’s well-known Theorem 3.4 in [9, p. 68] to the Galois groups $\Gamma_i = \text{Gal}(\mathbb{F}_3^1(N_i)|K)$ of the four little two-stage towers $K < \mathbb{F}_3^1(K) \leq \mathbb{F}_3^1(N_i)$ with abelian maximal normal subgroups $A_i = \text{Gal}(\mathbb{F}_3^1(N_i)|N_i) \simeq \text{Cl}_3(N_i)$ will show that, in general, the 3-class groups $\text{Cl}_3(N_i)$, $1 \leq i \leq 4$, are nearly homocyclic abelian 3-groups $A(3, u + v)$ of type $(3^u, 3^v)$ with $1 \leq v \leq u \leq v + 1$. The phrase ‘in general’ is made precise in the following theorems, where we distinguish second 3-class groups $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ of coclass $\text{cc}(G) = 1$ in § 3.1 and of coclass $\text{cc}(G) \geq 2$ in § 3.2, and use concepts and notation of our papers [31, 32], as recalled in the sequel.

3.1. Second 3-class groups G of coclass $\text{cc}(G) = 1$. Let G be a metabelian 3-group of order $|G| = 3^n$ and nilpotency class $\text{cl}(G) = m - 1$, where $n = m \geq 3$. Then G is of coclass $\text{cc}(G) = n - \text{cl}(G) = 1$ and the commutator factor group G/G' of G is of type $(3, 3)$ [9, 36]. The lower central series of G is defined recursively by $\gamma_1(G) = G$ and $\gamma_j(G) = [\gamma_{j-1}(G), G]$ for $j \geq 2$. In particular, $\gamma_2(G) = [G, G] = G'$ denotes the commutator subgroup.

The centralizer $\chi_2(G) = \{g \in G \mid [g, u] \in \gamma_4(G) \text{ for all } u \in \gamma_2(G)\}$ of the two-step factor group $\gamma_2(G)/\gamma_4(G)$, that is,

$$\chi_2(G)/\gamma_4(G) = \text{Centralizer}_{G/\gamma_4(G)}(\gamma_2(G)/\gamma_4(G)),$$

is the biggest subgroup of G such that $[\chi_2(G), \gamma_2(G)] \leq \gamma_4(G)$. It is characteristic, contains the commutator group $\gamma_2(G)$, and coincides with G if and only if $m = 3$. Let the isomorphism invariant $k = k(G)$, the *defect of commutativity* of G , be defined by

$$[\chi_2(G), \gamma_2(G)] = \gamma_{m-k}(G),$$

where $k = 0$ for $3 \leq m \leq 4$, and $0 \leq k \leq 1$ for $m \geq 5$, according to Miech [36, p. 331].

Suppose that generators of $G = \langle x, y \rangle$ are selected such that $x \in G \setminus \chi_2(G)$, if $m \geq 4$, and $y \in \chi_2(G) \setminus \gamma_2(G)$. We define the main commutator $s_2 = [y, x] \in \gamma_2(G)$ and the higher commutators $s_j = [s_{j-1}, x] = s_{j-1}^{x-1} \in \gamma_j(G)$ for $j \geq 3$. Then G satisfies two relations for third powers of the generators x and y of G ,

$$(3.1) \quad x^3 = s_{m-1}^w \quad \text{and} \quad y^3 s_2^3 s_3 = s_{m-1}^z$$

with exponents $-1 \leq w, z \leq 1$, according to Miech [36, Thm. 2, (3), p. 332]. Blackburn uses the notation $\delta = w$ and $\gamma = z$ for these relational parameters [9, (36)–(37), p. 84].

Additionally, the group G satisfies relations for third powers of higher commutators,

$$s_{j+1}^3 s_{j+2}^3 s_{j+3} = 1 \quad \text{for } 1 \leq j \leq m - 2,$$

and the main commutator relation of Miech [36, Thm. 2, (2), p. 332],

$$(3.2) \quad [y, s_2] = s_{m-1}^a \in [\chi_2(G), \gamma_2(G)] = \gamma_{m-k}(G),$$

with exponent $-1 \leq a \leq 1$. Blackburn uses the notation $\beta = a$ [9, (33), p. 82].

By $G_a^m(z, w)$ we denote the representative of an isomorphism class of metabelian 3-groups G of coclass $\text{cc}(G) = 1$ and of order $|G| = 3^m$, which satisfies the relations (3.2) and (3.1) with a fixed system of parameters a, w , and z . Obviously, the defect is $k = 0$ if and only if $a = 0$.

The maximal normal subgroups M_i of G contain the commutator group $\gamma_2(G)$ of G as a normal subgroup of index 3 and thus are of the shape $M_i = \langle g_i, \gamma_2(G) \rangle$. We define a fixed order by $g_1 = y, g_2 = x, g_3 = xy$ and $g_4 = xy^{-1}$. The commutator subgroups $\gamma_2(M_i)$ are of the general form $\gamma_2(M_i) = \langle s_2, \dots, s_{m-1} \rangle^{g_i^{-1}}$, according to [31, Cor. 3.1, p. 476], and in particular

$$(3.3) \quad \begin{aligned} \gamma_2(M_1) &= \begin{cases} 1, & \text{if } k = 0, \\ \gamma_{m-1}(G), & \text{if } k = 1, \end{cases} \\ \gamma_2(M_i) &= \gamma_3(G) \quad \text{for } 2 \leq i \leq 4. \end{aligned}$$

Theorem 3.1. (Transfer target type $\tau(G)$ for groups G with $\text{cc}(G) = 1$, $\text{cl}(G) \geq 5$)

The structure of the 3-class groups $\text{Cl}_3(N_i)$ of the four unramified cyclic cubic extension fields N_1, \dots, N_4 of an arbitrary base field K having 3-class group $\text{Cl}_3(K)$ of type $(3, 3)$ and second 3-class group $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ of coclass $\text{cc}(G) = 1$, order $|G| = 3^n$, and class $\text{cl}(G) = m - 1$, where $m = n \geq 3$, is given by the following nearly homocyclic abelian 3-groups.

(3.4)

$$\text{Cl}_3(N_1) \simeq \begin{cases} A(3, m - 1), & \text{if } [\chi_2(G), \gamma_2(G)] = 1, k = 0, m \geq 5, \\ A(3, m - 2), & \text{if } [\chi_2(G), \gamma_2(G)] = \gamma_{m-1}(G), k = 1, m \geq 6, \end{cases}$$

(3.5)

$$\text{Cl}_3(N_i) \simeq A(3, 2) \text{ for } 2 \leq i \leq 4, \text{ if } m \geq 4.$$

Proof. The metabelian 3-groups $\Gamma_i = \text{Gal}(\mathbb{F}_3^1(N_i)|K)$, having an abelian maximal subgroup $A_i \simeq \text{Cl}_3(N_i)$, are of coclass $\text{cc}(G) = 1$, according to Heider and Schmithals [23, Kor., p. 9] or also to [31, Cor. 3.1, p. 476, and Cor. 3.2, p. 480], since we are dealing with 3-groups here. This is the crucial condition for the applicability of Blackburn’s Theorem 3.4 [9, p. 68] (see our Appendix). It is also the reason why we need the connection between G and $\Gamma_1, \dots, \Gamma_4$.

We begin by investigating the distinguished extension N_1 .

Suppose first that $[\chi_2(G), \gamma_2(G)] = 1$, that is $k = 0$. Then $\gamma_2(M_1) = 1$, by [31, Cor. 3.1], the group Γ_1 is isomorphic to G , by formula (2.1), and has the order $|\Gamma_1| = 3^m$. For $m \geq 5$, the abelian normal subgroup A_1 of Γ_1 is isomorphic to $A(3, m - 1)$, according to Blackburn [9, Thm. 3.4], where the lower bound $p + 2 = 5$ for the index of nilpotency m is due to the specialization $p = 3$.

Next we consider the case $[\chi_2(G), \gamma_2(G)] = \gamma_{m-1}(G)$, that is $k = 1$. Then $\gamma_2(M_1) = \gamma_{m-1}(G)$, by [31, Cor. 3.1], the group Γ_1 is isomorphic to $G/\gamma_{m-1}(G)$, by formula (2.1), and is therefore the immediate predecessor of G [37, p. 182] on the coclass graph $\mathcal{G}(3, 1)$ in Figure 3.1, thus being isomorphic to the mainline group $G_0^{m-1}(0, 0)$ and of order $|\Gamma_1| = 3^{m-1}$. For $m \geq 6$, it follows that $A_1 \simeq A(3, m - 2)$, according to [9, Thm. 3.4].

However, Blackburn’s result cannot be applied to the other three extensions N_i with $2 \leq i \leq 4$, since the three isomorphic groups $\Gamma_i \simeq G/\gamma_3(G)$ [31, Cor. 3.1] are of order $|\Gamma_i| = 3^\mu$ with exponent $\mu = 3 < 5$. Therefore we must determine the structure of the abelian normal subgroup A_i of Γ_i by the following consideration. In the case $m \geq 4$, the group $\gamma_3(G) > 1$ is non-trivial. Since $\Gamma_i \simeq G/\gamma_3(G)$ is a predecessor of G [37, p. 182] on the coclass graph $\mathcal{G}(3, 1)$ in Figure 3.1, it can only be isomorphic to the extra special 3-group $G_0^3(0, 0)$ of exponent 3 on the mainline, whose four maximal

normal subgroups are all abelian of type $(3, 3)$ and are thus isomorphic to $A(3, 2)$. \square

The vertices of the coclass graph $\mathcal{G}(3, 1)$ in Figure 3.1 represent all isomorphism classes of finite 3-groups G with coclass $\text{cc}(G) = 1$. Two vertices are connected by a directed edge $H \rightarrow G$ if G is isomorphic to the last lower central quotient $H/\gamma_c(H)$ where $c = \text{cl}(H)$ denotes the nilpotency class of H , and $|H| = 3|G|$, i. e. $\gamma_c(H)$ is cyclic of order 3. The graph $\mathcal{G}(3, 1)$ has also been drawn in [4, Fig. 4.3, p. 63], [27, pp. 194–195], [37, p. 189 f.], [26, Prop. 8.2.4, p. 156], [12, p. 46], [14, § 9].

The two top vertices (contour squares) are abelian. C_9 is isolated and $C_3 \times C_3$ is the root of the unique coclass tree $\mathcal{T}(C_3 \times C_3)$ of $\mathcal{G}(3, 1)$. All other vertices (full discs) are metabelian, according to Blackburn [10, Thm. 6, p. 26]. Groups with defect $k = 0$ are represented by bigger discs than those with $k = 1$. Numbers in angles denote the identifiers of groups in the SmallGroup library [8] and in GAP 4.4 [20], where we omit the orders, which are given on the left hand scale.

The symbols Φ_s denote isoclinism families [22, 13, 24]. The principalization types or transfer kernel types, briefly TKT, [32, Thm. 2.5, p. 478, and Tbl. 6–7, pp. 492–493] in the bottom rectangle concern all vertices located vertically above, except indicated otherwise. Periodicity of length 2 [15, p. 275], $\mathcal{B}(j) \simeq \mathcal{B}(j + 2)$ for $j \geq 4$, sets in with branch $\mathcal{B}(4)$, having root of order 3^4 .

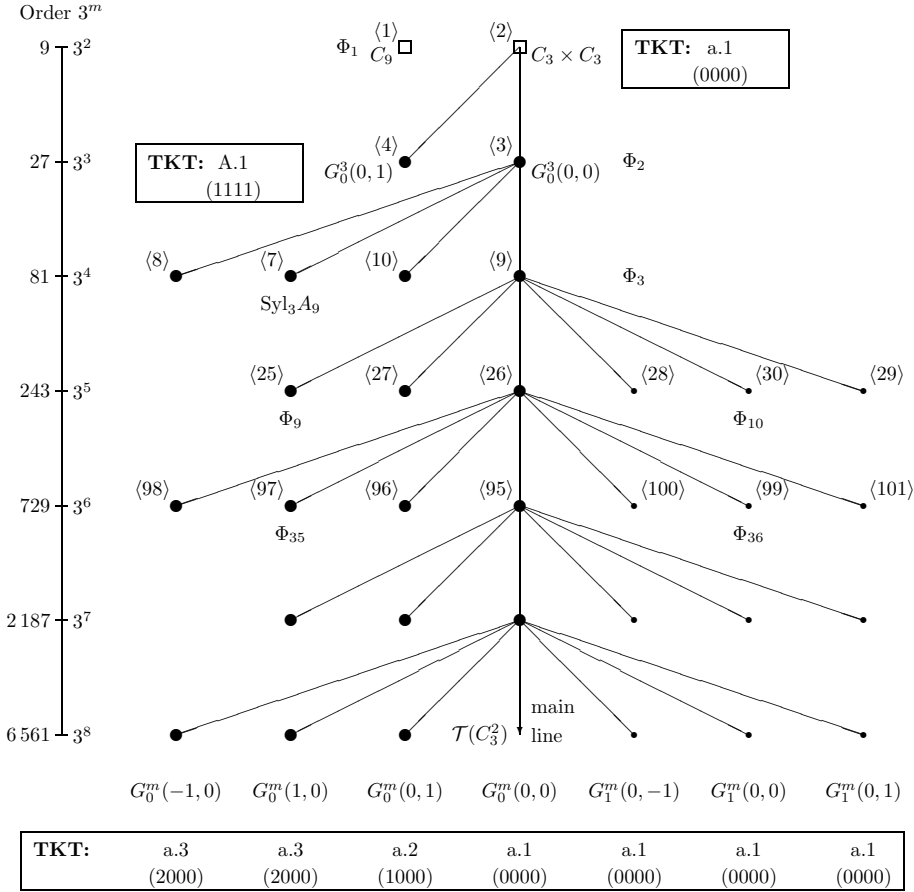
For G of coclass $\text{cc}(G) = 1$, it remains to investigate the structure of the following 3-class groups:

- $\text{Cl}_3(N_1)$ of order $3^{m-1} = 3^3$ for $m = 4$ (where $[\chi_2(G), \gamma_2(G)] = 1$, $k = 0$),
- $\text{Cl}_3(N_1)$ of order $3^{m-2} = 3^3$ for $m = 5$ and $[\chi_2(G), \gamma_2(G)] = \gamma_4(G) > 1$, $k = 1$,
- all four $\text{Cl}_3(N_i)$ with $1 \leq i \leq 4$ of order 3^2 for $m = 3$ (where $k = 0$).

3.2. Second 3-class groups G of coclass $\text{cc}(G) \geq 2$. Metabelian 3-groups G of coclass $\text{cc}(G) \geq 2$ must have at least one bicyclic factor $\gamma_3(G)/\gamma_4(G)$. Similarly as in § 2 of [31], we declare an isomorphism invariant $e = e(G)$ of G by $e + 1 = \min\{3 \leq j \leq m \mid 1 \leq |\gamma_j(G)/\gamma_{j+1}(G)| \leq 3\}$. This invariant $2 \leq e \leq m - 1$ characterizes the first cyclic factor $\gamma_{e+1}(G)/\gamma_{e+2}(G)$ of the lower central series of G , except $\gamma_2(G)/\gamma_3(G)$, which is always cyclic. We can calculate e from the 3-exponent n of the order $|G| = 3^n$ and the class $\text{cl}(G) = m - 1$, resp. the index m of nilpotency, of G by the formula $e = n - m + 2$. Since the coclass of G is given by $\text{cc}(G) = n - \text{cl}(G) = n - m + 1$, we have the relation $e = \text{cc}(G) + 1$.

For a group G of coclass $\text{cc}(G) \geq 2$ we need a generalization of the group $\chi_2(G)$. Denoting by m the index of nilpotency of G , we let $\chi_j(G)$ with

FIGURE 3.1. Root $C_3 \times C_3$ and branches $\mathcal{B}(j)$, $2 \leq j \leq 7$, on the coclass graph $\mathcal{G}(3, 1)$



$2 \leq j \leq m-1$ be the centralizers of two-step factor groups $\gamma_j(G)/\gamma_{j+2}(G)$ of the lower central series, that is, the biggest subgroups of G with the property $[\chi_j(G), \gamma_j(G)] \leq \gamma_{j+2}(G)$. They form an ascending chain of characteristic subgroups of G , $\gamma_2(G) \leq \chi_2(G) \leq \dots \leq \chi_{m-2}(G) < \chi_{m-1}(G) = G$, which contain the commutator subgroup $\gamma_2(G)$, and $\chi_j(G)$ coincides with G if and only if $j \geq m-1$. Similarly as in § 2 of [31], we characterize the smallest two-step centralizer different from the derived subgroup by an isomorphism invariant $s = s(G) = \min\{2 \leq j \leq m-1 \mid \chi_j(G) > \gamma_2(G)\}$.

The following assumptions for a metabelian 3-group G of coclass $\text{cc}(G) \geq 2$ with abelianization $G/\gamma_2(G)$ of type $(3, 3)$ can always be satisfied, according to [37, Satz 3.4.5, p. 94] (see our appendix).

Let G be a metabelian 3-group of coclass $\text{cc}(G) \geq 2$ with abelianization $G/\gamma_2(G)$ of type $(3, 3)$. Assume that G has order $|G| = 3^n$, class $\text{cl}(G) = m - 1$, and invariant $e = n - m + 2 \geq 3$, where $4 \leq m < n \leq 2m - 3$. Let generators of $G = \langle x, y \rangle$ be selected such that $\gamma_3(G) = \langle y^3, x^3, \gamma_4(G) \rangle$, $x \in G \setminus \chi_s(G)$ if $s < m - 1$, and $y \in \chi_s(G) \setminus \gamma_2(G)$. Suppose that a fixed order of the four maximal normal subgroups of G is defined by $M_i = \langle g_i, \gamma_2(G) \rangle$ with $g_1 = y, g_2 = x, g_3 = xy$, and $g_4 = xy^{-1}$. Let the main commutator of G be declared by $s_2 = t_2 = [y, x] \in \gamma_2(G)$ and higher commutators recursively by $s_j = [s_{j-1}, x], t_j = [t_{j-1}, y] \in \gamma_j(G)$ for $j \geq 3$. Starting with the powers $\sigma_3 = y^3, \tau_3 = x^3 \in \gamma_3(G)$, let $\sigma_j = [\sigma_{j-1}, x], \tau_j = [\tau_{j-1}, y] \in \gamma_j(G)$ for $j \geq 4$. With exponents $-1 \leq \alpha, \beta, \gamma, \delta, \rho \leq 1$ as parameters, let the following relations be satisfied

$$(3.6) \quad \begin{aligned} \sigma_2^3 &= \sigma_4 \sigma_{m-1}^{-\rho \beta} \tau_4^{-1}, & s_3 \sigma_3 \sigma_4 &= \sigma_{m-2}^{\rho \beta} \sigma_{m-1}^\gamma \tau_e^\delta, \\ t_3^{-1} \tau_3 \tau_4 &= \sigma_{m-2}^{\rho \delta} \sigma_{m-1}^\alpha \tau_e^\beta, & \tau_{e+1} &= \sigma_{m-1}^{-\rho}. \end{aligned}$$

Finally, let $[\chi_s(G), \gamma_e(G)] = \gamma_{m-k}(G)$ with the defect of commutativity $0 \leq k = k(G) \leq 1$ of G . Then, the defect is $k = 0$ if and only if $\rho = 0$. By $G_\rho^{m,n}(\alpha, \beta, \gamma, \delta)$ we denote the representative of an isomorphism class of metabelian 3-groups G with $G/\gamma_2(G)$ of type $(3, 3)$, of coclass $\text{cc}(G) = n - m + 1 \geq 2$, class $\text{cl}(G) = m - 1$, and order $|G| = 3^n$, which satisfies the relations (3.6) with a fixed system of parameters $(\alpha, \beta, \gamma, \delta, \rho)$.

Theorem 3.2. *(Incomplete TTT $(\tau_1(G), \tau_2(G))$ for groups G with $\text{cc}(G) \geq 3, \text{cl}(G) \geq 5$)*

The structure of the 3-class groups $\text{Cl}_3(N_i)$ of the four unramified cyclic cubic extension fields N_1, \dots, N_4 of an arbitrary base field K having 3-class group $\text{Cl}_3(K)$ of type $(3, 3)$ and second 3-class group $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ of coclass $\text{cc}(G) \geq 2$, order $|G| = 3^n$, class $\text{cl}(G) = m - 1$, and invariant $e = n - m + 2 \geq 3$, where $4 \leq m < n \leq 2m - 3$, is given by the following nearly homocyclic abelian 3-groups.

$$(3.7) \quad \text{Cl}_3(N_1) \simeq \begin{cases} \text{A}(3, m - 1), & \text{if } [\chi_s(G), \gamma_e(G)] = 1, k = 0, m \geq 5, \\ \text{A}(3, m - 2), & \text{if } [\chi_s(G), \gamma_e(G)] = \gamma_{m-1}(G), k = 1, m \geq 6, \end{cases}$$

$$(3.8) \quad \text{Cl}_3(N_2) \simeq \text{A}(3, e) \text{ for } e \geq 4,$$

$$(3.9) \quad \text{Cl}_3(N_i) \simeq \text{A}(3, 3) \text{ for } 3 \leq i \leq 4, \text{ if } \Gamma_i \not\cong G_0^4(1, 0) \simeq \text{Syl}_3 A_9.$$

Proof. For each $1 \leq i \leq 4$, equation (2.1) specifies the order of the Galois group of the i th little two-stage tower as $|\Gamma_i| = |G|/|\gamma_2(M_i)|$, where

$|G| = 3^n$ by assumption. According to [31, Cor. 3.2, p. 480], the orders of the commutator subgroups of the M_i are

$$|\gamma_2(M_i)| = \begin{cases} 3^{e-2} & \text{for } i = 1, [\chi_s(G), \gamma_e(G)] = 1, k = 0, \\ 3^{e-1} & \text{for } i = 1, [\chi_s(G), \gamma_e(G)] = \gamma_{m-1}(G), k = 1, \\ 3^{m-3} & \text{for } i = 2, \\ 3^{n-4} & \text{for } 3 \leq i \leq 4. \end{cases}$$

Using the relation $e = n - m + 2$, we obtain

$$|\Gamma_i| = \begin{cases} 3^m & \text{for } i = 1, [\chi_s(G), \gamma_e(G)] = 1, \\ 3^{m-1} & \text{for } i = 1, [\chi_s(G), \gamma_e(G)] = \gamma_{m-1}(G), \\ 3^{e+1} & \text{for } i = 2, \\ 3^4 & \text{for } 3 \leq i \leq 4. \end{cases}$$

Since Γ_i , having an abelian maximal subgroup $A_i \simeq \text{Cl}_3(N_i)$, is a metabelian 3-group of coclass $\text{cc}(G) = 1$ by [23, Kor., p. 9] or also by [31, Cor. 3.1–3.2, pp. 476 and 480], the structure of the abelian maximal normal subgroup A_i of Γ_i is given by

$$A_1 \simeq \begin{cases} \text{A}(3, m - 1), & \text{if } [\chi_s(G), \gamma_e(G)] = 1, k = 0, m \geq 5, \\ \text{A}(3, m - 2), & \text{if } [\chi_s(G), \gamma_e(G)] = \gamma_{m-1}(G), k = 1, m - 1 \geq 5, \end{cases}$$

$$A_2 \simeq \text{A}(3, e), \text{ for } e + 1 \geq 5,$$

$$A_i \simeq \text{A}(3, 3) \text{ for } 3 \leq i \leq 4, \text{ if } \Gamma_i \not\cong G_0^4(1, 0) \simeq \text{Syl}_3 A_9,$$

according to [9, Thm. 3.4], for $1 \leq i \leq 2$, and by an immediate analysis of the four isomorphism classes of metabelian 3-groups of order 3^m with index of nilpotency $m = 4$, for $3 \leq i \leq 4$. Representatives for these four isomorphism classes are three groups $G_0^4(0, 0)$, $G_0^4(0, 1)$, $G_0^4(-1, 0)$, whose abelian maximal normal subgroup is nearly homocyclic of type $(9, 3)$, and the exceptional group $G_0^4(1, 0) \simeq \text{Syl}_3 A_9$, which has the elementary abelian 3-group of type $(3, 3, 3)$ as its abelian maximal normal subgroup. See the level of order $3^4 = 81$, i. e. the stem of Φ_3 , in Figure 3.1. \square

For G of coclass $\text{cc}(G) \geq 2$, it remains to investigate the structure of the following 3-class groups:

- $\text{Cl}_3(N_1)$ of order $3^{m-k-1} = 3^3$ for small values of the index of nilpotency m , namely $m = 4$ (where $[\chi_s(G), \gamma_e(G)] = 1, k = 0$), and $m = 5$ if $[\chi_s(G), \gamma_e(G)] = \gamma_{m-1}(G), k = 1$,
- $\text{Cl}_3(N_2)$ of order $3^e = 3^3$ for groups G of coclass $\text{cc}(G) = 2$ (where $e = 3$),
- $\text{Cl}_3(N_i), 3 \leq i \leq 4$, of order 3^3 without restrictions for the parameters $m \geq 4$ and $e \geq 3$.

TABLE 3.1. 3-class groups to be investigated for certain parameters m, n, e, k of G

m	n	e	k	$\text{Cl}_3(N_1)$	$\text{Cl}_3(N_2)$	$\text{Cl}_3(N_3)$	$\text{Cl}_3(N_4)$
4	5	3	0	×	×	×	×
5	6	3	1	×	×	×	×
5	6	3	0		×	×	×
≥ 6	≥ 7	3	≥ 0		×	×	×
5	7	4	0			×	×
≥ 6	≥ 8	≥ 4	≥ 0			×	×

In Table 3.1, we give an overview of all systems (m, n, e, k) of parameters of the second 3-class group G of coclass $\text{cc}(G) \geq 2$, for which the 3-class groups $\text{Cl}_3(N_i)$, $1 \leq i \leq 4$, marked by the symbol \times have to be analyzed.

4. Searching for 3-class groups of 3-rank three

Several authors, namely Scholz [40, p. 218], Kisilevsky [25, Thm. 3, p. 205], Heider and Schmithals [23, Satz 7, p. 11], and Brink [11, pp. 51–52], have pointed out the theoretical possibility that the elementary abelian 3-group of order 27 can occur as the 3-class group $\text{Cl}_3(N)$ of an unramified cyclic cubic extension field N of a base field K with 3-class group $\text{Cl}_3(K)$ of type $(3, 3)$. The most explicit result among these statements is due to Heider and Schmithals. They prove that the occurrence of an elementary abelian 3-class group $\text{Cl}_3(N)$ of order 27 is restricted to extensions $N|K$ satisfying condition (B) in the sense of Taussky [43, p. 435], that is, having a partial principalization without fixed point [32, § 2.2, p. 476].

Since the nearly homocyclic abelian 3-groups $A(3, n)$ with $n \geq 2$, which generally occur as $\text{Cl}_3(N)$, according to the Theorems 3.1 and 3.2, are of 3-rank two, the elementary abelian 3-group of type $(3, 3, 3)$ is the unique possibility for $\text{Cl}_3(N)$ to be of 3-rank three. In contrast, many other possibilities arise when $\text{Cl}_3(K)$ is of type $(9, 3)$ [35].

Unfortunately, it was impossible to find a numerical example, let alone a general criterion, for the occurrence of 3-class groups of type $(3, 3, 3)$ in the bibliography, up to now. At first it was completely unknown, whether 3-class groups of type $(3, 3, 3)$ exist at all, if they appear sporadically or stochastically, or if their occurrence is ruled by deterministic laws.

In the present paper we systematically analyze this question by means of presentations for the second 3-class group $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$, which have been given by Blackburn [9, pp. 82–84] in the case of coclass $\text{cc}(G) = 1$ and by Nebelung [37, Satz 3.4.5, p. 94] (see our appendix) in the case of coclass $\text{cc}(G) \geq 2$. We arrive at the surprising result that the transfer target type (TTT) $\tau(K)$, resp. the number $\varepsilon = \varepsilon(K)$ of elementary abelian 3-groups

of type $(3, 3, 3)$ among the 3-class groups $\text{Cl}_3(N_i)$, $1 \leq i \leq 4$, is connected with the transfer kernel type (TKT) $\varkappa(K)$, that is the principalization type of 3-classes of K in N_i , $1 \leq i \leq 4$, by strict rules.

These connections offer new algorithmic possibilities for computing the principalization type of 3-classes of a *quadratic* base field $K = \mathbb{Q}(\sqrt{D})$ with 3-class group of type $(3, 3)$, independently from the classical algorithm for complex quadratic fields by Scholz and Taussky [41] and its modification for real quadratic fields by Heider and Schmithals [23]. The new algorithm is based on determining the structure of the 3-class groups $\text{Cl}_3(N)$ of the unramified cyclic cubic extensions $N|K$, that is, of four S_3 -fields N_1, \dots, N_4 of absolute degree six, and is described in § 5.

With the aid of an implementation of this algorithm in the number theoretical computer algebra system PARI/GP [39], we have computed the principalization type of 3-classes of all 4596 quadratic base fields K with 3-class group of type $(3, 3)$ and discriminant $-10^6 < D < 10^7$. The resulting extensive statistics of principalization types and second 3-class groups $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ will be presented in § 6.

As in § 3, we distinguish second 3-class groups $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ of coclass $\text{cc}(G) = 1$ in § 4.1 and of coclass $\text{cc}(G) \geq 2$ in § 4.2.

4.1. Second 3-class groups G of coclass $\text{cc}(G) = 1$.

Theorem 4.1. (*Transfer target type $\tau(G)$ for groups G with $\text{cc}(G) = 1$, $\text{cl}(G) \leq 4$)*

The structure of the 3-class groups $\text{Cl}_3(N_i)$ of the four unramified cyclic cubic extensions N_i ($1 \leq i \leq 4$) of an arbitrary base field K with 3-class group $\text{Cl}_3(K)$ of type $(3, 3)$ and second 3-class group $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ of coclass $\text{cc}(G) = 1$ and order $|G| = 3^m$ is given by the following array for small indices of nilpotency $3 \leq m = \text{cl}(G) + 1 \leq 5$.

In the case $m = 3$, it is supposed that the generating element y of the maximal normal subgroup $M_1 = \text{Gal}(\mathbb{F}_3^2(K)|N_1) = \langle y, s_2 \rangle$ of G has order 3.

$$(4.1) \quad \text{Cl}_3(N_1) \simeq \begin{cases} A(3, 2), & \text{if } m = 3, \\ A(3, 3), & \text{if } m = 4, G \not\simeq G_0^4(1, 0) \simeq \text{Syl}_3 A_9, \\ C_3 \times C_3 \times C_3, & \text{if } m = 4, G \simeq G_0^4(1, 0) \simeq \text{Syl}_3 A_9, \\ A(3, 3), & \text{if } k = 1, m = 5, \end{cases}$$

$$(4.2) \quad \text{Cl}_3(N_i) \simeq \begin{cases} A(3, 2), & \text{if } G \simeq G_0^3(0, 0), \\ C_9, & \text{if } G \simeq G_0^3(0, 1), \end{cases} \text{ for } 2 \leq i \leq 4, \text{ if } m = 3.$$

Proof. We make use of the well-known properties of the six isomorphism classes of metabelian 3-groups $G_a^m(z, w)$ of coclass $\text{cc}(G) = 1$ with the smallest indices of nilpotency $3 \leq m \leq 4$.

We start with the distinguished extension N_1 .

First, let $[\chi_2(G), \gamma_2(G)] = 1$, that is $k = 0$. Then the group Γ_1 is isomorphic to G and is of order $|\Gamma_1| = 3^m$, according to equation (2.1) and [31, Cor. 3.1, p. 476].

For $m = 4$, the abelian maximal normal subgroup $A_1 \simeq M_1$ is either nearly homocyclic of type (9, 3) and thus isomorphic to $A(3, 3)$, if G belongs to one of the three isomorphism classes $G_0^4(0, 0)$, $G_0^4(0, 1)$, $G_0^4(-1, 0)$, or elementary abelian of type (3, 3, 3), if G is isomorphic to $G_0^4(1, 0)$, the 3-Sylow subgroup $\text{Syl}_3 A_9$ of the alternating group of degree 9. See the stem of Φ_3 in Figure 3.1.

However, for $m = 3$ the structure of the abelian maximal normal subgroup $A_1 \simeq M_1$ can only be nearly homocyclic of type (3, 3) and thus isomorphic to $A(3, 2)$, in the case of the extra special group $G \simeq G_0^3(0, 0)$ of exponent 3 with four abelian maximal normal subgroups of type (3, 3), as well as in the case of the extra special group $G \simeq G_0^3(0, 1)$ of exponent 9 with one abelian maximal normal subgroup $A_1 \simeq M_1 = \langle y, s_2 \rangle$ of type (3, 3), which is distinguished by our choice of the generating element y , and three further cyclic maximal normal subgroups of order 9. See the level of order $3^3 = 27$, i. e. the stem of Φ_2 , in Figure 3.1.

Now we consider $[\chi_2(G), \gamma_2(G)] = \gamma_{m-1}(G)$, that is $k = 1$, with $m = 5$ and thus $\gamma_4(G) > 1$. According to equation (2.1) and [31, Cor. 3.1, p. 476], the group $\Gamma_1 \simeq G/\gamma_4(G)$ is the immediate predecessor of G [37, p. 182] on the coclass graph $\mathcal{G}(3, 1)$. Thus it is isomorphic to $G_0^4(0, 0)$ having the unique abelian maximal normal subgroup $A_1 \simeq A(3, 3)$. See the vertex $\langle 81, 9 \rangle$ in Figure 3.1.

For the other three extensions N_i with $2 \leq i \leq 4$, all groups $\Gamma_i \simeq G/\gamma_3(G) \simeq G$ [31, Cor. 3.1, p. 476] coincide with G , since $\gamma_3(G) = 1$ is trivial for $m = 3$. The structure of the abelian maximal normal subgroup $A_i \simeq M_i$ is either of type (3, 3) and thus isomorphic to $A(3, 2)$, if G is isomorphic to $G_0^3(0, 0)$, or cyclic of order 9, if G is isomorphic to $G_0^3(0, 1)$, taking into account that $M_i = \langle xy^{i-2}, s_2 \rangle = \langle xy^{i-2} \rangle$ with elements xy^{i-2} of order 9 whose third power coincides with s_2 . See the vertices $\langle 27, 3 \rangle$ and $\langle 27, 4 \rangle$ in Figure 3.1. □

Corollary 4.1.1. *(TTT $\tau(G)$ of $C_3 \times C_3$ and stem groups G in isoclinism families $\Phi_2, \Phi_3, \Phi_{10}$)*

Table 4.1 gives the structure of 3-class groups $\text{Cl}_3(N_i)$, $1 \leq i \leq 4$, and invariant $\varepsilon = \#\{1 \leq i \leq 4 \mid \text{Cl}_3(N_i) \simeq (3, 3, 3)\}$, for 3-groups $G \in \mathcal{G}(3, 1)$ of small nilpotency class $1 \leq \text{cl}(G) = m - 1 \leq 4$ in dependence on the principalization or transfer kernel type (TKT) \varkappa [32, Thm. 2.4–2.5, p. 478].

Example. First occurrences of second 3-class groups $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ of coclass $\text{cc}(G) = 1$ with invariants $m = n = 4$, $e = 2$ for real quadratic

TABLE 4.1. TTT $\tau(G)$ for G of coclass $\text{cc}(G) = 1$ and index of nilpotency $2 \leq m \leq 5$

m	k	Type	\varkappa	$\text{Cl}_3(\mathbb{F}_3^1(K))$	$\text{Cl}_3(N_1)$	$\text{Cl}_3(N_2)$	$\text{Cl}_3(N_3)$	$\text{Cl}_3(N_4)$	ε
2	0	a.1	(0000)	1	(3)	(3)	(3)	(3)	0
3	0	a.1	(0000)	(3)	(3, 3)	(3, 3)	(3, 3)	(3, 3)	0
3	0	A.1	(1111)	(3)	(3, 3)	(9)	(9)	(9)	0
4	0	a.1	(0000)	(3, 3)	(9, 3)	(3, 3)	(3, 3)	(3, 3)	0
4	0	a.2	(1000)	(3, 3)	(9, 3)	(3, 3)	(3, 3)	(3, 3)	0
4	0	a.3	(2000)	(3, 3)	(9, 3)	(3, 3)	(3, 3)	(3, 3)	0
4	0	a.3*	(2000)	(3, 3)	(3, 3, 3)	(3, 3)	(3, 3)	(3, 3)	1
5	1	a.1	(0000)	(9, 3)	(9, 3)	(3, 3)	(3, 3)	(3, 3)	0

fields $K = \mathbb{Q}(\sqrt{D})$ with discriminant $0 < D < 10^7$ and 3-class group of type (3, 3) turned out to be the following. The smallest discriminant D with TKT a.3, resp. a.2, is 32 009, resp. 72 329, according to [23, Tbl. 7, p. 24]. The smallest discriminant D with TKT a.3* is 142 097, known from [30, Part IV]. However, its special feature $\varepsilon = 1$, resp. $\text{Cl}_3(N_1) \simeq C_3 \times C_3 \times C_3$, was unknown up to 2009.

Conjecture. For quadratic fields, the TKT a.1 cannot occur with a second 3-class group of defect $k = 0$, i. e., an infinitely capable vertex located on the mainline of coclass graph $\mathcal{G}(3, 1)$. See § 6, Table 6.1.

4.2. Second 3-class groups G of coclass $\text{cc}(G) \geq 2$. Suppose the order of a 3-class group $\text{Cl}_3(N_i) \simeq M_i/\gamma_2(M_i)$ with $1 \leq i \leq 4$ has turned out to be 27. The basic idea for deciding whether this 3-class group is elementary abelian of type (3, 3, 3) or nearly homocyclic of type (9, 3) consists in estimating the order of the cosets $\bar{v} = v\gamma_2(M_i)$ of the generators $v \in \{g_i, s_2, \sigma_3, \dots, \sigma_{m-1}, \tau_3, \dots, \tau_e\}$ of $M_i = \langle g_i, \gamma_2(G) \rangle$ [31, Thm. 3.3, Proof, pp. 478–479] with respect to the commutator subgroup $\gamma_2(M_i)$. If all these orders are bounded from above by 3, then we have an elementary abelian 3-class group of type (3, 3, 3), otherwise a nearly homocyclic 3-class group of type (9, 3).

To reduce the investigation to the most important generator g_i , we first summarize general facts concerning the columns of Table 3.1, that is, the four 3-class groups $\text{Cl}_3(N_i)$, $1 \leq i \leq 4$, in the following three §§ 4.2.1–4.2.3.

4.2.1. The distinguished 3-class group $\text{Cl}_3(N_1) \simeq M_1/\gamma_2(M_1)$. According to [31, Cor. 3.2, p. 480], the maximal subgroup $M_1 < G = \langle x, y \rangle$ with generator y , distinguished by the conditions $\gamma_3(G) = \langle y^3, x^3, \gamma_4(G) \rangle$ and $y \in \chi_s(G) \setminus \gamma_2(G)$, has the following properties.

$$M_1 = \langle y, \gamma_2(G) \rangle = \langle y, s_2, \sigma_3, \dots, \sigma_{m-1}, \tau_3, \dots, \tau_e \rangle,$$

$$\gamma_2(M_1) = \langle t_3, \tau_4, \dots, \tau_{e+1} \rangle.$$

Since $\tau_4, \dots, \tau_e \in \gamma_2(M_1)$ for $e \geq 4$, the order of the cosets $\overline{\tau_4}, \dots, \overline{\tau_e}$ equals 1.

The relation $\tau_3^3 \tau_4^3 \tau_5 = 1$ for third powers [31, Thm. 3.3, Proof, p. 478] implies $\tau_3^3 = \tau_4^{-3} \tau_5^{-1} \in \gamma_2(M_1)$ and $\text{ord}(\overline{\tau_3}) \leq 3$, for any $e \geq 3$.

For the order of the cosets $\overline{s_2}, \overline{\sigma_3}, \dots, \overline{\sigma_{m-1}}$, we cannot ensure the upper bound 3, in general. However, in the two cases $m = 4, n = 5, \rho = 0$ and $m = 5, n = 6, \rho = \pm 1$ to be investigated, according to Table 3.1, this estimate is possible. For $m = 4, n = 5, \rho = 0$, we have the nilpotency relation $\sigma_4 = \sigma_m = 1, \sigma_3^3 = \sigma_4^{-3} \sigma_5^{-1} = 1$, and $s_2^3 = \sigma_4 \tau_4^{-1} = 1$ by (3.6), since $e = 3, \rho = 0$, and thus $\tau_4 = \tau_{e+1} = 1$. For $m = 5, n = 6, \rho = \pm 1$, we have the nilpotency relation $\sigma_5 = \sigma_m = 1, \sigma_4^3 = \sigma_5^{-3} \sigma_6^{-1} = 1, \sigma_3^3 = \sigma_4^{-3} \sigma_5^{-1} = 1$, and $s_2^3 = \sigma_4 \sigma_{m-1}^{-\rho} \tau_4^{-1} = \sigma_4^{1-\rho(\beta-1)}$ by (3.6), where $\sigma_4 = \sigma_{m-1} = \tau_{e+1}^{-\rho} = \tau_4^{-\rho} \in \gamma_2(M_1)$.

Consequently, we must only determine the order of the coset of the generator y with third power $y^3 = \sigma_3$.

4.2.2. The distinguished 3-class group $\text{Cl}_3(N_2) \simeq M_2/\gamma_2(M_2)$.

By [31, Cor. 3.2, p. 480], the maximal subgroup $M_2 < G = \langle x, y \rangle$ with generator x , distinguished by the conditions $\gamma_3(G) = \langle y^3, x^3, \gamma_4(G) \rangle$ and $x \in G \setminus \chi_s(G)$ if $s < m - 1$, has the following properties.

$$M_2 = \langle x, \gamma_2(G) \rangle = \langle x, s_2, \sigma_3, \dots, \sigma_{m-1}, \tau_3, \dots, \tau_e \rangle,$$

$$\gamma_2(M_2) = \langle s_3, \sigma_4, \dots, \sigma_{m-1} \rangle.$$

Since $\sigma_4, \dots, \sigma_{m-1} \in \gamma_2(M_2)$ for $m \geq 5$, the order of the cosets $\overline{\sigma_4}, \dots, \overline{\sigma_{m-1}}$ equals 1.

The relation $\sigma_3^3 \sigma_4^3 \sigma_5 = 1$ for third powers [31, Thm. 3.3, Proof, p. 478] implies $\sigma_3^3 = \sigma_4^{-3} \sigma_5^{-1} \in \gamma_2(M_1)$ and $\text{ord}(\overline{\sigma_3}) \leq 3$, for any $m \geq 4$.

Since only the case $e = 3$ is to be investigated, according to Table 3.1, we have $\tau_3^3 = \tau_4^{-3} = \tau_{e+1}^{-3} = \sigma_{m-1}^{3\rho} \in \gamma_2(M_2)$ by (3.6), because $\rho = 0$ for $m = 4$ and $\sigma_{m-1} \in \gamma_2(M_2)$ for $m \geq 5$.

Finally, we have $s_2^3 = \sigma_4 \sigma_{m-1}^{-\rho} \tau_4^{-1} = \sigma_4 \sigma_{m-1}^{-\rho(\beta-1)} \in \gamma_2(M_2)$, since $\sigma_4 \in \gamma_2(M_2)$ for $m \geq 4, \rho = 0$ for $m = 4$, and $\sigma_{m-1} \in \gamma_2(M_2)$ for $m \geq 5$, whence $\text{ord}(\overline{s_2}) \leq 3$.

Therefore, it only remains to investigate the order of the coset of the generator x with third power $x^3 = \tau_3$.

4.2.3. The other 3-class groups $\text{Cl}_3(N_i) \simeq M_i/\gamma_2(M_i)$, $3 \leq i \leq 4$. According to [31, Cor. 3.2, p. 480], the maximal subgroups $M_i < G = \langle x, y \rangle$ with $3 \leq i \leq 4$ with generators xy and xy^{-1} , having third powers $(xy)^3, (xy^{-1})^3 \in \zeta_{1+k}(G)$ in the first or second centre of G [37, Lem. 3.4.11, p. 105], have the following properties.

$$\begin{aligned} M_3 &= \langle xy, \gamma_2(G) \rangle = \langle xy, s_2, \sigma_3, \dots, \sigma_{m-1}, \tau_3, \dots, \tau_e \rangle, \\ \gamma_2(M_3) &= \langle s_3 t_3, \gamma_4(G) \rangle, \\ M_4 &= \langle xy^{-1}, \gamma_2(G) \rangle = \langle xy^{-1}, s_2, \sigma_3, \dots, \sigma_{m-1}, \tau_3, \dots, \tau_e \rangle, \\ \gamma_2(M_4) &= \langle s_3 t_3^{-1}, \gamma_4(G) \rangle, \\ \gamma_4(G) &= \langle \sigma_4, \dots, \sigma_{m-1}, \tau_4, \dots, \tau_e \rangle. \end{aligned}$$

Since $\sigma_4, \dots, \sigma_{m-1}, \tau_4, \dots, \tau_e \in \gamma_4(G) < \gamma_2(M_i)$ for $e \geq 4, m \geq 5$, the order of the cosets $\overline{\sigma_4}, \dots, \overline{\sigma_{m-1}}$ and $\overline{\tau_4}, \dots, \overline{\tau_e}$ equals 1.

Due to the relations $\sigma_3^3 \sigma_4^3 \sigma_5 = 1$ and $\tau_3^3 \tau_4^3 \tau_5 = 1$ for third powers [31, Thm. 3.3, Proof, p. 478], we have $\text{ord}(\overline{\sigma_3}) \leq 3$ and $\text{ord}(\overline{\tau_3}) \leq 3$.

Finally, we have $s_2^3 = \sigma_4 \sigma_{m-1}^{-\rho\beta} \tau_4^{-1} \in \gamma_2(M_i)$ by (3.6), since $\sigma_4, \tau_4 \in \gamma_2(M_i)$ for $m \geq 4, e \geq 3, \rho = 0$ for $m = 4$ and $\sigma_{m-1} \in \gamma_2(M_i)$ for $m \geq 5$, whence $\text{ord}(\overline{s_2}) \leq 3$.

Thus, it only remains to determine the order of the coset of the generator xy , resp. xy^{-1} , for $i = 3$, resp. $i = 4$. We summarize the results of §§ 4.2.1–4.2.3 in Lemma 4.1.

Lemma 4.1. *For each of the four 3-class groups $\text{Cl}_3(N_i) \simeq M_i/\gamma_2(M_i)$, $1 \leq i \leq 4$, the decision whether $\text{Cl}_3(N_i)$ is of type (9, 3) or of type (3, 3, 3), in the case of 3-class number $h_3(N_i) = 3^3$, exclusively depends on the order of the generator g_i of $M_i = \langle g_i, \gamma_2(G) \rangle$ with respect to the commutator subgroup $\gamma_2(M_i)$.*

The order of all the generators $s_2, \sigma_3, \dots, \sigma_{m-1}, \tau_3, \dots, \tau_e$ of $\gamma_2(G)$ with respect to $\gamma_2(M_i)$ is uniformly bounded from above by 3.

After the preliminaries in the last three sections we come to the details of the rows of Table 3.1 in the following four sections.

4.3. Groups G of coclass $\text{cc}(G) = 2$ with bicyclic centre and $m = 4, n = 5$. This section corresponds to the first row of Table 3.1. Here we must investigate the abelianizations of all four maximal subgroups $M_i, 1 \leq i \leq 4$. These 7 groups $\langle 243, i \rangle, 3 \leq i \leq 9$, form the stem of Hall's isoclinism family Φ_6 [22, p. 139], [24, 4.1, p. 618, and 4.5 (6), pp. 620–621], [6, pp. 182–183] and satisfy the following special relations, by (3.6):

$$\begin{aligned}
 n &= 5 = 2m - 3, \\
 s = e = n - m + 2 &= 3 = m - 1, \\
 [\chi_s(G), \gamma_e(G)] &= [G, \gamma_3(G)] = \gamma_4(G) = 1, \quad k = 0, \quad \rho = 0, \\
 \sigma_4 &= 1, \tau_4 = 1, \\
 \sigma_3^3 &= 1, \tau_3^3 = 1, \\
 s_2^3 &= 1, \\
 \gamma_2(G) &= \langle s_2 \rangle \times \langle \sigma_3 \rangle \times \langle \tau_3 \rangle \text{ of type } (3, 3, 3), \\
 \gamma_3(G) &= \langle \sigma_3 \rangle \times \langle \tau_3 \rangle = \zeta_1(G) \text{ of type } (3, 3).
 \end{aligned}$$

To get an adequate view of §§ 4.3–4.5 it is useful to visualize that part of coclass graph $\mathcal{G}(3, 2)$ which consists of 3-groups G of coclass $\text{cc}(G) = 2$ with abelianization $G/\gamma_2(G)$ of type $(3, 3)$ and small order $|G| = 3^n$ in Figure 4.1. The groups $C_3 \times C_3$ and $G_0^3(0, 0)$ form the top of the mainline of $\mathcal{G}(3, 1)$ in Figure 3.1. The edges of depth 2 neither belong to $\mathcal{G}(3, 1)$ nor to $\mathcal{G}(3, 2)$. The top of $\mathcal{G}(3, 2)$ at the level of order $3^5 = 243$ consists of two isolated vertices $\langle 5 \rangle, \langle 7 \rangle$, two roots $\langle 9 \rangle, \langle 4 \rangle$ of finite trees, a root $\langle 3 \rangle$ of an infinite tree, and two roots $\langle 6 \rangle, \langle 8 \rangle$ of coclass trees. Only the mainlines of infinite trees are shown. Vertices denoted by contour circles are metabelian [37, p. 189 ff.]. Groups with defect $k = 0$ are represented by bigger circles than those with $k = 1$. Vertices denoted by small contour squares are non-metabelian [4, Fig. 4.6–4.7, p. 74]. The symbol $n*$ denotes a batch of n siblings below a common parent. Numbers in angles denote the identifiers of groups in the SmallGroup library [8] and in GAP 4.4 [20], where we omit the orders, which are given on the left hand scale. The symbols Φ_s denote isoclinism families [22, 13, 24]. The principalization or transfer kernel types, briefly TKT, [32, Tbl. 6–7, p. 492–493] in rectangles concern the vertices located vertically above.

The commutator subgroups of the maximal normal subgroups are given by [31, Cor. 3.2, p. 480] and (3.6):

$$\begin{aligned}
 \gamma_2(M_1) &= \langle t_3 \rangle = \langle \sigma_3^{-\alpha} \tau_3^{1-\beta} \rangle, \\
 \gamma_2(M_2) &= \langle s_3 \rangle = \langle \sigma_3^{\gamma-1} \tau_3^\delta \rangle, \\
 \gamma_2(M_3) &= \langle s_3 t_3 \rangle = \langle \sigma_3^{\gamma-\alpha-1} \tau_3^{\delta-\beta+1} \rangle, \\
 \gamma_2(M_4) &= \langle s_3 t_3^{-1} \rangle = \langle \sigma_3^{\alpha+\gamma-1} \tau_3^{\beta+\delta-1} \rangle.
 \end{aligned}$$

In Table 4.2 we calculate the third powers g_i^3 of the generators $g_1 = y, g_2 = x, g_3 = xy, g_4 = xy^{-1}$ of the maximal normal subgroups $M_i = \langle g_i, \gamma_2(G) \rangle$ and the generators of the commutator subgroups $\gamma_2(M_i)$ with

FIGURE 4.1. Sporadic groups and roots of coclass trees on the coclass graph $\mathcal{G}(3, 2)$

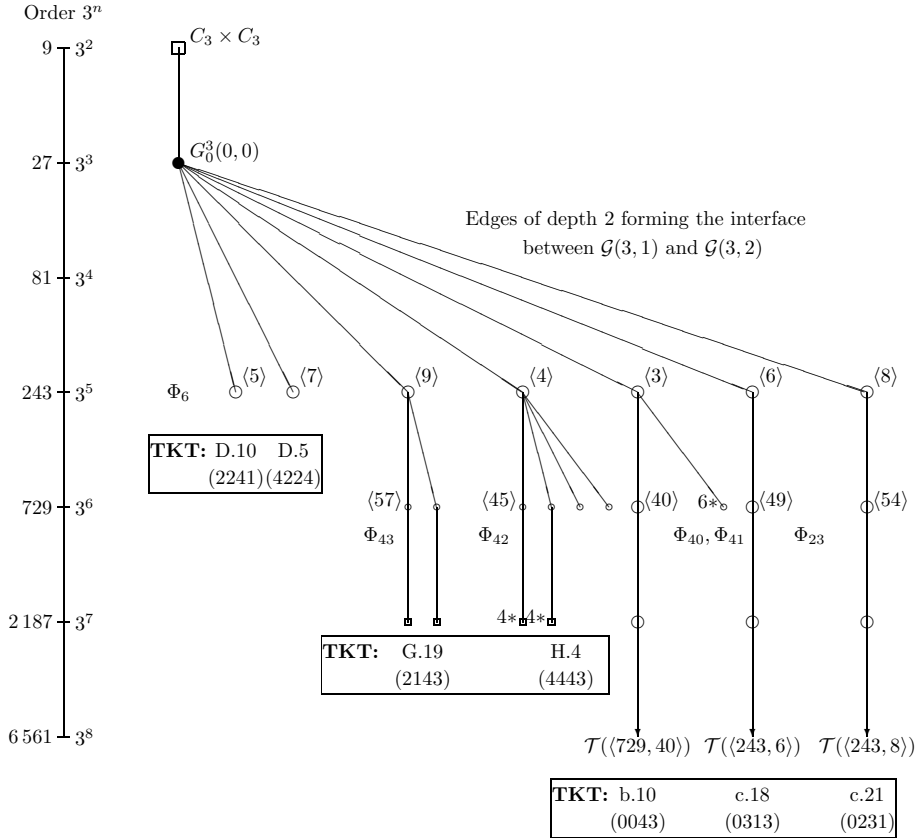


TABLE 4.2. Parameters, third powers, and generators for $m = 4, n = 5, e = 3, k = 0$

Type	α	β	γ	δ	y^3	x^3	$(xy)^3$	$(xy^{-1})^3$	t_3	s_3	$s_3 t_3$	$s_3 t_3^{-1}$
D.10	0	0	-1	1	σ_3	τ_3	$\sigma_3^{-1} \tau_3$	$\sigma_3 \tau_3^{-1}$	τ_3	$\sigma_3^{-2} \tau_3$	$\sigma_3^{-2} \tau_3^2$	σ_3^{-2}
D.5	1	1	-1	1	σ_3	τ_3	τ_3^2	σ_3^2	σ_3^{-1}	$\sigma_3^{-2} \tau_3$	τ_3	$\sigma_3^{-1} \tau_3$
G.19	0	-1	-1	0	σ_3	τ_3	$\sigma_3^{-1} \tau_3^{-1}$	$\sigma_3 \tau_3^{-1}$	τ_3^2	σ_3^{-2}	$\sigma_3^{-2} \tau_3^2$	$\sigma_3^{-2} \tau_3^{-2}$
H.4	1	1	1	1	σ_3	τ_3	$\sigma_3^2 \tau_3^2$	1	σ_3^{-1}	τ_3	$\sigma_3^{-1} \tau_3$	$\sigma_3 \tau_3$
b.10	0	0	0	0	σ_3	τ_3	1	1	τ_3	σ_3^{-1}	$\sigma_3^{-1} \tau_3$	$\sigma_3^{-1} \tau_3^{-1}$
c.18	0	-1	0	1	σ_3	τ_3	1	τ_3^{-2}	τ_3^2	$\sigma_3^{-1} \tau_3$	σ_3^{-1}	$\sigma_3^{-1} \tau_3^{-1}$
c.21	0	0	0	1	σ_3	τ_3	τ_3	τ_3^{-1}	τ_3	$\sigma_3^{-1} \tau_3$	$\sigma_3^{-1} \tau_3^2$	σ_3^{-1}

$1 \leq i \leq 4$ for each of the 7 isomorphism classes of groups G with $m = 4, n = 5$, and parameters $\alpha, \beta, \gamma, \delta$ given by [38, pp. 1–3]. The principalization

types [37, Satz 6.14, p. 208], [32, Tbl. 6–7, pp. 492–493] of these isomorphism classes are all different.

Generally, according to [37, Lem. 3.4.11, p. 105], the third powers of g_3 and g_4 are given by

$$(xy)^3 = \sigma_3^{\alpha+\gamma} \tau_3^{\beta+\delta} \text{ and } (xy^{-1})^3 = \sigma_3^{\alpha-\gamma} \tau_3^{\beta-\delta}.$$

The order of the coset of $g_i \in M_i$ with respect to $\gamma_2(M_i)$ is bounded from above by 3, if and only if the third power g_i^3 is contained in $\gamma_2(M_i)$.

Theorem 4.2. *(Transfer target type $\tau(G)$ of stem groups G in isoclinism family Φ_6)*

Let K be a number field with 3-class group $\text{Cl}_3(K)$ of type $(3, 3)$. Suppose that the second 3-class group $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ of K is of order $|G| = 3^n$ and of class $\text{cl}(G) = m - 1$, where $m = 4$ and $n = 5$, i. e., that G is one of the seven top vertices of coclass graph $\mathcal{G}(3, 2)$ in Figure 4.1, with invariant $e = 3$ and bicyclic centre $\zeta_1(G)$.

Then the structure of the 3-class groups of the first Hilbert 3-class field $\mathbb{F}_3^1(K)$ of K and of the four unramified cyclic cubic extensions N_1, \dots, N_4 of K is given by Table 4.3, in dependence on the principalization type \varkappa of K . The invariant $\varepsilon = \varepsilon(K)$ denotes the number of elementary abelian 3-class groups $\text{Cl}_3(N_i)$ of type $(3, 3, 3)$, for each principalization type.

TABLE 4.3. 3-class groups of type $(3, 3, 3)$ for $m = 4, n = 5, e = 3, k = 0$

Type	\varkappa	$\text{Cl}_3(\mathbb{F}_3^1(K))$	$\text{Cl}_3(N_1)$	$\text{Cl}_3(N_2)$	$\text{Cl}_3(N_3)$	$\text{Cl}_3(N_4)$	ε
D.10	(2241)	(3, 3, 3)	(9, 3)	(9, 3)	(3, 3, 3)	(9, 3)	1
D.5	(4224)	(3, 3, 3)	(3, 3, 3)	(9, 3)	(3, 3, 3)	(9, 3)	2
G.19	(2143)	(3, 3, 3)	(9, 3)	(9, 3)	(9, 3)	(9, 3)	0
H.4	(4443)	(3, 3, 3)	(3, 3, 3)	(3, 3, 3)	(9, 3)	(3, 3, 3)	3
b.10	(0043)	(3, 3, 3)	(9, 3)	(9, 3)	(3, 3, 3)	(3, 3, 3)	2
c.18	(0313)	(3, 3, 3)	(9, 3)	(9, 3)	(3, 3, 3)	(9, 3)	1
c.21	(0231)	(3, 3, 3)	(9, 3)	(9, 3)	(9, 3)	(9, 3)	0

Proof. The structure of the 3-class group $\text{Cl}_3(\mathbb{F}_3^1(K))$ of the first Hilbert 3-class field of K can be obtained from the parameters $m = 4$ and $e = 3$ by means of the following two isomorphisms from $\text{Cl}_3(\mathbb{F}_3^1(K))$ to the commutator subgroup $\gamma_2(G)$ of $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$, according to [1] and [37, Satz 4.2.4, p. 131]

$$\begin{aligned} \text{Cl}_3(\mathbb{F}_3^1(K)) &\simeq \text{Gal}(\mathbb{F}_3^2(K)|\mathbb{F}_3^1(K)) \\ &\simeq \gamma_2(G) \simeq \text{A}(3, m - 2) \times \text{A}(3, e - 2) = \text{A}(3, 2) \times \text{A}(3, 1) \end{aligned}$$

The structure of the 3-class groups $\text{Cl}_3(N_i)$ is a consequence of Table 4.2, since we have the following isomorphism, according to [1]

$$\begin{aligned} \text{Cl}_3(N_i) &\simeq \text{Gal}(\mathbb{F}_3^1(N_i)|N_i) \\ &\simeq \text{Gal}(\mathbb{F}_3^2(K)|N_i)/\text{Gal}(\mathbb{F}_3^2(K)|\mathbb{F}_3^1(N_i)) \simeq M_i/\gamma_2(M_i). \end{aligned}$$

Taking into consideration the preliminaries in §§ 4.2.1–4.2.3, resp. in Lemma 4.1, we use the equivalence of the following statements.

$$\begin{aligned} \text{Cl}_3(N_1) \simeq \text{A}(3, 3) &\iff g_1^3 = y^3 \notin \langle t_3 \rangle = \gamma_2(M_1), \\ \text{Cl}_3(N_2) \simeq \text{A}(3, 3) &\iff g_2^3 = x^3 \notin \langle s_3 \rangle = \gamma_2(M_2), \\ \text{Cl}_3(N_3) \simeq \text{A}(3, 3) &\iff g_3^3 = (xy)^3 \notin \langle s_3 t_3 \rangle = \gamma_2(M_3), \\ \text{Cl}_3(N_4) \simeq \text{A}(3, 3) &\iff g_4^3 = (xy^{-1})^3 \notin \langle s_3 t_3^{-1} \rangle = \gamma_2(M_4). \end{aligned}$$

□

Corollary 4.2.1. *For each of these seven isomorphism classes of the second 3-class group G , a 3-class group $\text{Cl}_3(N_i)$, $1 \leq i \leq 4$, is of type $(3, 3, 3)$ if and only if the norm class group $\text{Norm}_{N_i|K}(\text{Cl}_3(N_i))$ becomes principal either in none or in three of the extensions N_ℓ , $1 \leq \ell \leq 4$.*

The extensions with 3-class group of type $(3, 3, 3)$ always satisfy the condition (B) of Taussky [43], i. e., they have a partial principalization without fixed point, as predicted by [23, Satz 7, p. 11].

Proof. This is an immediate consequence of the principalization types \varkappa of these seven isomorphism classes. Let $j_{N_\ell|K} : \text{Cl}_p(K) \rightarrow \text{Cl}_p(N_\ell)$, $1 \leq \ell \leq 4$, denote the class extension homomorphisms [32, § 2.3, p. 477]. We demonstrate two cases.

For type H.4, $\varkappa = (4443)$, we have

$$\text{Norm}_{N_i|K}(\text{Cl}_3(N_i)) \cap \ker(j_{N_\ell|K}) = 1,$$

for $i = 1, 2$ and any $1 \leq \ell \leq 4$,

$$\text{Norm}_{N_3|K}(\text{Cl}_3(N_3)) = \ker(j_{N_4|K}), \text{ and } \text{Norm}_{N_4|K}(\text{Cl}_3(N_4)) = \ker(j_{N_\ell|K}),$$

for $1 \leq \ell \leq 3$.

For type b.10, $\varkappa = (0043)$, we have

$$\text{Norm}_{N_i|K}(\text{Cl}_3(N_i)) \subset \ker(j_{N_\ell|K}) = \text{Cl}_3(K),$$

for any $1 \leq i \leq 4$ and $\ell = 1, 2$, and additionally

$$\text{Norm}_{N_3|K}(\text{Cl}_3(N_3)) = \ker(j_{N_4|K}) \text{ and } \text{Norm}_{N_4|K}(\text{Cl}_3(N_4)) = \ker(j_{N_3|K}).$$

□

Corollary 4.2.2. *If K is a quadratic base field with $G \in \Phi_6$, then the three total principalization types b.10, c.21, c.18 are impossible, due to class number relations, and the remaining four partial principalization types D.10, G.19, H.4, D.5 are characterized uniquely by the invariant ε .*

Proof. For principalization types b.10, c.21, c.18 with singulet $\varkappa(1) = 0$ [32, § 2.2–2.3, p. 475–478], the entire 3-class group $\text{Cl}_3(K)$ becomes principal in N_1 . Hence, a quadratic base field K must be real, and the unramified cyclic cubic extension N_1 must be an S_3 -field of type α with even 3-exponent of the 3-class number $h_3(N_1)$, in contradiction to $\text{Cl}_3(N_1) \simeq A(3, 3)$ [31, Prop. 4.3–4.4, p. 484–485]. \square

Example. The first occurrences of second 3-class groups $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ of coclass $\text{cc}(G) = 2$ with invariants $m = 4$, $n = 5$, $e = 3$ and bicyclic centre among quadratic fields $K = \mathbb{Q}(\sqrt{D})$ with discriminant $-10^6 < D < 0$, resp. $0 < D < 10^7$, and 3-class group of type $(3, 3)$ turned out to be the following.

- The smallest value $|D|$ of the discriminant of a complex quadratic field K with principalization type D.10, resp. D.5, is 4027 [41, pp. 22–25], resp. 12 131 [23, Tbl. 3, p. 19].
- The smallest discriminant D of a real quadratic field K with principalization type D.10, resp. D.5, is 422 573, resp. 631 769. Both were unknown until 2006, resp. 2009.

Conjecture. For quadratic base fields, the principalization types G.19 and H.4 cannot occur with invariants $m = 4$, $n = 5$, and $k = 0$ of the corresponding second 3-class groups $\langle 243, 9 \rangle$, $\langle 243, 4 \rangle$, since they have terminal metabelian descendants (Figure 4.1) of the same principalization type with invariants $m = 5$, $n = 6$, and bigger defect $k = 1$. We call this the weak or restricted *leaf conjecture*. (Note added during revision: Meanwhile, this conjecture has been proved. See Thm. 1.4 and Cnj. 3.1 in [34].)

4.4. Groups G of coclass $\text{cc}(G) = 2$ with cyclic centre and $m = 5$, $n = 6$. This section corresponds to the second row of Table 3.1. Again, the abelianizations of all maximal subgroups M_i , $1 \leq i \leq 4$, have to be analyzed. These groups form the stem of Easterfield’s isoclinism families $\Phi_{40}, \Phi_{41}, \Phi_{42}, \Phi_{43}$ [13], [24, 4.1, p. 619, and 4.6 (40)–(43), p. 636] and satisfy the following special relations, by (3.6):

$$\begin{aligned}
 n &= 6 = 2m - 4, \\
 s &= 4 = m - 1, \\
 e &= n - m + 2 = 3 = m - 2, \\
 [\chi_s(G), \gamma_e(G)] &= [G, \gamma_3(G)] = \gamma_4(G) = \gamma_{m-1}(G) > 1, \quad k = 1, \quad \rho = \pm 1, \\
 \sigma_5 &= 1, \tau_5 = 1, \\
 \sigma_4^3 &= 1, \tau_4^3 = 1, \\
 \sigma_3^3 &= \sigma_4^{-3} = 1, \tau_3^3 = \tau_4^{-3} = 1, \\
 s_2^3 &= \sigma_4^{1-\rho(\beta-1)}, \\
 s_2^3 = 1 &\iff \beta = 0, \rho = -1 \text{ or } \beta = -1, \rho = 1, \\
 \tau_4 &= \sigma_4^{-\rho},
 \end{aligned}$$

$$\begin{aligned}
 \gamma_2(G) &= \langle s_2, \sigma_3, \sigma_4, \tau_3 \rangle \text{ of type } \begin{cases} (9, 3, 3), & \text{if } s_2^3 \neq 1, \\ (3, 3, 3, 3), & \text{if } s_2^3 = 1, \end{cases} \\
 \gamma_3(G) &= \langle \sigma_3, \sigma_4, \tau_3 \rangle \text{ of type } (3, 3, 3), \\
 \gamma_4(G) &= \langle \sigma_4 \rangle = \zeta_1(G) \text{ of type } (3).
 \end{aligned}$$

The commutator subgroups of the maximal normal subgroups are given by [31, Cor. 3.2, p. 480] and (3.6):

$$\begin{aligned}
 \gamma_2(M_1) &= \langle t_3, \sigma_4 \rangle = \langle \sigma_3^{-\rho\delta} \sigma_4^{\rho-\alpha} \tau_3^{1-\beta}, \sigma_4 \rangle = \langle \sigma_3^{-\rho\delta} \tau_3^{1-\beta}, \sigma_4 \rangle, \\
 \gamma_2(M_2) &= \langle s_3, \sigma_4 \rangle = \langle \sigma_3^{\rho\beta-1} \sigma_4^{\gamma-1} \tau_3^\delta, \sigma_4 \rangle = \langle \sigma_3^{\rho\beta-1} \tau_3^\delta, \sigma_4 \rangle, \\
 \gamma_2(M_3) &= \langle s_3 t_3, \sigma_4 \rangle = \langle \sigma_3^{\rho(\beta-\delta)-1} \tau_3^{\delta-\beta+1}, \sigma_4 \rangle, \\
 \gamma_2(M_4) &= \langle s_3 t_3^{-1}, \sigma_4 \rangle = \langle \sigma_3^{\rho(\beta+\delta)-1} \tau_3^{\beta+\delta-1}, \sigma_4 \rangle.
 \end{aligned}$$

The dependencies on the parameters α, γ disappear, since they occur in the exponent of $\sigma_4 \in \gamma_4(G)$, but each $\gamma_2(M_i)$ with $1 \leq i \leq 4$ contains $\gamma_4(G) = \langle \sigma_4 \rangle = \langle \tau_4 \rangle$.

In Table 4.4 we calculate the third powers g_i^3 of the generators $g_1 = y, g_2 = x, g_3 = xy, g_4 = xy^{-1}$ of the maximal normal subgroups $M_i = \langle g_i, \gamma_2(G) \rangle$ and the generators of the commutator subgroups $\gamma_2(M_i)$, $1 \leq i \leq 4$, modulo σ_4 for each of the 12 isomorphism classes of groups G with $m = 5, n = 6, \rho = \pm 1$ [38, pp. 4–7]. Several of these isomorphism classes have the same principalization type and the same parameters β, δ, ρ , as indicated by the second column of Table 4.4.

Generally, according to [37, Lem. 3.4.11, p. 105], the third powers of g_3 and g_4 modulo σ_4 are given by

$$(xy)^3 = \sigma_3^{\rho(\beta+\delta)} \sigma_4^{\alpha+\gamma+\rho(\beta+\delta)} \tau_3^{\beta+\delta} \equiv \sigma_3^{\rho(\beta+\delta)} \tau_3^{\beta+\delta}$$

$$\text{and } (xy^{-1})^3 = \sigma_3^{\rho(\delta-\beta)} \sigma_4^{\alpha-\gamma+\rho\beta} \tau_3^{\beta-\delta} \equiv \sigma_3^{\rho(\delta-\beta)} \tau_3^{\beta-\delta}.$$

TABLE 4.4. Parameters, third powers, and generators for $m = 5, n = 6, e = 3, k = 1$

Type	#isom.cl.	β	δ	ρ	y^3	x^3	$\overline{(xy)^3}$	$\overline{(xy^{-1})^3}$	$\overline{t_3}$	$\overline{s_3}$	$\overline{s_3 t_3}$	$\overline{s_3 t_3^{-1}}$
G.19	2	-1	0	1	σ_3	τ_3	$\sigma_3^{-1} \tau_3^{-1}$	$\sigma_3 \tau_3^{-1}$	τ_3^2	σ_3^{-2}	$\sigma_3^{-2} \tau_3^2$	$\sigma_3^{-2} \tau_3^{-2}$
H.4	4	1	1	1	σ_3	τ_3	$\sigma_3^2 \tau_3^2$	1	σ_3^{-1}	τ_3	$\sigma_3^{-1} \tau_3$	$\sigma_3 \tau_3$
b.10	3	0	0	-1	σ_3	τ_3	1	1	τ_3	σ_3^{-1}	$\sigma_3^{-1} \tau_3$	$\sigma_3^{-1} \tau_3^{-1}$
b.10	3	0	0	1	σ_3	τ_3	1	1	τ_3	σ_3^{-1}	$\sigma_3^{-1} \tau_3$	$\sigma_3^{-1} \tau_3^{-1}$

The order of the coset of $g_i \in M_i$ with respect to $\gamma_2(M_i)$ is bounded from above by 3, if and only if the third power g_i^3 is contained in $\gamma_2(M_i)$.

Theorem 4.3. (TTT $\tau(G)$ of stem groups G in isoclinism families Φ_s with $40 \leq s \leq 43$)

Let K be a number field with 3-class group $\text{Cl}_3(K)$ of type $(3, 3)$. Suppose that the second 3-class group $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ is of order $|G| = 3^n$ and class $\text{cl}(G) = m - 1$, where $n = 6$ and $m = 5$, such that $[\chi_s(G), \gamma_e(G)] = \gamma_{m-1}(G)$, $k = 1$, i.e., that G is one of the twelve vertices with defect $k = 1$ of coclass graph $\mathcal{G}(3, 2)$ in Figure 4.1, with invariant $e = 3$ and cyclic centre $\zeta_1(G)$.

Then the structure of the 3-class groups of the first Hilbert 3-class field $\mathbb{F}_3^1(K)$ of K and of the four unramified cyclic cubic extensions N_1, \dots, N_4 of K is given by Table 4.5, in dependence on the principalization type \varkappa of K and on the relational parameters β, ρ of G . The invariant ε denotes the number of 3-class groups $\text{Cl}_3(N_i)$ of type $(3, 3, 3)$, for each principalization type.

TABLE 4.5. 3-class groups of type $(3, 3, 3)$ for $m = 5, n = 6, e = 3, k = 1$

Type	\varkappa	β	ρ	$\text{Cl}_3(\mathbb{F}_3^1(K))$	$\text{Cl}_3(N_1)$	$\text{Cl}_3(N_2)$	$\text{Cl}_3(N_3)$	$\text{Cl}_3(N_4)$	ε
G.19	(2143)	-1	1	$(3, 3, 3, 3)$	(9, 3)	(9, 3)	(9, 3)	(9, 3)	0
H.4	(4443)	1	1	(9, 3, 3)	$(3, 3, 3)$	$(3, 3, 3)$	(9, 3)	$(3, 3, 3)$	3
b.10	(0043)	0	-1	$(3, 3, 3, 3)$	(9, 3)	(9, 3)	$(3, 3, 3)$	$(3, 3, 3)$	2
b.10	(0043)	0	1	(9, 3, 3)	(9, 3)	(9, 3)	$(3, 3, 3)$	$(3, 3, 3)$	2

Proof. Similarly as in the proof of Theorem 4.2, the structure of the 3-class group $\text{Cl}_3(\mathbb{F}_3^1(K))$ of the first Hilbert 3-class field of K is a consequence of $m = 5$, $e = 3$, and the isomorphisms in [1] and [37, Satz 4.2.4, p. 131]

$$\begin{aligned} &\text{Cl}_3(\mathbb{F}_3^1(K)) \simeq \gamma_2(G) \\ &\simeq \begin{cases} \text{A}(3, m - 2) \times \text{A}(3, e - 2) = \text{A}(3, 3) \times \text{A}(3, 1) & \text{for } \rho \not\equiv \beta - 1 \pmod{3}, \\ \text{A}(3, m - 3) \times \text{A}(3, e - 1) = \text{A}(3, 2) \times \text{A}(3, 2) & \text{for } \rho \equiv \beta - 1 \pmod{3}. \end{cases} \end{aligned}$$

The structure of the 3-class groups $\text{Cl}_3(N_i) \simeq M_i/\gamma_2(M_i)$ follows from Table 4.4, if we take into consideration the preparations in §§ 4.2.1–4.2.3, resp. in Lemma 4.1. □

Corollary 4.3.1. *For each of these twelve isomorphism classes of the second 3-class group G , a 3-class group $\text{Cl}_3(N_i)$, $1 \leq i \leq 4$, is of type $(3, 3, 3)$ if and only if the norm class group $\text{Norm}_{N_i|K}(\text{Cl}_3(N_i))$ becomes principal either in none or in three of the extensions N_ℓ , $1 \leq \ell \leq 4$.*

The extensions with 3-class group of type $(3, 3, 3)$ always satisfy the condition (B) of Taussky [43], i. e., they have a partial principalization without fixed point.

Proof. This is an immediate consequence of the principalization types \varkappa of these twelve isomorphism classes. □

Corollary 4.3.2. *If K is a quadratic base field with $G \in \Phi_s$, for some $s \in \{40, 41, 42, 43\}$, then the total principalization type b.10 is impossible, due to class number relations, and the remaining two partial principalization types G.19 and H.4 are characterised uniquely by the invariant ε .*

Furthermore, only the second 3-class groups $G = \langle 729, 57 \rangle$ and $G = \langle 729, 45 \rangle$ are possible.

Proof. For the principalization type b.10 with singulets $\varkappa(1) = \varkappa(2) = 0$, the entire 3-class group $\text{Cl}_3(K)$ becomes principal in N_1, N_2 . In the case of a quadratic base field K this yields a similar contradiction to $\text{Cl}_3(N_1) \simeq \text{Cl}_3(N_2) \simeq \text{A}(3, 3)$ as in the proof of Corollary 4.2.2. For the last assertion, we refer to Thm. 3.14 in [34]. □

Example. The first occurrences of second 3-class groups $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ of coclass $\text{cc}(G) = 2$ with invariants $m = 5$, $n = 6$, $e = 3$, $\rho = \pm 1$ and cyclic centre among quadratic fields $K = \mathbb{Q}(\sqrt{D})$ with discriminant $-10^6 < D < 0$, resp. $0 < D < 10^7$, and 3-class group of type $(3, 3)$ turned out to be the following.

- The smallest value $|D|$ of the discriminant of a complex quadratic field K with principalization type H.4, resp. G.19, is 3896, resp. 12067 [23, Tbl. 3, p. 19].
- The smallest discriminant D of a real quadratic field K with principalization type G.19, resp. H.4, is 214712, resp. 957013. Both were unknown until 2006, resp. 2009.

4.5. All other groups G of coclass $\text{cc}(G) = 2$ with $m \geq 5, n \geq 6$.

This section corresponds to the third and fourth row of Table 3.1. Here we have to analyze the abelianizations of three maximal subgroups $M_i, 2 \leq i \leq 4$. These groups satisfy the following general relations. For $m \geq 6$, we have $\sigma_{m-2}, \sigma_{m-1} \in \gamma_4(G)$. For $m = 5$, the case $\rho = \pm 1$ has been investigated in the preceding section already, and we only have to consider the remaining possibility $\rho = 0$. Since $\sigma_{m-1} \in \gamma_4(G)$, also for $m = 5$, the following congruences modulo $\gamma_4(G)$ are valid, generally. The apparent dependencies on parameters α, γ, ρ vanish. Here we use [37, Lem. 3.4.11, p. 105] and (3.6).

$$\begin{aligned} (xy)^3 &= \sigma_{m-2}^{\rho(\beta+\delta)} \sigma_{m-1}^{\alpha+\gamma+\rho(\beta+\delta)} \tau_3^{\beta+\delta} \equiv \tau_3^{\beta+\delta} \pmod{\gamma_4(G)}, \\ (xy^{-1})^3 &= \sigma_{m-2}^{\rho(\delta-\beta)} \sigma_{m-1}^{\alpha-\gamma+\rho\beta} \tau_3^{\beta-\delta} \equiv \tau_3^{\beta-\delta} \pmod{\gamma_4(G)}, \\ t_3 &= \tau_3 \tau_4 \tau_3^{-\beta} \sigma_{m-2}^{-\rho\delta} \sigma_{m-1}^{-\alpha} \equiv \tau_3^{1-\beta} \pmod{\gamma_4(G)}, \\ s_3 &= \sigma_3^{-1} \sigma_4^{-1} \sigma_{m-2}^{\rho\beta} \sigma_{m-1}^{\gamma} \tau_3^{\delta} \equiv \sigma_3^{-1} \tau_3^{\delta} \pmod{\gamma_4(G)}, \\ s_3 t_3 &\equiv \sigma_3^{-1} \tau_3^{\delta-\beta+1} \pmod{\gamma_4(G)}, \\ s_3 t_3^{-1} &\equiv \sigma_3^{-1} \tau_3^{\beta+\delta-1} \pmod{\gamma_4(G)}. \end{aligned}$$

In Table 4.6 we calculate the third powers g_i^3 of the generators $g_1 = y, g_2 = x, g_3 = xy, g_4 = xy^{-1}$ of the maximal normal subgroups $M_i = \langle g_i, \gamma_2(G) \rangle$ and the generators of the commutator subgroups $\gamma_2(M_i), 1 \leq i \leq 4$, modulo $\gamma_4(G)$ for each of the isomorphism classes of groups G with $m \geq 5, n \geq 6, e = 3$. Several of these classes have the same principalization type and the same parameters β, δ . The left number of isomorphism classes concerns the single case $m = 5$ [38, pp. 8–9], the right number odd values of $m \geq 7$ [38, pp. 8–9 and pp. 16–19] and the middle number even values of $m \geq 6$ [38, pp. 10–12 and pp. 13–15].

The order of the coset of $g_i \in M_i$ with respect to $\gamma_2(M_i)$ is bounded from above by 3 if and only if the third power g_i^3 is contained in $\gamma_2(M_i)$.

Theorem 4.4. *(TTT $\tau(G)$ of groups G on coclass trees of $\mathcal{G}(3, 2)$)*

Let K be a number field with 3-class group $\text{Cl}_3(K)$ of type $(3, 3)$. Suppose that the second 3-class group $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ of K is of order $|G| = 3^n$ and of class $\text{cl}(G) = m - 1$, where $n \geq 6$ and $m = n - 1$, i. e., that G is a vertex on one of the three coclass trees of coclass graph $\mathcal{G}(3, 2)$ in Figure 4.1, with invariant $e = 3$. In the case $m = 5, n = 6$ let $[\chi_s(G), \gamma_e(G)] = 1$.

Then the structure of the 3-class groups $\text{Cl}_3(N_i)$ of the four unramified cyclic cubic extensions N_i of K is given by Table 4.7, in dependence on the principalization type \varkappa of K . The invariant ε denotes the number of 3-class groups $\text{Cl}_3(N_i)$ of type $(3, 3, 3)$, for each principalization type. Generally, the first two 3-class groups $\text{Cl}_3(N_1)$ and $\text{Cl}_3(N_2)$ are nearly homocyclic.

TABLE 4.6. Parameters, third powers, and generators for $m \geq 5, n \geq 6, e = 3$

Type	#isom.cl.	β	δ	y^3	x^3	$(xy)^3$	$(xy^{-1})^3$	$\overline{t_3}$	$\overline{s_3}$	$\overline{s_3 t_3}$	$\overline{s_3 t_3^{-1}}$
b.10	1 7 9	0	0	σ_3	τ_3	1	1	τ_3	σ_3^{-1}	$\sigma_3^{-1} \tau_3$	$\sigma_3^{-1} \tau_3^{-1}$
d.19	1 2 1	0	0	σ_3	τ_3	1	1	τ_3	σ_3^{-1}	$\sigma_3^{-1} \tau_3$	$\sigma_3^{-1} \tau_3^{-1}$
d.23	1 1 1	0	0	σ_3	τ_3	1	1	τ_3	σ_3^{-1}	$\sigma_3^{-1} \tau_3$	$\sigma_3^{-1} \tau_3^{-1}$
d.25	1 2 1	0	0	σ_3	τ_3	1	1	τ_3	σ_3^{-1}	$\sigma_3^{-1} \tau_3$	$\sigma_3^{-1} \tau_3^{-1}$
c.18	1 1 1	-1	1	σ_3	τ_3	1	τ_3^{-2}	τ_3^2	$\sigma_3^{-1} \tau_3$	σ_3^{-1}	$\sigma_3^{-1} \tau_3^{-1}$
E.6	1 1 1	-1	1	σ_3	τ_3	1	τ_3^{-2}	τ_3^2	$\sigma_3^{-1} \tau_3$	σ_3^{-1}	$\sigma_3^{-1} \tau_3^{-1}$
E.14	1 2 1	-1	1	σ_3	τ_3	1	τ_3^{-2}	τ_3^2	$\sigma_3^{-1} \tau_3$	σ_3^{-1}	$\sigma_3^{-1} \tau_3^{-1}$
H.4	1 8 9	-1	1	σ_3	τ_3	1	τ_3^{-2}	τ_3^2	$\sigma_3^{-1} \tau_3$	σ_3^{-1}	$\sigma_3^{-1} \tau_3^{-1}$
c.21	1 1 1	0	1	σ_3	τ_3	τ_3	τ_3^{-1}	τ_3	$\sigma_3^{-1} \tau_3$	$\sigma_3^{-1} \tau_3^2$	σ_3^{-1}
E.8	1 1 1	0	1	σ_3	τ_3	τ_3	τ_3^{-1}	τ_3	$\sigma_3^{-1} \tau_3$	$\sigma_3^{-1} \tau_3^2$	σ_3^{-1}
E.9	1 2 1	0	1	σ_3	τ_3	τ_3	τ_3^{-1}	τ_3	$\sigma_3^{-1} \tau_3$	$\sigma_3^{-1} \tau_3^2$	σ_3^{-1}
G.16	1 8 9	0	1	σ_3	τ_3	τ_3	τ_3^{-1}	τ_3	$\sigma_3^{-1} \tau_3$	$\sigma_3^{-1} \tau_3^2$	σ_3^{-1}

TABLE 4.7. 3-class groups of type (3, 3, 3) for $m \geq 5, n \geq 6, e = 3$

Type	\varkappa	$\text{Cl}_3(N_1)$	$\text{Cl}_3(N_2)$	$\text{Cl}_3(N_3)$	$\text{Cl}_3(N_4)$	ε
b.10	(0043)	A(3, $m - 1$) or A(3, $m - 2$)	(9, 3)	(3, 3, 3)	(3, 3, 3)	2
d.19	(4043)	A(3, $m - 1$)	(9, 3)	(3, 3, 3)	(3, 3, 3)	2
d.23	(1043)	A(3, $m - 1$)	(9, 3)	(3, 3, 3)	(3, 3, 3)	2
d.25	(2043)	A(3, $m - 1$)	(9, 3)	(3, 3, 3)	(3, 3, 3)	2
c.18	(0313)	A(3, $m - 1$)	(9, 3)	(3, 3, 3)	(9, 3)	1
E.6	(1313)	A(3, $m - 1$)	(9, 3)	(3, 3, 3)	(9, 3)	1
E.14	(2313)	A(3, $m - 1$)	(9, 3)	(3, 3, 3)	(9, 3)	1
H.4	(3313)	A(3, $m - 1$) or A(3, $m - 2$)	(9, 3)	(3, 3, 3)	(9, 3)	1
c.21	(0231)	A(3, $m - 1$)	(9, 3)	(9, 3)	(9, 3)	0
E.8	(1231)	A(3, $m - 1$)	(9, 3)	(9, 3)	(9, 3)	0
E.9	(2231)	A(3, $m - 1$)	(9, 3)	(9, 3)	(9, 3)	0
G.16	(4231)	A(3, $m - 1$) or A(3, $m - 2$)	(9, 3)	(9, 3)	(9, 3)	0

Proof. The structure of the first 3-class group $\text{Cl}_3(N_1)$ is given here only for the sake of completeness and is contained in the statement of Theorem 3.2 already.

Similarly as in the proof of Theorem 4.2, the structure of the other 3-class groups $\text{Cl}_3(N_i) \simeq M_i/\gamma_2(M_i)$ with $2 \leq i \leq 4$ is a consequence of table

4.6, when the preparations in the §§ 4.2.2–4.2.3, resp. in Lemma 4.1, are taken into consideration. □

Corollary 4.4.1. (*ε as a tree invariant*)

All metabelian groups G on the coclass tree $\mathcal{T}(\langle 729, 40 \rangle)$, resp. $\mathcal{T}(\langle 243, 6 \rangle)$, resp. $\mathcal{T}(\langle 243, 8 \rangle)$, of coclass graph $\mathcal{G}(3, 2)$ in Figure 4.1 are characterized by the value $\varepsilon = 2$, resp. $\varepsilon = 1$, resp. $\varepsilon = 0$.

Proof. This is a consequence of Theorem 4.4 and the diagram [37, p. 189 ff.]. See also [34, Thm. 3.16–3.17]. □

Corollary 4.4.2. *As before, for these isomorphism classes of the second 3-class group G , the extensions with 3-class group of type $(3, 3, 3)$ satisfy the condition (B) of Taussky [43], i. e., they have a partial principalization without fixed point. However, here only the following weaker statement without admissible inversion is true: if a 3-class group $\text{Cl}_3(N_i)$, $3 \leq i \leq 4$, is of type $(3, 3, 3)$, then the norm class group $\text{Norm}_{N_i|K}(\text{Cl}_3(N_i))$ becomes principal in either two or three of the extensions N_ℓ , $1 \leq \ell \leq 4$.*

Proof. This follows by evaluating the principalization type \varkappa of the isomorphism classes. □

Corollary 4.4.3. *If K is a quadratic base field with G on a coclass tree of $\mathcal{G}(3, 2)$, then the four total principalization types b.10, d.19, d.23, d.25 with $\varepsilon = 2$, i. e., $G \in \mathcal{T}(\langle 729, 40 \rangle)$, are impossible, due to class number relations. The remaining eight principalization types cannot be determined uniquely by the invariant ε alone .*

- (1) *The total principalization types c.18 with $\varepsilon = 1$ and c.21 with $\varepsilon = 0$ are characterized by an even 3-exponent u of the first 3-class number $h_3(N_1) = 3^u$.*
- (2) *The partial principalization types E.6, E.14 with $\varepsilon = 1$ and E.8, E.9 with $\varepsilon = 0$ are determined by an odd 3-exponent w of the 3-class number $h_3(\mathbb{F}_3^1(K)) = 3^w$.*
- (3) *The partial principalization types H.4 with $\varepsilon = 1$ and G.16 with $\varepsilon = 0$ are characterized by an even 3-exponent w of the 3-class number $h_3(\mathbb{F}_3^1(K)) = 3^w$, provided the weak leaf conjecture 4.3 holds.*

Here, as before, $\mathbb{F}_3^1(K)$ denotes the first Hilbert 3-class field of K .

Proof. For the principalization types b.10, d.19, d.23, and d.25 having $\varkappa(2) = 0$, the entire 3-class group $\text{Cl}_3(K)$ becomes principal in N_2 . Therefore a quadratic base field K must be real and the unramified cyclic cubic extension N_2 must be an S_3 -field of type α with even 3-exponent of the 3-class number $h_3(N_2)$, in contradiction to $\text{Cl}_3(N_2) \simeq \text{A}(3, 3)$. See [31, Prop. 4.3–4.4, p. 484–485].

The parity of the 3-exponent of the first 3-class number $h_3(N_1) = 3^u$ turns out to be even, $u = m - 1 \equiv 0 \pmod{2}$, for the total principalization

types c.18 and c.21 with $\varkappa(1) = 0$, to be odd, $u = m - 1 \equiv 1 \pmod{2}$, for the partial principalization types E.6, E.14 and E.8, E.9 with $k = 0$ and thus $w = n - 2$ odd, and to be odd, $u = m - 2 \equiv 1 \pmod{2}$, for the partial principalization types H.4 and G.16 with conjectural $k = 1$ (see Conjecture 4.3) and thus $w = n - 2$ even.

Here, we use the relation $3^w = h_3(F_3^1(K)) = |\gamma_2(G)| = 3^{n-2}$ and [31, Thm. 5.2–5.3, p. 492–493]. \square

Example. The first occurrences of second 3-class groups $G = \text{Gal}(F_3^2(K)|K)$ on the coclass trees $\mathcal{T}(\langle 243, 6 \rangle)$ and $\mathcal{T}(\langle 243, 8 \rangle)$ with invariants $m \geq 5$, $n \geq 6$, $e = 3$ and bicyclic center, $k = 0$, resp. cyclic center, $k = 1$, over quadratic base fields $K = \mathbb{Q}(\sqrt{D})$ with 3-class group of type $(3, 3)$ and discriminant $-10^6 < D < 0$, resp. $0 < D < 10^7$, are summarised in Table 4.8. Here, $|D|$ denotes the smallest absolute value of the discriminant of a complex quadratic field K , and D the smallest discriminant of a real quadratic field K , of the corresponding principalization type. The earlier computations by Scholz and Taussky [41], Heider and Schmithals [23], and Brink [11], are confirmed. Cases without references, in particular all cases with real quadratic base fields, were unknown up to now.

TABLE 4.8. Examples for groups G on coclass trees with $m \geq 5$, $n \geq 6$, $e = 3$

Type	\varkappa	m	n	k	$ D $	ref.	D
c.18	(0313)	5	6	0	impossible		534 824
E.6	(1313)	6	7	0	15 544	[23]	5 264 069
E.6 \uparrow	(1313)	8	9	0	268 040		unknown
E.14	(2313)	6	7	0	16 627	[23]	3 918 837
E.14 \uparrow	(2313)	8	9	0	262 744		unknown
H.4 \uparrow	(3313)	7	8	1	21 668	[11, 29]	1 162 949
H.4 \uparrow^2	(3313)	9	10	1	446 788		unknown
c.21	(0231)	5	6	0	impossible		540 365
c.21 \uparrow	(0231)	7	8	0	impossible		1 001 957
E.8	(1231)	6	7	0	34 867		6 098 360
E.8 \uparrow	(1231)	8	9	0	370 740		unknown
E.9	(2231)	6	7	0	9 748	[41]	342 664
E.9 \uparrow	(2231)	8	9	0	297 079		unknown
G.16	(4231)	7	8	1	17 131	[23]	8 711 453
G.16 \uparrow	(4231)	9	10	1	819 743		unknown

Whereas the parameters $m = 4$, $n = 5$ for the principalization types D.10, D.5 in example 4.3, and the parameters $m = 5$, $n = 6$ for the *ground state* of principalization types G.19, H.4 in example 4.4, are determined

uniquely, we now have infinite coclass families of second 3-class groups $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ with strictly increasing nilpotency class $\text{cl}(G) = m - 1$, sharing the same principalization type. For this reason, we define *excited states* of principalization types, denoted by arrows \uparrow, \uparrow^2 , and so on. The index m of nilpotency can take all odd values $m \geq 5$ for the principalization types c.18, c.21, all even values $m \geq 6$ for the principalization types E.6, E.14, E.8, E.9, and all odd values $m \geq 7$ for principalization types H.4, G.16.

A special feature of the principalization types c.18 and c.21 is the location of their second 3-class groups $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ as infinitely capable vertices on mainlines of coclass trees $\mathcal{T}(\langle 243, 6 \rangle)$ and $\mathcal{T}(\langle 243, 8 \rangle)$.

Concrete numerical realisations are known for the ground state with minimal index of nilpotency m , for each of these infinite coclass families. Realisations for excited states with higher values of m are known for the principalization types E.6, E.14, E.8, E.9, H.4, and G.16 of complex quadratic fields and for the principalization type c.21 of real quadratic fields.

4.6. Groups G of coclass $\text{cc}(G) \geq 3$ with $m \geq 5$, $n \geq 7$. This section corresponds to the fifth and sixth row of Table 3.1. Here we must investigate the abelianizations of only two maximal subgroups M_i , $3 \leq i \leq 4$. The third powers of the generators of these groups satisfy the following general relations, according to [37, Lem. 3.4.11, p. 105].

$$\begin{aligned} (xy)^3 &= \sigma_{m-2}^{\rho(\beta+\delta)} \sigma_{m-1}^{\alpha+\gamma+\rho(\beta+\delta)} \tau_e^{\beta+\delta}, \\ (xy^{-1})^3 &= \sigma_{m-2}^{\rho(\delta-\beta)} \sigma_{m-1}^{\alpha-\gamma+\rho\beta} \tau_e^{\beta-\delta}. \end{aligned}$$

Therefore the order of the coset of $xy \in M_3$ with respect to $\gamma_2(M_3)$ and of the coset of $xy^{-1} \in M_4$ with respect to $\gamma_2(M_4)$ is certainly bounded from above by 3, when $m \geq 6$ and $e \geq 4$, and thus $\sigma_{m-2}, \sigma_{m-1}, \tau_e \in \gamma_4(G) < \gamma_2(M_i)$, for $3 \leq i \leq 4$.

It remains to investigate the 15 isomorphism classes of groups with $m = 5$, $n = 7$, $e = 4$, for which $[\chi_s(G), \gamma_e(G)] = 1$ and thus $\rho = 0$ [38, pp. 34–35]. For these isomorphism classes, third powers of the generators are given by

$$\begin{aligned} (xy)^3 &= \sigma_4^{\alpha+\gamma} \tau_4^{\beta+\delta}, \\ (xy^{-1})^3 &= \sigma_4^{\alpha-\gamma} \tau_4^{\beta-\delta}, \end{aligned}$$

and therefore $\text{ord}(\overline{xy}) \leq 3$ and $\text{ord}(\overline{xy^{-1}}) \leq 3$, independently from all parameters $\alpha, \beta, \gamma, \delta$, since $\sigma_4, \tau_4 \in \gamma_4(G) < \gamma_2(M_i)$, for $3 \leq i \leq 4$.

Theorem 4.5. *Let K be a number field with 3-classgroup $\text{Cl}_3(K)$ of type $(3, 3)$ and with second 3-class group $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ of order $|G| = 3^n$, $n \geq 7$, and of coclass $\text{cc}(G) \geq 3$, $m \leq n - 2$, $e \geq 4$.*

Then the structure of the 3-class groups of the four unramified cyclic cubic extensions N_i of K is nearly homocyclic for $\text{Cl}_3(N_1)$, $\text{Cl}_3(N_2)$, and elementary abelian of type $(3, 3, 3)$ for $\text{Cl}_3(N_3)$, $\text{Cl}_3(N_4)$, independently from the principalization type \varkappa of K . The number of 3-class groups $\text{Cl}_3(N_i)$ of type $(3, 3, 3)$ is always given by $\varepsilon = 2$. The extensions with 3-class group of type $(3, 3, 3)$ satisfy Taussky's condition (B) [43], that is, they have a partial principalization without fixed point.

Proof. The nearly homocyclic structure of the first and second 3-class group, $\text{Cl}_3(N_1)$, $\text{Cl}_3(N_2)$, is contained in the statement of Theorem 3.2 already.

Similarly as in the proof of Theorem 4.2, the elementary abelian structure of the third and fourth 3-class group, $\text{Cl}_3(N_i) \simeq M_i/\gamma_2(M_i)$, $3 \leq i \leq 4$, is a consequence of the considerations at the begin of this section, if we take into account the preparation in § 4.2.3. □

Example. The first occurrences of second 3-class groups $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ of coclass $\text{cc}(G) \geq 3$ with invariants $m \geq 6$, $n \geq 8$, $e \geq 4$, and bicyclic center, $k = 0$, resp. cyclic center, $k = 1$, over quadratic fields $K = \mathbb{Q}(\sqrt{D})$ with 3-class group of type $(3, 3)$ and discriminant $-10^6 < D < 0$, resp. $0 < D < 10^7$, are summarized in Table 4.9. Here, $|D|$ denotes the smallest absolute value of the discriminant of a complex quadratic field K and D the smallest discriminant of a real quadratic field K of the corresponding principalization type. The earlier computations by Brink [11] are confirmed.

Similarly as in example 4.5, these principalization types belong to infinite families of second 3-class groups $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$. Here, additionally to the nilpotency class $\text{cl}(G) = m - 1$, the coclass $\text{cc}(G) = e - 1 = n - m + 1$ can also take infinitely many values.

For the principalization types F.7, F.11, F.12, F.13 with defect $k = 0$, all even values $m \geq 6$ and all odd values $5 \leq e \leq m - 1$ are possible.

The very rare principalization types d.19, d.23, d.25, also having $k = 0$, play a unique exceptional role [32, Thm. 3.4, p. 491], since their second 3-class groups $G = \text{Gal}(\mathbb{F}_3^2(K)|K)$ can appear either as terminal metabelian vertices (leaves) with even values $m \geq 6$ and even values $4 \leq e \leq m - 1$ or as infinitely capable vertices on mainlines of coclass trees with odd values $m \geq 7$ and odd values $5 \leq e \leq m - 1$.

Assuming the weak leaf conjecture 4.3, the following types show up with defect $k = 1$ only. For the principalization type b.10 all even values $m \geq 6$ and all even values $4 \leq e \leq m - 2$ can occur, and for the principalization types G.16, G.19, and H.4 all odd values $m \geq 7$ and all odd values $5 \leq e \leq m - 2$ are admissible.

Concrete numerical realizations by complex quadratic fields are known for the ground state of these principalization types with minimal index of nilpotency m , but only partially by real quadratic base fields.

TABLE 4.9. Examples for groups G with $\text{cc}(G) \geq 3$, $m \geq 6$, $n \geq 8$, $e \geq 4$

Type	\varkappa	m	n	e	k	$\text{Cl}_3(\mathbb{F}_3^1(K))$	$ D $	ref.	D
b.10	(0043)	6	8	4	1	(9, 9, 3, 3)	impossible		710 652
d.19	(4043)	6	8	4	0	(9, 9, 3, 3)	impossible		2 328 721
d.23	(1043)	6	8	4	0	(9, 9, 3, 3)	impossible		1 535 117
d.25*	(0143)	7	10	5	0	(27, 9, 9, 3)	impossible		8 491 713
F.7	(3443)	6	9	5	0	(9, 9, 9, 3)	124 363		unknown
F.7 \uparrow	(3443)	8	11	5	0	(27, 27, 9, 3)	469 816		unknown
F.11	(1143)	6	9	5	0	(9, 9, 9, 3)	27 156	[11,29]	unknown
F.11 \uparrow	(1143)	8	11	5	0	(27, 27, 9, 3)	469 787		unknown
F.12	(1343)	6	9	5	0	(9, 9, 9, 3)	31 908	[11]	unknown
F.12 \uparrow	(1343)	8	11	5	0	(27, 27, 9, 3)	249 371		unknown
F.12 \uparrow^2	(1343)	8	13	7	0	(27, 27, 27, 9)	423 640		unknown
F.13	(3143)	6	9	5	0	(9, 9, 9, 3)	67 480	[11]	8 321 505
F.13 \uparrow	(3143)	8	11	5	0	(27, 27, 9, 3)	159 208		8 127 208
G.16r	(1243)	7	10	5	1	(27, 9, 9, 3)	290 703		unknown
G.16i	(1243)	7	10	5	1	(9, 9, 9, 9)	135 059		unknown
G.19r	(2143)	7	10	5	1	(27, 9, 9, 3)	96 827		unknown
G.19r \uparrow	(2143)	9	12	5	1	(81, 27, 9, 3)	509 160		unknown
G.19i	(2143)	7	10	5	1	(9, 9, 9, 9)	199 735		unknown
H.4r	(3343)	7	10	5	1	(27, 9, 9, 3)	256 935		unknown
H.4r \uparrow	(3343)	9	12	5	1	(81, 27, 9, 3)	678 804		unknown
H.4i	(3343)	7	10	5	1	(9, 9, 9, 9)	186 483		unknown

5. Implementing the principalization algorithm for quadratic fields

In this section we describe the computational techniques used to achieve the numerical results presented in § 6. The new principalization algorithm via class group structure has been implemented for quadratic fields, having a 3-class group of type (3, 3), with the aid of program scripts written for the number theoretical computer algebra system PARI/GP [7, 39]. We refer to the relevant methods of this software package by printing their names in typewriter font with trailing parentheses.

5.1. Generating polynomials for non-Galois cubic fields L .

5.1.1. Simply real cubic fields. Generating polynomials $p(X) = X^3 - bX^2 + cX - d$ of third degree for simply real cubic fields L of signature (1, 1) are obtained in the following manner. Suppose the intended upper bound for the absolute value of the field discriminant is $|d(L)| \leq U$. Then candidates for the coefficients b , c , and d run over three nested loops $1 \leq b \leq \lfloor 3 + 2\sqrt[4]{U} \rfloor$, $1 \leq c \leq \lfloor \frac{b^2 + \sqrt{U}}{3} \rfloor$, and $1 \leq d \leq \lfloor \frac{b^2 - 3 + 2\sqrt{U}}{6} \rfloor$, with bounds due to Godwin

and Angell, given by Fung and Williams [18, (2.5), p. 315]. For each triplet (b, c, d) the following tests are performed.

- (1) Reducible polynomials $p(X)$ are eliminated with the aid of `polisirreducible()`.
- (2) For irreducible polynomials the discriminant $d(L)$ of the cubic field $L = \mathbb{Q}(\vartheta)$, generated by the real zero ϑ of $p(X)$, is calculated by means of `nfdisc()` and tested for $-U \leq d(L) < 0$.
- (3) By `poldisc()` the discriminant $d(p)$ of the polynomial is computed and its index $i(p)$ is determined, using the formula $d(p) = i(p)^2 \cdot d(L)$. Polynomials with indices bigger than the bound

$$\frac{\sqrt{124b^2 + 432b + 4\sqrt{U} + 729}}{3\sqrt{3}}$$

are skipped, thus discouraging superfluous isomorphic fields [18, (2.6), p. 315].

- (4) Field discriminants $d(L) = f^2 \cdot d(K)$ [33, § 1, p. 832] are restricted to fundamental discriminants with conductor $f = 1$ by `isfundamental()`.
- (5) Finally, the 3-class group $\text{Cl}_3(K)$ of the complex quadratic subfield K of the Galois closure N of L , which is unramified with conductor $f = 1$ over K and has discriminant $d(N) = d(K)^3$ [33, Abstract, p. 831], is restricted to the type (3, 3) with the aid of `quadclassunit()`, thereby eliminating the numerous cyclic 3-class groups.

5.1.2. Totally real cubic fields. Trace free generating polynomials $p(X) = X^3 - cX - d$ for totally real cubic fields L of signature (3, 0) are collected in the following way. If the desired upper bound for the field discriminant is $d(L) \leq U$, then candidates for the coefficients c and d run over two nested loops $1 \leq c \leq \lfloor \sqrt{U} \rfloor$ and $1 \leq d \leq \lfloor \sqrt{\frac{4c^3}{27}} \rfloor$, with bounds given by Llorente and Quer [28, § 3, p. 586]. For each pair (c, d) the following tests are performed.

- (1) Reducible polynomials $p(X)$ are skipped with the aid of `polisirreducible()`.
- (2) For irreducible polynomials the discriminant $d(L)$ of the cubic field $L = \mathbb{Q}(\xi)$, generated by a zero ξ of $p(X)$, is calculated by means of `nfdisc()` and checked for $0 < d(L) \leq U$.
- (3) A further bound [28, Thm. 3, p. 584] is imposed on the linear coefficient

$$c \leq \begin{cases} \lfloor \frac{\sqrt{d(L)}}{3} \rfloor, & \text{if } 27 \mid d(L), \\ \lfloor \sqrt{d(L)} \rfloor, & \text{otherwise.} \end{cases}$$

- (4) By `poldisc()` the discriminant $d(p)$ of the polynomial is computed and its index $i(p)$ is determined, using the formula $d(p) = i(p)^2 \cdot d(L)$. Polynomials with index bigger than the bound

$$\begin{cases} \lfloor 2\sqrt{\frac{c}{3}} \rfloor, & \text{if } 27 \mid d(L), \\ \lfloor 2\sqrt{c} \rfloor, & \text{otherwise,} \end{cases}$$

are eliminated [28, Thm. 3, p. 584].

- (5) Field discriminants $d(L)$ are restricted to fundamental discriminants by `isfundamental()`.
- (6) Finally, the 3-class group $\text{Cl}_3(K)$ of the real quadratic subfield K of the Galois closure N of L is restricted to the type $(3, 3)$ with the aid of `quadclassunit()`.

5.2. Structure of 3-class groups $\text{Cl}_3(N)$ of S_3 -fields N . The generating polynomials of § 5.1 are stored as quadruplets (D, b, c, d) for $D < 0$, ordered by descending discriminants D , resp. as triplets (D, c, d) for $D > 0$, ordered by ascending discriminants D . The bounds in [18], resp. [28], provide a warranty that, for each discriminant $D = d(K) = d(L)$, generating polynomials for all four non-isomorphic cubic fields L sharing the same discriminant $d(L) = D$ [33, Cor. 3.1, p. 838] are contained in the list. Now the polynomials are iterated through the list in a single loop and for each of them the following steps are executed.

- (1) The regulator $R(L)$ and the class number $h(L)$ of L are calculated with the aid of `bnfinit()` using the flag 1, which ensures that a fundamental system of units of L is determined. An indicator is stored, if the class number $h(L)$ is divisible by 9 or 27.
- (2) By means of `polcompositum()`, applied to the cubic polynomial $p(X) = X^3 - bX^2 + cX - d$, resp. $p(X) = X^3 - cX - d$, and the quadratic polynomial $q(X) = X^2 - D$, a generating polynomial $s(X)$ of sixth degree for the Galois closure N of L is calculated.
- (3) The polynomial $s(X)$ of sixth degree is used to determine the structure of the class group $\text{Cl}(N)$ of the normal field N with the aid of `bnfinit()`, where the flag is set to 1. An indicator is stored, if the first three abelian type invariants (n_1, n_2, n_3, \dots) of the class group structure are all divisible by 3, that is, if the 3-class group $\text{Cl}_3(N)$ of N is elementary abelian of type $(3, 3, 3)$.

The results are evaluated in the following way. The structures of the 3-class groups $\text{Cl}_3(N_i)$, $1 \leq i \leq 4$, of quadruplets N_1, \dots, N_4 of S_3 -fields sharing the same discriminant D^3 , form the transfer target type $\tau(K)$ and determine the number $\varepsilon(K)$ of groups of type $(3, 3, 3)$. If some 3-class group $\text{Cl}_3(N_i)$ is of type $(3, 3)$, that is, isomorphic to $A(3, 2)$, then the second 3-class group G of K is of coclass $\text{cc}(G) = 1$, by Theorem 3.1, and the

algorithm can be terminated. Otherwise G is of coclass $\text{cc}(G) \geq 2$ and the algorithm can be terminated only if no indicator of $9 \mid h(L_i)$ has been stored, for any $1 \leq i \leq 4$. The transfer target type $\tau(K)$ and the invariant $\varepsilon(K)$ determine the principalization type (transfer kernel type) $\varkappa(K)$ and the structure of the second 3-class group $G = G_3^2(K) = \text{Gal}(F_3^2(K)|K)$ of the quadratic field K in two important special cases: uniquely for sporadic G of coclass $\text{cc}(G) = 2$, according to Theorems 4.2 and 4.3, and up to separation of types a.2 and a.3 for $\text{cc}(G) = 1$, according to Theorem 4.1.

5.3. First Hilbert 3-class field $F_3^1(K)$ of K . For quadruplets L_1, \dots, L_4 of cubic fields, sharing the same discriminant D , which have been marked by an indicator of $9 \mid h(L_i)$, for some $1 \leq i \leq 4$, a further step must be appended, when the second 3-class group $G = \text{Gal}(F_3^2(K)|K)$ of the quadratic field K is of coclass $\text{cc}(G) \geq 2$. A generating polynomial $f(X)$ of eighteenth degree for the first Hilbert 3-class field $F_3^1(K) = N_i \cdot L_j$ of K is calculated by means of `polcompositum()`, applied to the generating polynomial $s(X)$ of sixth degree of N_i and the cubic generating polynomial $p(X)$ of L_j , for some $1 \leq i \neq j \leq 4$. The polynomial $f(X)$ of eighteenth degree is used to determine the structure of the class group $\text{Cl}(F_3^1(K))$ of the first Hilbert 3-class field $F_3^1(K)$ of the quadratic field K with the aid of `bnfinit()`, where the flag is set to 1. Whereas computations for fields of third and sixth degree are usually a matter of less than a second in PARI/GP, the CPU time for a field of degree 18 may reach a few minutes, occasionally.

Together with the transfer target type $\tau(K)$ and the number $\varepsilon(K)$ of groups of type $(3, 3, 3)$, the structure of the 3-class group $\text{Cl}_3(F_3^1(K))$ determines the principalization type $\varkappa(K)$ and the structure of the second 3-class group $G = \text{Gal}(F_3^2(K)|K)$ of the quadratic field K in the following manner: up to separation of types E.6, E.14, resp. E.8, E.9, for $\text{cc}(G) = 2$ by means of Theorem 4.4, and up to separation of types F.7, F.11, F.12, F.13, resp. G.16, G.19, H.4, resp. d.19, d.23, d.25, for $\text{cc}(G) \geq 3$ by means of Theorem 4.5 and [31, Thm. 5.1–5.3, pp. 491–494].

In our paper [31], we have indicated another computational technique, trying to avoid the use of the highly sophisticated system PARI/GP. The invariants of the second 3-class group $G = \text{Gal}(F_3^2(K)|K)$, namely the class $\text{cl}(G) = m - 1$, coclass $\text{cc}(G) = e - 1$, and the order $|G| = 3^n$, $n = \text{cl}(G) + \text{cc}(G)$, are determined with the aid of 3-class numbers $h(L_i)$, $1 \leq i \leq 4$, of cubic fields, using [31, Thm. 5.1–5.3]. These class numbers can be computed by means of our own implementation of the classical Voronoi algorithms for calculating integral bases [44] and fundamental systems of units [45] for simply or totally real cubic fields and subsequent application of the analytic class number formula and Euler product method. The transfer kernel type \varkappa of G is assumed as an experimental input data in [31], produced by some

unspecified principalization algorithm, for instance the classical algorithm in [41, 23, 11, 29]. Consequently, the difficult determination of the defect k with the aid of the Hilbert 3-class field $F_3^1(K)$ of K is circumvented, when the weak leaf conjecture 4.3 is assumed to hold. We have successfully applied this cumbersome classical procedure to the restricted range $-10^5 < D < 10^6$ of quadratic discriminants, for which we needed 7 years from 2003 to 2009. However, the extension to $-10^6 < D < 10^7$ definitely requires the high speed performance of PARI/GP or MAGMA and the automatizability of our new principalization algorithm. Including the manual evaluation it was done within 5 months in 2010. CPU time added up to a total of a few weeks.

6. Numerical results on second 3-class groups of 4 596 quadratic fields

By means of the principalization algorithm, implemented in the PARI/GP programs of § 5, the principalization type $\varkappa(K)$ [32, Tbl. 6–7, pp. 492–493] and the structure of the second 3-class group $G = G_3^2(K) = \text{Gal}(F_3^2(K)|K)$ [31] has been determined for the 2020 complex quadratic fields $K = \mathbb{Q}(\sqrt{D})$ with discriminant $-10^6 < D < 0$ and for the 2576 real quadratic fields $K = \mathbb{Q}(\sqrt{D})$ with discriminant $0 < D < 10^7$, having a 3-class group $\text{Cl}_3(K)$ of type $(3, 3)$. The results of these extensive computations reveal reliable statistical tendencies concerning the distribution of the groups G on the sporadic part and various coclass trees of the coclass graphs $\mathcal{G}(3, r)$, $1 \leq r \leq 6$, [27, 26, 15] for a total of 4596 quadratic base fields.

Each of the following tables gives the characterizing *transfer target type* (TTT) τ , as a novelty which was unknown up to now, and the minimal discriminant and absolute frequency of various transfer kernel types (TKTs) \varkappa . Here, 11 tables are arranged according to the sign of the discriminant D and the *graph theoretic location* of the second 3-class group G , either on sporadic parts or on branches of coclass trees, which constitute the coclass graphs, thus providing necessary information for drawing graphical diagrams of coclass graphs in the subsequent paper [34]. In contrast, the four Tables 2–5 in our previous paper [31, § 6, pp. 496–499] were arranged according to the number ν of total principalizations.

Table 6.1 characterizes the most frequent types \varkappa of real quadratic fields with G of coclass 1 by means of $h_3(L_1)$ and $\text{Cl}_3(N_1)$. For the other fields with $2 \leq i \leq 4$, we always have $h_3(L_i) = 3$ and $\text{Cl}_3(N_i)$ of type $(3, 3)$. The second 3-class group G is a vertex of depth 1 on an odd branch $\mathcal{B}(j)$, $j \in \{3, 5, 7\}$, of the unique coclass tree $\mathcal{T}(C_3 \times C_3)$ of $\mathcal{G}(3, 1)$ in Figure 3.1. Mainline groups of depth 0 do not occur and are probably impossible for quadratic fields, as stated in our Conjecture 4.1. The types a.2 and a.3,

TABLE 6.1. Principalization types with $G \in \mathcal{G}(3, 1)$ for $D > 0$

type	\varkappa	j	$h_3(L_1)$	$Cl_3(N_1)$	ε	$Cl_3(F_3^1(K))$	min. D	ref.	freq.
a.2	(1000)	3	3	(9, 3)	0	(3, 3)	72 329	[23]	} 1 386
a.3	(2000)	3	3	(9, 3)	0	(3, 3)	32 009	[23]	
a.3*	(2000)	3	3	(3, 3, 3)	1	(3, 3)	142 097	[30]	697
a.1	(0000)	5	9	(9, 9)	0	(9, 9)	62 501	[23]	147
a.2 \uparrow	(1000)	5	9	(27, 9)	0	(9, 9)	790 085	}	72
a.3 \uparrow	(2000)	5	9	(27, 9)	0	(9, 9)	494 236		
a.1 \uparrow	(0000)	7	27	(27, 27)	0	(27, 27)	2 905 160		1
total:									2 303

resp. a.2 \uparrow and a.3 \uparrow , can only be separated with the aid of the classical principalization algorithm [41, 23]. Among the 2 576 real quadratic fields $K = \mathbb{Q}(\sqrt{D})$ with discriminant $0 < D < 10^7$ and 3-class group $Cl_3(K)$ of type (3, 3), the dominating part of 2 303 fields, that is 89.4%, has a second 3-class group G of coclass $cc(G) = 1$. Branch $\mathcal{B}(3)$ is populated most densely by $\frac{697}{2303} = 30.3\%$ groups of type a.3* and $\frac{1386}{2303} = 60.2\%$ groups of types a.2 and a.3.

TABLE 6.2. Principalization types with $G \in \mathcal{T}(\langle 2187, 64 \rangle) \subset \mathcal{G}(3, 3)$ for $D > 0$

type	\varkappa	$h_3(L_1)$	$h_3(L_2)$	$Cl_3(N_1)$	$Cl_3(N_2)$	$Cl_3(N_3)$	$Cl_3(N_4)$	ε	$Cl_3(F_3^1(K))$	min. D	fr.
b.10	(0043)	9	9	(9, 9)	(9, 9)	(3, 3, 3)	(3, 3, 3)	2	(9, 9, 3, 3)	710 652	8
d.19	(4043)	9	9	(27, 9)	(9, 9)	(3, 3, 3)	(3, 3, 3)	2	(9, 9, 3, 3)	2 328 721	1
d.23	(1043)	9	9	(27, 9)	(9, 9)	(3, 3, 3)	(3, 3, 3)	2	(9, 9, 3, 3)	1 535 117	1
total:											10

Table 6.2 characterizes the rare types \varkappa of real quadratic fields with G of coclass 3 by means of $h_3(L_i)$, $1 \leq i \leq 2$, and $Cl_3(N_i)$, $1 \leq i \leq 4$. For these cases we have $h_3(L_i) = 3$, for $3 \leq i \leq 4$, and the second 3-class group G is a vertex of depth 1 on the odd branch $\mathcal{B}(7)$ of the tree with root $\langle 2187, 64 \rangle \simeq G_0^{5,7}(0, 0, 0, 0)$ [37, p. 189 ff.] of $\mathcal{G}(3, 3)$. The types d.19, d.23, d.25 [32, Thm. 3.4, p. 491] can only be separated by the classical principalization algorithm [41, 23]. Among the 2 576 real quadratic fields under investigation, only a negligible part of 10 fields, that is 0.4%, has a second 3-class group G of coclass $cc(G) = 3$.

Whereas groups on $\mathcal{G}(3, 1)$, $\mathcal{G}(3, 3)$, and generally groups with odd coclass, are impossible for complex quadratic fields [31, Thm. 4.2, p. 489], we now proceed to coclass graphs which are populated by second 3-class groups of quadratic fields with either sign of the discriminant.

The second 3-class groups G for types \varkappa in Table 6.3 are sporadic vertices on $\mathcal{G}(3, 2)$ in Figure 4.1. They are determined uniquely by ε (Cor. 4.2.2).

TABLE 6.3. Principalization types with sporadic $G \in \mathcal{G}(3, 2)$ for $D > 0$

type	\varkappa	$h_3(L_1)$	$h_3(L_2)$	$Cl_3(N_1)$	$Cl_3(N_2)$	$Cl_3(N_3)$	$Cl_3(N_4)$	ε	$Cl_3(F_3^1(K))$	min. D	fr.
G.19	(2143)	3	3	(9, 3)	(9, 3)	(9, 3)	(9, 3)	0	(3, 3, 3, 3)	214 712	11
D.10	(2241)	3	3	(9, 3)	(9, 3)	(3, 3, 3)	(9, 3)	1	(3, 3, 3)	422 573	93
D.5	(4224)	3	3	(3, 3, 3)	(9, 3)	(3, 3, 3)	(9, 3)	2	(3, 3, 3)	631 769	47
H.4	(4443)	3	3	(3, 3, 3)	(3, 3, 3)	(9, 3)	(3, 3, 3)	3	(9, 3, 3)	957 013	27
total:											178

Among the 263 groups G of even coclass for real quadratic fields, a contribution of 178 groups, that is 67.7%, is sporadic of coclass $cc(G) = 2$. This is the adequate kind of relative frequencies for comparison with complex quadratic fields.

TABLE 6.4. Principalization types with sporadic $G \in \mathcal{G}(3, 2)$ for $D < 0$

type	\varkappa	$h(L_1)$	$h(L_2)$	$Cl_3(N_1)$	$Cl_3(N_2)$	$Cl_3(N_3)$	$Cl_3(N_4)$	ε	$Cl_3(F_3^1(K))$	min. $ D $	fr.
G.19	(2143)	3	3	(9, 3)	(9, 3)	(9, 3)	(9, 3)	0	(3, 3, 3, 3)	12 067	94
D.10	(2241)	3	3	(9, 3)	(9, 3)	(3, 3, 3)	(9, 3)	1	(3, 3, 3)	4 027	667
D.5	(4224)	3	3	(3, 3, 3)	(9, 3)	(3, 3, 3)	(9, 3)	2	(3, 3, 3)	12 131	269
H.4	(4443)	3	3	(3, 3, 3)	(3, 3, 3)	(9, 3)	(3, 3, 3)	3	(9, 3, 3)	3 896	297
total:											1 327

Table 6.4 is the complex counterpart of Table 6.3. Among the 2020 complex quadratic fields $K = \mathbb{Q}(\sqrt{D})$ with discriminant $-10^6 < D < 0$ and 3-class group $Cl_3(K)$ of type (3, 3) a considerable part of 1 327 fields, that is 65.7%, has a sporadic second 3-class group G of coclass $cc(G) = 2$. Type D.10 with a relative frequency of $\frac{667}{2020} = 33.0\%$ is the absolute high-champ for complex quadratic fields.

TABLE 6.5. Principalization types with $G \in \mathcal{T}(\langle 243, 6 \rangle) \subset \mathcal{G}(3, 2)$ for $D > 0$

type	\varkappa	j	$h_3(L_1)$	$h_3(L_2)$	$Cl_3(N_1)$	$Cl_3(N_2)$	$Cl_3(N_3)$	$Cl_3(N_4)$	ε	$Cl_3(F_3^1(K))$	min. D	freq.
c.18	(0313)	6	9	3	(9, 9)	(9, 3)	(3, 3, 3)	(9, 3)	1	(9, 3, 3)	534 824	29
E.6	(1313)	6	9	3	(27, 9)	(9, 3)	(3, 3, 3)	(9, 3)	1	(9, 9, 3)	5 264 069	} 7
E.14	(2313)	6	9	3	(27, 9)	(9, 3)	(3, 3, 3)	(9, 3)	1	(9, 9, 3)	3 918 837	
H.4 \uparrow	(3313)	6	9	3	(27, 9)	(9, 3)	(3, 3, 3)	(9, 3)	1	(27, 9, 3)	1 162 949	3
total:												39

The second 3-class groups G for types \varkappa in Table 6.5 are vertices on the coclass tree with root $\langle 243, 6 \rangle \simeq G_0^{4,5}(0, -1, 0, 1)$ of $\mathcal{G}(3, 2)$. G is a vertex of depth 0 for type c.18, of depth 1 for types E.6, E.14, and of depth 2 for type H.4, on the even branch $\mathcal{B}(6)$, [3, Tbl. 2, p. 266], [4, Fig. 4.8, p. 76], [37, p. 189 ff.]. The types E.6, E.14 can only be separated by the classical

principalization algorithm [41, 23]. Type H.4 can be identified either by the first Hilbert 3-class field or by the classical principalization algorithm. Among the 263 groups of even coclass for real quadratic fields, a fraction of 39 groups, that is 14.8%, populates this tree.

TABLE 6.6. Principalization types with $G \in \mathcal{T}(\langle 243, 6 \rangle) \subset \mathcal{G}(3, 2)$ for $D < 0$

type	\varkappa	j	$h(L_1)$	$h(L_2)$	$\text{Cl}_3(N_1)$	$\text{Cl}_3(N_2)$	$\text{Cl}_3(N_3)$	$\text{Cl}_3(N_4)$	ε	$\text{Cl}_3(\mathbb{F}_3^1(K))$	min. $ D $	freq.
E.6	(1313)	6	9	3	(27, 9)	(9, 3)	(3, 3, 3)	(9, 3)	1	(9, 9, 3)	15 544	} 186
E.14	(2313)	6	9	3	(27, 9)	(9, 3)	(3, 3, 3)	(9, 3)	1	(9, 9, 3)	16 627	
H.4 \uparrow	(3313)	6	9	3	(27, 9)	(9, 3)	(3, 3, 3)	(9, 3)	1	(27, 9, 3)	21 668	
E.6 \uparrow	(1313)	8	27	3	(81, 27)	(9, 3)	(3, 3, 3)	(9, 3)	1	(27, 27, 3)	268 040	} 15
E.14 \uparrow	(2313)	8	27	3	(81, 27)	(9, 3)	(3, 3, 3)	(9, 3)	1	(27, 27, 3)	262 744	
H.4 \uparrow^2	(3313)	8	27	3	(81, 27)	(9, 3)	(3, 3, 3)	(9, 3)	1	(81, 27, 3)	446 788	
total:											270	

Again, we continue opposing the complex analog to Table 6.5 in Table 6.6. Among the 2020 complex quadratic fields under investigation, a considerable fraction of 270 fields, that is 13.4%, has a second 3-class group G on the coclass tree $\mathcal{T}(\langle 243, 6 \rangle)$ of $\mathcal{G}(3, 2)$. However, the groups now populate two even branches $\mathcal{B}(j)$, $j \in \{6, 8\}$, of this tree.

TABLE 6.7. Principalization types with $G \in \mathcal{T}(\langle 243, 8 \rangle) \subset \mathcal{G}(3, 2)$ for $D > 0$

type	\varkappa	j	$h_3(L_1)$	$h_3(L_2)$	$\text{Cl}_3(N_1)$	$\text{Cl}_3(N_2)$	$\text{Cl}_3(N_3)$	$\text{Cl}_3(N_4)$	ε	$\text{Cl}_3(\mathbb{F}_3^1(K))$	min. D	freq.
c.21	(0231)	6	9	3	(9, 9)	(9, 3)	(9, 3)	(9, 3)	0	(9, 3, 3)	540 365	} 25
E.8	(1231)	6	9	3	(27, 9)	(9, 3)	(9, 3)	(9, 3)	0	(9, 9, 3)	6 098 360	
E.9	(2231)	6	9	3	(27, 9)	(9, 3)	(9, 3)	(9, 3)	0	(9, 9, 3)	342 664	} 14
G.16	(4231)	6	9	3	(27, 9)	(9, 3)	(9, 3)	(9, 3)	0	(27, 9, 3)	8 711 453	
c.21 \uparrow	(0231)	8	27	3	(27, 27)	(9, 3)	(9, 3)	(9, 3)	0	(27, 9, 3)	1 001 957	2
total:											43	

The second 3-class groups G for types \varkappa in Table 6.7 are vertices on the coclass tree with root $\langle 243, 8 \rangle \simeq G_0^{4,5}(0, 0, 0, 1)$ of $\mathcal{G}(3, 2)$. G is a vertex of depth 0 for type c.21, of depth 1 for types E.8, E.9, and of depth 2 for type G.16, on the even branches $\mathcal{B}(j)$, $j \in \{6, 8\}$, [3, Tbl. 2, p. 266], [4, Fig. 4.8, p. 76], [37, p. 189 ff.]. The types E.8, E.9 can only be separated by the classical principalization algorithm [41, 23]. Type G.16 can be identified either by the first Hilbert 3-class field or by the classical principalization algorithm. Among the 263 groups of even coclass, a fraction of 43 groups, that is 16.3%, populates this tree.

Table 6.8 is the complex analog of Table 6.7. Among the 2020 complex quadratic fields under investigation, a considerable fraction of 291 fields, that is 14.4%, has a second 3-class group G on the coclass tree $\mathcal{T}(\langle 243, 8 \rangle)$ of $\mathcal{G}(3, 2)$.

TABLE 6.8. Principalization types with $G \in \mathcal{T}((243, 8)) \subset \mathcal{G}(3, 2)$ for $D < 0$

type	\varkappa	j	$h(L_1)$	$h(L_2)$	$\text{Cl}_3(N_1)$	$\text{Cl}_3(N_2)$	$\text{Cl}_3(N_3)$	$\text{Cl}_3(N_4)$	ε	$\text{Cl}_3(\mathbb{F}_3^1(K))$	min. $ D $	freq.
E.8	(1231)	6	9	3	(27, 9)	(9, 3)	(9, 3)	(9, 3)	0	(9, 9, 3)	34 867	} 197
E.9	(2231)	6	9	3	(27, 9)	(9, 3)	(9, 3)	(9, 3)	0	(9, 9, 3)	9 748	
G.16	(4231)	6	9	3	(27, 9)	(9, 3)	(9, 3)	(9, 3)	0	(27, 9, 3)	17 131	
E.8 \uparrow	(1231)	8	27	3	(81, 27)	(9, 3)	(9, 3)	(9, 3)	0	(27, 27, 3)	370 740	} 13
E.9 \uparrow	(2231)	8	27	3	(81, 27)	(9, 3)	(9, 3)	(9, 3)	0	(27, 27, 3)	297 079	
G.16 \uparrow	(4231)	8	27	3	(81, 27)	(9, 3)	(9, 3)	(9, 3)	0	(81, 27, 3)	819 743	
total:											291	

TABLE 6.9. Principalization types with $G \in \mathcal{G}(3, 4)$ for $D > 0$

type	\varkappa	$h_3(L_1)$	$h_3(L_2)$	$\text{Cl}_3(N_1)$	$\text{Cl}_3(N_2)$	$\text{Cl}_3(N_3)$	$\text{Cl}_3(N_4)$	ε	$\text{Cl}_3(\mathbb{F}_3^1(K))$	min. D	fr.
F.13	(3143)	9	9	(27, 9)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(9, 9, 9, 3)	8 321 505	1
F.13 \uparrow	(3143)	27	9	(81, 27)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(27, 27, 9, 3)	8 127 208	1
d.25*	(0143)	27	9	(27, 27)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(27, 9, 9, 3)	8 491 713	1
total:											3

The extremely rare second 3-class groups G for types \varkappa in Table 6.9 are vertices on the coclass graph $\mathcal{G}(3, 4)$. The group G is either a sporadic vertex outside of coclass trees for type F.13, or a vertex on the even branch $\mathcal{B}(10)$ of one of the five metabelian coclass trees of $\mathcal{G}(3, 4)$, a vertex of depth 1 for type F.13 \uparrow , and a main line vertex of depth 0 for type d.25* [37, p. 189 ff.], [32, Thm. 3.4, p. 491]. The types F.13, F.13 \uparrow , and d.25* must be identified by the classical principalization algorithm [41, 23]. Among the 263 groups of even coclass for real quadratic fields, a very exotic fraction of only 3 groups, that is 1.1%, populates $\mathcal{G}(3, 4)$. The population sets in with remarkable delay at $D > 8 \cdot 10^6$.

In contrast, Table 6.10 shows that the coclass graph $G \in \mathcal{G}(3, 4)$ accommodates the second 3-class groups G of quite a notable portion of 131, that is 6.5%, among the 2020 complex quadratic fields under investigation. The group G is either a sporadic vertex outside of coclass trees, namely an isolated metabelian top vertex of $\mathcal{G}(3, 4)$ for the types F.7, F.11, F.12, F.13 and a terminal metabelian vertex of depth 1 on a finite tree for the types G.16r, G.19r, H.4r, G.16i, G.19i, H.4i, or a vertex on the even branch $\mathcal{B}(10)$ of one of the five metabelian coclass trees of $\mathcal{G}(3, 4)$, namely a vertex of depth 1 for the types F.7 \uparrow , F.11 \uparrow , F.12 \uparrow , F.13 \uparrow , and a vertex of depth 2 for the types G.19r \uparrow , H.4r \uparrow . We point out that the classical principalization algorithm [41, 23], which must be used to separate the types F.7, F.11, F.12, F.13, resp. G.16, G.19, H.4, is not able to reveal that the types G.16, G.19, and H.4 with odd index of nilpotency $m = 7$ and order 3^n , $n = 10 = 2m - 4$, can appear in a regular variant (r) with $\text{Cl}_3(\mathbb{F}_3^1(K))$ of type (27, 9, 9, 3) and

TABLE 6.10. Principalization types with $G \in \mathcal{G}(3, 4)$ for $D < 0$

type	\varkappa	$h(L_1)$	$h(L_2)$	$Cl_3(N_1)$	$Cl_3(N_2)$	$Cl_3(N_3)$	$Cl_3(N_4)$	ε	$Cl_3(F_3^1(K))$	$\min. D $	freq.
F.7	(3443)	9	9	(27, 9)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(9, 9, 9, 3)	124 363	} 78
F.11	(1143)	9	9	(27, 9)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(9, 9, 9, 3)	27 156	
F.12	(1343)	9	9	(27, 9)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(9, 9, 9, 3)	31 908	
F.13	(3143)	9	9	(27, 9)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(9, 9, 9, 3)	67 480	
F.7 \uparrow	(3443)	27	9	(81, 27)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(27, 27, 9, 3)	469 816	} 14
F.11 \uparrow	(1143)	27	9	(81, 27)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(27, 27, 9, 3)	469 787	
F.12 \uparrow	(1343)	27	9	(81, 27)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(27, 27, 9, 3)	249 371	
F.13 \uparrow	(3143)	27	9	(81, 27)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(27, 27, 9, 3)	159 208	
G.16r	(1243)	9	9	(27, 9)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(27, 9, 9, 3)	290 703	} 19
G.19r	(2143)	9	9	(27, 9)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(27, 9, 9, 3)	96 827	
H.4r	(3343)	9	9	(27, 9)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(27, 9, 9, 3)	256 935	
G.16i	(1243)	9	9	(27, 9)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(9, 9, 9, 9)	135 059	} 15
G.19i	(2143)	9	9	(27, 9)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(9, 9, 9, 9)	199 735	
H.4i	(3343)	9	9	(27, 9)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(9, 9, 9, 9)	186 483	
G.19r \uparrow	(2143)	27	9	(81, 27)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(81, 27, 9, 3)	509 160	2
H.4r \uparrow	(3343)	27	9	(81, 27)	(27, 9)	(3, 3, 3)	(3, 3, 3)	2	(81, 27, 9, 3)	678 804	3
total:											131

an irregular variant (i) with $Cl_3(F_3^1(K))$ of type (9, 9, 9, 9) [37, Satz 4.2.4, p. 131].

TABLE 6.11. Principalization type with $G \in \mathcal{G}(3, 6)$ for $D < 0$

type	\varkappa	$h(L_1)$	$h(L_2)$	$Cl_3(N_1)$	$Cl_3(N_2)$	$Cl_3(N_3)$	$Cl_3(N_4)$	ε	$Cl_3(F_3^1(K))$	$\min. D $	fr.
F.12 \uparrow^2	(1343)	27	27	(81, 27)	(81, 27)	(3, 3, 3)	(3, 3, 3)	2	(27, 27, 27, 9)	423 640	1
total:											1

Finally, in Table 6.11, type F.12 \uparrow^2 with a relative frequency of $\frac{1}{2020} = 0.05\%$ is the absolute low-champ for complex quadratic fields. The corresponding second 3-class group G is of the biggest order 3^{13} , known until now. It is the unique sporadic vertex outside of coclass trees which appeared on the coclass graph $\mathcal{G}(3, 6)$. Type F.12 \uparrow^2 must be identified by the classical principalization algorithm [41, 23].

Remark. Based on the statistical evaluation of all numerical results, we are able to conclude that the new principalization algorithm is significantly more efficient for real quadratic fields. The time consuming third step, § 5.3, of the algorithm can be avoided for $\frac{2548}{2576} = 98.9\%$ of the real quadratic fields (Tables 6.1, 6.2, 6.3 entirely and additionally the 29 cases of type c.18 in Table 6.5, the 27 cases of type c.21 in Table 6.7, and the single case of type d.25* in Table 6.9), but only for $\frac{1327}{2020} = 65.7\%$ of the complex quadratic fields (Table 6.4).

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8. Appendix

For the convenience of the reader, we literally cite some original results of Blackburn [9], and as a service to the mathematical community, we give a succinct survey of Nebelung's thesis [37, 38]. Both of these works are used essentially in our article.

8.1. A theorem by Blackburn. Let p be an arbitrary prime and observe that Blackburn denotes our two-step centralizer $\chi_2(G)$ of a p -group G (see § 3.1) by $\gamma_1(G)$, whereas his usage of $\gamma_i(G)$ for $i \geq 2$ coincides with ours, denoting the members of the lower central series. The following theorem shows that all subgroups $\gamma_i(G)$, $1 \leq i \leq m - p + 1$, of a p -group G of coclass $\text{cc}(G) = 1$ and class $\text{cl}(G) = m - 1 > p$ are regular and have the same invariants [21, pp. 78–81] as the nearly homocyclic abelian p -

group $A(p, m - i)$ of type $\left(\overbrace{p^{q_i+1}, \dots, p^{q_i+1}}^{r_i \text{ times}}, \overbrace{p^{q_i}, \dots, p^{q_i}}^{p-1-r_i \text{ times}} \right)$, where $m - i =$

$q_i(p - 1) + r_i$ by Euclidean division with quotient $q_i > 0$ and remainder $0 \leq r_i < p - 1$. In particular, if G is metabelian, then $\gamma_i(G) \simeq A(p, m - i)$, for $2 \leq i \leq m - p + 1$, and if G is metabelian with defect $k(G) = 0$, then also $\gamma_1(G) \simeq A(p, m - 1)$.

Theorem 8.1. (see [9, Thm. 3.4, p. 68]) *If G is a group of order p^m and class $m - 1$, where $m > 3$, then $\gamma_1(G)$ is a regular p -group. If $m > p + 1$, and for each $i = 1, 2, \dots, m - p + 1$, we write $m - i = (p - 1)q_i + r_i$ ($0 \leq r_i < p - 1$), then $\gamma_i(G)$ has r_i invariants equal to $q_i + 1$ and $p - r_i - 1$ invariants equal to q_i .*

8.2. Main theorems of Nebelung's thesis. Brigitte Nebelung completed her thesis [37, 38] in 1989 under supervision of Wolfram Jehne at Cologne. She had been introduced to computational group theory by

Charles R. Leedham-Green and Joachim Neubüser. Furthermore, she had studied the details concerning 3-groups of coclass 2 with two generators in Judith A. Ascione's thesis [4], written under supervision of Mike F. Newman.

Nebelung determined explicit parametrized presentations for all isomorphism classes of metabelian 3-groups with abelianization of type $(3, 3)$ and arbitrary coclass greater than or equal to 2, thereby extending Blackburn's results for coclass 1 in [9]. The lower and upper central series of all these groups were shown to have 3-elementary cyclic or bicyclic factors, the number of the latter being equal to the coclass. For all members of the lower central series, in particular for the abelian derived subgroup, the structure was given by abelian type invariants. Based on a computer calculation of a complete and irredundant set of isomorphism classes, listed in volume 2 [38] of her thesis, Nebelung proved that the metabelian coclass trees and their branches are arranged in periodic patterns. Finally, the number theoretic capitulation problem was solved for arbitrary number fields with 3-class group of type $(3, 3)$ by calculating the transfer kernels of any metabelian 3-group having abelianization of type $(3, 3)$ with respect to its maximal subgroups.

Throughout the sequel, let $4 \leq m \leq n$ be integers and G be a metabelian 3-group of order $|G| = 3^n$ and class $\text{cl}(G) = m - 1$ with derived quotient $G/G' \simeq (3, 3)$.

Nebelung's starting point for analyzing the structure of the lower and upper central series of G was the following inconspicuous result which proves to be very powerful. Observe that Nebelung denotes the members $\gamma_j(G)$ of the lower central series by G_j , for $j \geq 1$.

Theorem 8.2. (see [37, Satz 3.1.11, p. 57]) *There exist generators x, y of $G = \langle x, y \rangle$ such that $\gamma_3(G) = \langle x^3, y^3, \gamma_4(G) \rangle$.*

Among the generators of maximal subgroups $M_i < G$ modulo G' , x and y are indeed distinguished, in comparison to xy and xy^{-1} , provided the class of G is not too small. In the lattice of normal subgroups of G , the third powers x^3 and y^3 are lying near the top, namely in $\gamma_3(G) \setminus \gamma_4(G)$, whereas $(xy)^3$ and $(xy^{-1})^3$ are lying near the bottom, namely in the second centre $\zeta_2(G)$, and for groups with defect $k(G) = 0$ even in the centre $\zeta_1(G)$ of G , as we shall see in Theorem 8.7.

In the following results concerning the central series of a metabelian 3-group G with G/G' of type $(3, 3)$, Nebelung's usage of the invariants e and s coincides with ours in § 3.2, but she denotes our two-step centralizers $\chi_j(G)$ by C_j , for $j \geq 1$. We recall that the coclass of G in dependence on the invariant $e \geq 3$ is given by $\text{cc}(G) = e - 1 \geq 2$.

Theorem 8.3. (see [37, Satz 3.3.7, p. 70])

(1) *The single-step factors of the lower central series of G are given by*

$$\gamma_j(G)/\gamma_{j+1}(G) \simeq \begin{cases} (3, 3) & \text{for } j = 1 \text{ and } 3 \leq j \leq e, \\ (3) & \text{for } j = 2 \text{ and } e + 1 \leq j \leq m - 1. \end{cases}$$

(2) *Consequently, the exponent n of the order $|G| = 3^n$ is related to the index of nilpotency m by the inequalities $m \leq n \leq 2m - 3$ and the invariant e is given by $e = n + 2 - m$.*

(3) *The two-step factors $\gamma_j(G)/\gamma_{j+2}(G)$ of the lower central series of G are 3-elementary abelian, since $\gamma_j(G)^3 \leq \gamma_{j+2}(G)$, for $j \geq 1$.*

There are two extreme cases of the relations $4 \leq m \leq n \leq 2m - 3$. For the smallest possible exponent of the order, $n = m$, with given class $m - 1$, we have $e = 2$ and G is a CF-group of coclass $\text{cc}(G) = 1$ with *cyclic factors* $\gamma_j(G)/\gamma_{j+1}(G)$, except G/G' . For the biggest possible exponent of the order, $n = 2m - 3$, with given class $m - 1$, we have $e = m - 1$ and G is a BF-group of coclass $\text{cc}(G) = m - 2 \geq 2$ with *bicyclic factors* $\gamma_j(G)/\gamma_{j+1}$, except $G'/\gamma_3(G)$. BF-groups play the role of *interface* groups at the border between different coclass graphs in [35, § 3.3.5, Dfn. 3.3 and Thm. 3.11]. All groups, including the extreme cases, are BCF-groups with *bicyclic or cyclic factors*. (Nebelung uses the terminology ZEF-groups, which means *Rang Zwei oder Eins Faktoren*.)

From now on, we exclude groups of coclass $\text{cc}(G) = 1$, and we permanently rely on the following assumptions without repeating them explicitly. Let G be a metabelian 3-group with commutator factor group G/G' of type $(3, 3)$. Assume that G has order $|G| = 3^n$, class $\text{cl}(G) = m - 1$, and invariant $e = n + 2 - m \geq 3$, where $4 \leq m < n \leq 2m - 3$. Let generators x, y of $G = \langle x, y \rangle$ be selected such that $\gamma_3(G) = \langle x^3, y^3, \gamma_4(G) \rangle$, $x \in G \setminus \chi_s(G)$ if $s < m - 1$, and $y \in \chi_s(G) \setminus G'$. (Nebelung calls such a couple (x, y) an *admissible pair of normal generators*.) Let commutators of G be declared by $s_2 = t_2 = [y, x] \in \gamma_2(G)$ and recursively by $s_j = [s_{j-1}, x]$, $t_j = [t_{j-1}, y] \in \gamma_j(G)$ for $j \geq 3$. Starting with the powers $\sigma_3 = y^3$, $\tau_3 = x^3 \in \gamma_3(G)$, let $\sigma_j = [\sigma_{j-1}, x]$, $\tau_j = [\tau_{j-1}, y] \in \gamma_j(G)$ for $j \geq 4$.

Theorem 8.4. (see [37, Satz 3.3.7, p. 70])

(1) *The two-step centralizers of the lower central series of G are usually given by*

$$\chi_j(G) = \begin{cases} G' & \text{for } 1 \leq j \leq e - 1, \\ \langle y, G' \rangle & \text{for } e \leq j \leq m - 2, \\ G & \text{for } j \geq m - 1. \end{cases}$$

except in the special case that $e = m - 2$ and $[\chi_s(G), \gamma_e(G)] = \gamma_{m-1}(G)$, where

$$\chi_j(G) = \begin{cases} G' & \text{for } 1 \leq j \leq m - 2, \\ G & \text{for } j \geq m - 1. \end{cases}$$

- (2) For the invariants s and e it follows that usually $s = e$, except in the case $e = m - 2$, $[\chi_s(G), \gamma_e(G)] = \gamma_{m-1}(G)$, where $s = m - 1 > e$.

Let the upper central series of G be defined recursively by $\zeta_0(G) = 1$ and $\zeta_j(G)/\zeta_{j-1}(G) = \text{Centre}(G/\zeta_{j-1}(G))$, for $j \geq 1$. In particular, $\zeta_1(G)$ is the usual centre of G . Nebelung denotes $\zeta_j(G)$ by Z_j . The following theorem shows that the bicyclic factors of the upper central series of G are located near the bottom of G , whereas the bicyclic factors of the lower central series of G appear in a position near the top of G , by Theorem 8.3.

Theorem 8.5. (see [37, Satz 3.3.18, p. 82])

- (1) If $[\chi_s(G), \gamma_e(G)] = 1$, then the factors of the upper central series of G are given by

$$\zeta_j(G)/\zeta_{j-1}(G) \simeq \begin{cases} (3, 3) & \text{for } 1 \leq j \leq e - 2 \text{ and } j = m - 1, \\ (3) & \text{for } e - 1 \leq j \leq m - 2. \end{cases}$$

In particular, the centre $\zeta_1(G) = \langle \sigma_{m-1} \rangle \times \langle \tau_e \rangle$ of G is bicyclic of type $(3, 3)$.

- (2) If $[\chi_s(G), \gamma_e(G)] = \gamma_{m-1}(G)$, then the factors of the upper central series of G are given by

$$\zeta_j(G)/\zeta_{j-1}(G) \simeq \begin{cases} (3, 3) & \text{for } 2 \leq j \leq e - 1 \text{ and } j = m - 1, \\ (3) & \text{for } j = 1 \text{ and } e \leq j \leq m - 2. \end{cases}$$

In particular, the centre $\zeta_1(G) = \langle \sigma_{m-1} \rangle$ of G is cyclic of order 3.

The following main theorem gives parametrized presentations for all metabelian 3-groups G of coclass $\text{cc}(G) \geq 2$ having abelianization G/G' of type $(3, 3)$.

Theorem 8.6. (see [37, Thm. 3.4.5, p. 94]) There exists a unique matrix

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with entries in the finite field \mathbb{F}_3 , and a unique element $\rho \in \mathbb{F}_3$, such that the following relations are satisfied

$$\begin{aligned}
 (8.1) \quad & t_3^{-1} \tau_3 \tau_4 = \sigma_{m-1}^\alpha \sigma_{m-2}^{\rho\delta} \tau_e^\beta, \\
 & s_3 \sigma_3 \sigma_4 = \sigma_{m-1}^\gamma \sigma_{m-2}^{\rho\beta} \tau_e^\delta, \\
 & \tau_{e+1} = \sigma_{m-1}^{-\rho}, \\
 & [s_3, y] = [t_3, x] = \sigma_{m-1}^{-\rho\delta}, \\
 & s_2^3 (s_3 t_3)^3 s_4 t_4 = \sigma_{m-1}^{\rho\beta}, \\
 & s_2^3 = \sigma_4 \sigma_{m-1}^{-\rho\beta} \tau_4^{-1}.
 \end{aligned}$$

In the first two relations, put $\sigma_{m-2} = 1$ in the case of $m = 4$.

Concerning the parameter ρ , let $[\chi_s(G), \gamma_e(G)] = \gamma_{m-k}(G)$ with $0 \leq k \leq 1$. Then the defect of G is $k = k(G) = 0$ if and only if $\rho = 0$. In particular, a group G with $e = m - 1$ must have $\rho = 0$.

The parameters of a group G of coclass $cc(G) = 2$ are subject to various constraints:

- (1) If $m = 4, n = 5$, then $\rho = 0$ and $\begin{pmatrix} \alpha & \beta - 1 \\ \gamma - 1 & \delta \end{pmatrix} \in \text{GL}_2(\mathbb{F}_3)$.
- (2) If $m = 5, n = 6$, then $\begin{pmatrix} \rho\delta & \beta - 1 \\ \rho\beta - 1 & \delta \end{pmatrix} \in \text{GL}_2(\mathbb{F}_3)$.
- (3) If $5 < m = n - 1$, then $\beta - 1 \in \mathbb{F}_3 \setminus \{0\}$.

Consequently, a group G which satisfies the assumptions of the preceding theorem is exactly the representative $G_\rho^{m,n}(\alpha, \beta, \gamma, \delta)$ of an isomorphism class of metabelian 3-groups G with G/G' of type $(3, 3)$, which satisfies the relations (8.1) with a fixed system of parameters $-1 \leq \alpha, \beta, \gamma, \delta, \rho \leq 1$.

By means of a computer search for a complete and irredundant set of isomorphism classes of metabelian 3-groups G with G/G' of type $(3, 3)$, Nebelung successively determined all admissible families $(\alpha, \beta, \gamma, \delta, \rho)$ of parameters in the relations for groups of fixed coclass $e - 1$ and class $m - 1$, letting the invariant $e \geq 3$ increase independently and incrementing the index of nilpotency $m \geq e + 1$ in dependence on each fixed value of e . This revealed a *double periodicity* with respect to both, coclass and class, of primitive length 6 resp. 2. Consequently, it was possible to list the representatives $G_\rho^{m,n}(\alpha, \beta, \gamma, \delta)$ of isomorphism classes in volume 2 [38] of Nebelung's thesis in the form of finitely many parametrized presentations of infinite periodic coclass sequences where only m and $n = e + m - 2$ vary indefinitely for fixed coclass $e - 1$. Using the periodicity, Nebelung proved that the metabelian coclass trees and their branches are arranged in periodic patterns.

Theorem 8.7. (see [37, Lem. 3.4.11, p. 105]) *The third powers of xy and xy^{-1} are given by*

$$(xy)^3 = \sigma_{m-2}^{\rho(\beta+\delta)} \sigma_{m-1}^{\alpha+\gamma+\rho(\beta+\delta)} \tau_e^{\beta+\delta}$$

and

$$(xy^{-1})^3 = \sigma_{m-2}^{\rho(\delta-\beta)} \sigma_{m-1}^{\alpha-\gamma+\rho\beta} \tau_e^{\beta-\delta}.$$

In contrast to the powers $y^3 = \sigma_3$ and $x^3 = \tau_3$, which are elements of $\gamma_3(G) \setminus \gamma_4(G)$, the powers $(xy)^3$ and $(xy^{-1})^3$ are contained in the second centre $\langle \sigma_{m-1} \rangle \times \langle \sigma_{m-2} \rangle \times \langle \tau_e \rangle = \zeta_2(G)$ and for $\rho = 0$ even in the centre $\langle \sigma_{m-1} \rangle \times \langle \tau_e \rangle = \zeta_1(G)$. However, a uniform warranty that $(xy)^3, (xy^{-1})^3 \in \gamma_4(G)$ can only be given for a group G of coclass $\text{cc}(G) \geq 3$.

Theorem 8.8. (see [37, Satz 4.2.4, p. 131]) *The structure of the abelian commutator subgroup G' of G is given by the following direct product of nearly homocyclic abelian 3-groups.*

$$G' = \begin{cases} \text{A}(3, m-3) \times \text{A}(3, m-3) & \text{in the irregular case,} \\ \text{A}(3, m-2) \times \text{A}(3, e-2) & \text{otherwise.} \end{cases}$$

The irregular case is characterized by $n = 2m - 4$, $e = m - 2$, $m \equiv 1 \pmod{2}$, and

$$\begin{cases} 0 \neq \rho = \beta - 1 & \text{for } m = 5, \\ \rho = -1 & \text{for } m \geq 7. \end{cases}$$

9. Note added in proof

While this article was under review, the numerical results on second 3-class groups of complex quadratic fields $K = \mathbb{Q}(\sqrt{D})$ with 3-class rank 2 and discriminants $-10^6 < D < 0$ in our § 6 have been extended considerably to the range $-10^8 < D < 0$ in [BBH] N. Boston, M. R. Bush and F. Hajir, Heuristics for p -class towers of imaginary quadratic fields, to appear in Math. Ann. (see also arXiv: 1111.4679 v1 [math.NT] 20 Nov 2011). We are pleased that these authors essentially used the computational techniques described in our § 5, since the class field routines `AbelianExtension()` in MAGMA and `rnfkummer()` in PARI/GP became too time consuming in ranges outside of $-10^5 < D < 0$. As a partial result, our data concerning negative discriminants $-10^6 < D < 0$ in Tables 6.4 – 6.11 turned out to be in perfect accordance with the corresponding information in Tables 1 – 2 in Section 5 of [BBH], thus providing the first independent confirmation of our numerical discoveries. It should be pointed out that the authors of [BBH] use the terminology *IPAD* (index- p abelianization data) for our *TTT* (transfer target type). We also would like to emphasize that the joint experience of our present article and of the paper [BBH] established the first proof for the existence of p -class field towers having exactly 3 stages with an odd prime p in [BuMa] M. R. Bush and D. C. Mayer, 3-class field towers of exact length 3, to appear in J. Number Theory (see also arXiv: 1312.0251 v1 [math.NT] 01 Dec 2013).

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