Journal de Théorie des Nombres de Bordeaux

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 25, n° 2 (2013), p. 331-351.

<http://jtnb.cedram.org/item?id=JTNB_2013__25_2_331_0>

Journal de Théorie des Nombres de Bordeaux **25** (2013), 331–351

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Résumé. Nous discutons quelques théorèmes sur des valeurs moyennes dans le style du théorème de Bombieri-Vinogradov. Ils concernent des problèmes additifs binaires et ternaires avec des nombres premiers dans des progressions arithmétiques et des intervals courts. Nous donnons des estimations non-triviales pour certaines de ces valeurs moyennes. Comme application entre autres, nous démontrons que pour $n \not\equiv 1\,(6)$ grand et impair, le problème $n = p_1 + p_2 + p_3$ de Goldbach a une solution avec des nombres premiers p_1, p_2 dans des intervals courts : $p_i \in [X_i, X_i + Y]$, où $X_i^{\theta_i} = Y$ et $\theta_i \geq 0.933$ pour $i = 1, 2$, et tel que en plus, $(p_1 + 2)(p_2 + 2)$ a au plus 9 facteurs premiers.

Abstract. Some mean value theorems in the style of Bombieri-Vinogradov's theorem are discussed. They concern binary and ternary additive problems with primes in arithmetic progressions and short intervals. Nontrivial estimates for some of these mean values are given. As application inter alia, we show that for large odd $n \not\equiv 1\,(6)$, Goldbach's ternary problem $n = p_1 + p_2 + p_3$ is solvable with primes p_1, p_2 in short intervals $p_i \in [X_i, X_i+Y]$ with $X_i^{\theta_i} = Y$, $i = 1, 2$, and $\theta_1, \theta_2 \geq 0.933$ such that $(p_1 + 2)(p_2 + 2)$ has at most 9 prime factors.

Notations: By p, p_1, p_2, p_3 we denote prime numbers. The symbol $X \asymp Y$ means $X \ll Y \ll X$, and the symbol $n \sim N$ denotes the range $N \leq n < 2N$ for n. We write $a\ (q)$ for a residue class $a \mod q$. Throughout, a star at a residue sum or maximum means that the sum or maximum goes over all reduced residues. By $\tau(q)$ we denote the number of positive divisors of q, and by $\nu(q)$ the number of prime factors of q. The symbol P_s stands for a pseudo-prime of type s, that is a positive integer with at most s prime factors. Further, ε, ε_1 and ε_2 are small positive real constants. By $A > 0$ we denote a given positive constant, and $B = B(A) > 0$ denotes a positive constant depending only on A. All implicit constants may depend on A and ε, ε_1, ε_2.

Mots clefs. additive problems; circle method; sieve methods; short intervals.

By Q_1 and Q_2 we denote real numbers ≥ 1 serving as bounds for the moduli of the considered arithmetic progressions.

For the least common multiple of two integers a and b, we write $[a;b]$, and (a,b) denotes their greatest common divisor.

Considering intervals, we use the notation $[X, X+Y]$ for the set of integers n with $X < n \leq X+Y$, where X and Y denote real numbers ≥ 2. We call such an interval short, if $Y = X^\theta$ for a real number $0 < \theta < 1$.

The real numbers $R, Y, X, Y_1, Y_2, X_1, X_2 \geq 2$ are considered to be sufficiently large, being at least as big as some constant depending only on A and the ε, ε_1, ε_2. Further $L := \log Y$.

The statements of the Theorems 1,2,3,4,6 in this article begin with "For all $A > 0$ there is a $B = B(A) > 0$ such that for all $R, Y, X, X_1, \dots$ (list of occurring parameters) the following holds:". Theorems 5,7 begin with "For all $A > 0$ and all $R, Y, X, X_1, \dots$ (list of occurring parameters) the following holds:" This sentence is left out for an easier reading.

1. Introduction

1.1. Statement of results. This article examines binary and ternary additive problems with primes in arithmetic progressions (APs for short) and in short intervals. We study what kind of mixtures of such conditions on the prime summands are treatable and where the limits of current methods are, especially when treating ternary problems. We show two such ternary theorems resulting from two different approaches and give corollaries for additive problems with almost-twin primes in short intervals. Here we call a prime p almost-twin, if $p+2$ is an almost-prime P_s for some positive integer s.

The first approach works with an estimate that goes back to Kawada in [5] and leads to the following Theorem 1.1, in which we consider the ternary Goldbach problem with two primes in APs, both lying in short intervals being of the same length Y:

Theorem 1.1. *Let $n \geq X_1 + X_2 + 2Y$ be odd, let $n \ll X_1 Y$, let $X_2 \geq Y \gg (n - X_1)^{2/3+\varepsilon_1}$, let $X_1 \geq Y \gg X_1^{3/5+\varepsilon_2}$ and assume that $Q_i \ll Y X_i^{-1/2} L^{-B}$ for $i = 1,2$.*

Then for any fixed integers a_1, a_2 with $a_1 \leq n - X_1 - Y$ we have

$$\sum_{q_1 \leq Q_1} \sum_{q_2 \leq Q_2} \Bigg| \sum_{\substack{p_1+p_2+p_3=n \\ p_i \in [X_i, X_i+Y],\ i=1,2 \\ p_i \equiv a_i\ (q_i),\ i=1,2}} \log p_1 \log p_2 \log p_3 - \mathfrak{T}(n, q_1, a_1, q_2, a_2) Y^2 \Bigg| \ll Y^2 L^{-A}.$$

The singular series $\mathfrak{T}(n, q_1, a_1, q_2, a_2)$ contains the whole arithmetic information of the problem. It is given at the end of Section 3 in its Euler product form.

Using a sieve theorem for a two-dimensional sieve, we deduce from this result the following ternary corollary.

Corollary 1.1. *Let Y be large and consider any odd integer $n \not\equiv 1\,(6)$ with $n \geq X_1 + X_2 + 2Y$ and $n \ll X_1 Y$. Then the equation $n = p_1 + p_2 + p_3$ is solvable in primes p_1, p_2, p_3 such that $(p_1+2)(p_2+2) = P_9$, where $p_i \in [X_i, X_i + Y]$, $X_1^{\theta_1} = Y$, $(n - X_1)^{\theta_2} = Y$ with $\theta_i \geq 0.933$, $i = 1, 2$.*

Another variant of this corollary can be deduced as follows.

As a first step for this, we deduce in Section 5.1 a short interval version of Meng's result in [7]:

Corollary 1.2. *For all but $\mathrm{O}(YL^{-A})$ even integers $2k_1 \not\equiv 2\,(6)$ with $k_1 \in [X_1, X_1 + Y]$, the equation $2k_1 = p_2 + p_3$ is solvable in primes p_2, p_3 such that $p_2 + 2 = P_3$, $p_2 \in [X_2, X_2 + Y]$, where $X_2^{\theta} = Y$ with $\theta \geq 0.861$, and $X_2 + Y \leq 2X_1 \ll Y^{3/2-\varepsilon}$.*

By a counting argument, we infer in Section 5.1 a variant of Corollary 1.1, using Corollary 1.2 and a theorem of Wu in [18] on the number of Chen primes in short intervals.

Corollary 1.3. *Let $X_1^{\theta_1} = Y = X_2^{\theta_2}$ be large, where $\theta_1 \geq 0.971$ and $\theta_2 \geq 0.861$. Let n be an odd integer $n \not\equiv 1\,(6)$ with $X_1 + X_2 + 2Y \leq n \ll Y^{3/2-\varepsilon}$. Then the equation $n = p_1 + p_2 + p_3$ is solvable in primes p_1, p_2, p_3 such that $p_1 + 2 = P_2$, $p_2 + 2 = P_3$ and $p_i \in [X_i, X_i + Y]$, $i = 1, 2$.*

This corollary cannot be deduced from Corollary 1.1 before since the almost-prime conditions on $p_1 + 2$ and $p_2 + 2$ in Corollary 1.3 are stronger. And also not vice versa since the short interval condition in Corollary 1.1 is stronger due to $0.971 > 0.933$.

Now we state the results of the second approach, it leads to theorems of a similar kind. In this approach, we use an adaption of the theorem of Perelli and Pintz in [11].

For the ternary Goldbach problem with one prime in an arithmetic progression and two primes in given short intervals of different length, we show in Section 4 the following result.

Theorem 1.2. *Let n denote a large positive odd integer, let $X_1 \geq Y_1 \gg X_1^{3/5+\varepsilon_1}$, let $X_2 \geq Y_2 \gg X_2^{7/12+\varepsilon_2}$. Assume that $Y_2^{1/3+\varepsilon} \ll Y_1 \ll Y_2 \asymp n - X_1 - Y_1 - X_2 - Y_2 \geq 0$ holds. Then, for $Q \ll Y_1 X_1^{-1/2} L^{-B}$ with $Q \ll Y_1^{3/2}(n - X_1)^{-1/2}$ and any fixed integer a with $a \leq n - X_1 - Y_1$ we have*

$$\sum_{q \leq Q} \Bigg| \sum_{\substack{p_1+p_2+p_3=n \\ p_i \in [X_i, X_i+Y_i],\ i=1,2 \\ p_1 \equiv a\ (q)}} \log p_1 \log p_2 \log p_3 - \mathfrak{T}(n, q, a) Y_1 Y_2 \Bigg| \ll Y_1 Y_2 L^{-A}.$$

Further, if $Q \ll Y_1^{1/2}$ *is assumed instead of* $Q \ll Y_1^{3/2}(n-X_1)^{-1/2}$*, then* $\max^*{}_{a(q)}$ *can be inserted after the sum over* q*.*

The condition $n-X_1-Y_1-X_2-Y_2 \asymp Y_2$ is rather restrictive: If n, X_1, Y_1 are given, one has to choose X_2, Y_2 then appropriately, but still feasible in such a way that Y_1 and Y_2 may be of different magnitude. The theorem gives a sharp estimate then.

Here $\mathfrak{T}(n,q,a)$ denotes the ternary singular series with one prime in an arithmetic progression, namely

$$\mathfrak{T}(n,q,a) := \frac{1}{\varphi(q)} \prod_{\substack{p\mid n, p\nmid q \\ \text{or } p\nmid n-a, p\mid q}} \Big(1-\frac{1}{(p-1)^2}\Big) \prod_{p\nmid nq} \Big(1+\frac{1}{(p-1)^3}\Big) \prod_{p\mid n-a, p\mid q} \frac{p}{p-1}.$$

From Theorem 1.2 we deduce: Every large odd n can be written as $n = p_1+p_2+p_3$ with two primes in short intervals of different length, one of which lying in an arithmetic progression a modulo q for almost all admissible moduli $q \le Q$, where $Q \ll Y_1 X_1^{-1/2} L^{-B}$ and $Q \ll Y_1^{3/2}(n-X_1)^{-1/2}$.

A last corollary with almost-twin primes in short intervals of different interval lengths can be deduced from Theorem 1.2 above, again using sieve methods:

Corollary 1.4. *Let* $X_1^{\theta_1} = Y_1$ *be large, let* $X_2^{\theta_2} = Y_2$ *with* $\theta_2 > 3/5$*, and let* $Y_2^{\eta} \ll Y_1 \ll Y_2$*. Let* n *be an odd integer* $n \not\equiv 1\ (6)$ *with* $Y_2 \asymp n - X_1 - X_2 - Y_1 - Y_2 > 0$*. Then the equation* $n = p_1 + p_2 + p_3$ *is solvable in primes* p_1, p_2, p_3 *such that* $p_1 + 2 = P_3$ *and* $p_i \in [X_i, X_i + Y_i]$*,* $i = 1, 2$*, if*

- *either* $1 \ge \theta_1 \ge 0.861$*,* $1 \ge \eta > (1 + 1/\theta_1)^{-1} > 0.462$*,*
- *or* $0.5 \ge \eta \ge 0.463$*,* $1 \ge \theta_1 > 1/\min(1/\eta-1, (3-1/\eta)\Lambda_3/2) > 0.782$*, where* $\Lambda_3 := 4 - \log(27/7)/\log 3$*.*

1.2. A conjectured unification of the results. Now we ask what would be the strongest version of a theorem that combines both classes of results. All theorems above are deduced either by the Kawada-approach or by the Perelli-Pintz-approach. It would be interesting if there exists a slightly stronger theorem that would incorporate all such results. Such a unification, which seems to be unreachable by current methods, can be stated in the following way:

Conjecture 1.1. *There exist absolute constants* $0 < \theta, \theta_1, \theta_2 < 1$ *such that for* $X_1 \ge R \gg X_1^{\theta_1}$*,* $X_2 \ge Y \gg X_2^{\theta_2}$ *and* $R \ll Y^{\theta}$*, the estimate*

$$\sum_{\substack{r\in[X_1,X_1+R] \\ 2\mid r}} \sum_{q\le Q} \max_{\substack{a\ (q) \\ (a-r,q)=1}}{}^{*} \Big| \sum_{\substack{p_2\in[X_2,X_2+Y] \\ p_2\equiv a\ (q) \\ p_2+p_3=r}} \log p_2 \log p_3 - \mathfrak{S}(r,q,a) Y \Big| \ll \frac{RY}{L^A}$$

holds for all $Q \ll YX_2^{-1/2}L^{-B}$. Here the numbers r and p_2 are chosen from short intervals of length R respectively Y. In this statement, the Goldbach equation $p_2 + p_3 = r$ may be replaced by the twin equation $p_2 - p_3 = r$.

We note that a version of the conjecture with $Y \ll R^\theta$ would also be desirable, but this seems to be also hard.

Here the singular series $\mathfrak{S}(r,q,a)$ is the expression

$$\mathfrak{S}(r,q,a) := \begin{cases} \frac{1}{\varphi(q)}\mathfrak{S}(rq), & \text{if } 2 \mid r,\ (a,q) = (a-r,q) = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{1.1}$$

with

$$\mathfrak{S}(r) := \begin{cases} 2\prod_{p\neq 2}(1 - \frac{1}{(p-1)^2})\prod_{\substack{p\mid r\\ p\neq 2}} \frac{p-1}{p-2}, & \text{if } 2 \mid r, \\ 0, & \text{otherwise}. \end{cases} \tag{1.2}$$

Given as a series, it can be written as

$$\mathfrak{S}(r,q,a) = \sum_{s\geq 1} H_s(r,q,a)$$

with

$$H_s(r,q,a) := \frac{\mu(s)}{\varphi(s)\varphi([q;s])} \sideset{}{^*}\sum_{b(s)} e\Big(\frac{-rb}{s}\Big) \sideset{}{^*}\sum_{\substack{c(s)\\ (q,s)\mid c-a}} e\Big(\frac{bc}{s}\Big), \tag{1.3}$$

see e.g. [1, eq. (33)]. Further, we have

$$\mathfrak{S}(r) = \sum_{s\geq 1} \frac{\mu^2(s)}{\varphi^2(s)} \sideset{}{^*}\sum_{b(s)} e\Big(\frac{-rb}{s}\Big). \tag{1.4}$$

A number of theorems proved by several authors can be seen as special cases of this conjecture. Mikawa [9] considered the case for $R \asymp Y \asymp X_1 \asymp X_2$, and Laporta [6] the one for $Y \asymp R$ and short intervals for r. Meng [7],[8] considered the Goldbach variant for $R \asymp Y$ and non-short intervals. Kawada's estimate [5] for the special case $k = 2$, $a_0 = 1$, $b_0 = 0$, $a_1 = \pm 1$, $b_1 = r$ is contained in the conjecture with $R \asymp Y$, with short intervals for both p_2 and r. Perelli's and Pintz's result [11] is contained in the conjecture when taking no primes in arithmetic progressions, but such that r lies in a short interval.

By now, the paper [1] of A. Balog, A. Cojocaru and C. David contains an interesting result of this kind which can be seen closest to the conjecture, namely, that the Barban/Davenport/Halberstam-variant of the conjecture is true, that means, the corresponding estimate when max over a is replaced by sum over a. It suggests that also the conjecture, which is a sharper estimate of Bombieri-Vinogradov-type, could be true. It would be the next step for reaching stronger results in this area.

We remark that the major arc contribution of the conjecture can be shown with known standard methods. We obtain that $R \ll Y$ with $Y \gg X_1^{7/12+\varepsilon_1}$, $Y \gg X_2^{3/5+\varepsilon_2}$, $X_1 - X_2 \geq Y$ suffices. The problem to prove the conjecture lies in the minor arc contribution.

Theorems 1.2 and 1.1 show that we can put one AP-condition and two short-interval-conditions or two AP-conditions and one short-interval-condition (two with the same length are counted as one) on the primes when treating the ternary Goldbach problem. But stronger versions seem to be hard. The stated conjecture would lead to a version with two AP-conditions and two short-interval-conditions.

We would like to mention that a mean value theorem as Theorem 1.1 could also hold with all three primes in arithmetic progressions with large moduli. But this problem is of similar kind and seems to be unreachable, too. See also [14], [15],[16] and [4] for discussions of this conjecture.

2. Tools

As a corollary of the known inequalities of Halasz and Montgomery, see Satz 7.3.1 [2], we first state the following lemma.

Lemma 2.1. *Consider a fixed integer a. For a real bound $Q \geq 1$, positive integers M, N with $Q \leq M$, $N \leq M$, $M \geq a$ and $v_{M+1}, \dots, v_{M+N} \in \mathbb{C}$ we have*

$$\sum_{q \sim Q} \Big| \sum_{\substack{n \in [M, M+N] \\ n \equiv a\ (q)}} v_n \Big| \ll (N + Q^{2/3} M^{1/3})^{1/2} (\log(M+1))^{3/2} \cdot \cdot \Big(\sum_{n \in [M, M+N]} |v_n|^2 \Big)^{1/2}. \tag{2.1}$$

Proof. Write $v := (v_{M+1}, \dots, v_{M+N})^T \in \mathbb{C}^N$ and consider the usual scalar product $\langle v, w \rangle := \sum_{n \sim N} v_n \overline{w_n}$ on $\mathbb{C}^N$, where $v, w \in \mathbb{C}^N$. For every $q \sim Q$ let

$$\varphi_q(n) := \begin{cases} 1, & \text{if } n \equiv a\ (q) \\ 0, & \text{else,} \end{cases}$$

so that $\varphi_q \in \mathbb{C}^N$.

The left hand side of (2.1) then becomes $\sum_{q \sim Q} |\langle v, \varphi_q \rangle|$. Halasz-Montgomery's inequality states that this is

$$\leq \|v\| \Big(\sum_{q_1, q_2 \sim Q} |\langle \varphi_{q_1}, \varphi_{q_2} \rangle| \Big)^{1/2},$$

where $\|v\| := \Big(\sum_{n \in [M, M+N]} |v_n|^2 \Big)^{1/2}$.

Write $\tau_Q(n) := \sum_{d|n, d\sim Q} 1$. Now we have, since $M \geq a$ (also $Q, M \geq 2$ may be assumed w.l.o.g).

$$\begin{aligned}
(2.2)\quad & \sum_{q_1,q_2\sim Q} |\langle \varphi_{q_1}, \varphi_{q_2}\rangle| \\
&= \sum_{q_1,q_2\sim Q} \sum_{n\in[M,M+N]} \varphi_{q_1}(n)\varphi_{q_2}(n) = \sum_{n\in[M,M+N]} \sum_{\substack{q_1\sim Q\\ q_1|n-a}} \sum_{\substack{q_2\sim Q\\ q_2|n-a}} 1 \\
&= \sum_{s, 0\leq M-a<s\leq M+N-a} \sum_{\substack{q_1\sim Q\\ q_1|s}} \sum_{\substack{q_2\sim Q\\ q_2|s}} 1 \leq \sum_{s\in[M-a,M+N-a]} \tau_Q(s)^2 \\
&= \sum_{s\in[M-a,M+N-a]} \tau_Q(s) \sum_{s=uv, u\sim Q} 1 = \sum_{uv\in[M-a,M+N-a], u\sim Q} \tau_Q(uv) \\
&\leq \sum_{uv\in[M-a,M+N-a], u\sim Q} \tau(u)\tau(v) = \sum_{u\sim Q} \tau(u) \sum_{v\in[\frac{M-a}{u}, \frac{M+N-a}{u}]} \tau(v).
\end{aligned}$$

For the inner sum, we use Voronoï's estimate

$$\sum_{n\leq x} \tau(n) = x\log x + (2\gamma - 1)x + O(x^{1/3}\log x),$$

and continue (2.2) with

$$\begin{aligned}
&\ll \sum_{u\sim Q} \tau(u)\Big(\frac{N}{u}\log M + \frac{M^{1/3}}{u^{1/3}}\log M\Big) \\
&\ll N\log M \sum_{u\sim Q} \frac{\tau(u)}{u} + \sum_{u\sim Q} \frac{\tau(u)}{u^{1/3}} M^{1/3}\log M \\
&\ll N(\log M)^3 + Q^{2/3}M^{1/3}(\log M)^3,
\end{aligned}$$

where we used that $\sum_{u\leq Q} \frac{\tau(u)}{u} \ll (\log Q)^2 \leq (\log M)^2$. □

In a similar way, we can also prove the following variant which is uniform in the residues. We can expect nontrivial estimates from this for the range $Q \ll N^{1/2}$ for Q:

Lemma 2.2. *For a real $Q \geq 1$, integers $M, N \geq 2$ and $v_{M+1}, \ldots, v_{M+N} \in \mathbb{C}$ we have*

$$(2.3)\quad \sum_{q\sim Q} \max_{a\ (q)} \Big| \sum_{\substack{n\in[M,M+N]\\ n\equiv a\ (q)}} v_n \Big| \ll (N\log(Q+1) + Q^2)^{1/2} \Big(\sum_{n\in[M,M+N]} |v_n|^2 \Big)^{1/2}.$$

Proof. For a mod q define

$$\varphi_{q,a}(n) := \begin{cases} 1, & \text{if } n \equiv a\ (q), \\ 0, & \text{else,} \end{cases}$$

and let $a = a_q$ be a residue mod q such that $|\langle v, \varphi_{q,a}\rangle|$ is maximal. Then we have

$$\sum_{q_1,q_2\sim Q} |\langle \varphi_{q_1,a_{q_1}}, \varphi_{q_2,a_{q_2}}\rangle| = \sum_{q_1,q_2\sim Q} \sum_{n\in[M,M+N]} \varphi_{q_1,a_{q_1}}(n)\varphi_{q_2,a_{q_2}}(n)$$

$$= \sum_{q_1,q_2\sim Q} \sum_{\substack{n\in[M,M+N] \\ n\equiv a_{q_1}\ (q_1) \\ n\equiv a_{q_2}\ (q_2)}} 1 \leq \sum_{q_1,q_2\sim Q} \Big(\frac{N}{[q_1;q_2]} + 1\Big)$$

$$= Q^2 + \sum_{q_1\sim Q} \frac{N}{q_1} \sum_{d|q_1} \sum_{\substack{q_2\sim Q \\ (q_2,q_1)=d}} \frac{d}{q_2} \leq Q^2 + \frac{N}{Q} \sum_{q_1\sim Q} \sum_{d|q_1} \sum_{q_2'\sim Q/d} \frac{1}{q_2'}$$

$$\ll Q^2 + \frac{N}{Q} \sum_{q_1\sim Q} \tau(q_1) \ll Q^2 + N\log Q.$$

Then again Halasz-Montgomery's inequality shows the assertion. □

As a further tool, we use the theorem of Perelli, Pintz and Salerno in [12] in the following form, it is a Bombieri-Vinogradov theorem for short intervals.

Theorem 2.1. *Let $X \geq Y \gg X^{3/5+\varepsilon}$ and $Q \ll YX^{-1/2}(\log X)^{-B}$. Then*

$$\sum_{q\leq Q} \max_{(a,q)=1} \Big| \sum_{\substack{p\in[X,X+Y] \\ p\equiv a\ (q)}} \log p - \frac{Y}{\varphi(q)} \Big| \ll \frac{Y}{L^A}.$$

Further we use a special case of Kawada's Theorem in [5] in the following formulation:

Theorem 2.2. *Let $X_1^{2/3+\varepsilon} \ll Y \leq X_1$, let $Y \leq X_2$, let $X_2 \leq 2X_1 - Y$ and $Q_2 \ll YX_2^{-1/2}L^{-B}$. Then for any integer a_2, we have*

(2.4)
$$\sum_{q_2\leq Q_2} \sum_{k_1\in[X_1,X_1+Y]} \Big| \sum_{\substack{p_2+p_3=2k_1 \\ p_2\equiv a_2\ (q_2) \\ p_2\in[X_2,X_2+Y]}} \log p_2 \log p_3 - \mathfrak{S}(2k_1,q_2,a_2)Y \Big| \ll Y^2L^{-A},$$

with $\mathfrak{S}$ as given in (1.1).

This is by Kawada's result [5, Thm. 2] for the special case $k = 2, a_0 = 1, b_0 = 0, a_1 = -1, b_1 = 2k_1$, but with a small change made, namely the left boundaries X_1 and X_2 of the short intervals. Originally, the restriction $X_1 = X_2$ has been stated in [5], but Kawada's proof works also for any $X_2 \leq 2X_1 - Y$: In fact, for the minor arc contribution, the interval boundaries in the exponential sums do not play a role due to the use of Bessel's inequality at the end of §4 in [5], cp. also [5, (4.3)–(4.5)], where the case $k = 2$ is treated individually. For the minor arc contribution, just the assumption $Y \gg X_1^{2/3+\varepsilon}$ is used.

The condition $X_2 \leq 2X_1 - Y$ is added to avoid that the necessary interval for $p_3 = 2k_1 - p_2$ has a negative left boundary and hence a cut-off at 0. In Kawada's major arc treatment, the exponential sum range for p_3 has then still length $\asymp Y$ (there the sum is termed $P(\alpha)$ and is approximated by $\frac{\mu(q)}{\varphi(q)} T(\beta)$). Then the proof of (3.8) and (3.9) in §5 of [5] reads the same. The major arc treatment uses then the assumption $Q_2 \ll Y X_2^{-1/2} L^{-B}$ and $Y \gg X_2^{3/5+\varepsilon_2}$, but the latter estimate follows from the other assumptions in Theorem 2.2.

For the second approach, we use an adaption of the Theorem of Perelli and Pintz in [11]. It was independently found by Mikawa in [10]. Originally, this theorem states: If $X_1^{1/3+\varepsilon} \ll R \leq X_1$, then

$$\sum_{2k \in [X_1, X_1+R]} \Big| \sum_{p_2+p_3=2k} \log p_2 \log p_3 - \mathfrak{S}(2k) 2k \Big|^2 \ll R X_1^2 L^{-A}.$$

The proof in [11] can be adapted in such a way that an additional short interval-condition can be put on one of the primes, namely it can be shown (cp. [11, Thm. 3]):

Theorem 2.3. *Let $Y^{1/3+\varepsilon} \ll R \ll Y \asymp 2X_1 - X_2 - Y \geq 0$, let $R \leq X_1$ and $X_2^{7/12+\varepsilon_2} \ll Y \leq X_2$, then*

$$\sum_{k \in [X_1, X_1+R]} \Big| \sum_{\substack{p_2+p_3=2k \\ p_2 \in [X_2, X_2+Y]}} \log p_2 \log p_3 - \mathfrak{S}(2k) Y \Big|^2 \ll R Y^2 L^{-A}.$$

The condition $Y \asymp 2X_1 - X_2 - Y$ comes from the necessary interval condition for the prime p_3, in the original proof in [11] this plays an important role in the application of Gallagher's Lemma and an argument due to Saffari and Vaughan in [13]. It is not easy to delete this condition. In contrast, the replacement of $2k$ by Y in the main term is an obvious adaption.

3. Proof of Theorem 1.1–Kawada-approach

We start by proving the following binary theorem with one prime in a given arithmetic progression and lying in a short interval.

Theorem 3.1. *Let $X_1^{2/3+\varepsilon} \ll Y \leq X_1$, let $Y \leq X_2 \leq 2X_1 - Y$, let $Q_1 \ll Y^{3/2} X_1^{-1/2}$ and $Q_2 \ll Y X_2^{-1/2} L^{-B}$. Then, for any fixed integers a_1*

and a_2, where $a_1 \le X_1$, we have

$$\sum_{q_1 \le Q_1} \sum_{\substack{k_1 \in [X_1, X_1+Y] \\ 2k_1 \equiv a_1 \ (q_1)}} \sum_{q_2 \le Q_2} \Big| \sum_{\substack{p_2+p_3=2k_1 \\ p_2 \equiv a_2 \ (q_2) \\ p_2 \in [X_2, X_2+Y]}} \log p_2 \log p_3 - \mathfrak{S}(2k_1, q_2, a_2) Y \Big| \ll Y^2 L^{-A}.$$

If $Q_1 \ll Y^{3/2} X_1^{-1/2}$ is replaced by $Q_1 \ll Y^{1/2}$, then the estimate holds true with $\max_{a_1 \ (q_1)}$ inserted after $\sum_{q_1}$.

Proof: By Lemma 2.1, for $a_1 \le X_1$ and since $Q_1^{2/3} X_1^{1/3} \ll Y$, we deduce from Kawada's Theorem 2.2:

$$\begin{aligned}
&\sum_{q_1 \le Q_1} \sum_{\substack{k_1 \in [X_1, X_1+Y] \\ 2k_1 \equiv a_1 \ (q_1)}} \sum_{q_2 \le Q_2} \Big| \sum_{\substack{p_2+p_3=2k_1 \\ p_2 \equiv a_2 \ (q_2) \\ p_2 \in [X_2, X_2+Y]}} \log p_2 \log p_3 - \mathfrak{S}(2k_1, q_2, a_2) Y \Big| \\
\ll & Y^{1/2} L^{3/2} \\
&\cdot \Big(\sum_{k_1 \in [X_1, X_1+Y]} \Big(\sum_{q_2 \le Q_2} \Big| \sum_{\substack{p_2+p_3=2k_1 \\ p_2 \equiv a_2 \ (q_2) \\ p_2 \in [X_2, X_2+Y]}} \log p_2 \log p_3 - \mathfrak{S}(2k_1, q_2, a_2) Y \Big| \Big)^2 \Big)^{1/2} \\
\ll & Y^{1/2} L^{3/2} \\
&\cdot \Big(Y L^2 \sum_{k_1 \in [X_1, X_1+Y]} \sum_{q_2 \le Q_2} \Big| \sum_{\substack{p_2+p_3=2k_1 \\ p_2 \equiv a_2 \ (q_2) \\ p_2 \in [X_2, X_2+Y]}} \log p_2 \log p_3 - \mathfrak{S}(2k_1, q_2, a_2) Y \Big| \Big)^{1/2} \\
\ll & Y^{1/2} L^{3/2} (Y L^2 Y^2 L^{-2A-5})^{1/2} \ll Y^2 L^{-A} \text{ by (2.4).}
\end{aligned}$$

Note that for $Q_1 \ll Y^{1/2}$, we can apply Lemma 2.2 in the same way to get an estimate which is uniform over the residues a_1; we get then

$$\sum_{q_1 \le Q_1} \max_{a_1 \ (q_1)} \sum_{\substack{k_1 \in [X_1, X_1+Y] \\ 2k_1 \equiv a_1 \ (q_1)}} \sum_{q_2 \le Q_2} \Big| \sum_{\substack{p_2+p_3=2k_1 \\ p_2 \equiv a_2 \ (q_2) \\ p_2 \in [X_2, X_2+Y]}} \log p_2 \log p_3 - \mathfrak{S}(2k_1, q_2, a_2) Y \Big| \ll Y^2 L^{-A}.$$

So Theorem 3.1 follows. □

Proof of Theorem 1.1:

Write down the estimate of Theorem 3.1, but where the singular series in (1.3) is replaced by the partial sum for $s \le L^C$. The estimate is still true since Theorem 2.2 is true in this form (see the treatment of S_2 in [5, §6]).

Further, restrict the summation over k_1 to the k_1 of the form $2k_1 = n-p_1$ with $p_1 \equiv a_1\ (q_1)$, this gives then

$$\sum_{\substack{q_1\le Q_1\\ q_2\le Q_2}} \sum_{\substack{p_1\equiv a_1\ (q_1)\\ p_1\in[X_1,X_1+Y]}} \Big| \sum_{\substack{p_2+p_3=n-p_1\\ p_2\equiv a_2\ (q_2)\\ p_2\in[X_2,X_2+Y]}} \log p_2 \log p_3 - \sum_{s\le L^C} H_s(n-p_1,a_2,q_2)Y \Big| \ll Y^2L^{-A},$$

with H_s as in (1.3).

Since we used Theorem 3.1 for the $2k_1$-interval $[n-X_1-Y, n-X_1]$, the assumptions $n\ge X_1+X_2+2Y$, $Y \gg (n-X_1)^{2/3+\varepsilon_1}$, $a_1\le n-X_1-Y$, $Q_2 \ll YX_2^{-1/2}L^{-B}$ and $Q_1 \ll Y^{3/2}(n-X_1)^{-1/2}$ are used (the latter one holds true since $n \ll X_1Y$, so $Q_1 \ll Y^{3/2}X_1^{-1/2}Y^{-1/2}L^{-B} \ll Y^{3/2}(n-X_1)^{-1/2}$).

In the previous estimate, we insert the weight $\log p_1$ and deduce that

$$\begin{aligned}\sum_{\substack{q_1\le Q_1\\ q_2\le Q_2}} \Big| &\sum_{\substack{p_1+p_2+p_3=n\\ p_i\equiv a_i\ (q_i), i=1,2\\ p_i\in[X_i,X_i+Y_i], i=1,2}} \log p_1\log p_2\log p_3 \\ &- \sum_{\substack{p_1\equiv a_1\ (q_1)\\ p_1\in[X_1,X_1+Y]}} \log p_1 \sum_{s\le L^C} H_s(n-p_1,a_2,q_2)Y\Big| \ll Y^2L^{-A}.\end{aligned}$$

Now the main term is

$$\begin{aligned}&= Y \sum_{\substack{p_1\equiv a_1\ (q_1)\\ p_1\in[X_1,X_1+Y]}} \log p_1 \sum_{s\le L^C} \frac{\mu(s)}{\varphi(s)\varphi([q_2;s])} \\ &\qquad\cdot \sideset{}{^*}\sum_{b\ (s)} e\Big(-(n-p_1)\frac{b}{s}\Big) \sideset{}{^*}\sum_{\substack{c\ (s)\\ c\equiv a_2((q_2,s))}} e\Big(\frac{bc}{s}\Big) \\ &= Y \sum_{\substack{p_1\equiv a_1\ (q_1)\\ p_1\in[X_1,X_1+Y]}} \log p_1 \sum_{s\le L^C} \frac{\mu(s)}{\varphi(s)\varphi([q_2;s])} \\ &\qquad\cdot \sideset{}{^*}\sum_{b\ (s)} \sideset{}{^*}\sum_{\substack{c\ (s)\\ c\equiv a_2((q_2,s))}} \sideset{}{^*}\sum_{\substack{d\ (s)\\ d\equiv a_1((q_1,s))\\ d\equiv p_1\ (s)}} e\Big(-(n-d-c)\frac{b}{s}\Big)\end{aligned}$$

$$
\begin{aligned}
&= Y \sum_{s \le L^C} \frac{\mu(s)}{\varphi(s)\varphi([q_2;s])} \\
&\qquad \cdot \sideset{}{^*}\sum_{b\,(s)} \sideset{}{^*}\sum_{\substack{c\,(s)\\ c\equiv a_2((q_2,s))}} \sideset{}{^*}\sum_{\substack{d\,(s)\\ d\equiv a_1((q_1,s))}} e\Big(-(n-d-c)\frac{b}{s}\Big) \sum_{\substack{p_1\in[X_1,X_1+Y]\\ p_1\equiv a_1\,(q_1)\\ p_1\equiv d\,(s)}} \log p_1 \\
&= Y^2 \sum_{s \le L^C} \frac{\mu(s) G(n;a_1,q_1,a_2,q_2,s)}{\varphi(s)\varphi([q_2;s])\varphi([q_1;s])} \\
&\qquad\qquad + \mathrm{O}\Big(Y \sum_{s\le L^C} \frac{\mu^2(s)}{\varphi([q_2;s])} \max^*_{f([q_1;s])} |\Delta(X_1, X_1+Y; [q_1;s], f)|\Big),
\end{aligned}
$$

with

$$
G(n;a_1,q_1,a_2,q_2,s) := \sideset{}{^*}\sum_{b\,(s)} \sideset{}{^*}\sum_{\substack{c\,(s)\\ c\equiv a_2((q_2,s))}} \sideset{}{^*}\sum_{\substack{d\,(s)\\ d\equiv a_1((q_1,s))}} e\Big(-(n-d-c)\frac{b}{s}\Big).
$$

The O-term gives an admissible error due to Theorem 2.1 (the result of Perelli, Pintz and Salerno [12]). For this, the assumptions $Q_1 \ll Y X_1^{-1/2} L^{-B}$ and $X_1^{3/5+\varepsilon_2} \ll Y$ are used, cp. also [4, Sec. 2.1].

In the other term, the partial sum with $s \le L^C$ can be replaced by the full singular series, giving an admissible error. This can also be proved as in [4, Sec. 2.2], where we just have to set $q_3 = 1$. In addition, there the singular series is obtained in Euler product form being the term given here:

It equals 0 if $(q_1, q_2, n-(a_1+a_2)) > 1$, and in the notation of [4], it can be given in its Euler product form as

$$
\frac{1}{2\varphi(q_1)\varphi(q_2)} \prod_{p,(A)\text{ or }(D)} \Big(1 - \frac{1}{(p-1)^2}\Big) \cdot \prod_{p,(B)} \Big(1 + \frac{1}{(p-1)^3}\Big) \cdot \prod_{p,(C)\text{ or }(F)} \frac{p}{p-1},
$$

where

$$
\begin{aligned}
&(A) \Leftrightarrow p \mid n, p \nmid q_1, p \nmid q_2 \qquad (B) \Leftrightarrow p \nmid n, p \nmid q_1, p \nmid q_2 \\
&(C) \Leftrightarrow p \mid n - a_1, p \mid q_1, p \nmid q_2 \text{ or } p \mid n - a_2, p \mid q_2, p \nmid q_1 \\
&(D) \Leftrightarrow p \nmid n - a_1, p \mid q_1, p \nmid q_2 \text{ or } p \nmid n - a_2, p \mid q_2, p \nmid q_1 \\
&(F) \Leftrightarrow p \mid q_1, p \mid q_2, p \nmid n - (a_1 + a_2).
\end{aligned}
$$

So we are done with Theorem 1.1. $\square$

4. Proof of Theorem 1.2–Perelli/Pintz-approach

We show:

Theorem 4.1. *Let* $X_2 \geq Y \gg X_2^{7/12+\varepsilon_1}$*, let* $Y^{1/3+\varepsilon_2} \ll R \leq Y$*, let* $Q \ll R^{3/2}X_1^{-1/2}$ *and* $Y \asymp X_1 - X_2 - Y \geq 0$*. Then, for any fixed positive integer* a *with* $a \leq X_1$*, we have*

$$\sum_{q\leq Q}\sum_{\substack{k_1\in[X_1,X_1+R]\\ 2k_1\equiv a\ (q)}}\Big|\sum_{\substack{p_2+p_3=2k_1\\ p_2\in[X_2,X_2+Y]}}\log p_2\log p_3-\mathfrak{S}(2k_1)Y\Big|\ll RYL^{-A},$$

whereas for $Q \ll R^{1/2}$ *instead of* $Q \ll R^{3/2}X_1^{-1/2}$*, the estimate holds true with* $\max_{a\ (q)}$ *inserted after the sum over* q*.*

Proof: We start with the following estimate from Theorem 2.3.

By Lemma 2.1, for fixed $a \leq X_1$ and since $Q^{2/3}X_1^{1/3} \ll R$, we have

$$\begin{aligned}(4.1)\quad &\sum_{q\leq Q}\sum_{\substack{k_1\in[X_1,X_1+R]\\ 2k_1\equiv a\ (q)}}\Big|\sum_{\substack{p_2+p_3=2k_1\\ p_2\in[X_2,X_2+Y]}}\log p_2\log p_3-\mathfrak{S}(2k_1)Y\Big|\\ &\ll R^{1/2}L^{3/2}\Big(\sum_{k_1\in[X_1,X_1+R]}\Big|\sum_{\substack{p_2+p_3=2k_1\\ p_2\in[X_2,X_2+Y]}}\log p_2\log p_3-\mathfrak{S}(2k_1)Y\Big|^2\Big)^{1/2}.\end{aligned}$$

By Theorem 2.3 above, the expression in brackets is $\ll RY^2L^{-A-3}$, and we get the desired estimate RYL^{-A} for the left hand side in Theorem 4.1.

Note that for $Q \ll R^{1/2}$, we can apply Lemma 2.2 in order to get an estimate being uniform for all residues a, namely in the same way we get

$$\sum_{q\leq Q}\max_{a\ (q)}\sum_{\substack{k_1\in[X_1,X_1+R]\\ 2k_1\equiv a\ (q)}}\Big|\sum_{\substack{p_2+p_3=2k_1\\ p_2\in[X_2,X_2+Y]}}\log p_2\log p_3-\mathfrak{S}(2k_1)Y\Big|\ll RYL^{-A}.$$

□

Proof of Theorem 1.2: Let us write Y_1 for R and Y_2 for Y. The proof now follows the same idea as in the proof of Theorem 1.1: Use estimate (4.1) with $2k_1$ of the form $2k_1 = n - p_1$ and multiply with the weight $\log p_1$. The necessary conditions for using (4.1) have been formulated in Theorem 1.2.

But we use (4.1) with the singular series $\mathfrak{S}(2k_1)$ in the main term replaced by its partial sum for $s \leq L^C$ (cp. (1.4)). This is possible, the contribution of the series for $s > L^C$ gives an admissible error, as shown in [11, p. 45].

So we deduce the estimate

(4.2)

$$\sum_{q\le Q}\Big|\sum_{\substack{p_1+p_2+p_3=n\\ p_i\in[X_i,X_i+Y_i],i=1,2\\ p_1\equiv a\ (q)}}\log p_1\log p_2\log p_3$$

$$-Y_2\sum_{\substack{p_1\in[X_1,X_1+Y_1]\\ p_1\equiv a\ (q)}}\log p_1\sum_{s\le L^C}\frac{\mu^2(s)}{\varphi^2(s)}\sideset{}{^*}\sum_{b\ (s)} e\Big(-(n-p_1)\frac{b}{s}\Big)\Big|\ll Y_1Y_2L^{-A}.$$

Here, the main term is

$$Y_2\sum_{\substack{p_1\in[X_1,X_1+Y_1]\\ p_1\equiv a\ (q)}}\log p_1\sum_{s\le L^C}\frac{\mu^2(s)}{\varphi^2(s)}\sideset{}{^*}\sum_{b\ (s)}\ \sideset{}{^*}\sum_{\substack{c\ (s)\\ c\equiv a\ ((q,s))\\ c\equiv p_1\ (s)}} e\Big(-(n-c)\frac{b}{s}\Big)$$

$$=Y_2\sum_{s\le L^C}\frac{\mu^2(s)}{\varphi^2(s)}\sideset{}{^*}\sum_{b\ (s)}\ \sideset{}{^*}\sum_{\substack{c\ (s)\\ c\equiv a\ ((q,s))}}\ \sum_{\substack{p_1\in[X_1,X_1+Y_1]\\ p_1\equiv a\ (q)\\ p_1\equiv c\ (s)}}\log p_1\, e\Big(-(n-c)\frac{b}{s}\Big)$$

$$=\sum_{s\le L^C}\frac{\mu^2(s)}{\varphi^2(s)}F(n;a,q,s)\frac{Y_1Y_2}{\varphi([q;s])}$$

$$+\mathrm{O}\Big(Y_2\sum_{s\le L^C}\frac{\mu^2(s)}{\varphi(s)}\max_{f\ ([q;s])}{}^{*}\,|\Delta(X_1,X_1+Y_1;[q;s],f)|\Big),$$

where

$$F(n;a,q,s):=\sideset{}{^*}\sum_{b\ (s)}\ \sideset{}{^*}\sum_{\substack{c\ (s)\\ c\equiv a\ ((q,s))}} e\Big(-(n-c)\frac{b}{s}\Big)$$

and where we used $(q,a)=1$ in the last step, what we can assume w.l.o.g., else (4.2) holds true clearly. Note that the version of (4.2) can be shown with $\max_{a\ (q)}$ inserted after the sum over q by the use of the supplement of Theorem 4.1 coming from Lemma 2.2.

The O-term is $\ll Y_1Y_2L^{-A}$ and therefore admissible since we may apply Theorem 2.1 for $Y_1\gg X_1^{3/5+\varepsilon}$ and $Q\ll Y_1X_1^{-1/2}L^{-B}$.

The error that comes now from the replacement of the partial sum $\sum_{s\le L^C}$ by the full singular series can be estimated to be admissible in exactly the same way as in [4], Section 2.2: there, let $q_1=q, a_1=a, q_2=q_3=1$ and the same proof applies here, too. And since the singular series sums up to the given one $\mathfrak{T}(n,q,a)$, we are done.

Further, the same proof with Lemma 2.2 instead of Lemma 2.1 gives the supplement of the theorem, and $Q \ll Y_1^{1/2}$ is needed then as assumption. $\square$

5. Proof of the Corollaries with sieve methods

For the proof of these Corollaries we proceed as in the proof of Meng [7]. This works in exactly the same way for Corollaries 1.2 and 1.3. We give an indication of these proofs now. They rely on the application of Theorem 9.3 in [3], there $\Lambda_s := s + 1 - \frac{\log(4/(1+3^{-s}))}{\log 3}$ with a natural number s, and it is assumed that we sieve for a finite set $\mathcal{A}$, where ξ serves as an approximation for $\#\mathcal{A}$ (this corresponds X in [3]).

Proof of Corollary 1.2.
Here we work with the sequence $\mathcal{A}$ of all $p_2 + 2$ with $k_1 = p_2 + p_3$ such that p_2 lies in the short interval $p_2 \in [X_2, X_2 + Y]$ with $Y = X_2^{\theta}$, and k_1 lies in the short interval $k_1 \in [X_1, X_1 + Y]$ with $Y \gg X_1^{2/3+\varepsilon}$. We have $\xi \gg YL^{-2}$. We use then Theorem 9.3 in [3], its conditions can be checked in the same way as in [7], where for (d) we have to apply Theorem 3.1 in the version (5.3) below with $Q_1 = 1$ instead. This works with $\alpha = 1-1/2\theta$, and $|a| \le \xi^{\alpha(\Lambda_s-\delta)}$ (this is Condition (9.3.6) in [3]) holds if $1/\theta < (1-1/2\theta)\Lambda_s$. This is true for $s = 3$ since $\Lambda_3 \ge 2.771$, and $\theta \ge 0.861$. $\square$

Proof of Corollary 1.4.
Consider X_i, Y_i, $i = 1, 2$, and n as given. Now we work with the sequence $\mathcal{A}$ of all $p_2 + 2$ with $n = p_1 + p_2 + p_3$ such that p_i lies in the short interval $p_i \in [X_i, X_i + Y_i]$, $i = 1, 2$. We have $\xi \gg Y_1 L^{-3}$. Write $Y_1 = Y_2^{\eta}$ with $\frac{1}{3} < \eta \le 1$. Again Theorem 9.3 in [3] applies with $\alpha = \min(1 - \frac{1}{2\theta_1}, \frac{3}{2} - \frac{1}{2\eta})$, this time with Theorem 1.2. Now since $|a| \le X_1 + Y_1 + 2 \ll X_1 \ll Y_1^{1/\theta_1}$, the condition $|a| \le \xi^{\alpha(\Lambda_s-\delta)}$ is true if $1 < \theta_1 \alpha \Lambda_s$.

First case: If $1 - \frac{1}{2\theta_1} < \frac{3}{2} - \frac{1}{2\eta}$, we have as in Corollary 1.2 before $s = 3$, $\theta_1 \ge 0.861$, then we have to choose $\eta > (1 + 1/\theta_1)^{-1} \ge 0.463$. Second case: Here η has to be $\eta \le 0.5$, so that $\theta_1 \le 1$ can be such that $1/\theta_1 \le 1/\eta - 1$. Also $\alpha\Lambda_s = (3/2-1/2\eta)\Lambda_s > 1$, what gives $\eta > 0.439$ for $s = 3$. So with η in this range, one can choose θ_1 such that $1/\theta_1 < \min(1/\eta-1, (3/2-1/2\eta)\Lambda_3)$; we will have then that $\theta_1 > 0.782$. $\square$

Now Corollary 1.3 is a consequence of Corollary 1.2:

Proof of Corollary 1.3.
Consider the number of $n-p_1$ such that $p_1+2 = P_2$ and $p_1 \in [X_1, X_1+Y]$. By Wu's Theorem in [17], we know that this number is $\gg YL^{-2}$ if $Y = X_1^{\theta_1}$

for $\theta_1 \geq 0.971$. So not all of them can be exceptions in Corollary 1.2, so there is at least one of them being the sum of two primes p_2 and p_3, where p_2 lies in a short interval of length $Y = X_2^{\theta_2}$ with $\theta_2 \geq 0.861$, such that $p_2 + 2 = P_3$. Note that Corollary 1.2 is applicable since we assumed that $X_1 + X_2 + 2Y \leq n \ll Y^{3/2-\varepsilon}$, so for the lower bound $n - X_1 - Y$ of the numbers $n - p_1$ we have $X_2 + Y \leq n - X_1 - Y \ll Y^{3/2-\varepsilon}$. □

Now for Corollary 1.1 we have to work in a slightly different style; therefore we here give the proof of Corollary 1.1 in detail. As sieve method we need Theorem 10.3 of Halberstam and Richert [3], which we present first: For this let $\mathcal{A}$ be a finite sequence of integers, $\mathcal{P}$ an infinite set of primes, and $\mathcal{A}_d$ the sequence of all $a \in \mathcal{A}$ with $d \mid a$. Further, for the number of elements in $\mathcal{A}_d$ we write

$$\#\mathcal{A}_d = \frac{\omega(d)}{d}\xi + R_d$$

with a multiplicative arithmetic function ω such that $\omega(p) = 0$ for $p \notin \mathcal{P}$. Let $L := \log \xi$, $\xi \geq 2$. We assume that $(a, p) = 1$ for any prime $p \notin \mathcal{P}$ and any $a \in \mathcal{A}$.

Theorem 5.1. *(Theorem 10.3 of* [3]*) Assume that*

(a) there exist a constant $A_1 > 0$ such that

$$1 \leq \frac{1}{1 - \frac{\omega(p)}{p}} \leq A_1$$

for all $p \in \mathcal{P}$,

(b) for a constant $\kappa > 1$ (the sieve dimension), a constant $A_2 \geq 1$ and for all real v, w with $2 \leq v \leq w$ we have

$$\sum_{\substack{v \leq p \leq w \\ p \in \mathcal{P}}} \frac{\omega(p)}{p} \log p \leq \kappa \log \frac{w}{v} + A_2,$$

(c) for a constant $A_3 > 0$ and for all real z, y with $2 \leq z \leq y \leq \xi$ we have

$$\sum_{\substack{z \leq p < y \\ p \in \mathcal{P}}} \#\mathcal{A}_{p^2} \leq A_3\Big(\frac{\xi L}{z} + y\Big),$$

(Any fixed power of L is here possible, too, as remarked by Halberstam and Richert [3]*),*

(d) for constants $0 < \alpha < 1$ and $A_4, A_5 > 0$ we have

$$\sum_{d < \frac{\xi^\alpha}{L^{A_4}}} \mu^2(d) 3^{\nu(d)} |R_d| \leq A_5 \frac{\xi}{L^{\kappa+1}}.$$

Further assume that there exists a real $\mu > 0$ *such that* $|a| \le \xi^{\alpha\mu}$ *for all* $a \in \mathcal{A}$. *Let* $\zeta \in \mathbb{R}$, $0 < \zeta < \nu_\kappa$ *for a certain real* $\nu_\kappa > 1$ *depending on* κ *only, and let* $r \in \mathbb{N}$ *with*

$$r > (1+\zeta)\mu - 1 + (\kappa+\zeta)\log\frac{\nu_\kappa}{\zeta} - \kappa - \zeta\frac{\mu-\kappa}{\nu_\kappa}. \tag{5.1}$$

Then there exists a $\delta = \delta(r,\mu,\kappa,\zeta) > 0$ *with*

$$|\{P_r;\, P_r \in \mathcal{A}\}| \ge \delta\frac{\xi}{(\log\xi)^\kappa}\Big(1 - \frac{C}{\sqrt{\log X}}\Big) \tag{5.2}$$

for a constant $C > 0$ *depending at most on* r, μ, ζ *(as well as on the* A_i*'s,* κ *and* α*).*

We use this theorem in the case $\kappa = 2$, for which the numerical value $\nu_\kappa = 4.42\ldots$ is known (see [3, (7.4.9)]).

We are going to apply Theorem 3.1 and Theorem 1.1, but we need them in non-weighted version. With an obvious partial summation, we can transform the first estimate in Theorem 3.1 into

(5.3)

$$\sum_{\substack{q_1\le Q_1\\ q_2\le Q_2}}\ \sum_{\substack{k_1\in[X_1,X_1+R]\\ 2k_1\equiv a_1\ (q_1)}}\Big|\sum_{\substack{p_2+p_3=2k_1\\ p_2\equiv a_2\ (q_2)\\ p_2\in[X_2,X_2+Y]}} 1 - \mathfrak{S}(2k_1,q_2,a_2)\int_{X_2}^{X_2+Y}\frac{dt}{\log t\log(k_1-t)}\Big| \ll Y^2L^{-A}.$$

A non-weighted version of the estimate in Theorem 1.1 is the following:

$$\sum_{\substack{q_1\le Q_1\\ q_2\le Q_2}}\Big|\sum_{\substack{p_1+p_2+p_3=n\\ p_i\in[X_i,X_i+Y_i],i=1,2\\ p_i\equiv a_i\ (q_i),i=1,2}} 1 - \mathfrak{T}(n,q_1,a_1,q_2,a_2)H(X_1,X_2,Y,n)\Big| \ll Y^2L^{-A}, \tag{5.4}$$

where

$$H(X_1,X_2,Y,n) := \int_{X_1}^{X_1+Y}\frac{1}{\log v}\int_{X_2}^{X_2+Y}\frac{dt}{\log t\log(n-v-t)}dv. \tag{5.5}$$

This version can be obtained in the same way as the original version of the Theorem, but where estimate (5.3) and the π-version of the Theorem of Perelli, Pintz and Salerno [12] is used, namely

$$\sum_{q\le Q}\ {\max_{a\ (q)}}^{*}\ \max_{h\le Y}\Big|\sum_{\substack{p\in[X,X+h]\\ p\equiv a\ (q)}} 1 - \frac{1}{\varphi(q)}\int_X^{X+h}\frac{dt}{\log t}\Big| \ll YL^{-A}.$$

This version can be gained from the original one in the same way like Bombieri-Vinogradov's theorem can be transformed from a ψ-version into a π-version, as done in [2].

Proof of Corollary 1.1.

To prepare the application of Theorem 5.1, we set $\mathcal{P} := \{p;\, p \neq 2\}$ and for large odd $n \not\equiv 1\,(6)$, where $n \geq X_1+X_2+2Y$. Let $\mathcal{A}$ denote the sequence of all $(p_1+2)(p_2+2)$ such that $p_1+p_2+p_3=n$ and $p_i \in [X_i, X_i+Y]$ for $i=1,2$ is solvable. So this sequence assigns the number $(p_1+2)(p_2+2)$ to each pair (p_1,p_2), and these pairs may be given in some order.

Then $(a,2)=1$ holds for all $a \in \mathcal{A}$. Let $\mathcal{A}_d$ denote the sequence of all elements $a \in \mathcal{A}$ with $d \mid a$.

Now

$$|\mathcal{A}_d| = \sum_{\substack{p_i\in[X_i,X_i+Y],i=1,2\\ p_1+p_2+p_3=n\\ (p_1+2)(p_2+2)\equiv 0\ (d)}} 1 = \sum_{t\mid d} \sum_{\substack{p_i\in[X_i,X_i+Y],i=1,2\\ p_1+p_2+p_3=n\\ p_1\equiv -2\ (t)\\ p_2\equiv -2\ (d/t)\\ ((p_1+2)/t,d/t)=1}} 1$$

$$= \sum_{t\mid d} \sum_{\substack{p_i\in[X_i,X_i+Y],i=1,2\\ p_1+p_2+p_3=n\\ p_1\equiv -2\ (t)\\ p_2\equiv -2\ (d/t)}} \sum_{\substack{s\mid (p_1+2)/t,\\ s\mid d/t}} \mu(s) = \sum_{t\mid d} \sum_{s\mid d/t} \mu(s) \sum_{\substack{p_i\in[X_i,X_i+Y],i=1,2\\ p_1+p_2+p_3=n\\ p_1\equiv -2\ (st)\\ p_2\equiv -2\ (d/t)}} 1$$

which we want to approximate by

$$\sum_{t\mid d} \sum_{s\mid d/t} \mu(s)\mathfrak{T}(n, st, -2, d/t, -2) H(X_1, X_2, Y, n).$$

So in view of Theorem 5.1, we write $|\mathcal{A}_d| = \frac{\omega(d)}{d}\xi + R_d$ with

$$\xi := \frac{1}{2} H(X_1,X_2,Y,n) \prod_{p\mid n}\Bigl(1-\frac{1}{(p-1)^2}\Bigr) \prod_{p\nmid n}\Bigl(1+\frac{1}{(p-1)^3}\Bigr)$$

and

$$\omega(d) := d\sum_{t\mid d}\sum_{s\mid d/t} \frac{\mu(s)}{\varphi(st)\varphi(d/t)} \prod_{\substack{p\nmid n+2, p\mid st, p\nmid d/t\\ \text{or } p\nmid n+2, p\nmid st, p\mid d/t}} \Bigl(1-\frac{1}{(p-1)^2}\Bigr) \tag{5.6}$$

$$\cdot \prod_{p\mid n, p\mid ds}\Bigl(1-\frac{1}{(p-1)^2}\Bigr)^{-1} \prod_{p\nmid n, p\mid ds}\Bigl(1+\frac{1}{(p-1)^3}\Bigr)^{-1}$$

$$\cdot \prod_{\substack{p\mid n+2, p\mid st, p\nmid d/t\\ \text{or } p\mid n+2, p\mid d/t, p\nmid st\\ \text{or } p\nmid n+4, p\mid st, p\mid d/t}} \Bigl(1+\frac{1}{p-1}\Bigr).$$

From this formula it can be seen that ω is multiplicative in d.

Now we are going to check all conditions of Theorem 5.1 in this setting.

By a computation, from (5.6) we can deduce that, for a prime $\ell \neq 2$ we have:

$$\omega(\ell) = \begin{cases} \frac{2\ell}{\ell-2} - \frac{1}{\ell-1}, & \text{if } \ell \mid n,\ \ell \nmid n+2,\ \ell \nmid n+4, \\ \frac{\ell^2(2\ell-3)}{(\ell-1)^3+1}, & \text{if } \ell \nmid n,\ \ell \mid n+2,\ \ell \nmid n+4, \\ \ell\frac{2(\ell-1)^2-1-\ell}{(\ell-1)^3+1}, & \text{if } \ell \nmid n,\ \ell \nmid n+2,\ \ell \mid n+4, \\ \ell\frac{2(\ell-1)^2-2-\ell}{(\ell-1)^3+1}, & \text{if } \ell \nmid n,\ \ell \nmid n+2,\ \ell \nmid n+4. \end{cases}$$

So it follows that (a) holds (note that the second case, where $\omega(3)/3 = 1$, does not occur for $\ell = 3$ since $n \not\equiv 1\ (6)$).

For condition (b) we see that for all $w \geq v \geq 2$, we have

$$\sum_{v \leq p \leq w} \frac{\omega(p)}{p} \log p \leq 2 \sum_{v \leq p \leq w} \frac{\log p}{p-2} \leq 2 \log \frac{w}{v} + A_2$$

for a $A_2 \geq 1$ since

$$\sum_{p \leq x} \frac{\log p}{p-2} = \sum_{p \leq x} \frac{\log p}{p} + 2 \sum_{p \leq x} \frac{\log p}{(p-2)p} = \log x + \mathrm{O}(1).$$

So $\kappa = 2$ works as sieve dimension.

For condition (c) we see that $|\mathcal{A}_{p^2}| \ll \frac{Y^2}{p^2} + 1$, so

$$\sum_{z \leq p < y} |\mathcal{A}_{p^2}| \ll \sum_{z \leq q < y} \Big(\frac{Y^2}{q^2} + 1 \Big) \ll \frac{Y^2}{z} \log y + y \ll \frac{\xi}{z} L^4 + y.$$

We are going to check now condition (d). Let $D = Y^{1-1/2\theta} L^{-B} = \xi^{\alpha} L^{-B}$, where $\alpha = \frac{1}{2} - \frac{1}{4\theta}$ and $\theta = \min(\theta_1, \theta_2)$. Abbreviate

$$A(q_1, q_2) := \sum_{\substack{p_i \in [X_i, X_i+Y], i=1,2 \\ p_1+p_2+p_3=n \\ p_1 \equiv -2\ (q_1) \\ p_2 \equiv -2\ (q_2)}} 1 - \mathfrak{T}(n, q_1, -2, q_2, -2) H(X_1, X_2, Y, n).$$

Then we have

$$\sum_{d\sim D}\mu^2(d)3^{\nu(d)}|R_d| = \sum_{\substack{d\sim D\\(d,2)=1}}\mu^2(d)3^{\nu(d)}\Big|\sum_{t|d}\sum_{s|d/t}\mu(s)A(st,d/t)\Big|$$

$$\ll \sum_{\substack{d\sim D\\(d,2)=1}}\mu^2(d)3^{\nu(d)}\sum_{t|d}\sum_{s|d/t}\frac{Y}{(sd)^{1/2}}|A(st,d/t)|^{1/2}$$

$$\ll Y\sum_{q_1,q_2\le 2D}\Big(\sum_{\substack{t|q_1\\(t,q_2)=1}}\mu^2(tq_2)3^{\nu(tq_2)}q_1^{-1/2}q_2^{-1/2}\Big)|A(q_1,q_2)|^{1/2}$$

$$\ll Y\Big(\sum_{q_1,q_2\le 2D}\frac{1}{q_1q_2}\sum_{\substack{t|q_1\\(t,q_2)=1}}\mu^2(tq_2)3^{2\nu(tq_2)}\Big)^{1/2}\Big(\sum_{q_1,q_2\le 2D}|A(q_1,q_2)|\Big)^{1/2}$$

$$\ll Y^2L^{-A-3}\ll \xi L^{-A},$$

using Theorem 1.1 in the version (5.4), and since

$$\sum_{q_1,q_2}\frac{1}{q_1q_2}\sum_{t|q_1}3^{2\nu(t)}3^{2\nu(q_2)}\ll\sum_{q_1}\frac{\tau(q_1)^5}{q_1}\sum_{q_2}\frac{\tau(q_2)^4}{q_2}\ll L^{48}.$$

Now we need to know whether $|a|\le\xi^{\alpha\mu}$ for all $a\in\mathcal{A}$ and some $r\ge 2$. We have $a=(p_1+2)(p_2+2)\le(X_1+Y+2)(X_2+Y+2)\ll Y^{2/\theta}$, so we have to take $\mu>4/(2\theta-1)>4$. Write $\mu=4+\Delta$, then the right hand side of (5.1) is <9 for $\zeta=0.360$ and $\Delta=0.628$. This gives the value $\theta\ge 0.933$.

$r=9$ is the smallest possible value, and also $\theta\ge 0.933$ is optimal such that $r\ge 9$ can be chosen.

Therefore Theorem 5.1 applies with $r=9$. From this we deduce that there always exists a P_9 in $\mathcal{A}$, their number is at least $\gg\xi/(\log\xi)^2$. So we are done with the proof of Corollary 1.1. □

Acknowledgments. The author would like to thank the referee for a careful reading and many valuable comments.

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