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The number of solutions to the generalized Pillai equation $\pm ra^x \pm sb^y = c$.

par Reese SCOTT et Robert STYER

Résumé. Nous considérons N, le nombre de solutions (x, y, u, v) de l'équation $(-1)^u ra^x + (-1)^v sb^y = c$ en nombres entiers non négatifs x, y, et nombres entiers $u, v \in \{0, 1\}$, pour des entiers donnés $a > 1$, $b > 1$, $c > 0$, $r > 0$ et $s > 0$. Lorsque $\text{pgcd}(ra, sb) = 1$, nous montrons que $N \leq 3$, sauf pour un nombre fini de cas qui satisfont à $\max(a, b, r, s, x, y) < 2 \cdot 10^{15}$ pour chaque solution ; lorsque $\text{pgcd}(a, b) > 1$, nous montrons que $N \leq 3$ sauf pour trois familles infinies de cas exceptionnels. Nous trouvons plusieurs façons de générer un nombre infini de cas donnant $N = 3$ solutions.

Abstract. We consider N, the number of solutions (x, y, u, v) to the equation $(-1)^u ra^x + (-1)^v sb^y = c$ in nonnegative integers x, y and integers $u, v \in \{0, 1\}$, for given integers $a > 1$, $b > 1$, $c > 0$, $r > 0$ and $s > 0$. When $\gcd(ra, sb) = 1$, we show that $N \leq 3$ except for a finite number of cases all of which satisfy $\max(a, b, r, s, x, y) < 2 \cdot 10^{15}$ for each solution; when $\gcd(a, b) > 1$, we show that $N \leq 3$ except for three infinite families of exceptional cases. We find several different ways to generate an infinite number of cases giving $N = 3$ solutions.

1. Introduction

In this paper we consider N, the number of solutions (x, y, u, v) to the equation

$$(-1)^u ra^x + (-1)^v sb^y = c \tag{1.1}$$

in nonnegative integers x, y and integers $u, v \in \{0, 1\}$, for given integers $a > 1$, $b > 1$, $c > 0$, $r > 0$ and $s > 0$. When $x \geq 1$, $y \geq 1$, and $(u, v) = (0, 1)$, Equation (1.1) is the familiar Pillai equation which has been treated by many authors (for example [1], [3], [5], [6], [15], [17], [18]). In [15] we treated (1.1) with various additional restrictions on x, y, u, v, a, b. In this paper, we treat (1.1) with no additional restrictions. Brief histories of the problem

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are given in [3] and [15], but see [2] and [19] for a much more extended history.

Pillai [10] showed that (1.1) has only finitely many solutions (x, y) when $(u, v) = (0, 1)$. The case in which u and v are unrestricted was also considered by Pillai [11] when $(a, b, r, s) = (2, 3, 1, 1)$.

Shorey [17] showed that (1.1) has at most nine solutions in positive integers (x, y) when $(u, v) = (0, 1)$ and the terms on the left side of (1.1) are large relative to c.

More recent results are given by (A), (B), (C), (D), and (E) which follow:

(A) $N \le 3$ when $x \ge 2$, $y \ge 2$, $(u, v) = (0, 1)$, and $\gcd(ra, sb) = 1$, except possibly when either a or b is less than e^e (Le [6]).

(B) $N \le 2$ when $x \ge 2$, $y \ge 2$, $(u, v) = (0, 1)$, and $\gcd(ra, sb) = 1$, except possibly when (a, b) is one of 23 exceptions (Bo He and A. Togbé [5]).

(C) $N \le 2$ when $x \ge 1$, $y \ge 1$, u and v are unrestricted, and $\gcd(ra, sb) = 1$, except for a finite number of cases which can be found in a finite number of steps and for which $\max(a, b, r, s, x, y) < 8 \cdot 10^{14}$ for each solution [15].

(D) $N \le 3$ when $x \ge 1$, $y \ge 1$, $(u, v) = (0, 1)$, and $\gcd(ra, sb) = 1$, with no exceptions (Bo He and A. Togbé [5]).

(E) $N \le 3$ when $x \ge 0$, $y \ge 0$, $(u, v) = (0, 1)$, and $\gcd(ra, sb)$ is unrestricted, with no exceptions [15].

In this paper we show that (E) still holds even when (u, v) is unrestricted, with exceptions which are either completely designated or findable in a finite number of steps; although this has already been done for the case $rs = 1$ (see [14], which generalizes [1]), we found the case $rs > 1$ (with u and v unrestricted) to be of interest since (unlike (E)) it requires establishing more general results which not only help to obtain $N \le 3$ but also allow us to say a good deal about the infinite number of cases for which $N = 3$ (see Lemmas 2.1–2.11 in Section 2 and the discussion in Section 5). In particular, we find that anomalous exceptional cases found in previous treatments of the problem can be placed in the context of predictable infinite families.

For the case $\gcd(ra, sb) = 1$, see Theorem 1.2 below, the proof of which uses a theorem of Matveev (see Lemma 4.3 in Section 4). For the remaining cases, we will need some simple observations and definitions and also a preliminary lemma:

Observation 1.1. The choice of x and y uniquely determines the choice of u and v.

Following Observation 1.1, we will usually refer to a solution (x, y).

Observation 1.2. There are at most two solutions to (1.1) having the same value of x, similarly for y.

In what follows we will often refer to a *set of solutions* to (1.1) which we will write as

$$(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$$

and by which we mean the (unordered) set of ordered pairs $\{(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)\}$, with $N > 2$, where each pair (x_i, y_i) gives a solution to (1.1) for given integers a, b, c, r, and s.

We now define our use of the word *family*. We say that two sets of solutions $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ and $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, \ldots, X_N, Y_N)$ belong to the same family if there exists a positive rational number k such that $kc = C$, and for every i there exists a j such that $kra^{x_i} = RA^{X_j}$ and $ksb^{y_i} = SB^{Y_j}$, $1 \le i, j \le N$.

Observation 1.3. It follows from the above definition of family that, if $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ and $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, \ldots, X_N, Y_N)$ are in the same family with $kc = C$, then, by Observation 1.1, for every i there exists a unique j such that $kra^{x_i} = RA^{X_j}$ and $ksb^{y_i} = SB^{Y_j}$, and for every j there exists a unique i such that $kra^{x_i} = RA^{X_j}$ and $ksb^{y_i} = SB^{Y_j}$, $1 \le i, j \le N$. Further, by Observation 1.2 (recalling $N > 2$), a and A are both powers of the same integer, and b and B are both powers of the same integer.

Lemma 1.1. *Every family contains a unique member* $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ *with the following properties:* $\gcd(r, sb) = \gcd(s, ra) = 1$*;* $\min(x_1, x_2, \ldots, x_N) = \min(y_1, y_2, \ldots, y_N) = 0$*; and neither a nor b is a perfect power.*

If a set of solutions has the properties listed in Lemma 1.1, we say it is in *basic form.*

Proof of Lemma 1.1. Suppose a family contains a member $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ in basic form, that is, with the properties $\gcd(r, sb) = \gcd(s, ra) = 1$, $\min(x_1, x_2, \ldots, x_N) = \min(y_1, y_2, \ldots, y_N) = 0$, and neither a nor b is a perfect power. Then there must exist at least one i, $1 \le i \le N$, such that $\gcd(ra^{x_i}, sb^{y_i}) = 1$. Assume $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, \ldots, X_N, Y_N)$ is another set of solutions in basic form belonging to the same family so that there exists a positive rational number k such that $kc = C$. Since for at least one i we have no common factor dividing all of ra^{x_i}, sb^{y_i}, c, and since, by the definition of family, kra^{x_i}, ksb^{y_i}, and kc, are all integers, we must have k also an integer. But then, if $k > 1$, there does not exist a j such that $\gcd(RA^{X_j}, SB^{Y_j}) = 1$ so that $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, \ldots, X_N, Y_N)$ cannot be in basic form. So $k = 1$, and there must exist a j such that $r = RA^{X_j}$ and $X_j = \min(X_1, X_2, \ldots, X_N)$; but $X_j = 0$, so that $r = R$. Similarly $s = S$. But then, since none of a, A, b, B is a perfect power, $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, \dots, X_N, Y_N)$ is not distinct from $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$. So we have shown that a given family contains at most one basic form.

It remains to show that each family contains at least one basic form. Now suppose a given family contains a set of solutions $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ which is *not necessarily* in basic form. For any two solutions (x_i, y_i) and (x_j, y_j), $1 \le i, j \le N$, we have

$$ra^{\min(x_i, x_j)}(a^{|x_j - x_i|} + (-1)^\gamma) = sb^{\min(y_i, y_j)}(b^{|y_j - y_i|} + (-1)^\delta) \tag{1.2}$$

where $\gamma, \delta \in \{0, 1\}$. We can choose (i, j) so that $\min(x_i, x_j) = \min(x_1, x_2, \dots, x_N) = x_0$ and $\min(y_i, y_j) = \min(y_1, y_2, \dots, y_N) = y_0$. For this choice of (i, j), we have

$$ra^{x_0}(a^t + (-1)^\gamma) = sb^{y_0}(b^w + (-1)^\delta) > 0$$

where $t = |x_j - x_i|$, $w = |y_j - y_i|$, and $\gamma, \delta \in \{0, 1\}$. Let $g = \gcd(ra^{x_0}, sb^{y_0})$ and $h = \gcd(a^t + (-1)^\gamma, b^w + (-1)^\delta)$. Then, taking $R = \frac{ra^{x_0}}{g} = \frac{b^w + (-1)^\delta}{h}$, $S = \frac{sb^{y_0}}{g} = \frac{a^t + (-1)^\gamma}{h}$, $C = \frac{c}{g}$, we obtain a set of solutions $(a, b, C, R, S; x_1 - x_0, y_1 - y_0, \dots, x_N - x_0, y_N - y_0)$ in the same family as the original set. In this set of solutions we can easily adjust a and b, if necessary, so that neither a nor b is a perfect power. The resulting set of solutions is in basic form except possibly when $\min(t, w) = 0$, in which case without loss of generality we can take $t = 0$, $w > 0$, and $\gamma = 0$ (recalling Observation 1.1). We need to show $\gcd(a, S) = 1$. Assume $\gcd(a, S) > 1$. Then $S = 2$, $2|a$, and R is odd. For each i, $1 \le i \le N$, we have $|Ra^{x_i - x_0} \pm Sb^{y_i - y_0}| = C$. Choosing i so that $x_i = x_0$, we get C odd, while choosing i so that $x_i > x_0$ (which we can do by Observation 1.2, noting that we have $N > 2$ by the definition of a set of solutions), we get C even, a contradiction which completes the proof of Lemma 1.1. $\square$

Now we define the *associate of a set of solutions* $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ to be the set of solutions $(b, a, c, s, r; y_1, x_1, y_2, x_2, \dots, y_N, x_N)$.

We are now ready to state the following results:

Theorem 1.1. *When* $\gcd(a, b) > 1$, (1.1) *has at most three solutions* (x, y, u, v) *where* x *and* y *are nonnegative integers and* $u, v \in \{0, 1\}$, *except for sets of solutions (or associates of sets of solutions) which are members of families containing one of the following basic forms:* $(2, 2, 4, 1, 3; 0, 0, 1, 1, 3, 2, 4, 2)$, $(2, 2, 3, 1, 1; 0, 1, 0, 2, 1, 0, 2, 0)$, $(6, 2, 8, 1, 7; 0, 0, 1, 1, 2, 2, 3, 5)$.

There are an infinite number of cases of three solutions to (1.1) *when* $\gcd(a, b) > 1$, *even if we consider only sets of solutions in basic form.*

Theorem 1.2. *When* $\gcd(ra, sb) = 1$*,* (1.1) *has at most three solutions* (x, y, u, v) *where* x *and* y *are nonnegative integers and* $u, v \in \{0, 1\}$*, except for a finite number of cases all of which can be found in a finite number of steps.*

More precisely, if (1.1) *has more than three solutions* (x, y, u, v) *when* $\gcd(ra, sb) = 1$*, then*

$$\max(a, b, r, s, x, y) < 2 \times 10^{15}$$

for each solution, and, further,

$$\max(r, s, x, y) < 8 \times 10^{14}, \min(\max(a, b), \max(b, c), \max(a, c)) < 8 \times 10^{14}$$

for each solution.

Theorem 1.3. *There are an infinite number of cases of exactly three solutions to* (1.1) *with* $\gcd(ra, sb) = 1$*, even if we consider only sets of solutions in basic form.*

In the case $\gcd(a, b) > 1$, Theorem 1.1 completely designates all exceptions. In the case $\gcd(a, b) = 1$, since any basic form for which $\gcd(a, b) = 1$ must also satisfy $\gcd(ra, sb) = 1$, Theorem 1.2 reduces the problem to a finite search for basic forms; another (somewhat lengthy) paper [16] completes the search using not only the methods of [4] and [18] as in previous work by the authors but also using LLL basis reduction. The completion of the search in [16] proves Theorem A below, for which we need one more definition:

We define a *subset of a set of solutions* $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ to be any set of solutions with the same a, b, c, r, s and with all its pairs (x, y) among the pairs (x_i, y_i), $1 \le i \le N$. Note that this subset may be (and, in our usage, usually is) the set of solutions $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ itself.

Theorem A. [16] Any set of solutions $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ to (1.1) with $N > 3$ must be in the same family as a subset (or an associate of a subset) of one of the following:

$$\begin{aligned}
&(3,2,1,1,2;0,0,1,0,1,1,2,2)\\
&(3,2,5,1,2;0,1,1,0,1,2,2,1,3,4)\\
&(3,2,7,1,2;0,2,2,0,1,1,2,3)\\
&(5,2,3,1,2;0,0,0,1,1,0,1,2,3,6)\\
&(5,3,2,1,1;0,0,0,1,1,1,2,3)\\
&(7,2,5,3,2;0,0,0,2,1,3,3,9)\\
&(6,2,8,1,7;0,0,1,1,2,2,3,5)\\
&(2,2,3,1,1;0,1,0,2,1,0,2,0)\\
&(2,2,4,3,1;0,0,1,1,2,3,2,4)
\end{aligned} \tag{1.3}$$

2. Preliminary Lemmas

Now we prove a number of results which deal with exceptional cases and which culminate in Lemma 2.11 at the end of this section. The treatment is essentially self-contained except for frequent use of results from [14].

Lemma 2.1. *If* $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ *is a set of solutions to* (1.1)*, then* $\min(x_i, y_i) = 0$ *for at most two choices of* i*,* $1 \le i \le N$*, except when* $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ *is in the same family as a subset (or an associate of a subset) of one of the following:*

$$(3, 3, 2, 1, 1; 0, 0, 0, 1, 1, 0),$$
$$(5, 2, 3, 1, 2; 0, 0, 0, 1, 1, 0, 1, 2, 3, 6),$$
$$(2, 2, 5, 1, 3; 0, 1, 1, 0, 3, 0),$$
$$(2, 2, 3, 1, 1; 0, 1, 0, 2, 1, 0, 2, 0).$$

Proof. It suffices to show that, if $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ is a set of solutions to (1.1) such that there are more than two values of i for which $\min(x_i, y_i) = 0$, the unique basic form in the same family is a subset (or an associate of a subset) of one of the listed exceptions in Lemma 2.1. By Observation 1.3, we can assume this basic form or its associate has a subset $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, X_3, Y_3)$ for which either

$$X_1 = Y_1 = X_2 = Y_3 = 0, X_3 > 0, Y_2 > 0, \tag{2.1}$$

or

$$X_1 = Y_2 = Y_3 = 0, X_3 > X_2 > 0, Y_1 > 0, \tag{2.2}$$

where $X_3 > X_2$ follows from Observation 1.1.

If (2.1) holds, then, considering (1.2) with $(A, B, R, S, X_i, Y_i, X_j, Y_j)$ replacing $(a, b, r, s, x_i, y_i, x_j, y_j)$ and with $(i, j) = (1, 2)$ and $(1, 3)$, respectively, and noting $\gcd(R, S) = 1$ by the definition of basic form, we obtain $S \le 2$ and $R \le 2$ (note that neither side of (1.2) can be zero). Suppose $R = S = 1$. Then $C = 2$ and, considering the solution (X_2, Y_2), we get $B = 3$ and $Y_2 = 1$, and, considering the solution (X_3, Y_3), we get $A = 3$ and $X_3 = 1$, giving the first exceptional case listed in Lemma 2.1; clearly there are no further solutions. If $R \ne S$, then by symmetry we can take $R = 1$ and $S = 2$. Then $C = 1$ or 3. If $C = 1$, then, considering the solution (X_2, Y_2), we have a contradiction to (2.1). So $C = 3$, and, considering the solution (X_2, Y_2), we get $B = 2$ and $Y_2 = 1$, and, considering the solution (X_3, Y_3), we get $A = 5$ and $X_3 = 1$; for this choice of (A, B, C, R, S), any further solutions (X, Y) must have $\min(X, Y) > 0$, so that, by Theorem 1 of [14], the only further solutions are $(X, Y) = (1, 2)$ and $(3, 6)$, giving the second exceptional case listed in the formulation of Lemma 2.1.

Now suppose (2.2) holds. Considering (1.2) as before with $(i,j) = (2,3)$, we get $R = 1$, $A = 2$, and $X_2 = 1$. Then considering (1.2) with $(i,j) = (1,2)$, we get either

$$S(B^{Y_1} + (-1)^{\delta}) = 3 \tag{2.3}$$

or

$$S(B^{Y_1} - 1) = 1. \tag{2.4}$$

Suppose (2.3) holds with $S = 3$. Then $B = 2$, $Y_1 = 1$, and $\delta = 1$. Considering the solution (X_2, Y_2), we get $C = 1$ or 5. Considering the solution (X_1, Y_1), we get $C = 7$ or 5. So $C = 5$, so that $X_3 = 3$, giving the third exceptional case listed in the formulation of Lemma 2.1; clearly no further solutions are possible.

Now suppose (2.3) holds with $S = 1$. Then $(B, Y_1, \delta) = (2, 1, 0)$ or $(2, 2, 1)$. $(B, Y_1, \delta) = (2, 1, 0)$ requires $C = 1$, making the solution (X_3, Y_3) impossible by Observation 1.1. $(B, Y_1, \delta) = (2, 2, 1)$ requires $C = 3$ and $X_3 = 2$. In this case there is a fourth solution $(X, Y) = (0, 1)$, giving the fourth exceptional case listed in the formulation of Lemma 2.1; clearly no further solutions are possible.

If (2.4) holds, then $S = 1$, $B = 2$, and $Y_1 = 1$. As in the immediately preceding case, we must have $C = 3$ and $X_3 = 2$, and there exists the further solution $(X, Y) = (0, 2)$, again giving the fourth exceptional case in the formulation of Lemma 2.1, which has no further solutions. □

From Lemma 2.1 we immediately have the following:

Corollary 2.1. If a set of solutions to (1.1) is not in the same family as a subset (or an associate of a subset) of one of the entries listed in Lemma 2.1, and, further, if this set of solutions has at least one x value equal to zero and at least one y value equal to zero, then, letting $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ be this set of solutions or its associate, we can assume one of the following holds:

$$x_1 = y_1 = y_2 = 0, x_i > 0 \text{ for } i > 1, y_i > 0 \text{ for } i > 2, \tag{2.5}$$

$$x_1 = y_2 = 0, x_i > 0 \text{ for } i > 1, y_1 > 0, y_i > 0 \text{ for } i > 2, \tag{2.6}$$

$$x_1 = y_1 = 0, x_i > 0 \text{ for } i > 1, y_i > 0 \text{ for } i > 1. \tag{2.7}$$

Lemma 2.2. *Let* $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ *be a set of solutions to* (1.1) *for which* $\gcd(a, b) > 1$ *and* $\min(x_1, x_2, \dots, x_N) = \min(y_1, y_2, \dots, y_N) = 0$. *If* $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ *is not in the same family as a subset (or an associate of a subset) of one of the entries listed in Lemma 2.1, then we must have* (2.7).

Proof. Let $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ be a set of solutions to (1.1) for which $\gcd(a, b) > 1$ and $\min(x_1, x_2, \dots, x_N) = \min(y_1, y_2, \dots, y_N) = 0$. Suppose $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ is not in the same family as a subset (or an associate of a subset) of one of the entries listed in Lemma 2.1. If Lemma 2.2 holds for a given set of solutions, it also holds for the associate of that set of solutions, so, by Corollary 2.1, we can assume that (2.5), (2.6), or (2.7) holds. Suppose that (2.5) or (2.6) holds. Then by Observation 1.3 we can assume $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ is a member of a family containing a basic form which has as a subset a basic form $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, X_3, Y_3)$ for which either

$$X_1 = Y_1 = Y_2 = 0, X_2 > 0, X_3 > 0, Y_3 > 0 \tag{2.8}$$

or

$$X_1 = Y_2 = 0, X_2 > 0, X_3 > 0, Y_1 > 0, Y_3 > 0. \tag{2.9}$$

Regardless of which of (2.8) or (2.9) holds, we have, considering (1.2) with $(i, j) = (2, 3)$,

$$RA^{\min(X_2, X_3)}(A^{|X_3 - X_2|} + (-1)^{\gamma}) = S(B^{Y_3} + (-1)^{\delta}). \tag{2.10}$$

Let p be a prime dividing both A and B. Then p divides the left side of (2.10) but not the right side of (2.10), since $\gcd(RA, S) = 1$ by the definition of basic form. This contradiction proves Lemma 2.2. $\square$

Observation 2.1. If p is a prime such that $p \mid a$ and $p \mid b$, and t is the greatest integer such that $p^t \mid c$, then, if (x, y) is a solution to (1.1), we must have $\min(x, y) \le t$.

Lemma 2.3. *Let $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ be a set of solutions satisfying* (2.7)*. Then no two of x_1, x_2, $\dots$, x_N are equal and no two of y_1, y_2, $\dots$, y_N are equal, except when $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ is in the same family as a subset (or an associate of a subset) of one of the following:*

$$(2, 2, 6, 1, 5; 0, 0, 2, 1, 4, 1),$$
$$(3, 3, 3, 1, 2; 0, 0, 1, 1, 2, 1),$$
$$(2, 6, 2, 1, 1; 0, 0, 2, 1, 3, 1),$$
$$(2, 2, 4, 1, 3; 0, 0, 3, 2, 4, 2, 1, 1).$$

Proof. By Observation 1.3, any set of solutions contradicting Lemma 2.3 must occur in the same family as a basic form (or the associate of a basic form) which has as a subset a basic form $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, X_3, Y_3)$ for which

$$X_1 = Y_1 = 0, X_3 > X_2 > 0, Y_2 = Y_3 > 0. \tag{2.11}$$

To prove Lemma 2.3, it suffices to find each possible basic form $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, X_3, Y_3)$ satisfying (2.11), determine its further solutions if any, and confirm that the list of exceptions in the formulation of Lemma 2.3 is complete.

Consider (1.2) with $(A, B, R, S, X_i, Y_i, X_j, Y_j)$ replacing $(a, b, r, s, x_i, y_i, x_j, y_j)$, taking $1 \le i, j \le 3$. Combining (1.2) for $(i, j) = (1, 2)$ with (1.2) for $(i, j) = (1, 3)$, we get

$$\frac{A^{X_3} + (-1)^{\gamma_3}}{A^{X_2} + (-1)^{\gamma_2}} = \frac{B^{Y_2} + (-1)^{\delta_3}}{B^{Y_2} + (-1)^{\delta_2}}, \tag{2.12}$$

where γ_2, γ_3, δ_2, δ_3 are in the set $\{0, 1\}$. Noting we cannot have $RA^{X_i} + SB^{Y_i} = C$ for $i > 1$, we see that we must have

$$|\gamma_2 - \delta_2| = |\gamma_3 - \delta_3|. \tag{2.13}$$

The left side of (2.12) must be greater than or equal to 1, with equality when and only when $A = 2$, $X_2 = 1$, $X_3 = 2$, $\gamma_2 = 0$, and $\gamma_3 = 1$. Therefore

$$(\delta_2, \delta_3) = (1, 0) \tag{2.14}$$

by (2.13) so that we have $\gamma_2 \ne \gamma_3$ and, considering the right side of (2.12) as a reduced fraction and letting m and n be positive integers, we find that the right side of (2.12) must equal either $\frac{n+2}{n}$ (when B is even) or $\frac{m+1}{m}$ (when B is odd). If $A^{X_3 - X_2} \ge 5$, then both sides of (2.12) are greater than 3, which is impossible. So we must have $2 \le A^{X_3 - X_2} \le 4$.

Assume first $A^{X_3 - X_2} = 4$. If $A^{X_2} \ge 8$, then both sides of (2.12) are greater than 3 which is impossible. If $A^{X_2} = 2$, then the left side of (2.12) must be either 9 or $7/3$, neither of which is a possible value for the right side of (2.12). So $A^{X_2} = 4$. If $(\gamma_2, \gamma_3) = (1, 0)$ then the left side of (2.12) equals $17/3$, which is again impossible. So we are left with $A = 2$, $X_2 = 2$, $X_3 = 4$, $\gamma_2 = 0$, $\gamma_3 = 1$, $B = 2$, $Y_2 = 1$, $\delta_2 = 1$, and $\delta_3 = 0$. Considering (1.2) with $(i, j) = (1, 2)$ and recalling $\gcd(R, S) = 1$ since we are dealing with a basic form, we find $R = 1$, $S = 5$, and $C = 6$, giving the first exceptional case in the formulation of the lemma, which has no further solutions by Observation 2.1.

Now assume $A^{X_3 - X_2} = 3$. If $A^{X_2} \ge 9$, then the left side of (2.12) is greater than 2 but not equal to 3, which is impossible by (2.14). So we have $A = 3$, $X_2 = 1$, and $X_3 = 2$. Since neither side of (2.12) can be greater than 3, we must have $(\gamma_2, \gamma_3) = (0, 1)$, giving $B = 3$ and $Y_2 = 1$. As in the previous paragraph, we find R, S, and C, obtaining the second exceptional case in the formulation of the lemma; applying Observation 2.1, we find that there are no further solutions.

It remains to consider $A^{X_3 - X_2} = 2$. Consider first the case $(\gamma_2, \gamma_3) = (0, 1)$. If $A^{X_2} \ge 16$, the left side of (2.12) is greater than $5/3$ and less than 2, which is impossible by (2.14). If $X_2 = 1$, the left side of (2.12) is equal to 1, contradicting (2.14). So we are left with either $X_2 = 2$ or $X_2 = 3$, so that, respectively, $B^{Y_2} = 6$ or $B^{Y_2} = 4$. In each of these two cases, we find R, S, and C as in the preceding cases: when $X_2 = 2$, we obtain the third exceptional case in the formulation of the lemma, which has no further solutions by Observation 2.1; when $X_2 = 3$, we obtain a subset of the fourth exceptional case, which has no further solutions by Observation 2.1.

Finally consider the case $A^{X_3-X_2} = 2$ with $(\gamma_2, \gamma_3) = (1, 0)$. If $A^{X_2} \geq 8$, then the left side of (2.12) is greater than 2 and less than 3, contradicting (2.14). If $X_2 = 1$, the left side of (2.12) equals 5, again contradicting (2.14). So we are left with $X_2 = 2$, so that $B^{Y_2} = 2$, and, proceeding as in the preceding paragraphs, we obtain $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, X_3, Y_3) = (2, 2, 2, 1, 3; 0, 0, 2, 1, 3, 1)$, which has no further solutions by Observation 2.1 and which is in the same family as $(2, 2, 4, 1, 3; 1, 1, 3, 2, 4, 2)$, which is a subset of the fourth exceptional case listed in the formulation of the lemma. ☐

To obtain results similar to that of Lemma 2.3 for the cases (2.5) and (2.6), we will need the following

Lemma 2.4. *Suppose* $\gcd(ra, sb) = 1$ *and* (1.1) *has two solutions* (x_1, y_1) *and* (x_2, y_2) *with* $x_1 = 0$ *and* $x_2 > 0$*. Then, if* a *is even,* r *must be even.*

Proof. If a is even, then the solution (x_2, y_2) requires sb odd and c odd, so that the solution (x_1, y_1) requires r even. ☐

Corollary 2.2. Suppose $\gcd(ra, sb) = 1$ and (1.1) has three solutions (x_1, y_1), (x_2, y_2), (x_3, y_3) with $x_1 < x_2$ and $x_1 < x_3$. Then $y_2 \neq y_3$.

Proof. Suppose $\gcd(ra, sb) = 1$ and (1.1) has three solutions (x_1, y_1), (x_2, y_2), (x_3, y_3) with $x_1 < x_2$ and $x_1 < x_3$. Suppose $y_2 = y_3$. Then considering (1.2) with $(i, j) = (2, 3)$, we find $ra^{\min(x_2, x_3)} | 2$, so that $a = 2$, $r = 1$, $\min(x_2, x_3) = 1$, and $x_1 = 0$, contradicting Lemma 2.4. ☐

We are now ready to prove Lemmas 2.5 and 2.6 which follow.

Lemma 2.5. *Let* $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ *be a set of solutions for which* (2.5) *holds. Then no two of* x_1, x_2, $\ldots$, x_N *are equal and, except for* $y_1 = y_2$*, no two of* y_1, y_2, $\ldots$, y_N *are equal, unless* $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ *is in the same family as a subset of one of the following:*

$$(3, 5, 2, 1, 1; 0, 0, 1, 0, 1, 1, 3, 2),$$
$$(3, 2, 1, 1, 2; 0, 0, 1, 0, 1, 1, 2, 2),$$
$$(5, 2, 3, 1, 2; 0, 0, 1, 0, 1, 2, 3, 6),$$
$$(2, 7, 3, 2, 1; 0, 0, 1, 0, 1, 1).$$

Proof. Let $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ be a set of solutions satisfying (2.5), and assume there exists at least one pair (i, j) where $2 \le i < j \le N$ such that either $x_i = x_j$ or $y_i = y_j$. Then by Observation 1.3 $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ must be in the same family as a basic form $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, \ldots, X_N, Y_N)$ for which $X_1 = Y_1 = Y_2 = 0$, $X_i > 0$ for $i > 1$, $Y_i > 0$ for $i > 2$, and there exists at least one pair (i, j), where $2 \le i < j \le N$, for which $X_i = X_j$ or $Y_i = Y_j$. By Lemma 2.2 we can assume $\gcd(a, b) = \gcd(A, B) = 1$. By the definition of basic form, we have $\gcd(RA, SB) = 1$. By Corollary 2.2, we can assume $X_i \ne X_j$ and $Y_i \ne Y_j$ for every pair (i, j) such that $\min(i, j) \ge 3$. Thus we can assume without loss of generality that $X_2 = X_3$, so that $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, \ldots, X_N, Y_N)$ has as a subset a basic form $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, X_3, Y_3)$ for which

$$X_1 = Y_1 = Y_2 = 0, X_2 = X_3 > 0, Y_3 > 0. \tag{2.15}$$

It suffices to find each possible basic form $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, X_3, Y_3)$ satisfying (2.15), determine its further solutions if any, and thus verify the list of exceptions in the formulation of Lemma 2.5.

Assume (2.15) holds, and consider (1.2) with $(A, B, R, S, X_i, Y_i, X_j, Y_j)$ replacing $(a, b, r, s, x_i, y_i, x_j, y_j)$. Take $1 \le i, j \le 3$. Then considering (1.2) with $(i, j) = (2, 3)$, we have $S \le 2$. And considering (1.2) with $(i, j) = (1, 2)$, we have $R \le 2$.

If $R = S = 1$, then, considering the solution (X_1, Y_1) to (1.1), we find $C = 2$. Considering the solution (X_2, Y_2), we find $A = 3$ and $X_2 = 1$. Considering the solution (X_3, Y_3), we find $B = 5$ and $Y_3 = 1$. Clearly, when $(A, B, C, R, S) = (3, 5, 2, 1, 1)$, any further solution to

$$(-1)^U RA^X + (-1)^V SB^Y = C, \tag{2.16}$$

where $U, V \in \{0, 1\}$, must have

$$X > 0, Y > 0, \tag{2.17}$$

and

$$(U, V) \ne (0, 0). \tag{2.18}$$

By Theorem 7 of [14], the only further solution is $(X_4, Y_4) = (3, 2)$, giving the first exceptional case listed in the formulation of Lemma 2.5.

If $R = 1$ and $S = 2$, then, considering (2.16) with $(X, Y) = (X_1, Y_1)$, we have either $C = 1$ or $C = 3$. If $C = 1$ then, considering the solution (X_2, Y_2), we find $A = 3$ and $X_2 = 1$. Considering the solution (X_3, Y_3), we find $B = 2$ and $Y_3 = 1$. Any further solution must satisfy (2.17) and (2.18). By Theorem 7 of [14], the only remaining solution is $(X_4, Y_4) = (2, 2)$, giving the second exceptional case in the formulation of Lemma 2.5. So we now consider $C = 3$. Considering the solution (X_2, Y_2), we find $A = 5$ and $X_2 = 1$. Considering (X_3, Y_3), we find $B = 2$ and $Y_3 = 2$. Any further solution must satisfy (2.18), and, since we are dealing with (2.5), it suffices to consider only further solutions which satisfy (2.17). Theorem 7 of [14] shows that the only further solution with $\min(X, Y) > 0$ is $(X_4, Y_4) = (3, 6)$, giving the third exceptional case in the formulation of Lemma 2.5.

Finally, considering $R = 2$ and $S = 1$, we find $C = 1$ or $C = 3$ again. If $C = 1$, then, considering the solution (X_2, Y_2), we find $X_2 = 0$, a contradiction. So we must have $C = 3$. Considering the solution (X_2, Y_2), we find $A = 2$ and $X_2 = 1$. Considering (X_3, Y_3), we find $B = 7$ and $Y_3 = 1$. Any further solution must satisfy (2.17) and (2.18). Once again we can use Theorem 7 of [14], this time finding there are no further solutions; we obtain the fourth exceptional case listed in the formulation of Lemma 2.5. □

Lemma 2.6. *Let $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ be a set of solutions for which* (2.6) *holds. Then no two of x_1, x_2, $\ldots$, x_N are equal and no two of y_1, y_2, $\ldots$, y_N are equal, unless $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ is in the same family as a subset (or an associate of a subset) of one of the following:*

$$(3, 2, 5, 1, 2; 0, 1, 1, 0, 1, 2, 2, 1, 3, 4), \tag{2.19}$$

$$(5, 3, 4, 1, 1; 0, 1, 1, 0, 1, 2), \tag{2.20}$$

$$(3, 2, 7, 1, 2; 0, 2, 2, 0, 2, 3, 1, 1), \tag{2.21}$$

$$(5, 2, 3, 1, 2; 0, 1, 1, 0, 1, 2, 3, 6), \tag{2.22}$$

$$(2^g + (-1)^\epsilon, 2, 2^g - (-1)^\epsilon, 1, 2; 0, g-1, 1, 0, 1, g). \tag{2.23}$$

where $\epsilon \in \{0, 1\}$ and $g > 1$ is a positive integer.

Proof. Proceeding as in the proof of Lemma 2.5, we see that it suffices to find each possible basic form $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, X_3, Y_3)$ with $\gcd(RA, SB) = 1$ for which

$$X_1 = Y_2 = 0, X_2 = X_3 > 0, Y_1 > 0, Y_3 > 0, \tag{2.24}$$

determine further solutions if any, and thus verify the list of exceptions in the formulation of Lemma 2.6. Note that we can use (2.24) without loss of generality, since, if Lemma 2.6 holds for the associate of a set of solutions, it holds for the set of solutions itself.

Suppose (2.24) holds. Considering (1.2) with $(i, j) = (2, 3)$, we get

$$RA^{X_2} = \frac{S}{2}\left(B^{Y_3} + (-1)^\delta\right) \tag{2.25}$$

where $\delta \in \{0, 1\}$, so that

$$C = RA^{X_2} - (-1)^\delta S = SB^{Y_3} - RA^{X_2}. \tag{2.26}$$

Note that (2.25) implies $S \le 2$.

Combining (2.25) and (2.26), we find

$$C = \frac{S}{2}\left(B^{Y_3} - (-1)^\delta\right). \tag{2.27}$$

Suppose $Y_1 \geq Y_3$. Then $SB^{Y_1} \geq SB^{Y_3} > RA^{X_2} > R$, so that $C = SB^{Y_1} \pm R > SB^{Y_3} - RA^{X_2} = C$, a contradiction. So we have a positive integer $d = Y_3 - Y_1$. Considering the solution (X_1, Y_1) and using (2.27), we find that one of the following three equations must hold:

$$R = \frac{S}{2}\left(\left(B^d - 2\right)B^{Y_1} - (-1)^\delta\right), \tag{2.28}$$

$$R = \frac{S}{2}\left(\left(B^d + 2\right)B^{Y_1} - (-1)^\delta\right), \tag{2.29}$$

$$R = \frac{S}{2}\left(\left(2 - B^d\right)B^{Y_1} + (-1)^\delta\right). \tag{2.30}$$

Suppose (2.28) holds. Then, since $\frac{2}{S}R$ divides $\frac{2}{S}RA^{X_2} - \frac{2}{S}R$, we have, using (2.25) and (2.28),

$$\left(B^d - 2\right)B^{Y_1} - (-1)^\delta \mid 2B^{Y_1} + (-1)^\delta 2 > 0. \tag{2.31}$$

From this we get

$$\left(B^d - 4\right)B^{Y_1} \leq 3,$$

so that $B^d \leq 5$. If $B^d = 5$, then $B^{Y_1} = 2$ or 3, which is impossible. So we have $2 \leq B^d \leq 4$.

If $B^d = 2$, then, by Lemma 2.4, $S = 2$, and by (2.28), $R = 1$ and $\delta = 1$; further, by (2.25) we have $A^{X_2} = 2^{Y_1+1} - 1$ and by (2.27) we have $C = 2^{Y_1+1} + 1$. It is a well known elementary result that we must have $X_2 = 1$. Thus we find that $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, X_3, Y_3)$ must be (2.23) with $\epsilon = 1$ and $g = Y_1 + 1$. To see if there are any further solutions, we consider solutions (X_i, Y_i) with $i > 1$ and apply Theorem 1 of [14], noting that c must be a Fermat number greater than 3; we find that the only cases with further solutions are given by subsets of (2.19).

Now suppose $B^d = 3$. Then, from (2.31), we get

$$3^{Y_1} - (-1)^\delta \mid 2 \cdot 3^{Y_1} + (-1)^\delta 2 = 2 \cdot 3^{Y_1} - (-1)^\delta 2 + (-1)^\delta 4,$$

so that $3^{Y_1} - (-1)^\delta \mid 4$, so that $Y_1 = 1$. If $\delta = 0$, we find, using (2.28), (2.27), and (2.25), that $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, X_3, Y_3)$ must be (2.20), which has no further solutions by Theorem 7 of [14] (noting that we cannot have $RA^{X_4} + SB^{Y_4} = C$). If $\delta = 1$, we find that $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, X_3, Y_3) = (2, 3, 5, 2, 1; 0, 1, 1, 0, 1, 2)$; by Theorem 1 of [14], we see that there are only two further solutions, giving (2.19) with the roles of A and B reversed.

Now suppose $B^d = 4$. From (2.31) we have

$$2B^{Y_1} - (-1)^\delta \mid 2B^{Y_1} + (-1)^\delta 2 = 2B^{Y_1} - (-1)^\delta + (-1)^\delta 3,$$

so that $2B^{Y_1} - (-1)^\delta \mid 3$, so that, since $Y_1 > 0$, we must have $(B, Y_1, \delta) = (2, 1, 0)$, from which we find that $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, X_3, Y_3) =$

$(3,2,7,3,2;0,1,1,0,1,3)$, which has no further solutions by Theorem 1 of [14], and which is in the same family as $(3,2,7,1,2;1,1,2,0,2,3)$, which is a subset of (2.21). So it suffices to consider (2.29) and (2.30).

Suppose (2.29) holds. Then $0 < \frac{2}{S}RA^{X_2} - \frac{2}{S}R = -2B^{Y_1} + (-1)^{\delta}2$, which is impossible.

Finally, suppose (2.30) holds. Then $B^d = 2$, $S = 2$, $R = 1$, and $\delta = 0$. Further, by (2.25) we have $A^{X_2} = 2^{Y_1+1} + 1$ and by (2.27) we have $C = 2^{Y_1+1} - 1$. Here we have the possibility $X_2 = 2$, which gives $Y_1 = 2$, $A = 3$, and $C = 7$ and we obtain $(A,B,C,R,S;X_1,Y_1,X_2,Y_2,X_3,Y_3) = (3,2,7,1,2;0,2,2,0,2,3)$, which has as its only further solution $(X,Y) = (1,1)$ by Theorem 1 of [14], so we obtain (2.21). If $X_2 \neq 2$, then we have $X_2 = 1$ and find that $(A,B,C,R,S;X_1,Y_1,X_2,Y_2,X_3,Y_3)$ must be (2.23) with $\epsilon = 0$ and $g = Y_1 + 1$; we apply Theorem 1 of [14] as above to see that, when $\epsilon = 0$ and $2^g + 1$ is not a perfect power, the only case of (2.23) allowing further solutions is given by (2.22). (Note that (2.22) also has the solution $(0,0)$ which does not need to be taken into account here, since we are dealing with (2.6).) □

Lemma 2.7. *Let $(a,b,c,r,s;x_1,y_1,x_2,y_2,\dots,x_N,y_N)$ be a basic form satisfying* (2.5) *with no two of x_2, x_3, …, x_N equal, and no two of y_3, …, y_N equal. Then $x_2 = \min(x_2,x_3,\dots,x_N)$ except when $(a,b,c,r,s;x_1,y_1,x_2,y_2,\dots,x_N,y_N)$ is a subset of either* $(3,2,5,1,4;0,0,2,0,1,1,3,3)$ *or* $(2,3,5,2,3;0,0,2,0,1,1,4,2)$.

Proof. Let $(a,b,c,r,s;x_1,y_1,x_2,y_2,\dots,x_N,y_N)$ be a basic form satisfying (2.5) with no two of x_2, x_3, …, x_N equal, and no two of y_3, …, y_N equal, and suppose $x_2 \neq \min(x_2,x_3,\dots,x_N)$. Without loss of generality take $x_3 = \min(x_2,x_3,\dots,x_N)$. Combining (1.2) for $(i,j) = (1,2)$ and (1.2) for $(i,j) = (1,3)$, we get

$$\frac{a^{x_3} + (-1)^{\gamma_3}}{a^{x_2} + (-1)^{\gamma_2}} = \frac{b^{y_3} + (-1)^{\delta_3}}{2}, \tag{2.32}$$

where γ_2, γ_3, and δ_3 are in the set $\{0,1\}$. Since $0 < x_3 < x_2$, the left side of (2.32) is less than or equal to one, with equality when and only when $a = 2$, $x_2 = 2$, $x_3 = 1$, $\gamma_2 = 1$, and $\gamma_3 = 0$. The right side of (2.32) is an integer or half integer greater than or equal to 1/2. So we have both sides of (2.32) equal either to 1/2 or to 1.

If both sides of (2.32) equal 1/2, we must have $b^{y_3} = 2$ and $\delta_3 = 1$. If also $\gamma_3 = 1$, the left side of (2.32) is less than 1/2, so we have $\gamma_3 = 0$. If $a^{x_2 - x_3} = 2$, the left side of (2.32) is greater than 1/2; if $a^{x_2-x_3} \geq 4$, the left side of (2.32) is less than 1/2. So we have $a^{x_2-x_3} = 3$, in which case, if $x_3 > 1$, the left side of (2.32) is less than 1/2. This leaves as the only possibility $a = 3$, $x_3 = 1$, $x_2 = 2$, $\gamma_3 = 0$, $\gamma_2 = 1$, $b = 2$, $y_3 = 1$, $\delta_3 = 1$. Considering (1.2) with $(i,j) = (1,2)$, we find $r/s = 1/4$. By the definition of basic form, $\gcd(r,s) = 1$, so $r = 1$, $s = 4$, and $c = 5$, giving $(a,b,c,r,s;x_1,y_1,x_2,y_2,x_3,y_3) = (3,2,5,1,4;0,0,2,0,1,1)$; if there exists a further solution (x_4,y_4) we must have $x_4 > 0$, so that Theorem 1 of [14], in combination with the fact that $3+2=5$, shows that the only possible further solution is $(x_4,y_4) = (3,3)$, giving the first exceptional case listed in the formulation of the lemma.

Now suppose both sides of (2.32) are equal to 1. Then we must have $a = 2$, $x_2 = 2$, $x_3 = 1$, $\gamma_2 = 1$, $\gamma_3 = 0$, $b = 3$, $y_3 = 1$, $\delta_3 = 1$. Considering (1.2) with $(i,j) = (1,2)$, we find $r/s = 2/3$, so that $r = 2$, $s = 3$, and $c = 5$, giving $(a,b,c,r,s;x_1,y_1,x_2,y_2,x_3,y_3) = (2,3,5,2,3;0,0,2,0,1,1)$; by Theorem 1 of [14] there is only one possible further solution $(x_4,y_4) = (4,2)$, giving the second exceptional case listed in the lemma. □

Lemma 2.8. *Let* $(a,b,c,r,s;x_1,y_1,x_2,y_2,\dots,x_N,y_N)$ *be a basic form satisfying* (2.6) *with no two of* x_2, x_3, $\dots$, x_N *equal and no two of* y_1, y_3, y_4, $\dots$, y_N *equal. Then* $x_2 = \min(x_2,x_3,\dots,x_N)$ *and* $y_1 = \min(y_1,y_3,y_4,\dots,y_N)$, *except when* $(a,b,c,r,s;x_1,y_1,x_2,y_2,\dots,x_N,y_N)$ *or its associate is* $(3,2,7,1,2;0,2,2,0,1,1)$.

Proof. Assume $(a,b,c,r,s;x_1,y_1,x_2,y_2,\dots,x_N,y_N)$ is a basic form satisfying (2.6) with no two of x_2, x_3, $\dots$, x_N equal and no two of y_1, y_3, y_4, $\dots$, y_N equal, and assume further that $x_2 \neq \min(x_2,x_3,\dots,x_N)$. Without loss of generality we can take $x_3 = \min(x_2,x_3,\dots,x_N)$. Choose $i \in \{1,3\}$ so that $y_i = \max(y_1,y_3)$, and let $n = 1$ or 2 according as $i = 3$ or 1. Let $k = s/c$. Then

$$kc = s \geq ra^{x_2} - c \geq a^n ra^{x_i} - c \geq a^n(sb^{y_i} - c) - c \geq a^n(b^2kc - c) - c$$

so

$$kc \geq (a^n b^2 k - (a^n+1))c \tag{2.33}$$

so that

$$k \leq \frac{a^n+1}{a^n b^2 - 1}. \tag{2.34}$$

Note that

$$c > \left(\frac{1+k}{a}\right)c = \frac{c+s}{a} \geq \frac{ra^{x_2}}{a} \geq ra^{x_3} \tag{2.35}$$

Assume $i = 3$. Then

$$b \leq \frac{sb^{y_3}}{sb^{y_1}} \leq \frac{c + ra^{x_3}}{c - r} \leq \frac{c + \frac{ra^{x_2}}{a}}{c - \frac{ra^{x_2}}{a^2}} \leq \frac{c + \frac{(1+k)c}{a}}{c - \frac{(1+k)c}{a^2}} = \frac{1 + \frac{1+k}{a}}{1 - \frac{1+k}{a^2}}. \tag{2.36}$$

If $a \geq 3$ then (2.34) gives $k \leq 4/11$ so that (2.36) gives $b < 2$, a contradiction. So $a = 2$, (2.34) gives $k \leq 3/17$, so that (2.36) gives $b < 3$, again a contradiction.

So we must have $i = 1$, $n = 2$, giving

$$b \leq \frac{sb^{y_1}}{sb^{y_3}} \leq \frac{c+r}{c-ra^{x_3}} \leq \frac{c+\frac{ra^{x_2}}{a^2}}{c-\frac{ra^{x_2}}{a}} \leq \frac{c+\frac{(1+k)c}{a^2}}{c-\frac{(1+k)c}{a}} = \frac{1+\frac{1+k}{a^2}}{1-\frac{1+k}{a}}. \tag{2.37}$$

Suppose $a \geq 3$. Then (2.34) gives $k \leq 2/7$ and (2.37) gives $b \leq 2$, with equality only when $a = 3$ and $s = (2/7)c$, so that, since $\gcd(s, c) = 1$, we have $s = 2$ and $c = 7$, so that, considering the solution (x_2, y_2) and noting $x_2 > 1$, we get $r = 1$ and $x_2 = 2$, so that $x_3 = 1$, and, considering the solution (x_1, y_1), we get $y_1 = 2$, so that $y_3 = 1$, giving the exceptional set of solutions in the formulation of the lemma.

If $a = 2$, (2.34) gives $k \leq 1/7$ and (2.37) gives $b \leq 3$, with equality only when $s = (1/7)c$, so that $s = 1$, $c = 7$, and, considering the solution (x_2, y_2) and noting that $x_2 > 1$ and $2 \mid r$ (by Lemma 2.4), we get $r = 2$ and $x_2 = 2$, so that $x_3 = 1$, and, considering the solution (x_1, y_1), we get $y_1 = 2$, $y_3 = 1$, giving the set of solutions in the formulation of the lemma with the roles of a and b reversed.

In each of these two exceptional cases, Theorem 1 of [14] shows there is only one possible further solution (x, y), given by $16 - 9 = 7$, which violates the assumption that no two positive x values are equal and no two positive y values are equal. So we must have $x_2 = \min(x_2, x_3, \dots, x_N)$ except when $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N) = (3, 2, 7, 1, 2; 0, 2, 2, 0, 1, 1)$ or $(2, 3, 7, 2, 1; 0, 2, 2, 0, 1, 1)$.

The same argument shows $y_1 = \min(y_1, y_3, y_4, \dots, y_N)$ except when $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N) = (2, 3, 7, 2, 1; 0, 2, 2, 0, 1, 1)$ or $(3, 2, 7, 1, 2; 0, 2, 2, 0, 1, 1)$. $\square$

Lemma 2.9. *Let $(a, b, c, r, s; x_t, y_t, x_h, y_h, x_k, y_k)$ be a set of solutions for which $x_t < x_h < x_k$, $y_t < y_h$, $y_t < y_k$, and $y_h \neq y_k$. Then $y_h < y_k$, except when $(a, b, c, r, s; x_t, y_t, x_h, y_h, x_k, y_k)$ is in the same family as $(2, 2, 2, 1, 1; 0, 0, 1, 2, 2, 1)$.*

Proof. Assume $(a, b, c, r, s; x_t, y_t, x_h, y_h, x_k, y_k)$ is a set of solutions for which $x_t < x_h < x_k$, $y_t < y_h$, and $y_t < y_k$, and assume further $y_k < y_h$. Combining (1.2) for $(i, j) = (t, h)$ with (1.2) for $(i, j) = (t, k)$, we have

$$\frac{a^{x_k - x_t} + (-1)^{\gamma_2}}{a^{x_h - x_t} + (-1)^{\gamma_1}} = \frac{b^{y_k - y_t} + (-1)^{\delta_2}}{b^{y_h - y_t} + (-1)^{\delta_1}}, \tag{2.38}$$

where $\gamma_1, \gamma_2, \delta_1, \delta_2 \in \{0, 1\}$. The left side of (2.38) must be greater than or equal to one, with equality if and only if $a = 2$, $x_h - x_t = 1$, $x_k - x_t = 2$, $\gamma_1 = 0$, $\gamma_2 = 1$. But the right side of (2.38) must be less than or equal to one, with equality if and only if $b = 2$, $y_h - y_t = 2$, $y_k - y_t = 1$, $\delta_1 = 1$, $\delta_2 = 0$, so we have $(a, b, c, r, s; x_t, y_t, x_h, y_h, x_k, y_k)$ in the same family as the basic form $(2, 2, 2, 1, 1; 0, 0, 1, 2, 2, 1)$. $\square$

Lemma 2.10. *Let* $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ *be a set of solutions for which* $\min(x_1, x_2, \dots, x_N) = \min(y_1, y_2, \dots, y_N) = 0$*, no two solutions* (x, y) *have the same positive* x*, and no two solutions* (x, y) *have the same positive* y*. Let* i *and* j *be distinct integers,* $1 \le i, j \le N$*. Then* $0 < x_i < x_j$ *implies* $y_i < y_j$*, and* $0 < y_i < y_j$ *implies* $x_i < x_j$*, except when* $(a, b, c, r, s; x_1, y_1, x_2, \dots, x_N, y_N)$ *is in the same family as a subset (or an associate of a subset) of one of the following:*

$$
\begin{aligned}
&(3, 2, 7, 1, 2; 0, 2, 2, 0, 1, 1),\\
&(3, 2, 5, 1, 2; 0, 1, 1, 0, 1, 2, 2, 1, 3, 4),\\
&(2, 2, 2, 1, 1; 0, 0, 1, 2, 2, 1),\\
&(2, 2, 5, 1, 3; 0, 1, 1, 0, 3, 0),\\
&(2, 2, 3, 1, 1; 0, 1, 0, 2, 1, 0, 2, 0).
\end{aligned}
$$

Proof. Assume $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ is a set of solutions for which $\min(x_1, x_2, \dots, x_N) = \min(y_1, y_2, \dots, y_N) = 0$, no two solutions (x, y) have the same positive x, and no two solutions (x, y) have the same positive y.

Note that, if Lemma 2.10 holds for a given set of solutions, it holds for the associate of that set of solutions. Note also that, of the exceptional sets of solutions in Lemma 2.1, the first two cases (and any subset of the second case) have no members of their families contradicting Lemma 2.10, and the last two are listed in Lemma 2.10. Thus, by Corollary 2.1, it suffices to show that Lemma 2.10 holds for $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ under the assumption that (2.5), (2.6), or (2.7) holds.

If (2.7) holds, let $t = 1$, $2 \le h \le N$, $2 \le k \le N$, and take $x_h < x_k$. Then we have (2.38) with the conditions given in Lemma 2.9, so that Lemma 2.9 applies to prove Lemma 2.10 unless $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ has a subset in the same family as $(2, 2, 2, 1, 1; 0, 0, 1, 2, 2, 1)$; since $x_t = y_t = 0$, this subset has no further solutions since $(2, 2, 2, 1, 1; 0, 0, 1, 2, 2, 1)$ has no further solutions by Observation 2.1 (immediately preceding Lemma 2.3). Thus $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ itself is in the same family as $(2, 2, 2, 1, 1; 0, 0, 1, 2, 2, 1)$. This proves Lemma 2.10 when (2.7) holds.

Suppose (2.5) holds with $x_2 \ne \min(x_2, x_3, \dots, x_N)$. Then, by Lemma 2.7 and Observation 1.3, $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ must be in the same family as a subset of one of the basic forms $(3, 2, 5, 1, 4; 0, 0, 2, 0, 1, 1, 3, 3)$ or $(2, 3, 5, 2, 3; 0, 0, 2, 0, 1, 1, 4, 2)$, and therefore in the same family as a subset (or an associate of a subset) of the second exceptional case of the formulation of Lemma 2.10. So we can assume $x_2 = \min(x_2, x_3, \dots, x_N)$.

Suppose (2.6) holds with either $x_2 \ne \min(x_2, x_3, \dots, x_N)$ or $y_1 \ne \min(y_1, y_3, \dots, y_N)$. Then, by Lemma 2.8 and Observation 1.3, $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ must be in the same family as the basic form $(3, 2, 7, 1, 2;$

$0, 2, 2, 0, 1, 1)$ or its associate; since $(3, 2, 7, 1, 2; 0, 2, 2, 0, 1, 1)$ is the first exception listed in Lemma 2.10, we can assume $x_2 = \min(x_2, x_3, \dots, x_N)$ and $y_1 = \min(y_1, y_3, \dots, y_N)$.

Thus, when (2.5) or (2.6) holds it suffices to show that Lemma 2.10 holds for $3 \le i, j \le N$. Take $t \le 2$, $h \ge 3$, $k \ge 3$, and take $x_h < x_k$. We have (2.38) as in Lemma 2.9, so that Lemma 2.9 proves Lemma 2.10 since, by Lemma 2.2, $\gcd(a, b) = 1$ and the exceptional case of Lemma 2.9 is impossible. □

Lemma 2.11. *Assume a set of solutions is not in the same family as a subset (or an associate of a subset) of one of the following:*

$$\begin{aligned}
&(3,2,1,1,2;0,0,1,0,1,1,2,2)\\
&(5,3,2,1,1;0,0,0,1,1,1,2,3)\\
&(5,2,3,1,2;0,0,0,1,1,0,1,2,3,6)\\
&(3,2,5,1,2;0,1,1,0,1,2,2,1,3,4)\\
&(3,2,7,1,2;0,2,2,0,1,1,2,3)\\
&(5,3,4,1,1;0,1,1,0,1,2)\\
&(7,2,3,1,2;0,0,0,1,1,1)\\
&(2,2,2,1,1;0,0,1,2,2,1)\\
&(2,2,4,3,1;0,0,1,1,2,3,2,4)\\
&(6,2,2,1,1;0,0,1,2,1,3)\\
&(2,2,6,5,1;0,0,1,2,1,4)\\
&(3,3,3,2,1;0,0,1,1,1,2)\\
&(2,2,3,1,1;0,1,0,2,1,0,2,0)\\
&(2,2,5,3,1;0,1,1,0,0,3)\\
&(3,3,2,1,1;0,0,0,1,1,0)\\
&(2^g+(-1)^\epsilon,2,2^g-(-1)^\epsilon,1,2;0,g-1,1,0,1,g).
\end{aligned} \tag{2.39}$$

where $\epsilon \in \{0, 1\}$ and g is an integer with $g+\epsilon > 3$. Then, letting $(a, b, c, r, s; x_1, y_1, x_2, y_2, \dots, x_N, y_N)$ be this set of solutions or its associate, we can assume one of the following must hold:

$$x_1 < x_2 < \cdots < x_N, y_1 = y_2 < y_3 < \cdots < y_N, \tag{2.40}$$

$$x_1 < x_2 < \cdots < x_N, y_2 < y_1 < y_3 < \cdots < y_N. \tag{2.41}$$

$$x_1 < x_2 < \cdots < x_N, y_1 < y_2 < \cdots < y_N, \tag{2.42}$$

Proof. Each exceptional case (and therefore each subset of each exceptional case) in the formulations of Lemmas 2.1, 2.3, 2.5, 2.6, and 2.10 is in the same family as a subset (or an associate of a subset) of one of the entries in (2.39).

On the other hand, the basic form $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, \ldots, X_N, Y_N)$ in the same family as $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ must *not* be in the same family as a subset (or an associate of a subset) of one of the entries in (2.39). Therefore, $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, \ldots, X_N, Y_N)$ cannot be in the same family as a subset (or an associate of a subset) of one of the exceptional cases of Lemmas 2.1, 2.3, 2.5, 2.6, or 2.10. By Lemmas 2.1, 2.3, 2.5, 2.6, and 2.10, we can assume that we have one of the following:

(2.43) $$0 = X_1 < X_2 < \cdots < X_N, 0 = Y_1 = Y_2 < Y_3 < \cdots < Y_N,$$

(2.44) $$0 = X_1 < X_2 < \cdots < X_N, 0 = Y_2 < Y_1 < Y_3 < \cdots < Y_N.$$

(2.45) $$0 = X_1 < X_2 < \cdots < X_N, 0 = Y_1 < Y_2 < \cdots < Y_N,$$

Applying Observation 1.3 proves the lemma. □

3. Proof of Theorem 1.1

Proof of Theorem 1.1. To handle the first paragraph of Theorem 1.1, it suffices to prove that there are no sets of solutions in basic form with $N > 3$ when $\gcd(a, b) > 1$, except for cases (or associates of cases) listed in the formulation of the theorem.

Assume there exists a set of solutions $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ in basic form with $N \geq 4$ and $\gcd(a, b) > 1$. The first two exceptional cases in the formulation of the theorem are the only entries (or associates of entries) in (2.39) with $\gcd(a, b) > 1$ and $N > 3$, so from here on we can assume, by Lemma 2.11 and Lemma 2.2, that (2.42) holds, and we must have

(3.1) $$0 = x_1 < x_2 < \cdots < x_N, 0 = y_1 < y_2 < \cdots < y_N.$$

Suppose for some prime p we have $p^\alpha || a$ and $p^\beta || b$ with $\alpha, \beta > 0$. Recall that, by the definition of basic form, we have $\gcd(r, sb) = \gcd(s, ra) = 1$. Let i and j be integers such that $1 \leq i < j \leq N$. Then, since $x_i < x_j$ and $y_i < y_j$, we have from (1.2)

(3.2) $$\alpha x_i = \beta y_i.$$

Since $x_2 > 0$ and $y_2 > 0$, we have, for $i = 2$ and 3,

(3.3) $$\frac{x_i}{y_i} = \frac{\beta}{\alpha}.$$

Let $\alpha_0 = \alpha / \gcd(\alpha, \beta)$ and $\beta_0 = \beta / \gcd(\alpha, \beta)$. Then for $i = 2$ or 3 we have

(3.4) $$x_i = k_i \beta_0, y_i = k_i \alpha_0$$

where k_i is a positive integer. Now combining (1.2) for $(i,j) = (1,2)$ with (1.2) for $(i,j) = (1,3)$ we obtain

$$\frac{a^{\beta_0 k_3} + (-1)^{\gamma_3}}{a^{\beta_0 k_2} + (-1)^{\gamma_2}} = \frac{b^{\alpha_0 k_3} + (-1)^{\delta_3}}{b^{\alpha_0 k_2} + (-1)^{\delta_2}} \tag{3.5}$$

where γ_2, γ_3, δ_2 and δ_3 are in the set $\{0,1\}$.

Since $c \le r + s$, we must have

$$u_i \ne v_i, \qquad 2 \le i \le 4, \tag{3.6}$$

where u_i and v_i are the values of u and v in (1.1) when $(x,y) = (x_i, y_i)$.

If $a^{\beta_0} = b^{\alpha_0}$, then we have $\gamma_3 = \delta_3$ and $\gamma_2 = \delta_2$ in (3.5) and, from (3.4), $a^{x_2} = b^{y_2}$. Considering (1.2) with $(i,j) = (1,2)$ and letting $h = \gcd(a^{x_2} + (-1)^{\gamma_2}, b^{y_2} + (-1)^{\delta_2})$, we have $r = (b^{y_2} + (-1)^{\delta_2})/h$ and $s = (a^{x_2} + (-1)^{\gamma_2})/h$ so that $r = s$. But then, by (3.6), we have $c = 0$, a contradiction. So $a^{\beta_0} \ne b^{\alpha_0}$. For convenience of notation let $A = a^{\beta_0}$, $B = b^{\alpha_0}$, $n = k_2$, $m = k_3$. Then (3.5) becomes

$$\frac{A^m + (-1)^{\gamma_3}}{A^n + (-1)^{\gamma_2}} = \frac{B^m + (-1)^{\delta_3}}{B^n + (-1)^{\delta_2}}. \tag{3.7}$$

We have already shown $A = B$ implies $c = 0$ so $A \ne B$. Assume $A > B$. (The remainder of the proof works also for $B > A$. Indeed, since we have (3.1), the roles of a and b are interchangeable.)

(3.7) implies

$$\frac{A^m - 1}{A^n + 1} \le \frac{B^m + 1}{B^n - 1}. \tag{3.8}$$

If $n \ge 2$ then (3.8) implies

$$\frac{(B+2)^3 - 1}{(B+2)^2 + 1} \le \frac{B^3 + 1}{B^2 - 1} \tag{3.9}$$

noting that $A \ge B + 2$ since $\gcd(A,B) > 1$. But (3.9) does not hold for $B \ge 2$, so we must have $n = 1$ so that

$$\frac{A^m - 1}{A + 1} \le \frac{B^m + 1}{B - 1}$$

which implies, when $m \ge 3$,

$$\frac{(B+2)^3 - 1}{B + 3} \le \frac{B^3 + 1}{B - 1}. \tag{3.10}$$

But (3.10) does not hold for $B \ge 2$ so we must have $m = 2$ and $n = 1$. By (3.6), we can let $D = |\gamma_3 - \delta_3| = |\gamma_2 - \delta_2|$ (note $D = 0$ when $u_1 \ne v_1$, and $D = 1$ when $(u_1, v_1) = (0,0)$, where u_1 and v_1 are the values of u and v

in (1.1) when $(x,y) = (x_1,y_1)$). If $D = 0$ then the only possible choice of signs is given by

$$\frac{A^2+1}{A-1} = \frac{B^2+1}{B-1}. \tag{3.11}$$

Since the function $(w^2+1)/(w-1)$ is monotone increasing for $w > 1+\sqrt{2}$, the only possible solution to (3.11) is $(A,B) = (3,2)$ which is not under consideration here since we are taking $\gcd(a,b) > 1$.

So $D = 1$. If $\gamma_2 = \gamma_3$ then we have

$$\frac{A^2+1}{A+1} = \frac{B^2-1}{B-1}$$

which, since the right side is the integer $B+1$, requires $A = 0$ or 1, a contradiction. So we must have $\gamma_2 \neq \gamma_3$ so that

$$\frac{A^2-1}{A+1} = \frac{B^2+1}{B-1} \tag{3.12}$$

which, since the left hand side is the integer $A-1$, requires $B = 2$ or 3. If $B = 2$ then from (3.12) we see that $A = 6$ and we obtain $a = 6$, $b = 2$, $r = (2-1)/\gcd((2-1),(6+1)) = 1$, $s = (6+1)/\gcd((2-1),(6+1)) = 7$, $c = 8$ (since $D = 1$ implies $(u_1,v_1) = (0,0)$), $x_1 = 0$, $y_1 = 0$, $x_2 = 1$, $y_2 = 1$, $x_3 = 2$, $y_3 = 2$. We find there is a fourth solution $x_4 = 3$, $y_4 = 5$. Noting that $2^3 || c$, we easily see there can be no further solutions (by Observation 2.1 immediately preceding Lemma 2.3). We obtain the third of the exceptions in the formulation of Theorem 1.1. If $B = 3$, then, from (3.12), we find $A = 6$ and we obtain $a = 6$, $b = 3$, $r = (3-1)/\gcd((6+1),(3-1)) = 2$, $s = (6+1)/\gcd((6+1),(3-1)) = 7$, $c = 9$, $x_1 = 0$, $y_1 = 0$, $x_2 = 1$, $y_2 = 1$, $x_3 = 2$, $y_3 = 2$. Noting $3^2 || c$, we easily see there are no further solutions.

This ends the proof of the first part of Theorem 1.1.

For the proof of the second part of Theorem 1.1, we note that there are an infinite number of choices of a no two of which have a common factor and for each of which there are an infinite number of choices of m such that a set of solutions to (1.1) with $N = 3$ is given by $(a,b,c,r,s;x_1,y_1,x_2,y_2,x_3,y_3) =$

$$\left(a, ta, \frac{a(t+(-1)^{u+v+1})}{h}, \frac{ta+(-1)^v}{h}, \frac{a+(-1)^u}{h}; 0,0,1,1,m+1,2\right) \tag{3.13}$$

where $m \geq 0$ is an integer, $t = \dfrac{a^m+(-1)^v}{a+(-1)^u}$ is an integer, $h = \gcd(ta+(-1)^v, a+(-1)^u)$, and u and v are in the set $\{0,1\}$. Thus there are an infinite number of families with $N = 3$. (Note that we do not actually need the fact that there are an infinite number of choices of m for each choice of a, although this is easily verified.) This completes the proof of Theorem 1.1. $\square$

A further example giving an infinite number of sets of solutions with $N = 3$ is the following $(a, b, c, r, s; x_1, y_1, x_2, y_2, x_3, y_3)$ (closely related to (3.13)):

$$(2, 4t, \frac{4t+4}{h_1}, \frac{4t+1}{h_1}, \frac{3}{h_1}; 0, 0, 2, 1, m_1+2, 2) \tag{3.14}$$

where $m_1 \geq -1$ is an odd integer, $t = \frac{2^{m_1}+1}{3}$, $h_1 = 3$ or 1 according as $m_1 \equiv 5 \bmod 6$ or not, and $u, v \in \{0, 1\}$.

4. Proof of Theorem 1.2

To prove Theorem 1.2 we will show that if (1.1) has more than three solutions (x, y, u, v) when $\gcd(ra, sb) = 1$, then for each solution

$$\max(a, b, r, s, x, y) < 2 \times 10^{15} \tag{4.1}$$

and

(4.2)

$$\max(r, s, x, y) < 8 \times 10^{14}, \min(\max(a, b), \max(b, c), \max(a, c)) < 8 \times 10^{14}.$$

To do this, we require a few additional lemmas.

Lemma 4.1. *Suppose* $\gcd(ra, sb) = 1$ *and suppose* (1.1) *has four solutions* (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) *with* $x_1 < x_2 < x_3 < x_4$. *Let* $Z = \max(x_4, y_1, y_2, y_3, y_4)$. *Then*

$$a^{x_3 - x_2} \leq Z, \qquad s \leq Z + 1,$$

and, if $a > b$,

$$x_2 a^{x_3 - x_2} \leq Z.$$

Proof. By Corollary 2.2, we see that no two of y_2, y_3, y_4 are equal. By considering (1.2) with $(i, j) = (2, 3)$ and $\delta = \delta_1 \in \{0, 1\}$, we find that $\left(b^{|y_3 - y_2|} + (-1)^{\delta_1}\right) / (ra^{x_2})$ is an integer prime to a. Considering (1.2) with $(i, j) = (3, 4)$ and $\delta = \delta_2 \in \{0, 1\}$, we find $\left(b^{|y_4 - y_3|} + (-1)^{\delta_2}\right) / (ra^{x_3})$ is an integer. There must be a least positive integer n such that $(b^n \pm 1)/(ra^{x_2})$ is an integer prime to a. Now we can apply Lemma 1 of [15] to get

$$n \frac{a^{x_3 - x_2}}{2^{g+h-1}} \leq \max(y_3, y_4) \leq Z,$$

where $g = 1$ and $h = 0$ unless r is odd, $a \equiv 2 \bmod 4$, and $x_2 = 1$, in which case $x_1 = 0$. But by Lemma 2.4 we cannot have r odd and $a \equiv 2 \bmod 4$ when $x_1 = 0$, so that we can simply take $g + h - 1 = 0$. Also if $a > b$, then $n > x_2$.

Finally, considering (1.2) with $(i, j) = (2, 3)$ we find $s \leq a^{x_3 - x_2} + 1 \leq Z + 1$. □

Lemma 4.2. *Suppose* $\gcd(ra, sb) = 1$ *and suppose* (1.1) *has 2 solutions* (x_1, y_1) *and* (x_2, y_2)*, with* $x_1 < x_2$*. Then, if* $r > 1$ *or if* $x_1 > 0$*,*

$$ra^{x_2} > c/2,$$

and, if $r = 1$ *and* $x_1 = 0$*,*

$$a^{x_2} > (c-2)/2.$$

Proof. Suppose $ra^{x_2} \le c/2$. Then $sb^{y_2} \ge c/2$ and also $ra^{x_1} < c/2$ so that $sb^{y_1} > c/2$. But we have, considering (1.2) with $(i,j) = (1,2)$,

$$ra^{x_1}(a^{x_2-x_1} + (-1)^\gamma) = sb^{\min(y_1,y_2)}\left(b^{|y_2-y_1|} + (-1)^\delta\right) > 0.$$

If $r > 1$, or if $x_1 > 0$, this gives $c/2 \ge ra^{x_2} > a^{x_2-x_1}+1 \ge sb^{\min(y_1,y_2)} \ge c/2$, a contradiction, so $ra^{x_2} > c/2$ when $r > 1$ or $x_1 > 0$.

If $r = 1$ and $x_1 = 0$, suppose $a^{x_2} \le c/2 - 1$, so that $sb^{y_2} \ge c/2 + 1$. Proceeding in the same way we obtain $c/2 \ge a^{x_2}+1 \ge sb^{\min(y_1,y_2)} \ge c/2+1$, a contradiction. So $a^{x_2} > c/2 - 1$ when $r = 1$ and $x_1 = 0$. $\square$

Lemma 4.3. *Suppose* (1.1) *has a solution* (x, y, u, v) *for some* (a, b, c, r, s)*, and suppose we have the following conditions:*

$$\min(x,y) > 0, (u,v) \ne (0,0), \gcd(ra, sb) = 1.$$

Let $Z = \max(x, y)$*,* $J = \max(a, b)$*, and* $d = \min(ra^x, sb^y)$*. Then one of the following inequalities must hold:*

$$Z < \frac{\log(1 + c/d) + \log(c)}{\log(2)} + 1.6901816335 \cdot 10^{10} \log(\max(r,s,2)) \log(J) \log(1.5eZ), \tag{4.3}$$

$$Z < \frac{\log(1 + c/d) + \log(c)}{\log(2)} + 22.997 \left(\log\left(\frac{Z}{\log(2)}\right) + 2.405\right)^2 \log(J). \tag{4.4}$$

When $rs = 1$*, then either* (4.4) *holds or* $Z < 2409.08 \log(J)$*.*

Proof. Suppose $\gcd(ra, sb) = 1$. Let $Z = \max(x, y)$ where (x, y, u, v) is a solution in positive integers to (1.1). Let $D = \max(ra^x, sb^y)$, $d = \min(ra^x, sb^y)$, $J = \max(a, b)$, and $j = \min(a, b)$. Assume $(u, v) \ne (0, 0)$ so that

$$|ra^x - sb^y| = D - d = c.$$

Assume $rs > 1$ and also assume there do not exist integers a_0, b_0, m, n, t, and w such that $r = a_0^m$, $a = a_0^t$, $s = b_0^n$, and $b = b_0^w$. Let

$$\Lambda = |\log(r/s) + x\log(a) - y\log(b)| = \log(D) - \log(d) = \log(1 + c/d) < c/d$$

so that

$$\log(1/\Lambda) > \log(d) - \log(c). \tag{4.5}$$

Now we can apply a result of Matveev [7], as given by Mignotte in Theorem 1 of [9], with $(\alpha_1, \alpha_2, \alpha_3) = (r/s, a, b)$, $(A_1, A_2, A_3) = (\log(\max(r, s)), \log(a), \log(b))$, and $B = Z$ to get

$$\log(1/\Lambda) < K A_1 A_2 A_3 \log(1.5eB), \tag{4.6}$$

where $K = 1.6901816335 \cdot 10^{10}$. Combining (4.5) and (4.6) we get

$$\log(d) < \log(c) + K \log(\max(r, s)) \log(a) \log(b) \log(1.5eZ). \tag{4.7}$$

Also $\Lambda = \log(1 + c/d)$, so that, adding Λ to both sides of (4.7), we get

$$\log(D) < \log(1 + c/d) + \log(c) + K \log(\max(r, s)) \log(a) \log(b) \log(1.5eZ).$$

From this, noting that $Z \log(j) \leq \log(D)$, we have

$$Z \log(j) < \log(1 + c/d) + \log(c) + K \log(\max(r, s)) \log(a) \log(b) \log(1.5eZ).$$

Dividing through by $\log(j)$ and noting $j \geq 2$, we get (4.3).

Now suppose $rs = 1$. Let $G = \max(x/\log(b), y/\log(a))$ and let $\Lambda = |x \log(a) - y \log(b)|$. Using a theorem of Mignotte [8] as given in Section 3 of [1] and using the parameters chosen by Bennett in Section 6 of [1], we see that we must have either

$$G < 2409.08 \tag{4.8}$$

or

$$\log(\Lambda) > -22.997(\log(G) + 2.405)^2 \log(a) \log(b). \tag{4.9}$$

If (4.8) holds then $Z < 2409.08 \log(J)$, which implies (4.3). When (4.9) holds, proceeding as in the case $rs > 1$ above and noting $G \leq Z/\log(2)$, we obtain (4.4).

Suppose $rs > 1$ and there exist integers a_0, b_0, m, n, t, and w such that $r = a_0^m$, $a = a_0^t$, $s = b_0^n$, and $b = b_0^w$. Then we can rewrite (1.1) as

$$(-1)^u a_0^{tx+m} + (-1)^v b_0^{wy+n} = c.$$

From this we obtain, using the same method as above when $rs = 1$, and letting $Z_0 = \max(tx + m, wy + n)$, and $J_0 = \max(a_0, b_0)$,

$$Z \leq Z_0 < 2409.08 \log(J_0) \leq 2409.08 \log(J) \tag{4.10}$$

or

(4.11)

$$Z_0 < \frac{\log(1 + c/d) + \log(c)}{\log(2)} + 22.997 \left(\log\left(\frac{Z_0}{\log(2)} \right) + 2.405 \right)^2 \log(J_0).$$

(4.10) implies (4.3), and, if (4.11) holds, then (4.4) holds since $Z \leq Z_0$ and $J \geq J_0$. $\square$

Lemma 4.4. *Let* $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ *and* $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, \ldots, X_N, Y_N)$ *be any two sets of solutions in the same family for which* $\gcd(ra, sb) = \gcd(RA, SB) = 1$. *Then for every* i *there exists a unique* j *and for every* j *there exists a unique* i *such that* $ra^{x_i} = RA^{X_j}$ *and* $sb^{y_i} = SB^{Y_j}$.

Proof. Suppose $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ and $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, \ldots, X_N, Y_N)$ are any two sets of solutions in the same family, so that $C = kc$ for some positive rational k. Suppose $\gcd(ra, sb) = \gcd(RA, SB) = 1$. Since $\gcd(ra, sb) = 1$, by the definition of family k is an integer and, since $\gcd(RA, SB) = 1$, we must have $k = 1$. By Observation 1.3, the lemma holds. $\square$

Proof of Theorem 1.2: It suffices to prove that (4.1) and (4.2) hold when $N \geq 4$ and $\gcd(ra, sb) = 1$.

If (4.1) or (4.2) fails to hold for a given set of solutions for which $N > 4$, then that set of solutions has a subset for which $N = 4$ and for which (4.1) or (4.2) fails to hold. So it suffices to consider only $N = 4$. By Lemma 4.4, both (4.1) and (4.2) hold for any set of solutions with $\gcd(ra, sb) = 1$ in the same family as a subset (or an associate of a subset) of one of the entries listed in (2.39), so by Lemma 2.11 it suffices to consider a set of solutions $(a, b, c, r, s; x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$ for which one of (2.40), (2.41), or (2.42) holds (note that if (4.1) or (4.2) holds for the associate of a set of solutions, it holds for the set of solutions itself). If (2.42) holds, or if (2.40) holds with $y_1 = y_2 > 0$, we can apply Theorem 1 of [15] to the solutions (x_2, y_2), (x_3, y_3), and (x_4, y_4) to obtain (4.1) and (4.2). Similarly, if (2.41) holds with either $x_1 > 0$ or $y_2 > 0$, we can apply Theorem 1 of [15] to obtain (4.1) and (4.2). So it suffices to consider only two cases:

$$x_1 < x_2 < x_3 < x_4, 0 = y_1 = y_2 < y_3 < y_4, \tag{4.12}$$

$$0 = x_1 < x_2 < x_3 < x_4, 0 = y_2 < y_1 < y_3 < y_4. \tag{4.13}$$

Case 1: (4.12) *holds.*

Let $(a, b, c, r, s; x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$ be a set of solutions satisfying (4.12) with $\gcd(ra, sb) = 1$. Let $Z = \max(x_4, y_4)$. Let $R = ra^{x_1}$.

We can apply Lemma 4.1 to obtain

$$a^{x_3 - x_2} \leq Z, \qquad s \leq Z + 1.$$

From this we have $ra^{x_3} \leq ra^{x_2} Z \leq (c + s)Z \leq (2s + R)Z$. Considering the solution (x_3, y_3), we find $sb^{y_3} \leq ra^{x_3} + c \leq (2s + R)Z + R + s$, $b^{y_3} \leq (2 + (R/s))Z + 1 + R/s \leq (8/3)Z + 5/3$ if $R/s \leq 2/3$. Considering (1.2) with $(i, j) = (1, 2)$, we find $R \leq 2$, so that $R/s > 2/3$ implies $(R, s) = (2, 1)$ or $(1, 1)$.

So assume $(R,s) \neq (2,1)$ or $(1,1)$, so that $R/s \leq 2/3$. Then, by the results in the preceding paragraph, we have

$$a \leq Z, b \leq (8/3)Z + 5/3, r \leq R \leq 2, s \leq Z+1, c \leq Z+3. \tag{4.14}$$

By (4.12) $\min(x_4, y_4) > 0$, and, since $c \leq R+s$, $(u_4, v_4) \neq (0,0)$, where u_i and v_i are the values of u and v in (1.1) when $(x,y) = (x_i, y_i)$. We can apply Lemma 4.3 with $(x,y) = (x_4, y_4)$ to see that (4.3) or (4.4) holds with $J = \max(a,b)$ and $d = \min(ra^{x_4}, sb^{y_4})$. Note that $1 + c/d \leq \max(2,c)$. Now, using (4.14) in (4.3) and (4.4), we obtain $Z < 7.4 \times 10^{14}$, and using (4.14) again we obtain (4.1) and (4.2), completing the proof of Theorem 1.2 for Case 1 when $(R,s) \neq (2,1)$ or $(1,1)$.

Now assume $(R,s) = (2,1)$, so that we have $c = 1$ or 3. If $c = 1$, then, considering the solution (x_2, y_2), we have $x_2 = x_1$, a contradiction. So $c = 3$, $a^{x_2 - x_1} = 2$, $x_3 > 1$, and $y_3 > 0$. Take $3 \leq i \leq 4$. We have $u_i \neq v_i$ since $c \leq R+s$. $(u_i, v_i) = (1,0)$ implies $b^{y_i} \equiv 3 \bmod 8$, while $(u_i, v_i) = (0,1)$ implies $b^{y_i} \equiv 5 \bmod 8$, so y_i is odd and we must have $(u_3, v_3) = (u_4, v_4)$. $(u_3, v_3) = (u_4, v_4) = (1,0)$ contradicts Lemma 2 of [13] since $y_4 > 1$. So $(u_3, v_3) = (u_4, v_4) = (0,1)$, in which case, using Theorem 3 of [12] and recalling y_3 and y_4 are both odd, we must have $(b, y_3, y_4) = (5,1,3)$ so

$$(a,b,c,r,s; x_1, y_1, x_2, y_2, \ldots, x_4, y_4) = (2,5,3,2,1; 0,0,1,0,2,1,6,3)$$

or

$$(a,b,c,r,s; x_1, y_1, x_2, y_2, \ldots, x_4, y_4) = (2,5,3,1,1; 1,0,2,0,3,1,7,3).$$

Both these cases certainly satisfy (4.1) and (4.2).

Now assume $(R,s) = (1,1)$, so that $r = R = 1$. We have $c = 2$, $a^{x_2} = 3$, $x_3 > 1$, and $y_3 > 0$. Take $3 \leq i \leq 4$. We have $u_i \neq v_i$. If $(u_i, v_i) = (0,1)$ and $2 \mid y_i$, then, considering the coefficients of the real term and the imaginary term of $(1+\sqrt{-2})^{x_i}$ modulo 9, we find that we must have $3 \mid x_i$, and, using Lemma 1 of [12], we find that the only possibility is $x_i = 3$, giving $b = 5$ and $y_i = 2$, in which case, by Theorem 7 of [14], the only further solution is $(x,y) = (1,1)$, which contradicts (4.12). So $(u_i, v_i) = (0,1)$ implies y_i odd. But then, considering the solutions (x_3, y_3) and (x_4, y_4) modulo 8, we find that we must have $(u_3, v_3) = (u_4, v_4) = (1,0)$ or $(u_3, v_3) = (u_4, v_4) = (0,1)$. If $(u_3, v_3) = (u_4, v_4) = (1,0)$ we have a contradiction to Lemma 2 of [13]. If $(u_3, v_3) = (u_4, v_4) = (0,1)$, then, since y_3 and y_4 are both odd, we can apply Theorem 3 of [12] to obtain a contradiction. So Theorem 1.2 holds for Case 1.

Case 2: (4.13) *holds.*

Without loss of generality we can take $a > b$. Let $Z = \max(x_4, y_4)$. We can apply Lemma 4.1 to obtain

$$b < a \leq Z, \qquad \max(r,s) \leq Z+1. \tag{4.15}$$

We have $\min(x_4, y_4) > 0$, and, since $c \le r + sb^{y_1}$, we have $(u_4, v_4) \ne (0,0)$. Let $D = \max(ra^{x_4}, sb^{y_4})$ and $d = \min(ra^{x_4}, sb^{y_4})$. Now we can apply Lemma 4.3 with $(x, y) = (x_4, y_4)$ to see that we must have either

$$(4.16) \quad Z < \frac{\log(1 + c/d) + \log(c)}{\log(2)} + 1.6901816335 \cdot 10^{10} \log(\max(r, s, 2)) \log(a) \log(1.5eZ),$$

or

$$(4.17) \quad Z < \frac{\log(1 + c/d) + \log(c)}{\log(2)} + 22.997 \left(\log\left(\frac{Z}{\log(2)}\right) + 2.405\right)^2 \log(a).$$

We first show that $\log(1 + c/d)$ is small. By Lemma 4.2,

$$d > b^2(c-2)/2 \ge 2c - 4.$$

So

$$(4.18) \qquad 1 + \frac{c}{d} < 1.75$$

since $d \ge 8$. Now applying (4.15) and (4.18) to (4.16) and (4.17), and letting $K = 1.6901816335 \cdot 10^{10}$, we get either

$$(4.19) \qquad Z < 0.9 + \frac{\log(c)}{\log(2)} + K \log(Z+1) \log(Z) \log(1.5eZ),$$

or

$$(4.20) \qquad Z < 0.9 + \frac{\log(c)}{\log(2)} + 22.997 \left(\log\left(\frac{Z}{\log(2)}\right) + 2.405\right)^2 \log(Z).$$

(4.20) implies (4.19), so from here on we consider only (4.19).

Suppose $c \le (Z+1)^{10^{11}}$. Then (4.19) gives $Z < 7.9 \times 10^{14}$, so that applying (4.15), we get (4.1) and (4.2), completing the proof of Theorem 1.2 when $c \le (Z+1)^{10^{11}}$. So from here on we can assume that, by (4.15),

$$(4.21) \qquad c > (Z+1)^{10^{11}} \ge (\max(r, s))^{10^{11}}.$$

We can apply Lemma 4.1 to obtain

$$(4.22) \qquad x_2 a \le Z.$$

Now $c \le ra^{x_2} + s$ so that, taking $t = \frac{10^{11}}{10^{11}-1}$, we have, from (4.21), noting that (4.21) also implies $c > 10^{11}s$,

$$c < t^t a^{t x_2}$$

which gives

$$(4.23) \qquad \frac{\log(c)}{\log(2)} < \frac{t\log(t) + t x_2 \log(a)}{\log(2)}.$$

We certainly have

$$\frac{t\log(t)}{\log(2)} < 0.1 \tag{4.24}$$

and we also have

$$\frac{tx_2\log(a)}{\log(2)} < \frac{x_2 a}{1.89} \le \frac{Z}{1.89} \tag{4.25}$$

where the first inequality follows from taking $a > 2$ and the second inequality follows from (4.22). Now we have, using (4.19), (4.24), and (4.25),

$$Z < 0.9 + 0.1 + \frac{Z}{1.89} + K\log(Z+1)\log(Z)\log(1.5eZ)$$

from which we get

$$0.47Z < 1 + K\log(Z+1)\log(Z)\log(1.5eZ), \tag{4.26}$$

from which we obtain a bound on Z, which, with (4.15), gives (4.1).

So it remains to prove (4.2) for Case 2 when $c > (Z+1)^{10^{11}}$. To do this, we must significantly improve the value 0.47 in (4.26), so we must significantly improve the value 1.89 in (4.25). One way to do this is to take a large: in fact, it suffices to take $a > 50$ to get the necessary improvements, and we obtain (4.2), completing the proof of Theorem 1.2 for $a > 50$. To handle $a \le 50$, we will need two more lemmas:

Lemma 4.5. *Let $a > 1$ and $b > 1$ be relatively prime integers. For $1 \le i \le m$, let p_i be one of the m distinct prime divisors of a. Let $p_i^{g_i} || b^{n_i} \pm 1$, where n_i is the least positive integer for which there exists a positive integer k such that $|b^{n_i} - kp_i| = 1$, and $\pm$ is read as the sign that maximizes g_i.*

Write

$$\sigma = \sum_i g_i \log(p_i)/\log(a).$$

Then, if

$$a^x \mid b^y \pm 1,$$

where the $\pm$ sign is independent of the above, we must have

$$a^x \mid a^\sigma y.$$

Proof. Let $a = \prod_i p_i^{\alpha_i}$. If $a^x | b^y \pm 1$, then for each i, $p_i^{x\alpha_i} | b^y \pm 1$, so that $p_i^{x\alpha_i - g_i} | y$ (in the case $x\alpha_i < g_i$, $p_i^{x\alpha_i - g_i}$ is a fraction that evenly divides y). Thus, y is divisible by

$$\prod_i p_i^{x\alpha_i - g_i} = a^{x-\sigma}.$$

□

Lemma 4.6. *If $a > 2$ and $(a,b) \neq (3,2)$, then, in the notation of Lemma 4.5,*

$$\sigma < \frac{a \log b}{2 \log a}.$$

If $(a,b) = (3,2)$, then $\sigma = 1$.

Proof. We assume $a > 2$ and $(a,b) \neq (3,2)$. Then if a is odd, $\prod_i p_i^{g_i} \leq b^{\phi(a)/2} + 1 \leq b^{(a-1)/2} + 1 < b^{a/2}$, verifying Lemma 4.6 when a is odd. If $a > 4$ is even, then $\prod_i p_i^{g_i} < b^{\phi(a/2)} < b^{a/2}$ verifying the lemma in this case also. Finally, when $a = 4$, define g so that $2^g || b \pm 1$, where the sign is chosen to maximize g. Then the lemma holds unless $\frac{g \log(2)}{\log(4)} \geq \frac{4 \log(b)}{2 \log(4)}$, that is, unless $2^g \geq b^2$, which is impossible.

Finally, if $(a,b) = (3,2)$, $\sigma = 1$ by the definition of σ in Lemma 4.5. □

Returning to the proof of Theorem 1.2, it remains to handle Case 2 when $c > (Z+1)^{10^{11}}$ and $a \leq 50$. By (4.19), (4.23), and (4.24), we have

$$Z < 1 + \frac{t x_2 \log(a)}{\log(2)} + K \log(Z+1) \log(Z) \log(1.5eZ). \tag{4.27}$$

By Lemma 4.5 and Lemma 4.6 we have $a^{x_2} \leq a^\sigma Z$ where $\sigma < a/2$. Considering the second term on the right side of (4.27), we have

$$\begin{aligned} t x_2 \log(a)/\log(2) &\leq (t/\log(2))(\log(Z) + \sigma \log(a)) \\ &< 1.443(\log(Z) + a \log(a)/2). \end{aligned}$$

Substituting this into (4.27), and recalling $a \leq 50$, we get

$$Z < 143 + 1.443 \log(Z) + K \log(Z) \log(Z+1) \log(1.5eZ). \tag{4.28}$$

From (4.28) we obtain a bound on Z which, combined with (4.15), gives (4.2). This completes the proof of Theorem 1.2. □

Comment on Lemmas 4.5 and 4.6: Lemmas 3 and 4 of [14] can be replaced by Lemmas 4.5 and 4.6 respectively, giving a shorter, simpler presentation in which Lemma 5 of [14] would be replaced by the following:

Lemma 4.7. *Let $a > 2$, $b > 1$, and $c > 0$ be integers with $\gcd(a,b) = 1$. If $(-1)^u a^x + (-1)^v b^y = c$ has two solutions (x_1, y_1, u_1, v_1) and (x_2, y_2, u_2, v_2), with $x_2 \geq x_1 \geq 1$, $y_2 \geq y_1 \geq 1$, and u_1, u_2, v_1, $v_2 \in \{0,1\}$, and if further $a^{x_1} > c/2$, then*

$$x_1 < \sigma + k,$$

where σ is defined as in Lemma 4.5, and $k = \frac{8.1 + \log\log(a)}{\log(a)}$ when $a < 5346$ and $k = 1.19408$ otherwise.

The proof of Lemma 4.7 is essentially identical to that of Lemma 5 of [14]: we can use a result of Mignotte [8] as in Proposition 4.4 of Bennett [1], noting that we do not need to consider the cases $(a,b)=(3,2)$ and $(a,b)=(5,2)$, since these cases are handled by the elementary methods of [11] along with the Theorem 4 of [12]. Note that in the proof of Lemma 4.7 we will use the bound in Lemma 4.6 rather than the bound in Lemma 4 of [14]: also note that we will obtain $\frac{y_2\log(b)}{\log(c)} > 10.519$ rather than $\frac{y_2\log(b)}{\log(c)} > 34$ as in [14]. Lemmas 3, 4, and 5 of [14] are used to prove Theorem 3 of [14]; if we use instead Lemmas 4.5, 4.6, and 4.7 of the present paper, both the formulation and the proof of Theorem 3 of [14] remain completely unchanged.

5. The Case $N \le 3$ with $\gcd(ra, sb) = 1$

The question remains: what can be said of cases for which $N \le 3$?

It follows directly from Theorem 2 of [15] that cases of exactly two solutions to (1.1) with $\gcd(ra, sb) = 1$ are commonplace and easy to construct, even if we restrict consideration to basic forms only (here we are allowing $N = 2$ in the definition of "set of solutions", and noting that we can easily adjust the two solutions to get $\min(x_1, x_2) = \min(y_1, y_2) = 0$).

When $N = 3$, we find many examples. Here we list several types of sets of solutions, each one of which generates an infinite number of families giving three solutions to (1.1). We list these sets of solutions in the form $(a, b, c, r, s; x_1, y_1, x_2, y_2, x_3, y_3)$:

(5.1)
$$\left(a, \frac{a^{kd}+(-1)^{u+v}}{a^d+(-1)^u}, \frac{a^d b+(-1)^{u+v+1}}{h}, \frac{b+(-1)^v}{h}, \frac{a^d+(-1)^u}{h}; 0, 1, d, 0, kd, 2\right)$$

where a and $b = \frac{a^{kd}+(-1)^{u+v}}{a^d+(-1)^u}$ are integers greater than 1, d and k are positive integers, $h = \gcd(a^d + (-1)^u, b + (-1)^v)$, and u and v are in the set $\{0, 1\}$. When $u = 0$, we take $k - v$ odd; when $(u, v) = (1, 1)$, we take $a^d \le 3$. When $a = d = 2$ and $(u, v) = (1, 1)$, we can take k to be a half integer ($k = 3/2$ gives the exceptional case $(3, 2, 11)$ in Theorem 7 of [13]). When $k = 2$ and $u - v$ is odd, the same choice of (a, b, r, s) as in (5.1) gives the additional set of solutions

(5.2)
$$\left(a, a^d+(-1)^v, \frac{2a^d+(-1)^v}{h}, \frac{a^d+(-1)^v 2}{h}, \frac{a^d+(-1)^{v+1}}{h}; 0, 0, d, 1, 3d, 3\right).$$

Other sets of solutions can be constructed with specified values of a. For example, when $a = 3$ we have

(5.3)
$$\left(3, \frac{3^g+(-1)^v}{2}, \frac{3^{g+1}+(-1)^v}{2^{2+v-\alpha}}, \frac{3(3^{g-1}+(-1)^v)}{2^{2+v-\alpha}}, 2^{1-v+\alpha}; 0, 1, 1, 0, 2g, 3\right)$$

where $v \in \{0,1\}$, g is a positive integer, $\alpha = 0$ when $2 \mid g-v$, $\alpha = 1$ when g is odd and $v = 0$, and $\alpha = 2$ when g is even and $v = 1$. Note that, when g is odd and $v = 1$, (5.3) corresponds to (7) in Theorem 1 of [12] and, more specifically, the case $(a,b,c) = (13,3,10)$ in (1.2) of [1]. Note also that the cases $(g,v) = (1,0)$ and $(2,1)$ correspond to the cases $(3,2,5)$ and $(3,2,13)$ in the exceptional cases of Theorem 7 of [13]; again see also (1.2) of [1].

When $a = 2$, we have

$$(2, 2^g + (-1)^v, 2^g + (-1)^{v+1}, 2, 1; 0, 1, g-1, 0, g, 1) \tag{5.4}$$

and

$$(2, 2^g + (-1)^v, 2^{g+1} + (-1)^v, 2^g, 1; 0, 1, 1, 0, g, 2) \tag{5.5}$$

where $v \in \{0,1\}$ and g is a positive integer. Note that (5.2) with $a = d = 2$ and $v = 0$ combines with (5.4) to give the exceptional case $(5,2,3)$ in Theorem 7 of [13] (again see also (1.2) of [1]). Also (5.4) and (5.5) combine to give the case $(3,2,7)$ of Theorem 7 of [13].

Also, it is easy to construct sets of solutions for which $x_1 = y_1 = y_2 = 0$. For example, we have, for a even and $x > 0$,

$$(a, 2a^x \pm 1, a^x \pm 1, 2, a^x \mp 1; 0, 0, x, 0, 2x, 1), \tag{5.6}$$

or, more generally,

$$\left(a, b, \frac{a^{x_2} + (-1)^t}{2^m}, 2^{1-m}, \frac{a^{x_2} + (-1)^{t+1}}{2^m}; 0, 0, x_2, 0, x_3, 1\right), \tag{5.7}$$

where $b = \frac{2a^{x_3} + (-1)^{t+w+1} a^{x_2} + (-1)^{w+1}}{a^{x_2} + (-1)^{t+1}}$, and where $x_2 > 0$ and $x_3 > 0$ are chosen so that $a^{x_3} \equiv (-1)^w \bmod \frac{a^{x_2} + (-1)^{t+1}}{2^m}$, $t \in \{0,1\}$, $w \in \{0,1\}$, and $m = 1$ or 0 according as a is odd or even. Taking the lower sign in (5.6), the case $a = 2$ with $x = 1$ gives the well known case $(a,b,c) = (2,3,1)$ (again see (1.2) of [1] and Theorem 7 of [13]).

Since each of (5.1)–(5.7) generates an infinite number of families for which $N = 3$, we have, after Theorem 1.2, a proof of Theorem 1.3.

Comment on anomalous cases (not necessarily with $\gcd(ra, sb) = 1$):

If we exclude from consideration any set of solutions in the same family as a set of solutions given by either (3.13) or (3.14) or any of (5.1)–(5.7), then we are aware of only 14 essentially different cases of (a,b,c,r,s) giving exactly three solutions to (1.1), the largest of which is

$$(56744, 1477, 83810889, 1478, 56743; 0, 1, 1, 0, 3, 4). \tag{5.8}$$

Easily derived from (5.8) is $(56745, 1477, 41906182, 739, 28373; 0, 1, 1, 0, 3, 4)$.

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