Journal de Théorie des Nombres de Bordeaux

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 25, n$^{\text{o}}$ 3 (2013), p. 565-578.

<http://jtnb.cedram.org/item?id=JTNB_2013__25_3_565_0>

Journal de Théorie des Nombres
de Bordeaux **25** (2013), 565–578

The factorization of $f(x)x^n + g(x)$ with $f(x)$ monic and of degree ≤ 2.

par Joshua HARRINGTON, Andrew VINCENT et Daniel WHITE

Résumé. Dans cet article, nous étudions la factorisation des polynômes $f(x)x^n + g(x) \in \mathbb{Z}[x]$ dans le cas particulier où $f(x)$ est un polynôme quadratique unitaire avec discriminant négatif. Nous mentionnons également des résultats similaires dans le cas où $f(x)$ est unitaire et linéaire.

Abstract. In this paper we investigate the factorization of the polynomials $f(x)x^n + g(x) \in \mathbb{Z}[x]$ in the special case where $f(x)$ is a monic quadratic polynomial with negative discriminant. We also mention similar results in the case that $f(x)$ is monic and linear.

1. Introduction

Factorization of polynomials of the form $f(x)x^n + g(x)$, where $f(x)$ and $g(x)$ are fixed and n is large, has been considered by Schinzel in [6] and [7], and later by Filaseta, Ford, and Konyagin in [4]. In this paper we consider the special case $f(x) = x^2 + bx + c \in \mathbb{Z}[x]$ with $c \geq 2$ and $|b| < 2\sqrt{c-1}$. We additionally impose certain restrictions on $g(x)$. In particular we prove the following two main theorems.

Theorem 1. *Let $f(x) = x^2 + bx + c \in \mathbb{Z}[x]$ with $c \geq 2$ and $|b| < 2\sqrt{c-1}$. Let ϵ be such that $0 < \epsilon < \sqrt{c} - 1$ and $\epsilon \leq 1$. If $g(x) = \sum_{j=0}^{t} a_j x^j \in \mathbb{Z}[x]$ with*

$$1 + |b| + c + \sum_{j=1}^{t} |a_j| < |a_0| < 2(\sqrt{c} - \epsilon)^2,$$

then the polynomial $f(x)x^n + g(x)$ is irreducible for all

$$n > \max\left\{ t, \frac{t\log\left(\sqrt{c}+\epsilon\right) + \log\left|\frac{a_0}{\sqrt{4c-b^2}-\epsilon}\right| + \log\left(\frac{\min\{t+1,2\}}{\epsilon}\right)}{\log(\sqrt{c}-\epsilon)} \right\}.$$

Manuscrit reçu le 30 juillet 2012, révisé le 28 mai 2013.
Mots clefs. polynomials, trinomials, irreducible, factorization.
Classification math. 11C08, 12E05, 26C10.

For the second theorem we need the following definition.

Definition. We define the non-cyclotomic part of a non-zero polynomial $w(x) \in \mathbb{Z}[x]$ to be $w(x)$ with all of its cyclotomic factors removed. That is, $k(x)$ is the non-cyclotomic part of $w(x)$ if we can write $w(x) = h(x)k(x)$ where $h(x)$ is identically 1 or a product of cyclotomic polynomials and $k(x)$ has no cyclotomic factors.

Theorem 2. *Let $f(x) = x^2 + bx + c \in \mathbb{Z}[x]$ with $c \geq 2$ and $|b| < 2\sqrt{c-1}$. Let ϵ be such that $0 < \epsilon < \sqrt{c} - 1$ and $\epsilon \leq 1$. Let $g(x) = \sum_{j=0}^{t} a_j x^j \in \mathbb{Z}[x]$ with*

$$1 + |b| + c + \sum_{j=1}^{t} |a_j| = |a_0| < 2(\sqrt{c} - \epsilon)^2.$$

Then for any integer $n \geq t+1$, any cyclotomic factor of $P(x) = f(x)x^n + g(x)$ must be in $\{x+1, x-1\}$. Furthermore, if

$$n > \max\left\{ t, \frac{t\log(\sqrt{c}+\epsilon) + \log\left|\frac{a_0}{\sqrt{4c-b^2}-\epsilon}\right| + \log\left(\frac{\min\{t+1,2\}}{\epsilon}\right)}{\log(\sqrt{c}-\epsilon)} \right\},$$

then the non-cyclotomic part of the polynomial $P(x) = f(x)x^n + g(x)$ is irreducible.

We follow up each of these results with several corollaries. Similar results in the case that $f(x)$ is monic and linear are mentioned in the concluding remarks of the paper.

2. Three preliminary lemmas

Before proving Theorems 1 and 2 we first establish 3 lemmas. To establish the first lemma, we make use of the following simplified version of a classical result due to Rouché (see [3], p. 125).

Theorem 3 (Rouché's Theorem)**.** *Let $f(x)$ and $g(x)$ be polynomials in $\mathbb{C}[x]$. For any $\alpha \in \mathbb{C}$ and positive real number r, if $|g(z_0)| < |f(z_0)|$ for all $z_0 \in \{z \in \mathbb{C} : |z-\alpha| = r\}$, then $f(x)$ and $f(x)+g(x)$ have the same number of roots in $\{z \in \mathbb{C} : |z-\alpha| < r\}$.*

Lemma 1. *Let $f(x)$ and $g(x)$ be non-zero polynomials in $\mathbb{Z}[x]$ of degrees r and t respectively. Let a be the leading coefficient of $f(x)$, and let $\alpha_1, \ldots, \alpha_r$ be the roots of $f(x)$. Let $H(g)$ be the height of $g(x)$; in other words, $H(g)$ is the maximum of the absolute values of the coefficients of $g(x)$. Fix $\epsilon > 0$ and $j \in \{1, \ldots, r\}$. If $|\alpha_j| > 1 + \epsilon$ and $f(x)$ has no roots in the set $\{z \in \mathbb{C} : |z - \alpha_j| = \epsilon\}$, then the polynomial $P(x) = f(x)x^n + g(x)$ has at least one root in the set $\{z \in \mathbb{C} : |z - \alpha_j| < \epsilon\}$ for all*

$$n > \frac{t \log(|\alpha_j| + \epsilon) + \log\left(\dfrac{H(g)}{\prod\limits_{\substack{1 \le i \le r \\ i \ne j}} ||\alpha_j - \alpha_i| - \epsilon|}\right) + \log\left(\frac{t+1}{|a\epsilon|}\right)}{\log(|\alpha_j| - \epsilon)}.$$

Proof. Let $f(x)$ and $g(x)$ be as in the statement of the lemma and let $\epsilon > 0$. For $1 \le j \le r$, suppose that $f(x)$ has no roots in the set $\{z \in \mathbb{C} : |z - \alpha_j| = \epsilon\}$. Then for $0 \le \theta < 2\pi$,

$$\begin{aligned} \left|f(\alpha_j + \epsilon e^{i\theta})\right| &= \left|a \prod_{i=1}^{r} (\alpha_j + \epsilon e^{i\theta} - \alpha_i)\right| \\ &= |a\epsilon| \prod_{\substack{1 \le i \le r \\ i \ne j}} \left|\alpha_j - \alpha_i + \epsilon e^{i\theta}\right| \\ &\ge |a\epsilon| \prod_{\substack{1 \le i \le r \\ i \ne j}} ||\alpha_j - \alpha_i| - \epsilon| \,. \end{aligned}$$

We also know that for $0 \le \theta < 2\pi$,

$$\left|g(\alpha_j + \epsilon e^{i\theta})\right| \le (t+1)H(g)(|\alpha_j| + \epsilon)^t.$$

So let

$$n > \frac{t \log(|\alpha_j| + \epsilon) + \log\left(\dfrac{H(g)}{\prod\limits_{\substack{1 \le i \le r \\ i \ne j}} ||\alpha_j - \alpha_i| - \epsilon|}\right) + \log\left(\frac{t+1}{|a\epsilon|}\right)}{\log(|\alpha_j| - \epsilon)}.$$

Then the result follows from Rouché's Theorem since

$$|P(z_0) - f(z_0)z_0^n| = |g(z_0)| < |f(z_0)|\,|z_0|^n$$

for all $z_0 \in \{z \in \mathbb{C} : |z - \alpha_j| = \epsilon\}$. $\square$

Remark. If $g(x) = \sum_{k=0}^{t} a_k x^k$, then the theorems in this paper require $\sum_{k=1}^{t} |a_k| < |a_0|$. Notice then that under the assumptions of Lemma 1

$$\begin{aligned}
\left|g(\alpha_j + \epsilon e^{i\theta})\right| &\leq \sum_{k=0}^{t} |a_k| \left(|\alpha_j| + \epsilon\right)^k \\
&\leq \sum_{k=0}^{t} |a_k| \left(|\alpha_j| + \epsilon\right)^t \\
&= (|\alpha_j| + \epsilon)^t \left(H(g) + \sum_{k=1}^{t} |a_k|\right) \\
&< (|\alpha_j| + \epsilon)^t (H(g) + H(g)) \\
&= 2H(g)(|\alpha_j| + \epsilon)^t.
\end{aligned}$$

Thus, for the purposes of the theorems in this paper we may take

$$n > \frac{t\log(|\alpha_j| + \epsilon) + \log\left(\frac{H(g)}{\prod\limits_{\substack{1\leq i\leq r \\ i\neq j}} ||\alpha_j - \alpha_i| - \epsilon|}\right) + \log\left(\frac{\min\{t+1, 2\}}{|a\epsilon|}\right)}{\log(|\alpha_j| - \epsilon)}$$

in the statement of Lemma 1.

Lemma 2. *Let $f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{Q}[x]$ and suppose that $a_i a_j \neq 0$ for some $0 \leq i < j \leq n$. Suppose further that*

$$\sum_{\substack{0\leq k\leq n \\ k\neq t}} |a_k| \leq q^t \cdot |a_t|$$

for some $0 \leq t \leq n$ with $t \neq i$ and $t \neq j$ and $q \in \mathbb{R}$ with $0 < q \leq 1$. If $f(x)$ has a root α in the set $\{z \in \mathbb{C} : q \leq |z| \leq 1\}$, then it must be the case that

$$\sum_{\substack{0\leq k\leq n \\ k\neq t}} |a_k| = q^t \cdot |a_t|$$

and $|\alpha| = 1$ with $\alpha^{2(j-i)} - 1 = 0$.

Proof. Let $f(x)$ be as in the statement of the theorem. Suppose $\alpha \in \mathbb{C}$ is a root of $f(x)$ with $q \leq |\alpha| \leq 1$. Then

$$\begin{aligned}
q^t \cdot |a_t| &\leq \left|a_t\alpha^t\right| \\
&= \left|a_j\alpha^j + a_i\alpha^i + \sum_{\substack{0\leq k\leq n \\ k\neq i, k\neq j, k\neq t}} a_k\alpha^k\right| \\
&\leq \left|a_j\alpha^j + a_i\alpha^i\right| + \sum_{\substack{0\leq k\leq n \\ k\neq i, k\neq j, k\neq t}} \left|a_k\alpha^k\right| \\
&\leq \sum_{\substack{0\leq k\leq n \\ k\neq t}} \left|a_k\alpha^k\right| \\
&\leq \sum_{\substack{0\leq k\leq n \\ k\neq t}} |a_k| \\
&\leq q^t \cdot |a_t|\,.
\end{aligned}$$

Thus, we see immediately that

$$\sum_{\substack{0\leq k\leq n \\ k\neq t}} |a_k| = q^t \cdot |a_t|$$

and $|\alpha| = 1$. It also follows that $\alpha^{2(j-i)} - 1 = 0$ since

$$\left|a_j\alpha^{j-i} + a_i\right| = \left|a_j\alpha^j + a_i\alpha^i\right| = \left|a_j\alpha^j\right| + \left|a_i\alpha^i\right|.$$

This completes the proof. $\square$

Remark. The lemma implies that if $i \neq 0$, $j \neq 0$, and $t = 0$, then f has no roots in $\{z \in \mathbb{C} : 0 \leq |z| < 1\}$. To see this, let $0 < q < 1$ get arbitrarily close to 0. This shows that $f(x)$ has no roots in $\{z \in \mathbb{C} : 0 < |z| < 1\}$. Now by assumption, $a_i \neq 0$, $a_j \neq 0$, and $a_0 = a_t \geq |a_i| + |a_j|$. Thus, $f(0) \neq 0$.

Lemma 3. *Let $f(x) \in \mathbb{Z}[x]$ be monic with no roots in $\{z \in \mathbb{C} : |z| \leq 1\}$If $f(x)$ has a root $\alpha \in \mathbb{C} \setminus \mathbb{R}$ with $|\alpha| > \sqrt{\frac{|f(0)|}{2}}$, then $f(x)$ is irreducible.*

Proof. Let $f(x) \in \mathbb{Z}[x]$ be monic with no roots in $\{z \in \mathbb{C} : |z| \leq 1\}$ and let $f(\alpha) = 0$ for some $\alpha \in \mathbb{C} \setminus \mathbb{R}$ with $|\alpha| > \sqrt{\frac{|f(0)|}{2}}$. Suppose that $f(x)$ is not irreducible. Then we can write $f(x) = h(x)k(x)$ for some monic $h(x) \in \mathbb{Z}[x]$ and monic $k(x) \in \mathbb{Z}[x]$, each of positive degree. Since $f(\alpha) = 0$ we may assume without loss of generality that $k(\alpha) = 0$. Since $\alpha \in \mathbb{C} \setminus \mathbb{R}$, this then implies $k(\overline{\alpha}) = 0$. Now let r be the degree of $h(x)$ and let $\alpha_1, \ldots, \alpha_r$ be the roots of $h(x)$. Since $f(x)$ has no roots in $\{z \in \mathbb{C} : |z| \leq 1\}$, we know that $|\alpha_i| > 1$ for $1 \leq i \leq r$. Thus, $|h(0)| \geq 2$, since $h(x) \in \mathbb{Z}[x]$. Similarly, we see that $|k(0)| \geq |\alpha\overline{\alpha}| > \frac{|f(0)|}{2}$. Hence, $|f(0)| = |h(0)|\,|k(0)| > 2 \cdot \frac{|f(0)|}{2} = |f(0)|$. This contradiction proves the lemma. $\square$

3. Theorem 1 and its corollaries

With the lemmas in the previous section established, we now prove Theorem 1 and provide several corollaries illustrating how the theorem can be used.

Proof of Theorem 1. Let $f(x) = x^2 + bx + c \in \mathbb{Z}[x]$ with $c \geq 2$ and $|b| < 2\sqrt{c-1}$. Then the two roots of $f(x)$ are

$$\alpha_1 = \frac{-b + \sqrt{b^2 - 4c}}{2} \qquad \text{and} \qquad \alpha_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}.$$

Since $|b| < 2\sqrt{c-1}$ we see that $b^2 - 4c < -4$. Thus, α_1 and α_2 are non-real and have absolute value $\sqrt{c}$. Now let ϵ be such that $0 < \epsilon < \sqrt{c} - 1$ and $\epsilon \leq 1$. Then

$$|\alpha_j| = \sqrt{c} > 1 + \epsilon \quad \text{for } j \in \{1, 2\}$$

and

$$|\alpha_1 - \alpha_2| = \left|\sqrt{b^2 - 4c}\right| > 2 > \epsilon.$$

Thus, by Lemma 1 and the remark after, for $g(x) = \sum_{j=0}^{t} a_j x^j \in \mathbb{Z}[x]$, if

$$n > \max\left\{t, \frac{t\log(\sqrt{c}+\epsilon) + \log\left|\frac{a_0}{\sqrt{4c-b^2}-\epsilon}\right| + \log\left(\frac{\min\{t+1,2\}}{\epsilon}\right)}{\log(\sqrt{c}-\epsilon)}\right\},$$

then the polynomial $P(x) = f(x)x^n + g(x)$ has a root α with $|\alpha - \alpha_j| < \epsilon$ for each $j \in \{1, 2\}$. Notice that such an α must be non-real since $b^2 - 4c < -4$ implies that

$$\begin{aligned}
|\mathrm{Im}(\alpha)| &= |\mathrm{Im}(\alpha_1 + \alpha - \alpha_1)| \\
&\geq |\mathrm{Im}(\alpha_1)| - |\mathrm{Im}(\alpha - \alpha_1)| \\
&\geq \frac{\left|\sqrt{b^2 - 4c}\right|}{2} - |\alpha - \alpha_2| \\
&> 1 - \epsilon \geq 0.
\end{aligned}$$

Now let

$$1 + |b| + c + \sum_{j=1}^{t} |a_j| < |a_0| < 2(\sqrt{c} - \epsilon)^2.$$

Since $|\alpha_j| = \sqrt{c}$, we see that

$$|\alpha| > |\alpha_1 + \alpha - \alpha_1| \geq |\alpha_1| - |\alpha - \alpha_1| > \sqrt{c} - \epsilon > \sqrt{\frac{|a_0|}{2}} = \sqrt{\frac{|P(0)|}{2}}.$$

Also, we deduce from Lemma 2 that $P(x)$ has no roots in $\{z \in \mathbb{C} : |z| \leq 1\}$. Hence, $P(x)$ is irreducible by Lemma 3. $\square$

Remark. Let $\epsilon = 1$, $c \geq 16$, $|b| < 2\sqrt{c-1}$, and $|a_0| < 2(\sqrt{c}-\epsilon)^2$. Then

$$\begin{aligned}
&\frac{t\log(\sqrt{c}+\epsilon) + \log|a_0| + \log(\min\{t+1,2\}) - \log(\epsilon) - \log\left|\sqrt{4c-b^2}-\epsilon\right|}{\log(\sqrt{c}-\epsilon)} \\
&\qquad \leq \frac{t\log(\sqrt{c}+1) + \log(2(\sqrt{c}-1)^2) + \log(2)}{\log(\sqrt{c}-1)} \\
&\qquad = 2 + \log(2) + \log(2) + \frac{t\log(\sqrt{c}+1)}{\log(\sqrt{c}-1)} \\
&\qquad < 4 + \frac{t\log(\sqrt{c}+1)}{\log(\sqrt{c}-1)}.
\end{aligned}$$

Letting

$$A(c) = \frac{\log(\sqrt{c}+1)}{\log(\sqrt{c}-1)}$$

it can be checked that $A(c)$ is a decreasing function of c and $A(16) = 1.46\ldots < \frac{3}{2}$. Thus, if $c \geq 16$ in Theorem 1, then one can take $\epsilon = 1$ and the result holds for $n \geq 4 + \frac{3t}{2}$.

Now, letting $\epsilon = 1$ we use Theorem 1 to prove the following four corollaries. We note here that the results in these corollaries can be improved slightly by letting $0 < \epsilon < 1$.

Corollary 1. *Let n and c be positive integers, and let $d \in \mathbb{Z}$ such that $c+1 < |d| < 2(\sqrt{c}-1)^2$. Then the trinomial $x^{n+2} + cx^n + d$ is irreducible.*

Proof. Let n and c be positive integers, and let $d \in \mathbb{Z}$ such that $c+1 < |d| < 2(\sqrt{c}-1)^2$. Notice then that $c \geq 16$. Now we write $P(x) = f(x)x^n + g(x)$ where $f(x) = x^2 + c$ and $g(x) = d$. Since $t = \deg g = 0$, we deduce from Theorem 1 that if

$$\frac{\log|d| - \log(2\sqrt{c}-1)}{\log(\sqrt{c}-1)} < \frac{\log(2(\sqrt{c}-1)^2) - \log(2\sqrt{c}-1)}{\log(\sqrt{c}-1)} < 1 \leq n,$$

then $P(x)$ is irreducible. $\square$

Corollary 2. *Let n and c be positive integers with $n \geq 3$, and let d and $\ell \neq 0$ be integers such that $1 + c + |\ell| < |d| < 2(\sqrt{c}-1)^2$. Then the quadrinomial $x^{n+2} + cx^n + \ell x + d$ is irreducible.*

Proof. Let n and c be positive integers with $n \geq 3$, and let d and $\ell \neq 0$ be integers such that $1 + c + |\ell| < |d| < 2(\sqrt{c}-1)^2$. Since $1 + c + |\ell| < |d| < 2(\sqrt{c}-1)^2$ it must be the case that $c \geq 17$. Now we write $P(x) = f(x)x^n +$

$g(x)$ where $f(x) = x^2 + c$ and $g(x) = \ell x + d$. Since $t = \deg\, g = 1$, it follows from Theorem 1 that if

$$\frac{\log(\sqrt{c}+1) + \log(2) + \log(2(\sqrt{c}-1)^2) - \log(2\sqrt{c}-1)}{\log(\sqrt{c}-1)} < 3 \leq n,$$

then $P(x)$ is irreducible. □

Corollary 3. *Let n and c be positive integers with $n \geq 3$, and let d and b be integers such that $0 < |b| \leq 2\sqrt{c-1}$ and $1 + c + |b| < |d| < 2(\sqrt{c}-1)^2$. Then the quadrinomial $x^{n+2} + bx^{n+1} + cx^n + d$ is irreducible.*

Proof. Let n and c be positive integers with $n \geq 3$ and let d and b be integers such that

$$0 < |b| \leq 2\sqrt{c-1} \quad \text{and} \quad 1 + c + |b| < |d| < 2(\sqrt{c}-1)^2.$$

We deduce that $c \geq 17$. Now we write $P(x) = f(x)x^n + g(x)$ where $f(x) = x^2 + bx + c$ and $g(x) = d$. Since $t = \deg\, g = 0$, we deduce from Theorem 1 that if

$$\frac{\log|d| - \log(\sqrt{4c-b^2}-1)}{\log(\sqrt{c}-1)} < \frac{\log(2(\sqrt{c}-1)^2)}{\log(\sqrt{c}-1)} < 3 \leq n,$$

then $P(x)$ is irreducible. □

Corollary 4. *Let b and t be integers with $t \geq 1$ and let $g(x) = \sum_{j=1}^{t} a_j x^j \in \mathbb{Z}[x]$. There exists a positive integer λ so that for all integers $c \geq \lambda$ and all integers $n \geq t+3$, if d is an integer with $1 + |b| + |c| + \sum_{j=1}^{t} |a_j| < |d| < 2(\sqrt{c}-1)^2$, then the polynomial $(x^2 + bx + c)x^n + g(x) + d$ is irreducible.*

Proof. Let b and t be integers with $t \geq 1$ and let $g(x) = \sum_{j=1}^{t} a_j x^j \in \mathbb{Z}[x]$. Choose λ_1 so that $\max\left\{5, \frac{b^2}{4}+1\right\} < \lambda_1$. Then for any integers c and d with $c \geq \lambda_1$ and $|d| < 2(\sqrt{c}-1)^2$,

$$\begin{aligned}&\frac{t\log(\sqrt{c}+1) + \log(\min\{t+1,2\}) + \log|d| - \log\left|\sqrt{4c-b^2}-1\right|}{\log(\sqrt{c}-1)}\\ &\qquad < \frac{t\log(\sqrt{c}+1) + \log(2) + \log(2(\sqrt{c}-1)^2)}{\log(\sqrt{c}-1)}.\end{aligned}$$

Now notice that

$$\lim_{c\to\infty} \frac{t\log(\sqrt{c}+1) + \log(2) + \log(2(\sqrt{c}-1)^2)}{\log(\sqrt{c}-1)} = t+2.$$

Thus, λ_2 can be chosen so that for all $c \geq \lambda_2$,

$$\frac{t\log(\sqrt{c}+1) + \log(2) + \log(2(\sqrt{c}-1)^2)}{\log(\sqrt{c}-1)} < t+3.$$

Now by Theorem 1, letting $\epsilon = 1$ and $\lambda = \max\{\lambda_1, \lambda_2\}$ proves the result. □

4. Theorem 2 and its corollaries

Next we prove Theorem 2 and provide several corollaries illustrating how the theorem can be used.

Proof of Theorem 2. Let $f(x) = x^2 + bx + c \in \mathbb{Z}[x]$ with $c \geq 2$ and $|b| < 2\sqrt{c-1}$. Let

$$g(x) = \sum_{j=0}^{t} a_j x^j \in \mathbb{Z}[x]$$

with $|a_0| < 2(\sqrt{c} - \epsilon)^2$. Now choose ϵ so that $0 < \epsilon < \sqrt{c} - 1$ and $\epsilon \leq 1$. Following the proof of Theorem 1 we see that if

$$n > \max\left\{t, \frac{t\log(\sqrt{c}+\epsilon) + \log\left|\frac{a_0}{\sqrt{4c-b^2}-\epsilon}\right| + \log\left(\frac{\min\{t+1,2\}}{\epsilon}\right)}{\log(\sqrt{c}-\epsilon)}\right\},$$

then the polynomial $P(x) = f(x)x^n + g(x)$ has a root $\alpha \in \mathbb{C} \setminus \mathbb{R}$ with $|\alpha| > \sqrt{c} - \epsilon > \sqrt{\frac{|a_0|}{2}}$.

Now suppose that

$$1 + |b| + c + \sum_{j=1}^{t} |a_j| = |a_0|.$$

Then for $n \geq t+1$ we deduce from Lemma 2 that $P(x)$ has no roots in $\{z \in \mathbb{C} : |z| < 1\}$. Furthermore, Lemma 2 implies that if $P(x)$ has a root $\beta \in \{z \in \mathbb{C} : |z| = 1\}$, then $\beta^4 = 1$ since $c \neq 0$. Lemma 2 further implies that if $b \neq 0$, then $\beta^2 = 1$. Notice however that in the case $b = 0$,

$$\left|(-1+c)i^n + \sum_{j=1}^{t} a_j i^j\right| \leq |c-1| + \sum_{j=1}^{t} |a_j| < c + 1 + \sum_{j=1}^{t} |a_j| = |a_0|.$$

From this we deduce that $P(i) \neq 0$. Thus, β is a root of some cyclotomic polynomial in $\{x+1, x-1\}$. This proves the first implication of the theorem.

Now write $P(x) = h(x)k(x)$ so that $h(x)$ is a product of cyclotomic polynomials and $k(x)$ has no cyclotomic factors. It then follows that $|k(0)| = |a_0|$ and $k(x)$ has no roots in $\{z \in \mathbb{C} : |z| \leq 1\}$. Also, since $h(x)$ is the product of cyclotomic polynomials and $|\alpha| > \sqrt{c} - \epsilon > 1$, we know that $k(\alpha) = 0$. Since

$$|\alpha| > \sqrt{\frac{|a_0|}{2}} = \sqrt{\frac{|k(0)|}{2}},$$

we deduce from Lemma 3 that $k(x)$ must be irreducible. $\square$

Remark. Notice that Theorem 2 implies that $P(x)$ is reducible if and only if $P(x)$ has a root in $\{-1, 1\}$.

Corollary 5. *Let n and c be positive integers with $c \geq 2$. Then for $\nu \in \{-1, 1\}$, the following are true for the trinomial $P(x) = x^{n+2} + cx^n + \nu \cdot (c+1)$:*

(1) *If n is odd and $\nu = 1$, then $P(x) = (x+1)k(x)$ for some irreducible $k(x) \in \mathbb{Z}[x]$.*
(2) *If n is even and $\nu = 1$, then $P(x)$ is irreducible unless $P(x) = x^4 + 3x^2 + 4$ or $P(x) = x^4 + 5x^2 + 6$.*
(3) *If n is odd and $\nu = -1$, then $P(x) = (x-1)k(x)$ for some irreducible $k(x) \in \mathbb{Z}[x]$.*
(4) *If n is even and $\nu = -1$, then $P(x) = (x+1)(x-1)k(x)$ for some irreducible $k(x) \in \mathbb{Z}[x]$ unless $P(x) = x^6 + 3x^4 - 4$.*

Proof. Let n and c be positive integers with $c \geq 2$ and let $\nu \in \{-1, 1\}$. It follows from Theorem 2 that if $P(x) = x^{n+2} + cx^n + \nu \cdot (c+1)$ has a cyclotomic factor, then it must be in the set $\{x+1, x-1\}$. Furthermore, since $P'(x) = (n+2)x^{n+1} + cnx^{n-1}$, we see that any roots of $P(x)$ in $\{-1, 1\}$ must be of multiplicity one. Hence, if $c \geq 16$, then the result follows from Theorem 2 by letting $\epsilon = 1$. Now suppose that $2 \leq c \leq 15$. Let

$$\epsilon = \frac{\sqrt{c} - \sqrt{\frac{c+1}{2}}}{2} < \sqrt{c} - 1$$

so that $c + 1 < 2(\sqrt{c} - \epsilon)^2$ and $\epsilon \leq 1$. A computation gives that

$$\frac{\log(c+1) - \log(\epsilon) - \log(2\sqrt{c} - \epsilon)}{\log(\sqrt{c} - \epsilon)} < 9.$$

Thus, the result follows from Theorem 2 with $\epsilon = \frac{\sqrt{c} - \sqrt{\frac{c+1}{2}}}{2}$ for $2 \leq c \leq 15$ and $n \geq 9$. The remaining cases can easily be checked computationally. $\square$

Corollary 6. *Let n and c be positive integers with $c \geq 2$. Then the polynomials*

$$f(x) = x^{2n} - x^{2n-1} + c(x^{2n-2} - x^{2n-3} + x^{2n-4} - \cdots - x + 1)$$

$$g(x) = x^{2n} + x^{2n-1} + c(x^{2n-2} + x^{2n-3} + \cdots + x + 1)$$

$$\text{and} \qquad h(x) = x^{2n} + c(x^{2(n-1)} + x^{2(n-2)} + \cdots + x^2 + 1)$$

are all irreducible, with the exception $h(x) = x^4 + 4x^2 + 4$.

Proof. Let k and c be positive integers with $c \geq 2$. The result follows by observing that

$$\begin{aligned}
&x^{k+2} + cx^k + (c+1) = (x+1)f(x), && \text{whenever } k = 2n-1,\\
&x^{k+2} + cx^k - (c+1) = (x-1)g(x), && \text{whenever } k = 2n-1,\\
&x^{k+2} + cx^k - (c+1) = (x+1)(x-1)h(x), && \text{whenever } k = 2n
\end{aligned}$$

and applying Corollary 5. $\square$

Corollary 7. *Let n, c, and ℓ be integers with $n \geq 4$, $c \geq 3$, and $0 < |\ell| \leq c-2$. Then for $\nu \in \{-1, 1\}$, the following are true for the quadrinomial $P(x) = x^{n+2} + cx^n + \ell x + \nu \cdot (c + 1 + |\ell|)$:*

(1) *If n is odd, $\ell > 0$, and $\nu = 1$, then $P(x) = (x+1)k(x)$ for some irreducible $k(x) \in \mathbb{Z}[x]$.*
(2) *If n is even, $\ell > 0$, and $\nu = 1$, then $P(x)$ is irreducible.*
(3) *If n is even, $\ell < 0$, and $\nu = -1$, then $P(x) = (x+1)k(x)$ for some irreducible $k(x) \in \mathbb{Z}[x]$.*
(4) *If n is odd, $\ell < 0$, and $\nu = -1$, then $P(x)$ is irreducible.*
(5) *If $\ell > 0$ and $\nu = -1$, then $P(x) = (x-1)k(x)$ for some irreducible $k(x) \in \mathbb{Z}[x]$.*
(6) *If $\ell < 0$ and $\nu = 1$, then $P(x)$ is irreducible.*

Proof. Let $n \geq 4$, $c \geq 3$ and $0 < |\ell| \leq c-2$. It follows from Theorem 2 that if

$$P(x) = x^{n+2} + cx^n + \ell x + \nu \cdot (c+1)$$

has a cyclotomic factor, then it must be in the set $\{x+1, x-1\}$. Furthermore, since

$$P'(x) = (n+2)x^{n+1} + cnx^{n-1} + \ell$$

and $|\ell| \leq c-2$, we see that any roots of $P(x)$ in $\{-1, 1\}$ must be of multiplicity one. Now let $\epsilon = \frac{1}{4\sqrt{c}}$ so that $0 < \epsilon < \sqrt{c} - 1$ and $\epsilon \leq 1$. Notice then that $c + |\ell| + 1 \leq 2c - 1 < 2(\sqrt{c} - \epsilon)^2$. Also notice that

$$\begin{aligned}
&\frac{\log(\sqrt{c}+\epsilon) + \log(c+|\ell|+1) + \log(2) - \log(\epsilon) - \log(2\sqrt{c}-\epsilon)}{\log(\sqrt{c}-\epsilon)} \\
&\quad \leq \frac{\log(\sqrt{c}+\epsilon) + \log(2(\sqrt{c}-\epsilon)^2) + \log(2) - \log(\epsilon) - \log(2(\sqrt{c}-\epsilon))}{\log(\sqrt{c}-\epsilon)} \\
&\quad \leq 1 + \frac{\log(\sqrt{c}+\epsilon)}{\log(\sqrt{c}-\epsilon)} + \frac{\log(2)}{\log(\sqrt{c}-\epsilon)} - \frac{\log(\epsilon)}{\log(\sqrt{c}-\epsilon)} \\
&\quad \leq 1 + \frac{\log(8c+2)}{\log\left(\sqrt{c} - \frac{1}{2\sqrt{3}}\right)}.
\end{aligned}$$

Letting

$$A(c) = 1 + \frac{\log(8c+2)}{\log\left(\sqrt{c} - \frac{1}{2\sqrt{3}}\right)},$$

one can check that $A(c)$ is a decreasing function of c. Thus, when $c \geq 79$ the result follows from Theorem 2 since $A(79) < 4$. We then check computationally that $A(c) < 10$ for $c \in \{3, \ldots, 78\}$. With this the result follows from Theorem 2 for $n \geq 10$ and $c \in \{3, \dots, 78\}$. The remaining cases can then be checked computationally. $\square$

5. Concluding remarks

The methods used to prove Theorems 1 and 2 are similar to methods found in [5]. There the first author focused his attention on the factorization of trinomials of the form

$$x^{n+1} + cx^n + d = (x + c)x^n + d \in \mathbb{Z}[x]$$

with certain restrictions on n, c, and d. The following is a result given in that paper.

Lemma 4. *Let K be a positive integer and let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial with no roots in the set $\{z \in \mathbb{C} : |z| \leq K\}$. If $f(x)$ has a root α with $|\alpha| > \frac{|f(0)|}{K+1}$, then $f(x)$ is irreducible in $\mathbb{Z}[x]$.*

Using this lemma with $K = 1$ along with Lemma 1 and the remark after, as well as Lemma 2, one can prove the following two theorems. We omit proofs here, as the results follow similarly to the proofs of Theorems 1 and 2.

Theorem 4. *Let $|c| \geq 2$ and let $0 < \epsilon < |c| - 1$. If $g(x) = \sum_{j=0}^{t} a_j k^j \in \mathbb{Z}[x]$ with*

$$1 + |c| + \sum_{j=1}^{t} |a_j| < |a_0| < 2(|c| - \epsilon),$$

then the polynomial $(x + c)x^n + g(x)$ is irreducible for all

$$n > \max\left\{t, \frac{t\log(|c| + \epsilon) + \log|a_0| + \log(\min\{t + 1, 2\}) - \log(\epsilon)}{\log(|c| - \epsilon)}\right\}.$$

Theorem 5. *Let $|c| \geq 2$ and let $0 < \epsilon < |c| - 1$. Let $g(x) = \sum_{j=0}^{t} a_j k^j \in \mathbb{Z}[x]$ with*

$$1 + |c| + \sum_{j=1}^{t} |a_j| = |a_0| < 2(|c| - \epsilon).$$

For

$$n > \max\left\{t, \frac{t\log(|c| + \epsilon) + \log|a_0| + \log(\min\{t + 1, 2\}) - \log(\epsilon)}{\log(|c| - \epsilon)}\right\},$$

the non-cyclotomic part of the polynomial $P(x) = (x + c)x^n + g(x)$ is irreducible. Furthermore, any cyclotomic factor of $P(x)$ must be in $\{x + 1, x - 1\}$.

We note here that the results in this paper rely heavily on the location of the roots of the polynomials in question. In particular, the two main theorems enforce restrictions on the third coefficient of the polynomial so that all of the roots of the polynomial lie in $\{z \in \mathbb{C} : |z| \geq 1\}$. Putting size restrictions on the coefficients of a polynomial to force its roots to lie in certain areas of the complex plane is certainly not a new idea. An example of particular importance to this paper is a paper of A. Brauer's titled "On the Irreducibility of Polynomials with large Third Coefficient" [1]. There, Brauer enforces restrictions on the third coefficient of polynomials to force all but exactly two roots to lie in $\{z \in \mathbb{C} : |z| < 1\}$. One can immediately see how the two papers might help complement each other. For this reason, the authors encourage the interested reader to read Brauer's paper [1] and its follow-up [2].

6. Acknowledgments

The authors would like to thank Lenny Jones for his contributions in regard to Lemma 2. The authors would also like to thank Michael Filaseta for his many suggestions in regards to this paper.

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