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Diophantine approximation in Banach spaces

par Lior FISHMAN, David SIMMONS et Mariusz URBAŃSKI

Résumé. Dans cet article, nous étendons à la dimension infinie la théorie de l'approximation diophantienne simultanée. De plus, nous discutons des théorèmes de type Dirichlet dans un cadre très général et nous définissons ce que signifie étre optimal pour un tel théorème. Nous montrons que l'optimalité est impliquée par, mais n'implique pas, l'existence de points mal approchables.

Abstract. In this paper, we extend the theory of simultaneous Diophantine approximation to infinite dimensions. Moreover, we discuss Dirichlet-type theorems in a very general framework and define what it means for such a theorem to be optimal. We show that optimality is implied by but does not imply the existence of badly approximable points.

1. Introduction

Definition 1.1. A *Diophantine space* is a triple $(X, \mathcal{Q}, H)$, where X is a complete metric space, $\mathcal{Q}$ is a dense subset of X, and $H : \mathcal{Q} \to (0, +\infty)$.

The prototypical example is the triple $(\mathbb{R}^d, \mathbb{Q}^d, H_{\mathrm{std}})$, where H_{std} is the standard height function on $\mathbb{Q}^d$, i.e. $H_{\mathrm{std}}(\mathbf{p}/q) = q$ assuming that $\gcd(p_1, \ldots, p_d, q) = 1$. Other (mostly implicit) examples may be found in [2, 3, 4, 5, 6, 7, 10] and the references therein.

This paper has two goals. The first is to clarify the theory of Dirichlet-type theorems on an abstract Diophantine space. Until now, it seems that there is no generally accepted definition of what it means for a Dirichlet-type theorem to be optimal; in each case where a Dirichlet-type theorem is proved, its optimality is demonstrated by producing points which are badly approximable with respect to the approximation function of the Dirichlet-type theorem. However, in Section 2 we make a case for a wider notion of optimality, which is implied by but does not imply the existence of badly approximable points.

The second goal of this paper is to provide a complete theory of Diophantine approximation in the Diophantine space $(X, \mathbb{Q}\Lambda, H_{\mathrm{std}})$, where X is a Banach space, $\Lambda \leq X$ is a lattice, and H_{std} is the standard height function on $\mathbb{Q}\Lambda$ (precise definitions given below). This is related to the first goal since

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it turns out that when Λ is a non-cobounded lattice, the optimal Dirichlet function of $(X, \mathbb{Q}\Lambda, H_{\mathrm{std}})$ does not possess badly approximable points. Thus the theory of Diophantine approximation in Banach spaces gives a natural example of optimality failing to imply the existence of badly approximable points, justifying the clarification made in the first part.

Convention 1. In the introduction, propositions which are proven later in the paper will be numbered according to the section they are proven in. Propositions numbered as 1.# are either straightforward, proven in the introduction, or quoted from the literature.

Convention 2. $x_n \underset{n}{\to} x$ means $x_n \to x$ as $n \to +\infty$.

Convention 3. $\mathrm{HD}(S)$ is the Hausdorff dimension of a set S. $\mathcal{H}^f(S)$ is the Hausdorff f-measure of a set S.

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1.1. Dirichlet-type theorems on Diophantine spaces.

Definition 1.2. A *Dirichlet-type theorem* on a Diophantine space $(X, \mathcal{Q}, H)$ is a true statement of the form

$$\forall \mathbf{x} \in X \;\; \exists C_{\mathbf{x}} > 0 \;\; \exists (\mathbf{r}_n)_1^\infty \text{ in } \mathcal{Q} \text{ such that}$$
$$\begin{cases} \mathbf{r}_n \underset{n}{\to} \mathbf{x} \text{ and} \\ d(\mathbf{x}, \mathbf{r}_n) \leq C_{\mathbf{x}} \psi \circ H(\mathbf{r}_n) \;\; \forall n \in \mathbb{N} \end{cases},$$

where $\psi : (0, +\infty) \to (0, +\infty)$. The function ψ is called a *Dirichlet function.* If the constant $C_{\mathbf{x}}$ can be chosen to be independent of $\mathbf{x}$, then the function ψ is called *uniformly Dirichlet.*[1]

The prototypical example is Dirichlet's theorem, which states that for the Diophantine space $(\mathbb{R}^d, \mathbb{Q}^d, H_{\mathrm{std}})$, the function $\psi(q) = q^{-(1+1/d)}$ is uniformly Dirichlet (the constant $C_{\mathbf{x}}$ is 1 for every $\mathbf{x} \in \mathbb{R}^d$).

Dirichlet-type theorems are common in treatments of various Diophantine spaces; cf. the references given above. However, a Dirichlet-type theorem is usually not considered important unless it is *optimal*, or unable to be improved by more than a constant factor. The optimality of a Dirichlet function is usually established by demonstrating the existence of *badly approximable* points.

Definition 1.3. Let $(X, \mathcal{Q}, H)$ be a Diophantine space, and let $\psi : (0, +\infty) \to (0, +\infty)$. A point $\mathbf{x} \in X$ is said to be *badly approximable*

[1] Most known Dirichlet functions are uniformly Dirichlet; however, a Dirichlet function which is not uniformly Dirichlet is discussed in [4], Theorem 8.1.

with respect to ψ if

$$\exists \varepsilon > 0 \ \ \forall \mathbf{r} \in \mathcal{Q} \ \ d(\mathbf{r}, \mathbf{x}) \geq \varepsilon \psi \circ H(\mathbf{r}). \tag{1.1}$$

The set of points in X which are badly approximable with respect to ψ will be denoted BA_ψ, and its complement will be denoted WA_ψ.

The intuitive reason that the existence of badly approximable points implies optimality is that "if there were a Dirichlet-type theorem which improved the Dirichlet-type theorem corresponding to ψ by more than a constant, it would contradict the existence of badly approximable points". We can make this intuition into a theorem, specifically the following theorem:

Theorem 2.6. *Let $(X, \mathcal{Q}, H)$ be a Diophantine space. If $\psi : (0, +\infty) \to (0, +\infty)$ is any nonincreasing function and if $\mathrm{BA}_\psi \neq \varnothing$ and if $\phi : (0, +\infty) \to (0, +\infty)$ satisfies*

$$\frac{\phi}{\psi} \to 0, \tag{2.4}$$

then ϕ is not a Dirichlet function.

In this paper, we take the point of view that the conclusion of Theorem 2.6 rather than its hypothesis is the true definition of optimality of a Dirichlet function in a Diophantine space. In other words, a Dirichlet function $\psi : (0, +\infty) \to (0, +\infty)$ is *optimal* if there is no Dirichlet function $\phi : (0, +\infty) \to (0, +\infty)$ satisfying (2.4). The inequivalence of optimality and the existence of badly approximable points will be demonstrated in Theorem 3.5 below. However, their equivalence in the case where X is σ-compact will be demonstrated in Proposition 2.7.

One could also conceivably define a Dirichlet-type theorem to be optimal if it implies all other Dirichlet-type theorems. This notion will be made rigorous in Section 2; however, it turns out to be too strong, and even in $(\mathbb{R}, \mathbb{Q}, H_{\mathrm{std}})$ there are no Dirichlet functions which satisfy this strong notion of optimality. However, the notion can be refined by requiring that ψ and ϕ lie in a Hardy field (see §2.2); in this case, the notion turns out to be equivalent to the notion of optimality defined above.

1.2. The four main questions in Diophantine approximation. Given any Diophantine space $(X, \mathcal{Q}, H)$, we will be interested in the following questions:

1. (Dirichlet-type theorem) Find an optimal Dirichlet function for the Diophantine space. Is the set of badly approximable points for this Dirichlet function nonempty?
2. (Jarník–Schmidt type theorem) Given $\psi : (0, +\infty) \to (0, +\infty)$, what is the Hausdorff dimension of BA_ψ?

3. (Jarník–Besicovitch type theorem) Given $\psi : (0, +\infty) \to (0, +\infty)$, what is the Hausdorff dimension of WA_ψ?
4. (Khinchin-type theorem) Given $\psi : (0, +\infty) \to (0, +\infty)$, what are the measures of BA_ψ and WA_ψ?

Note that the last question assumes the existence of a natural measure on the space X.

We will usually be satisfied if questions 2-4 can be answered for functions ψ satisfying reasonable hypotheses, e.g. for ψ in a Hardy field (see §2.2).

Remark 1.4. One can also ask whether BA_ψ or WA_ψ is generic in a topological sense, i.e. comeager. However, the question is trivial, as shown by the following proposition:

Proposition 1.5. *Let $(X, \mathcal{Q}, H)$ be a Diophantine space. Then for any function $\psi : (0, +\infty) \to (0, +\infty)$, WA_ψ is comeager.*

Proof. By writing

$$\mathrm{WA}_\psi = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{r} \in \mathcal{Q}} B\left(\mathbf{r}, \frac{1}{n}\psi \circ H(\mathbf{r})\right),$$

we see that WA_ψ is the intersection of countably many open dense sets. □

Remark 1.6. An example of a Diophantine space with no (reasonable) optimal Dirichlet function is given in [6], Theorem 1.3. Even if an optimal Dirichlet function exists, we should not expect it to be unique without additional constraints; cf. Remark 2.11.

1.3. Diophantine approximation in Banach spaces.

Definition 1.7. Let X be a Banach space. A *lattice* in X is a subgroup $\Lambda \leq X$ such that

(I) Λ is (topologically) discrete, or equivalently,

$$\varepsilon_\Lambda := \min_{\mathbf{p} \in \Lambda \setminus \{\mathbf{0}\}} \|\mathbf{p}\| > 0, \text{ and}$$

(II) $\mathbb{R}\Lambda$ is dense in X, or equivalently, no proper closed subspace of X contains Λ.

If $\Lambda \leq X$ is a lattice, the *standard height function* $H_{\mathrm{std}} : \mathbb{Q}\Lambda \to X$ is the function

$$H_{\mathrm{std}}(\mathbf{r}) = \min\{q \in \mathbb{N} : q\mathbf{r} \in \Lambda\},$$

i.e. $H_{\mathrm{std}}(\mathbf{p}/q) = q$ if $\mathbf{p}/q$ is in reduced form.

Remark 1.8. Suppose that X is separable. Then for a closed subgroup $\Lambda \leq X$, the following are equivalent (see [1, Theorem 1.1]):

(A) Λ is discrete,

(B) Λ is locally compact and does not contain any one-dimensional subspace of X,
(C) Λ is countable,
(D) Λ is isomorphic to a (finite or infinite) direct sum of copies of $\mathbb{Z}$,
(E) Λ is a free abelian group.

Clearly, if $\Lambda \leq X$ is a lattice then $(X, \mathbb{Q}\Lambda, H_{\mathrm{std}})$ is a Diophantine space. If $X = \mathbb{R}^d$ and $\Lambda = \mathbb{Z}^d$, then this Diophantine space is just the usual space $(\mathbb{R}^d, \mathbb{Q}^d, H_{\mathrm{std}})$ studied in simultaneous Diophantine approximation. This example generalizes to infinite dimension in several different ways:

Example 1.9. Fix $1 \leq p < \infty$. Then

$$\mathbb{Z}^\infty := \{\mathbf{p} \in \mathbb{Z}^{\mathbb{N}} : p_i = 0 \text{ for all but finitely many } i \in \mathbb{N}\}$$

is a lattice in $\ell^p(\mathbb{N})$.

Remark 1.10. In Example 1.9, we are *not* approximating a point $\mathbf{x} \in \ell^p(\mathbb{N})$ by an arbitrary rational point $\mathbf{r} \in \mathbb{Q}^{\mathbb{N}} \cap \ell^p(\mathbb{N})$; rather, we are only approximating $\mathbf{x}$ by those rational points with only finitely many nonzero coordinates. The reason for this is that there is no appropriate analogue of the “LCM of the denominators” for a rational point with infinitely many nonzero coordinates.

Note that for $p = \infty$, $\mathbb{Z}^\infty$ is not a lattice in $\ell^\infty(\mathbb{N})$, since it is contained in $c_0(\mathbb{N})$, the set of all sequences in $\ell^\infty(\mathbb{N})$ which tend to zero, which is a proper closed subspace of $\ell^\infty(\mathbb{N})$. To get an example in $\ell^\infty(\mathbb{N})$, we have two options: shrink the space or expand the lattice.

Example 1.11. $\mathbb{Z}^\infty$ is a lattice in $c_0(\mathbb{N})$.

Example 1.12. $\mathbb{Z}^{\mathbb{N}}$ is a lattice in $\ell^\infty(\mathbb{N})$.

We remark that although the space $\ell^\infty(\mathbb{N})$ is not separable, this does not cause any additional complications in our arguments, which apply equally well to separable and non-separable Banach spaces.

It turns out that the theory of Diophantine approximation in $(X, \mathbb{Q}\Lambda, H_{\mathrm{std}})$ depends on one crucial dichotomy: whether or not the lattice Λ is cobounded. A lattice $\Lambda \leq X$ is *cobounded* if its *codiameter*

$$\mathrm{codiam}(\Lambda) := \sup\{d(\mathbf{x}, \Lambda) : \mathbf{x} \in X\}$$

is finite. In the above, Examples 1.11 and 1.12 are cobounded, whereas Example 1.9 is not cobounded.

1.3.1. ***Prevalence.*** It is not clear what measure would be natural on an infinite-dimensional Banach space. In [9], B. R. Hunt, T. D. Sauer, and J. A. Yorke argued that asking for a measure is too much, and one should be satisfied with being able to give a good definition of “full measure” and "measure zero". They introduced the notions of *shy* and *prevalent* subsets of a Banach space:

Definition 1.13. Let X be a Banach space. A measure μ is *transverse* to a set $S \subseteq X$ if $\mu(S + \mathbf{v}) = 0$ for all $\mathbf{v} \in X$. S is said to be *shy* if it is transverse to some compactly supported probability measure, and *prevalent* if its complement is shy.

If X is finite-dimensional, then a set is shy if and only if it has Lebesgue measure zero; it is prevalent if and only if its complement has Lebesgue measure zero. Moreover, the set of shy sets form a σ-ideal (i.e. the countable union of shy sets is shy, and any set contained in a shy set is shy). These facts together with several others (see [9]) give support to the idea that "shy" is the appropriate analogue of "measure zero" in infinite dimensions and that "prevalent" is the appropriate analogue of "full measure".

In the sequel we will need the following proposition:

Proposition 1.14. *Non-shy sets (and in particular prevalent sets) have full Hausdorff dimension.*

Proof. Since the proposition is obvious if $\dim(X) < +\infty$, assume that $\dim(X) = +\infty$. Let $S \subseteq X$ be a non-shy set. Fix $n \in \mathbb{N}$, let $X_n \subseteq X$ be an n-dimensional subspace, and let μ_n be Lebesgue measure on the unit ball of X_n. Since S is not shy, there exists $\mathbf{v} \in X$ such that $\mu_n(S + \mathbf{v}) > 0$. Since μ_n gives measure zero to any set of Hausdorff dimension strictly less than n, we have $\mathrm{HD}(S) = \mathrm{HD}(S + \mathbf{v}) \geq n$. Since n was arbitrary, $\mathrm{HD}(S) = +\infty$. ☐

1.4. Main theorems. We now present the theory of Diophantine approximation in the space $(X, \mathbb{Q}\Lambda, H_{\mathrm{std}})$, where X is a Banach space and $\Lambda \leq X$ is a lattice. The theory breaks down into three major cases: finite-dimensional, infinite-dimensional cobounded, and infinite-dimensional non-cobounded. (In finite dimensions, every lattice is cobounded.)

Notation 1.15. For $s \geq 0$, let

$$\psi_s(q) = q^{-s}.$$

1.4.1. ***Finite-dimensional case.*** Assume that Λ is a lattice in a d-dimensional Banach space X, with $d < +\infty$. Then there exists a linear isomorphism $T : \mathbb{R}^d \to X$ such that $T[\mathbb{Z}^d] = \Lambda$. This demonstrates that the classical results quoted below hold for any lattice in any finite-dimensional Banach space, not just for $\mathbb{Z}^d \leq \mathbb{R}^d$.

Theorem 1.16 (Dirichlet 1842 ($d \in \mathbb{N}$); optimality by Liouville 1844 ($d = 1$), Perron 1921 ($d \in \mathbb{N}$)). *For every* $\mathbf{x} \in X$ *and* $Q \in \mathbb{N}$*, there exists* $\mathbf{p} \in \Lambda$ *and* $q \leq Q$ *such that*

$$\left\| \mathbf{x} - \frac{\mathbf{p}}{q} \right\| \leq \frac{C}{qQ^{1/d}},$$

where $C > 0$ *is independent of* $\mathbf{x}$*. In particular, the function* $\psi_{1+1/d}$ *is uniformly Dirichlet, and in fact,* $\psi_{1+1/d}$ *is optimal.*

Theorem 1.17 (Jarník 1928 ($d = 1$), Schmidt 1969 ($d \in \mathbb{N}$)). *We have* $\mathrm{HD}(\mathrm{BA}_{\psi_{1+1/d}}) = d$.

Theorem 1.18 (Jarník 1929 ($d = 1$), Jarník 1931 ($d \in \mathbb{N}$), Besicovitch 1934 ($d = 1$)). *For all* $s \geq 1 + 1/d$*, we have* $\mathrm{HD}(\mathrm{WA}_{\psi_s}) = (d+1)/s$.

Theorem 1.19 (Khinchin 1924 ($d = 1$), Khinchin 1926 ($d \in \mathbb{N}$), Groshev 1938 ($d \in \mathbb{N}$)). *If* $q \mapsto q^d \psi(q)$ *is nonincreasing, then* WA_ψ *is of full Lebesgue measure if the series* $\sum_{q=1}^{\infty} q^d \psi(q)$ *diverges; if the series converges, then* WA_ψ *is of Lebesgue measure zero.*

1.4.2. ***Infinite-dimensional non-cobounded case.*** Assume that Λ is a non-cobounded lattice in an infinite-dimensional Banach space X.

Theorem 3.5 (Dirichlet-type theorem). *The function* $\psi_0 \equiv 1$ *is an optimal uniformly Dirichlet function. However,* $\mathrm{BA}_{\psi_0} = \varnothing$.

Theorem 3.1 (Khinchin-type theorem, Jarník–Schmidt type theorem). *For any function* $\psi \to 0$, BA_ψ *is prevalent. In particular,* $\mathrm{HD}(\mathrm{BA}_\psi) = +\infty$.

To state the Jarník–Besicovitch type theorem in the non-cobounded case, we introduce the notion of *strong discreteness.*

Definition 1.20. A lattice $\Lambda \leq X$ is *strongly discrete* if

$$\#(\Lambda \cap B(\mathbf{0}, C)) < +\infty \;\; \forall C > 0.$$

All three of the examples given in §1.3 are not strongly discrete.

Theorem 3.6 (Jarník–Besicovitch type theorem).

(i) *For any* $s \geq 0$*, we have* $\mathrm{HD}(\mathrm{WA}_{\psi_s}) = +\infty$.
(ii) *Suppose that* Λ *is not strongly discrete. Then for any nonincreasing function* $\psi \to 0$, $\mathrm{HD}(\mathrm{WA}_\psi) = +\infty$*. In fact, for any nondecreasing function* $f : (0, +\infty) \to (0, +\infty)$, $\mathcal{H}^f(\mathrm{WA}_\psi) = +\infty$.

1.4.3. ***Infinite-dimensional cobounded case.*** Assume that Λ is a cobounded lattice in an infinite-dimensional Banach space X.

Theorem 4.5 (Dirichlet-type theorem). *Fix $\varepsilon > 0$. For every $\mathbf{x} \in X$ and for every $q \in \mathbb{N}$, there exists $\mathbf{p} \in \Lambda$ such that*

$$\left\|\mathbf{x} - \frac{\mathbf{p}}{q}\right\| \leq \frac{\operatorname{codiam}(\Lambda) + \varepsilon}{q}.$$

In particular, the function $\psi_1(q) = 1/q$ is uniformly Dirichlet, and in fact, ψ_1 is optimal.

Theorem 4.6 (Jarník–Besicovitch type theorem). *For any nonincreasing function $\psi \to 0$, $\mathrm{HD}(\mathrm{WA}_\psi) = +\infty$. In fact, for any nondecreasing function $f : (0, +\infty) \to (0, +\infty)$, $\mathcal{H}^f(\mathrm{WA}_\psi) = +\infty$.*

Theorem 4.1 (Khinchin-type theorem, Jarník–Schmidt type theorem). *The set BA_{ψ_1} is prevalent. In particular, $\mathrm{HD}(\mathrm{BA}_{\psi_1}) = +\infty$.*

Remark 1.21. Based on the finite-dimensional case, it is natural to expect that $\psi_1(q) = 1/q$ is an optimal Dirichlet function in the infinite-dimensional case, as it is the limit of the optimal Dirichlet functions $\psi_{1+1/d}$ of the finite-dimensional cases. However, according to the theorems above this is only true if the lattice is cobounded, whereas if the lattice is not cobounded then $\psi_0 \equiv 1$ is the optimal Dirichlet function. A possible explanation for this can be found in the fact that in $\mathbb{R}^d$, the function $\psi_{1+1/d}$ is uniformly Dirichlet with the constant $C_d = 1$ if $\mathbb{R}^d$ is equipped with the ℓ^∞ norm; this suggests that if the ℓ^∞ norm is used, then there can be stability as $d \to \infty$. If an ℓ^p norm is used with $1 \leq p < \infty$, then the constant C_d will degenerate as $d \to \infty$, and the limit function will no longer be Dirichlet. (To look at in another way, in order to "take the limit of Dirichlet's theorem" o ne would need to take the limit of the functions $C_d\psi_{1+1/d}$ as $d \to \infty$, and if $C_d \underset{d}{\to} \infty$ fast enough, then this sequence does not converge.)

2. Optimal Dirichlet functions

In this section we discuss and motivate the notion of an optimal Dirichlet function introduced in §1.1. We begin with the following observation:

Observation 2.1. Let $(X, \mathcal{Q}, H)$ be a Diophantine space. Suppose that $\psi \leq C\phi$, with ψ Dirichlet. Then ϕ is Dirichlet.

Based on this observation, one is tempted to say that a Dirichlet function ψ is optimal if it is maximal in the partial order on Dirichlet functions defined by

$$\psi \succ \phi \quad \Leftrightarrow \quad \exists C > 0 \;\; \psi \leq C\phi,$$

or equivalently, if every other Dirichlet function ϕ can be proved to be Dirichlet as a result of applying Observation 2.1.

Definition 2.2. A Dirichlet function ψ is *strongly optimal* if $\psi \succ \phi$ for every Dirichlet function ϕ.

This definition makes rigorous the idea that a Dirichlet-type theorem is optimal if it "implies all other Dirichlet-type theorems (via Observation 2.1)". However, the definition is too strong even for the most canonical Diophantine space $(\mathbb{R}, \mathbb{Q}, H_{\mathrm{std}})$. Indeed, we have the following:

Proposition 2.3. *There is no strongly optimal Dirichlet function on* $(\mathbb{R}, \mathbb{Q}, H_{\mathrm{std}})$*. In particular, the Dirichlet function* ψ_2 *is not strongly optimal on* $(\mathbb{R}, \mathbb{Q}, H_{\mathrm{std}})$*.*

Proof.

Lemma 2.4. *For any sequence* $\mathbf{Q} = (Q_n)_1^\infty$ *increasing to infinity, the function*

$$\psi_{\mathbf{Q}}(q) = \frac{1}{qQ(q)}, \tag{2.1}$$

where

$$Q(q) = \min\{Q_n : Q_n \geq q\}$$

is uniformly Dirichlet for $(\mathbb{R}, \mathbb{Q}, H_{\mathrm{std}})$*.*

Proof. Fix $x \in \mathbb{R}$ and let $C = 1$. By Theorem 1.16, for each $n \in \mathbb{N}$ there exists $r_n = p_n/q_n \in \mathbb{Q}$ with $q_n \leq Q_n$ such that

$$|x - r_n| \leq \frac{1}{q_n Q_n}. \tag{2.2}$$

Since $Q_n \geq q_n$, we have

$$Q(q_n) \leq Q_n$$

and thus

$$|x - r_n| \leq \frac{1}{q_n Q(q_n)} = \psi(q_n). \tag{2.3}$$

Since $Q_n \underset{n}{\to} +\infty$, (2.2) implies that $r_n \underset{n}{\to} x$. Thus the function (2.1) is uniformly Dirichlet.

$\triangleleft$

To complete the proof, we will find two sequences $\mathbf{Q}_0 = (Q_n^{(0)})_1^\infty$ and $\mathbf{Q}_1 = (Q_n^{(1)})_1^\infty$ such that the minimum of the two functions $\psi_{\mathbf{Q}_0}$ and $\psi_{\mathbf{Q}_1}$ is not a Dirichlet function. We choose the sequences

$$Q_n^{(i)} = 2^{2^{2n+i}}, \quad i = 1, 2$$

and leave it to the reader to verify that the function $\phi = \min(\psi_{\mathbf{Q}_0}, \psi_{\mathbf{Q}_1})$ satisfies

$$\phi \leq \psi_3.$$

(It suffices to check the inequality for the worst-case scenario $q \in \mathbf{Q}_0 \cup \mathbf{Q}_1$.) Now suppose that ψ is an optimal Dirichlet function for $(\mathbb{R}, \mathbb{Q}, H_{\mathrm{std}})$. Then since $\psi_{\mathbf{Q}_0}$ and $\psi_{\mathbf{Q}_1}$ are Dirichlet, we have $\psi \leq C\psi_{\mathbf{Q}_i}$ for some $C > 0$. Thus

$\psi \leq C\phi \leq C\psi_3$, so by Observation 2.1, ψ_3 is Dirichlet. This contradicts the optimality of ψ_2, since $\frac{\psi_3}{\psi_2} \to 0$. □

Having ruled out strong optimality as a notion of optimality, we turn to the weaker notion of optimality given in §1.1. We repeat it here for convenience:

Definition 2.5. A Dirichlet function ψ is *optimal* (with respect to a Diophantine space $(X, \mathcal{Q}, H)$) if there is no Dirichlet function ϕ satisfying

$$\frac{\phi}{\psi} \to 0. \tag{2.4}$$

How do we know that this is the "correct" definition? We give two reasons:

1. In the case of a σ-compact Diophantine space, for example a finite-dimensional Banach space, our new definition agrees with the more classical criterion of the existence of badly approximable points. Even in the non σ-compact case, the existence of badly approximable points implies optimality.
2. The notion of optimality agrees with the notion of strong optimality if the class of functions is restricted to a suitable class of "non-pathological" functions.

We now proceed to elaborate on each of these reasons.

2.1. Optimality versus BA. Traditionally, the existence of badly approximable points has been thought to demonstrate that Dirichlet's function is optimal (up to a constant). In our terminology, this intuition becomes a theorem:

Theorem 2.6 (Existence of BA implies optimality)**.** *Let* $(X, \mathcal{Q}, H)$ *be a Diophantine space. If* $\psi : (0, +\infty) \to (0, +\infty)$ *is any nonincreasing function and if* $\mathrm{BA}_\psi \neq \varnothing$ *and if* $\phi : (0, +\infty) \to (0, +\infty)$ *satisfies* (2.4)*, then* ϕ *is not a Dirichlet function.*

Proof. Fix $\mathbf{x} \in \mathrm{BA}_\psi$. If ϕ is Dirichlet, then there exist $C_\mathbf{x} > 0$ and a sequence $(\mathbf{r}_n)_1^\infty$ such that

$$d(\mathbf{r}_n, \mathbf{x}) \leq C_\mathbf{x}\phi(q_n) \text{ and } \mathbf{r}_n \underset{n}{\to} \mathbf{x},$$

where $q_n := H(\mathbf{r}_n)$. Combining with (1.1) gives

$$\varepsilon\psi(q_n) \leq C_\mathbf{x}\phi(q_n);$$

rearranging yields

$$\frac{\phi(q_n)}{\psi(q_n)} \geq \frac{\varepsilon}{C_\mathbf{x}} > 0. \tag{2.5}$$

On the other hand, we have

$$\varepsilon\psi(q_n) \leq d(\mathbf{r}_n, \mathbf{x}) \underset{n}{\to} 0;$$

since ψ is positive and nonincreasing this implies that $q_n \underset{n}{\to} +\infty$. Together with (2.5), this contradicts (2.4). $\square$

The converse to Theorem 2.6 does not hold in such generality (cf. Theorem 3.5), but rather holds only under the hypothesis that the underlying Diophantine space is σ-compact.

Proposition 2.7 (Optimality implies existence of BA)**.** *Let* $(X, \mathcal{Q}, H)$ *be a* σ*-compact Diophantine space. Then if* ψ *is a bounded optimal Dirichlet function, then* $\mathrm{BA}_\psi \neq \varnothing$*.*

Proof. Let $(K_n)_1^\infty$ be an increasing sequence of compact sets whose union is X.

Suppose by contradiction that $\mathrm{BA}_\psi = \varnothing$. Then for each $n \in \mathbb{N}$ and for each $\mathbf{x} \in X$, there exists $\mathbf{r} \in \mathcal{Q}$ such that

$$d(\mathbf{r}, \mathbf{x}) < \frac{1}{n}\psi \circ H(\mathbf{r}).$$

Let $U_{\mathbf{r},n}$ be the set of all $\mathbf{x}$ satisfying the above; then for each $n \in \mathbb{N}$, $(U_{\mathbf{r},n})_{\mathbf{r}}$ is an open cover of X, and in particular an open cover of K_n. Let $(U_{\mathbf{r},n})_{\mathbf{r}\in F_n}$ be a finite subcover, and let $Q_n = \max_{F_n}(H)$. Let

$$\phi(q) = \psi(q)\max\{1/n : q \leq Q_n\}.$$

Clearly $\phi(q)/\psi(q) \underset{q}{\to} 0$. We claim that ϕ is a Dirichlet function. Indeed, fix $\mathbf{x} \in X$, and let $C_{\mathbf{x}} = 1$. For all $n \in \mathbb{N}$ sufficiently large, we have $\mathbf{x} \in K_n$. Fix such an n, and choose $\mathbf{r}_n \in F_n$ so that $\mathbf{x} \in U_{\mathbf{r}_n,n}$. Then $q_n := H(\mathbf{r}_n) \leq Q_n$. It follows that

$$\phi(q_n) \geq \frac{1}{n}\psi(q_n) > d(\mathbf{r}_n, \mathbf{x}).$$

Since ψ is bounded, this implies that $\mathbf{r}_n \underset{n}{\to} \mathbf{x}$. Thus $\mathbf{x}$ is ϕ-approximable. Thus ϕ is a Dirichlet function, and so ψ is not an optimal Dirichlet function. $\square$

2.2. Hardy fields. One possible reaction to the phenomenon of Lemma 2.4 is to insist that the functions $\psi_{\mathbf{Q}}$ defined in that lemma are pathological. One way to make this rigorous is to consider the notion of a *Hardy field.*

Definition 2.8. A *germ at infinity* is an equivalence class of C^∞ functions from $[0, +\infty)$ to $\mathbb{R}$, where two functions are considered equivalent if they agree on all sufficiently large values.

A *Hardy field* is a field of germs at infinity which is closed under differentiation.

Remark 2.9. If $\psi \not\equiv 0$ is an element of a Hardy field, then by definition, there is a C^∞ function from $[0,+\infty)$ to $\mathbb{R}$ which agrees with $1/\psi$ on all sufficiently large values. This implies that $\psi \neq 0$ on all sufficiently large values; since ψ is continuous, either $\psi > 0$ or $\psi < 0$ on all sufficiently large values.

A standard example of a Hardy field is the class of *Hardy L-functions*, which is the class all functions which can be written using the symbols $+, -, \times, \div$, exp and log together with the constants and the identity function; cf. [8, Chapter III]. From now on, we will consider functions to be "non-pathological" if their germs at infinity are elements of some fixed Hardy field.

Observation 2.10. If the germs of ψ and ϕ are elements of the same Hardy field, then either $\psi \prec \phi$ or $\phi \prec \psi$. Moreover, $\psi \not\succ \phi$ if and only if $\frac{\phi}{\psi} \to 0$.

Proof. Both assertions follow from the well-known fact that $\lim_{q\to\infty} \frac{\phi}{\psi}(q)$ exists. $\square$

The second part of Observation 2.10 can be taken as a motivation for Definition 2.5. Indeed, it shows that if a Hardy field is fixed and all functions are assumed to be elements of that Hardy field, then the notions of strong optimality and optimality agree.

Remark 2.11. The first part of Observation 2.10 shows that if a Hardy field is fixed and all functions are assumed to be elements of that Hardy field, then any two optimal Dirichlet functions ϕ and ψ "agree up to a constant", i.e. their ratio $\frac{\phi}{\psi}$ is bounded from above and below. If the restriction to a Hardy field is not made, then Lemma 2.4 can be used to show that there are uncountably many optimal Dirichlet functions on $(\mathbb{R}, \mathbb{Q}, H_{\mathrm{std}})$, no two of which are comparable. (The Dirichlet function $\psi_{\mathbf{Q}}$ is optimal because $\psi_{\mathbf{Q}} \leq \psi_2$.)

Remark 2.12. Restricting to elements of a Hardy field is also useful in answering questions 2-4 of §1.2. To see this, note that the map $\psi \mapsto \mathrm{BA}_\psi$ is order-preserving, i.e. $\psi \prec \phi$ implies $\mathrm{BA}_\psi \subseteq \mathrm{BA}_\phi$. Similarly, the map $\psi \mapsto \mathrm{WA}_\psi$ is order-reversing. Since in a Hardy field, $\prec$ is a total order (Observation 2.10), it is possible to prescribe the values of $\mathrm{HD}(\mathrm{BA}_\psi)$ and $\mathrm{HD}(\mathrm{WA}_\psi)$ on all ψ in a Hardy field by prescribing the values of $\mathrm{HD}(\mathrm{BA}_\psi)$ and $\mathrm{HD}(\mathrm{WA}_\psi)$ for a relatively small collection of ψs. Using this principle, in the case of Banach spaces it is possible to answer questions 2-4 completely (except for question 3 in the case of strongly discrete lattices) based on the information given in §1.4. Details are left to the reader.

3. Infinite-dimensional non-cobounded case

In this section, we assume that Λ is a non-cobounded lattice in an infinite-dimensional Banach space X, and we consider the Diophantine space $(X, \mathbb{Q}\Lambda, H_{\mathrm{std}})$. We begin by proving the following:

Theorem 3.1 (Khinchin-type theorem, Jarník–Schmidt type theorem)**.** *For any function $\psi \to 0$, BA_ψ is prevalent. In particular, $\mathrm{HD}(\mathrm{BA}_\psi) = +\infty$.*

Proof. We will need the following lemma:

Lemma 3.2. *For any $0 < \varepsilon < R < +\infty$, there exists $\mathbf{w} \in X$ so that $\|\mathbf{w}\| = R$ and*

$$d(\mathbf{w}, \Lambda) \geq R - \varepsilon.$$

Proof. Since Λ is not cobounded, there exists $\mathbf{x} \in X$ such that $S := d(\mathbf{x}, \Lambda) \geq R$. By the definition of distance, there exists $\mathbf{p} \in \Lambda$ such that

$$S \leq \|\mathbf{x} - \mathbf{p}\| \leq S + \varepsilon. \tag{3.1}$$

Let

$$\mathbf{w} = R\frac{\mathbf{x} - \mathbf{p}}{\|\mathbf{x} - \mathbf{p}\|}.$$

Clearly $\|\mathbf{w}\| = R$. On the other hand,

$$\begin{aligned} d(\mathbf{w}, \Lambda) &\geq d(\mathbf{x} - \mathbf{p}, \Lambda) - d(\mathbf{w}, \mathbf{x} - \mathbf{p}) \\ &= S - \|\mathbf{x} - \mathbf{p}\| \left|1 - \frac{R}{\|\mathbf{x} - \mathbf{p}\|}\right| \\ &= S - \big|\|\mathbf{x} - \mathbf{p}\| - R\big| \\ &\geq S - |S - R| - \varepsilon && \text{(by (3.1))} \\ &= R - \varepsilon. && \text{(since } S \geq R\text{)} \end{aligned}$$

$\triangleleft$

Let $(\rho_n)_1^\infty$ be the unique sequence satisfying $\rho_1 = 1$ and

$$\rho_{n+1} = \frac{\rho_n}{2^{n+5}}. \tag{3.2}$$

For each $n \in \mathbb{N}$, let $N_n \in \mathbb{N}$ be large enough so that

$$\psi(q) \leq \frac{\rho_{n+1}}{8} \quad \forall q \geq N_n. \tag{3.3}$$

Let

$$M_n = 2^n N_n!$$

By Lemma 3.2, there exists $\mathbf{w}_n \in X$ be such that $\|\mathbf{w}_n\| = M_n$ and

$$d(\mathbf{w}_n, \Lambda) \geq M_n - \rho_n/4. \tag{3.4}$$

Let $\mathbf{v}_n = \frac{\rho_n}{M_n}\mathbf{w}_n$, so that $\|\mathbf{v}_n\| = \rho_n$.

Claim 3.3. *For each $n \in \mathbb{N}$ and for each $\mathbf{x} \in X$,*

$$\#\left\{i = 0, \ldots, 2^n - 1 : B(\mathbf{x} + i\mathbf{v}_n, \rho_n/4) \cap \frac{\Lambda}{N_n!} \neq \varnothing\right\} \leq 1.$$

Proof. By contradiction, suppose there exist $0 \leq i_1 < i_2 < 2^n$ and $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that

$$\mathbf{x}_j \in B(\mathbf{x} + i_j\mathbf{v}_n, \rho_n/4) \cap \frac{\Lambda}{N_n!}, \quad j = 1, 2.$$

Thus

$$\begin{aligned}\frac{\rho_n}{2} &\geq \|(\mathbf{x}_2 - \mathbf{x} - i_2\mathbf{v}_n) - (\mathbf{x}_1 - \mathbf{x} - i_1\mathbf{v}_n)\| \\ &= \|(\mathbf{x}_2 - \mathbf{x}_1) - (i_2 - i_1)\mathbf{v}_n\| \\ &\geq d\left((i_2 - i_1)\mathbf{v}_n, \frac{\Lambda}{N_n!}\right) \\ &= \frac{1}{N_n!} d((i_2 - i_1)N_n!\mathbf{v}_n, \Lambda) \\ &= \frac{1}{N_n!} d\left((i_2 - i_1)N_n!\frac{\rho_n}{M_n}\mathbf{w}_n, \Lambda\right).\end{aligned}$$

Now

$$(i_2 - i_1)N_n!\frac{\rho_n}{M_n} = (i_2 - i_1)\frac{\rho_n}{2^n} \leq \rho_n \leq 1,$$

and so by the triangle inequality

$$\begin{aligned}N_n!\frac{\rho_n}{2} &\geq d\left((i_2 - i_1)N_n!\frac{\rho_n}{M_n}\mathbf{w}_n, \Lambda\right) \\ &\geq d(\mathbf{w}_n, \Lambda) - \left\|\mathbf{w}_n - (i_2 - i_1)N_n!\frac{\rho_n}{M_n}\mathbf{w}_n\right\| \\ &\geq (M_n - \rho_n/4) - (M_n - (i_2 - i_1)N_n!\rho_n) && \text{(by (3.4))} \\ &\geq (N_n! - 1/4)\rho_n, && \text{(since } i_2 - i_1 \geq 1)\end{aligned}$$

a contradiction. ◁

For each $n \in \mathbb{N}$, let

$$\mu_n = \frac{1}{2^n}\sum_{i=0}^{2^n-1} \delta_{i\mathbf{v}_n};$$

then μ_n is a compactly supported probability measure on $B(\mathbf{0}, 2^n\rho_n)$. Define

$$\Sigma : \prod_{n=1}^{\infty} B(\mathbf{0}, 2^n\rho_n) \to X$$

by

$$\Sigma((\mathbf{x}_n)_1^\infty) = \sum_{n=1}^{\infty} \mathbf{x}_n, \tag{3.5}$$

and let $\mu = \Sigma[\prod_{n=1}^\infty \mu_n]$. Note that if K_n is the support of μ_n, then μ gives full measure to $\Sigma(\prod_{n=1}^\infty K_n)$, which is compact, so μ is compactly supported.

To complete the proof, we will show that μ is transverse to WA_ψ. To this end, fix $\mathbf{v} \in X$, and we will show that $\mu(\mathrm{WA}_\psi + \mathbf{v}) = 0$.

Fix $n \in \mathbb{N}$; for each sequence $(\mathbf{x}_j)_1^{n-1}$, applying Claim 3.3 with $\mathbf{x} = \sum_{j=1}^{n-1} \mathbf{x}_j - \mathbf{v}$ shows that

$$\mu_n \left\{ \mathbf{x}_n : B\left(\sum_{j=0}^n \mathbf{x}_j - \mathbf{v}, \rho_n/4 \right) \cap \frac{\Lambda}{N_n!} \neq \varnothing \right\} \leq \frac{1}{2^n},$$

and Fubini's theorem gives

$$\left(\prod_{j=1}^\infty \mu_j \right) \left\{ (\mathbf{x}_j)_1^\infty : B\left(\sum_{j=0}^n \mathbf{x}_j - \mathbf{v}, \rho_n/4 \right) \cap \frac{\Lambda}{N_n!} \neq \varnothing \right\} \leq \frac{1}{2^n}.$$

Thus by the easy direction of the Borel–Cantelli lemma, the set

$$N = \left\{ (\mathbf{x}_j)_1^\infty : \exists^\infty n \in \mathbb{N} \;\; B\left(\sum_{j=0}^n \mathbf{x}_j - \mathbf{v}, \rho_n/4 \right) \cap \frac{\Lambda}{N_n!} \neq \varnothing \right\}$$

is a $\prod_{j=1}^\infty \mu_j$-nullset. So to complete the proof, it suffices to show the following:

Claim 3.4. $\Sigma^{-1}(\mathrm{WA}_\psi + \mathbf{v}) \subseteq N$, *i.e. if a sequence* $(\mathbf{x}_n)_1^\infty \in \prod_{n=1}^\infty B(\mathbf{0}, 2^n \rho_n)$ *satisfies*

$$\mathbf{x} = \sum_{n=0}^\infty \mathbf{x}_n \in \mathrm{WA}_\psi + \mathbf{v},$$

then there exist infinitely many $n \in \mathbb{N}$ *for which*

$$B\left(\sum_{j=0}^n \mathbf{x}_j - \mathbf{v}, \rho_n/4 \right) \cap \frac{\Lambda}{N_n!} \neq \varnothing. \tag{3.6}$$

Proof. Let $(\mathbf{r}_k)_1^\infty$ be a sequence of rational points whose limit is $\mathbf{x} - \mathbf{v}$ and which satisfy

$$\|\mathbf{x} - \mathbf{v} - \mathbf{r}_k\| \leq \psi(q_k),$$

where $q_k = H_{\mathrm{std}}(\mathbf{r}_k)$.

Fix $k \in \mathbb{N}$, and let $n = n_k$ be minimal so that $N_n \geq q_k$. Then $N_{n-1} < q_k$, and so by (3.3),

$$\psi(q_k) \leq \frac{\rho_n}{8}.$$

On the other hand, by (3.2),

$$\left\| \sum_{j=n+1}^{\infty} \mathbf{x}_j \right\| \leq \sum_{j=n+1}^{\infty} 2^j \rho_j \leq \frac{\rho_n}{8}.$$

Combining the three preceding equations gives

$$\left\| \sum_{j=0}^{n} \mathbf{x}_j - \mathbf{v} - \mathbf{r}_k \right\| \leq \frac{\rho_n}{4},$$

i.e.

$$\mathbf{r}_k \in B\left(\sum_{j=0}^{n} \mathbf{x}_j - \mathbf{v}, \rho_n/4\right) \cap \frac{\Lambda}{N_n!}.$$

Since the sequence $(n_k)_1^\infty$ is clearly unbounded, this demonstrates that (3.6) holds for infinitely many n. $\triangleleft$

Thus μ is transverse to WA_ψ and so BA_ψ is prevalent; thus $\mathrm{HD}(\mathrm{BA}_\psi) = +\infty$ by Proposition 1.14. $\square$

Next, we deduce Theorem 3.5 as a corollary of Theorem 3.1.

Theorem 3.5 (Dirichlet-type theorem)**.** *The function $\psi_0 \equiv 1$ is an optimal uniformly Dirichlet function. However, $\mathrm{BA}_{\psi_0} = \varnothing$.*

Proof. The fact that $\psi \equiv 1$ is a Dirichlet function which has no badly approximable points is true of every Diophantine space, simply because $\mathcal{Q}$ is dense in X. It remains to show optimality. Let $\phi : [0, +\infty) \to (0, +\infty)$ be a function such that $\frac{\phi}{\psi} = \phi \to 0$. Then $\sqrt{\phi} \to 0$, and so by Theorem 3.1, the set $\mathrm{BA}_{\sqrt{\phi}}$ is prevalent and in particular nonempty. Then by Theorem 2.6, the function ϕ cannot be a Dirichlet function, since $\frac{\phi}{\sqrt{\phi}} \to 0$. $\square$

Finally, we prove the infinite-dimensional version of the Jarník–Besicovitch theorem.

Theorem 3.6 (Jarník–Besicovitch type theorem)**.**

(i) *For any $s \geq 0$, we have $\mathrm{HD}(\mathrm{WA}_{\psi_s}) = +\infty$.*

(ii) *Suppose that Λ is not strongly discrete. Then for any nonincreasing function $\psi \to 0$, $\mathrm{HD}(\mathrm{WA}_\psi) = +\infty$. In fact, for any nondecreasing function $f : (0, +\infty) \to (0, +\infty)$, $\mathcal{H}^f(\mathrm{WA}_\psi) = +\infty$.*

Remark 3.7. In this theorem, the hypothesis that Λ is not cobounded is not used; cf. Theorem 4.6.

Proof of Theorem 3.6.

(i) For each $d \in \mathbb{N}$, let $X_d \subseteq X$ be a subspace of dimension d such that $X_d \cap \Lambda$ is a lattice in X_d. Then by Theorem 1.18,

$$\mathrm{HD}(\mathrm{WA}_{\psi_s}(X_d, \mathbb{Q}\Lambda \cap X_d, H_{\mathrm{std}})) = \frac{d+1}{s}\cdot$$

But clearly $\mathrm{WA}_{\psi_s}(X, \mathbb{Q}\Lambda, H_{\mathrm{std}}) \supseteq \mathrm{WA}_{\psi_s}(X_d, \mathbb{Q}\Lambda \cap X_d, H_{\mathrm{std}})$, whence

$$\mathrm{HD}(\mathrm{WA}_{\psi_s}(X, \mathbb{Q}\Lambda, H_{\mathrm{std}})) \geq \frac{d+1}{s} \underset{d}{\to} +\infty.$$

(ii) Let $\psi \to 0$ be a nonincreasing function, and let $f : (0, +\infty) \to (0, +\infty)$ be a nondecreasing function. Let $\varepsilon_\Lambda = \min_{\mathbf{p} \in \Lambda \setminus \{\mathbf{0}\}} \|\mathbf{p}\| > 0$, and let $C_\Lambda > 0$ be larve enough so that $\#(\Lambda \cap B(\mathbf{0}, C_\Lambda)) = +\infty$. Choose a sequence $(q_n)_1^\infty$ by induction as follows: Let $q_0 = 1$, and if q_n has been chosen, let $q_{n+1} \in q_n\mathbb{N} \setminus \{q_n\}$ be large enough so that $2C_\Lambda/q_{n+1} \leq \min(\psi(q_n)/n, \varepsilon_\Lambda/(3q_n))$. Let $S = \Lambda \cap B(\mathbf{0}, C_\Lambda)$, and define $\pi : S^{\mathbb{N}} \to X$ by

$$\pi((\mathbf{p}_n)_1^\infty) = \sum_{n \in \mathbb{N}} \frac{\mathbf{p}_n}{q_n}\cdot$$

Claim 3.8. $\pi(S^{\mathbb{N}}) \subseteq \mathrm{WA}_\psi$.

Proof. Fix $(\mathbf{p}_n)_1^\infty \in S^{\mathbb{N}}$, and let $\mathbf{x} = \pi((\mathbf{p}_n)_1^\infty) = \sum_{n \in \mathbb{N}} \mathbf{p}_n/q_n$. Then for each $N \in \mathbb{N}$

$$\left\|\mathbf{x} - \sum_{n \leq N} \frac{\mathbf{p}_n}{q_n}\right\| \leq \sum_{n > N} \frac{\|\mathbf{p}_n\|}{q_n} \leq C_\Lambda \sum_{n > N} \frac{1}{q_n} \leq \frac{2C_\Lambda}{q_{n+1}} \leq \psi(q_n)/n.$$

On the other hand, since $q_1 \mid q_2 \mid \cdots \mid q_N$, we have

$$H_{\mathrm{std}}\left(\sum_{n \leq N} \frac{\mathbf{p}_n}{q_n}\right) \leq q_N,$$

and so since ψ is nonincreasing, we have

$$\left\|\mathbf{x} - \sum_{n \leq N} \frac{\mathbf{p}_n}{q_n}\right\| \leq \frac{1}{n}\psi \circ H_{\mathrm{std}}\left(\sum_{n \leq N} \frac{\mathbf{p}_n}{q_n}\right)$$

and thus $\mathbf{x} \in \mathrm{WA}_\psi$. $\triangleleft$

Claim 3.9. *If $\mathcal{C}$ is any collection of subsets of X of diameter less than $\varepsilon_\Lambda/3$ which covers $\pi(S^{\mathbb{N}})$, then $\mathcal{C}$ contains an infinite collection of sets whose diameters are bounded from below.*

Proof. By contradiction, suppose not; then for each n, the set

$$\mathcal{C}_n := \{A \in \mathcal{C} : \varepsilon_\Lambda/(3q_{n+1}) \leq \mathrm{diam}(A) < \varepsilon_\Lambda/(3q_n)\}$$

is finite. We now choose a sequence $(\mathbf{p}_n)_1^\infty$ in $S^{\mathbb{N}}$ by induction. If $\mathbf{p}_1, \ldots, \mathbf{p}_{N-1}$ have been chosen, then for each $\mathbf{p} \in S$ let

$$C_{N,\mathbf{p}} = \sum_{n<N} \frac{\mathbf{p}_n}{q_n} + \frac{\mathbf{p}}{q_N} + B(\mathbf{0}, \varepsilon_\Lambda/(3q_N)).$$

The sets $(C_{N,\mathbf{p}})_{\mathbf{p}\in S}$ are disjoint; in fact, the distance between $C_{N,\mathbf{p}}$ and $C_{N,\widetilde{\mathbf{p}}}$ for $\mathbf{p} \neq \widetilde{\mathbf{p}}$ is always at least $\varepsilon_\Lambda/(3q_N)$. Thus each $A \in \mathcal{C}_N$ can intersect at most one of the sets $C_{N,\mathbf{p}}$, so since $\#(S) = +\infty$ there exists $\mathbf{p}_N \in S$ such that $C_{N,\mathbf{p}_N}$ is disjoint from $\bigcup(\mathcal{C}_N)$. This completes the inductive step.

Calculation (based on the inequality $2C_\Lambda/q_{N+1} \leq \varepsilon_\Lambda/(3q_N)$) shows that the point $\mathbf{x} = \pi((\mathbf{p}_n)_1^\infty)$ is in each of the sets $C_{N,\mathbf{p}_N}$, and so it is not in any of the sets $\bigcup(\mathcal{C}_N)$. This contradicts that $\mathcal{C}$ covers $\pi(S^{\mathbb{N}})$. ◁

Now if $f : (0, +\infty) \to (0, +\infty)$ is nondecreasing, then the equation $\mathcal{H}^f(\pi(S^{\mathbb{N}})) = +\infty$ is evident from the claim. Finally, setting $f(t) = t^s$ with s arbitrary shows that $\mathrm{HD}(\mathrm{WA}_\psi) = +\infty$. □

4. Infinite-dimensional cobounded case

In this section, we assume that Λ is a cobounded lattice in an infinite-dimensional Banach space X, and we consider the Diophantine space $(X, \mathbb{Q}\Lambda, H_{\mathrm{std}})$. We begin by proving the following:

Theorem 4.1 (Khinchin-type theorem, Jarník–Schmidt type theorem)**.** *The set* BA_{ψ_1} *is prevalent. In particular,* $\mathrm{HD}(\mathrm{BA}_{\psi_1}) = +\infty$*.*

The proof will follow the same lines as the proof of Theorem 3.1, but with some modifications.

Proof of Theorem 4.1.

Lemma 4.2. *There exists a sequence of unit vectors* $(\mathbf{e}_i)_1^\infty$ *satisfying*

$$\|\mathbf{e}_j - \mathbf{e}_i\| \geq 1 \text{ whenever } i \neq j. \tag{4.1}$$

Proof. We construct the sequence $(\mathbf{e}_i)_1^\infty$ by induction. Suppose that $(\mathbf{e}_i)_1^{n-1}$ have been defined, and let $V = \sum_{i=1}^{n-1} \mathbb{R}\mathbf{e}_i$. Let $\mathbf{w}$ be a unit vector in X/V, and let $\mathbf{e}_n \in X$ be a unit vector representing $\mathbf{w}$. Then for all $i < n$,

$$\|\mathbf{e}_n - \mathbf{e}_i\| \geq d(\mathbf{e}_n, V) = \|\mathbf{w}\| = 1.$$

This demonstrates (4.1). ◁

Let $\varepsilon_\Lambda = \min_{\mathbf{p}\in\Lambda} \|\mathbf{p}\| > 0$, let $\lambda = 16$, and for each $n \in \mathbb{N}$ and $i = 1, \ldots, \lambda^{2n}$ let

$$\mathbf{v}_{i,n} = \frac{\varepsilon_\Lambda \mathbf{e}_i}{4\lambda^n}.$$

Claim 4.3. *For any point* $\mathbf{x} \in X$,

$$\#\left\{ i = 1, \ldots, \lambda^{2n} : B\left(\mathbf{x} + \mathbf{v}_{i,n}, \frac{\varepsilon_\Lambda}{16\lambda^n}\right) \cap \left(\bigcup_{q=1}^{\lambda^n} \frac{\Lambda}{q} \right) \neq \varnothing \right\} \leq \lambda^n. \tag{4.2}$$

Proof. Fix $q = 1, \ldots, \lambda^n$, and by contradiction suppose there exist $1 \leq i_1 < i_2 \leq \lambda^{2n}$ such that $B\left(\mathbf{x} + \mathbf{v}_{i,n}, \frac{\varepsilon_\Lambda}{16\lambda^n}\right) \cap \frac{\Lambda}{q} \neq \varnothing$. Then there exist $\mathbf{p}_1, \mathbf{p}_2 \in \Lambda$ so that

$$\left\| \mathbf{v}_{i_j,n} + \mathbf{x} - \frac{\mathbf{p}_j}{q} \right\| \leq \frac{\varepsilon_\Lambda}{16\lambda^n},$$

which implies that

$$\|\mathbf{v}_{i_1,n} - \mathbf{v}_{i_2,n}\| - \frac{\varepsilon_\Lambda}{8\lambda^n} \leq \frac{1}{q}\|\mathbf{p}_1 - \mathbf{p}_2\| \leq \|\mathbf{v}_{i_1,n} - \mathbf{v}_{i_2,n}\| + \frac{\varepsilon_\Lambda}{8\lambda^n}.$$

By (4.1) we have

$$\frac{\varepsilon_\Lambda}{4\lambda^n} \leq \|\mathbf{v}_{i_1,n} - \mathbf{v}_{i_2,n}\| \leq \frac{\varepsilon_\Lambda}{2\lambda^n},$$

and so

$$\frac{\varepsilon_\Lambda}{8\lambda^n} \leq \frac{1}{q}\|\mathbf{p}_1 - \mathbf{p}_2\| \leq \frac{5\varepsilon_\Lambda}{8\lambda^n}.$$

The lower bound implies that $\mathbf{p}_1 - \mathbf{p}_2 \neq \mathbf{0}$, so since $\mathbf{p}_1 - \mathbf{p}_2 \in \Lambda$ we have $\|\mathbf{p}_1 - \mathbf{p}_2\| \geq \varepsilon_\Lambda$; thus

$$\frac{5\varepsilon_\Lambda}{8\lambda^n} \geq \frac{\varepsilon_\Lambda}{q} \geq \frac{\varepsilon_\Lambda}{\lambda^n},$$

a contradiction. Thus

$$\#\left\{ i = 1, \ldots, \lambda^{2n} : B\left(\mathbf{v}_{i,n} + \mathbf{x}, \frac{\varepsilon_\Lambda}{16\lambda^n}\right) \cap \frac{\Lambda}{q} \neq \varnothing \right\} \leq 1,$$

and summing over $q = 1, \ldots, \lambda^n$ yields (4.2). ◁

At this point, the proof follows much the same structure as the proof of Theorem 3.1. For each $n \in \mathbb{N}$, let

$$\mu_n = \frac{1}{\lambda^{2n}} \sum_{i=1}^{\lambda^{2n}} \delta_{\mathbf{v}_{i,n}};$$

then μ_n is a compactly supported probability measure on $B(\mathbf{0}, \varepsilon_\Lambda/(4\lambda^n))$. Let

$$\Sigma : \prod_{n=1}^{\infty} B(\mathbf{0}, \varepsilon_\Lambda/(4\lambda^n)) \to X$$

be defined by (3.5), and let $\mu = \Sigma\left[\prod_{n=1}^{\infty} \mu_n\right]$. As in the proof of Theorem 3.1, μ is compactly supported; fix $\mathbf{v} \in X$, and we will show that $\mu(\mathrm{WA}_\psi + \mathbf{v}) = 0$.

Fix $n \in \mathbb{N}$; for each sequence $(\mathbf{x}_j)_1^{n-1}$, applying Claim 3.3 with $\mathbf{x} = \sum_{j=1}^{n-1} \mathbf{x}_j - \mathbf{v}$ shows that

$$\mu_n \left\{ \mathbf{x}_n : B\left(\sum_{j=0}^{n} \mathbf{x}_j - \mathbf{v}, \varepsilon_\Lambda/16\lambda^n\right) \cap \left(\bigcup_{q=1}^{\lambda^n} \frac{\Lambda}{q}\right) \neq \varnothing \right\} \leq \frac{1}{\lambda^n},$$

and Fubini and Borel–Cantelli imply that

$$N = \left\{ (\mathbf{x}_j)_1^\infty : \exists^\infty n \in \mathbb{N} \;\; B\left(\sum_{j=0}^{n} \mathbf{x}_j - \mathbf{v}, \varepsilon_\Lambda/16\lambda^n\right) \cap \left(\bigcup_{q=1}^{\lambda^n} \frac{\Lambda}{q}\right) \neq \varnothing \right\}$$

is a $\prod_{j=1}^\infty \mu_j$-nullset. So to complete the proof, it suffices to show the following:

Claim 4.4. $\Sigma^{-1}(\mathrm{WA}_\psi + \mathbf{v}) \subseteq N$, *i.e. if a sequence* $(\mathbf{x}_n)_1^\infty \in \prod_{n=1}^\infty B(\mathbf{0}, \varepsilon_\Lambda/(4\lambda^n))$ *satisfies*

$$\mathbf{x} = \sum_{n=0}^{\infty} \mathbf{x}_n \in \mathrm{WA}_\psi + \mathbf{v},$$

then there exist infinitely many $n \in \mathbb{N}$ *for which*

$$B\left(\sum_{j=0}^{n} \mathbf{x}_j - \mathbf{v}, \frac{\varepsilon_\Lambda}{16\lambda^n}\right) \cap \left(\bigcup_{q=1}^{\lambda^n} \frac{\Lambda}{q}\right) \neq \varnothing.$$

Proof. Let $(\mathbf{r}_k)_1^\infty$ be a sequence of rational points whose limit is $\mathbf{x} - \mathbf{v}$ and which satisfy

$$\|\mathbf{x} - \mathbf{v} - \mathbf{r}_k\| \leq \psi(q_k)/k,$$

where $q_k = H_{\mathrm{std}}(\mathbf{r}_k)$.

Fix $k \in \mathbb{N}$, and let $n = n_k$ be minimal so that $\lambda^n \geq q_k$. Then $\lambda^{n-1} < q_k$, and so,

$$\psi(q_k) = \frac{1}{q_k} \leq \frac{1}{\lambda^{n-1} k}.$$

On the other hand, by (3.2),

$$\left\| \sum_{j=n+1}^{\infty} \mathbf{x}_j \right\| \leq \sum_{j=n+1}^{\infty} \frac{\varepsilon_\Lambda}{4\lambda^j} \leq \frac{\varepsilon_\Lambda}{2\lambda^{n+1}}.$$

Combining the three preceding equations gives

$$\left\| \sum_{j=0}^{n} \mathbf{x}_j - \mathbf{v} - \mathbf{r}_k \right\| \leq \frac{1}{\lambda^{n+1}} \max(\varepsilon_\Lambda, \lambda^2/k),$$

and if $k \geq \lambda^2/\varepsilon_\Lambda$, then

$$\left\| \sum_{j=0}^{n} \mathbf{x}_j - \mathbf{v} - \mathbf{r}_k \right\| \leq \frac{1}{\lambda^{n+1}} \varepsilon_\Lambda = \frac{\varepsilon_\Lambda}{16\lambda}.$$

i.e.

$$\mathbf{r}_k \in B\left(\sum_{j=0}^{n} \mathbf{x}_j - \mathbf{v}, \frac{\varepsilon_\Lambda}{16\lambda^n}\right) \cap \left(\bigcup_{q=1}^{\lambda^n} \frac{\Lambda}{q}\right).$$

Since the sequence $(n_k)_1^\infty$ is clearly unbounded, this demonstrates that (3.6) holds for infinitely many n. $\triangleleft$

$\square$

Next, we deduce Theorem 4.5 as a consequence of Theorem 4.1.

Theorem 4.5 (Dirichlet-type theorem)**.** *Fix $\varepsilon > 0$. For every $\mathbf{x} \in X$ and for every $q \in \mathbb{N}$, there exists $\mathbf{p} \in \Lambda$ such that*

$$\left\|\mathbf{x} - \frac{\mathbf{p}}{q}\right\| \leq \frac{\operatorname{codiam}(\Lambda) + \varepsilon}{q}. \tag{4.3}$$

In particular, the function $\psi_1(q) = 1/q$ is uniformly Dirichlet, and in fact, ψ_1 is optimal.

Proof. Note that

$$d(\mathbf{x}, \Lambda/q) = \frac{1}{q} d(q\mathbf{x}, \Lambda) \leq \frac{\operatorname{codiam}(\Lambda)}{q}.$$

Thus for every $\varepsilon > 0$, there exists $\mathbf{p}/q = \mathbf{p}_q/q \in \Lambda/q$ such that (4.3) holds. Clearly $\mathbf{p}_q/q \underset{q}{\to} \mathbf{x}$, which demonstrates that ψ_1 is uniformly Dirichlet (with $C = \operatorname{codiam}(\Lambda) + \varepsilon$). Finally, Theorem 4.1 implies that ψ_1 is optimal. $\square$

We conclude by deducing Theorem 4.6 as a corollary of Theorem 3.6.

Theorem 4.6 (Jarník–Besicovitch type theorem)**.** *For any nonincreasing function $\psi \to 0$, $\mathrm{HD}(\mathrm{WA}_\psi) = +\infty$. In fact, for any nondecreasing function $f : (0, +\infty) \to (0, +\infty)$, $\mathcal{H}^f(\mathrm{WA}_\psi) = +\infty$.*

Proof. Since the proof of Theorem 3.6 did not assume that Λ was not cobounded, to complete the proof it suffices to show that every infinite-dimensional cobounded lattice is not strongly discrete. Suppose that Λ is a cobounded lattice, and let $C = 5\operatorname{codiam}(\Lambda)$. Letting $(\mathbf{e}_i)_1^\infty$ be a sequence of unit vectors with the property (4.1), we see that the collection $(B(3C\mathbf{e}_i/4, C/4))_{i=1}^\infty$ is a disjoint collection of subsets of $B(\mathbf{0}, C)$, and thus if Λ is strongly discrete then there exists $i \in \mathbb{N}$ such that $\Lambda \cap B(3C\mathbf{e}_i/4, C/4) = \varnothing$, which implies that $\operatorname{codiam}(\Lambda) \geq C/4$, a contradiction. $\square$

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