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Oleg KARPENKOV

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Résumé. Dans cet article, nous décrivons l'ensemble des classes de conjugaison dans le groupe $\mathrm{SL}(n,\mathbb{Z})$. Nous étendons la théorie de réduction de Gauss géométrique qui résout le problème pour $\mathrm{SL}(2,\mathbb{Z})$ au cas multidimensionnel, où les matrices de Hessenberg ς-réduites jouent le rôle de matrices réduites. Ensuite, nous trouvons des invariants complets des classes de conjugaison dans $\mathrm{GL}(n,\mathbb{Z})$ en termes fractions continues multidimensionnelles de Klein-Voronoi.

Abstract. In this paper we describe the set of conjugacy classes in the group $\mathrm{SL}(n,\mathbb{Z})$. We expand geometric Gauss Reduction Theory that solves the problem for $\mathrm{SL}(2,\mathbb{Z})$ to the multidimensional case, where ς-reduced Hessenberg matrices play the role of reduced matrices. Further we find complete invariants of conjugacy classes in $\mathrm{GL}(n,\mathbb{Z})$ in terms of multidimensional Klein-Voronoi continued fractions.

1. Introduction

Two matrices A and B in $\mathrm{SL}(n,\mathbb{Z})$ are *integer conjugate* if there exists a matrix C in $\mathrm{GL}(n,\mathbb{Z})$ such that

$$B = CAC^{-1}.$$

In this paper we study the following problem.

Problem. Describe the set of integer conjugacy classes in $\mathrm{SL}(n,\mathbb{Z})$.

One of the most common strategies to solve this kind of problem is to find complete invariants to distinguish the classes, and further if possible to give a normal form for each conjugacy class. For instance, in the similar problem for $\mathrm{SL}(n,F)$, for an algebraically closed field F, one has the Jordan normal forms classifying conjugacy classes.

A complete description of the set of integer conjugacy classes in $\mathrm{SL}(2,\mathbb{Z})$ is given by Gauss Reduction Theory [8, 12]. It turns out that it is natural to consider several normal forms for an integer conjugacy class instead of one.

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For the case of $\mathrm{SL}(n,\mathbb{Z})$ so far only an algorithm deciding if two matrices are conjugate was known [1, 5].

The classical approach to the above problem is algebraic. It is based on splitting $\mathrm{GL}(n,\mathbb{Q})$ conjugacy classes into $\mathrm{GL}(n,\mathbb{Z})$ conjugacy classes, then the problem is reduced to certain problems related to orders of algebraic fields. In this paper we introduce an alternative geometric approach based on generalization of Gauss Reduction Theory.

Description of the paper. The current paper presents the following main two results. We work only with matrices whose characteristic polynomials are irreducible over $\mathbb{Q}$. This is the first and simplest open case to study.

I. We consider Hessenberg matrices as a multidimensional analogue of the reduced matrices in Gauss Reduction Theory. In Section 2 we show that *each integer conjugacy class of irreducible matrices in* $\mathrm{SL}(n,\mathbb{Z})$ *contains a finite number of Hessenberg matrices with minimal complexity.* (Theorem 2.1).

II. In Section 3 we introduce complete geometric invariants of Dirichlet groups: periodic multidimensional continued fractions in the sense of Klein-Voronoi (Theorem 3.1). Further we deduce the complete invariants of integer conjugacy classes of matrices in $\mathrm{GL}(n,\mathbb{Z})$: the classes are represented by periodic shifts of the above-mentioned periodic continued fractions (Theorem 3.2).

2. Hessenberg matrices and conjugacy classes

In this section we study the question of reduction to so-called ς-reduced matrices, and we investigate families of perfect Hessenberg matrices in general.

2.1. Notions and definition. First, we introduce matrices that generalize the reduced matrices in Gauss Reduction Theory for $\mathrm{SL}(2,\mathbb{Z})$. We confine our study to those matrices in $\mathrm{SL}(n,\mathbb{Z})$ whose characteristic polynomial is irreducible over $\mathbb{Q}$.

A matrix A of the form

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-2} & a_{2,n-1} & a_{2,n} \\ 0 & a_{3,2} & \cdots & a_{3,n-2} & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

is called an (*upper*) *Hessenberg* matrix. The $n \times (n-1)$-matrix obtained from A by deleting the last column is called the *Hessenberg type* of A and

denoted by A^{red}. The integer

$$\varsigma(A) := \prod_{j=1}^{n-1} |a_{j+1,j}|^{n-j}$$

is called the *Hessenberg complexity* of A. It equals to the volume of the parallelepiped spanned by $e_1 = (1, 0, \dots, 0)^t$, $A(e_1)$, $A^2(e_1), \dots, A^{n-1}(e_1)$.

Definition. A Hessenberg matrix $A \in \mathrm{SL}(n, \mathbb{Z})$ is said to be *perfect* if for any pair of integers (i, j) satisfying $1 \le i < j+1 \le n$ the inequalities $0 \le a_{i,j} < a_{j+1,j}$ hold. Denote the set of all $n \times n$ perfect Hessenberg matrices by $\mathrm{PH}(n, \mathbb{Z})$. Denote all matrices in $\mathrm{PH}(n, \mathbb{Z})$ integer conjugate to A by $\mathrm{PH}(A)$.

Definition. We say that a perfect Hessenberg matrix A is ς*-reduced* if its Hessenberg complexity is minimal in $\mathrm{PH}(A)$. Denote the set of all $n \times n$ ς-reduced perfect Hessenberg matrices by $\mathrm{RPH}(n, \mathbb{Z})$. Denote all matrices in $\mathrm{RPH}(n, \mathbb{Z})$ integer conjugate to A by $\mathrm{RPH}(A)$.

In Theorem 2.1 below we show that the number of ς-reduced matrices is finite and nonzero in any integer conjugacy class.

2.2. Perfect Hessenberg matrices conjugate to a given one. In the following an integer vector is *primitive* if its coordinates are relatively prime.

Proposition 2.1. *Assume that $A \in \mathrm{SL}(n, \mathbb{Z})$ has a characteristic polynomial irreducible over $\mathbb{Q}$. Then for any integer primitive vector v there exists a unique matrix $C \in \mathrm{GL}(n, \mathbb{Z})$ such that*

— $C(e_1) = v$;

— $CAC^{-1} \in \mathrm{PH}(n, \mathbb{Z})$.

Proof suggested by the referee. Existence. Consider the spaces

$$V_i = \langle v, A(v), A^2(v), \dots, A^{i-1}(v) \rangle \quad \text{for } i = 1, \dots, n.$$

Since the characteristic polynomial of A is irreducible, $\dim V_i = i$ and the spaces V_i form a complete flag in $\mathbb{R}^n$. Since $A^1(v), \dots, A^{n-1}(v) \in \mathbb{Z}^n$, $\mathrm{rk}(V_i \cap \mathbb{Z}^n) = i$ for $i = 1, \dots, n$.

Let us inductively construct an integer basis $\{\tilde{e}_i\}$ of $\mathbb{R}^n$ such that:

— for $i = 1, \dots, n$, the vectors $\tilde{e}_1, \dots, \tilde{e}_i$ form a basis of the sublattice $V_i \cap \mathbb{Z}^n$;

— $CAC^{-1} \in \mathrm{PH}(n, \mathbb{Z})$, where C is the transition matrix to the basis $(\tilde{e}_1, \dots, \tilde{e}_n)$.

Base of induction. The vector $\tilde{e}_1 := v$ generates $V_1 \cap \mathbb{Z}^n$.

Induction step. Consider two lattices: $L = L_{k+1} = V_{k+1} \cap \mathbb{Z}^n$ and the sublattice $L' := \langle \tilde{e}_1, \dots, \tilde{e}_k, A(\tilde{e}_k) \rangle_{\mathbb{Z}}$. Since $L_k := \langle \tilde{e}_1, \dots, \tilde{e}_k \rangle_{\mathbb{Z}}$ is a pure sublattice of L and L' has finite index in L, the quotient L/L' is cyclic of

some finite order, say, a. Hence there are unique $b_1, \ldots, b_k \in \{0, \ldots, a-1\}$ such that

$$\frac{1}{a}\Big(A(\tilde{e}_k) - \sum_{i=1}^{k} b_i \tilde{e}_i\Big) =: \tilde{e}_{k+1} \in L. \tag{2.1}$$

Then $L = \langle \tilde{e}_1, \ldots, \tilde{e}_{k+1} \rangle_{\mathbb{Z}}$. This concludes the induction step.

Denote by C the transition matrix to the basis $\{\tilde{e}_i\}$. Then we have $C(e_1) = v$ and $CAC^{-1} \in \mathrm{PH}(n, \mathbb{Z})$.

Uniqueness. Let C_1 and C_2 satisfy the conditions of the theorem. Suppose that C_1 and C_2 are the transition matrices to the bases $\{\tilde{e}_i\}$ and $\{\hat{e}_i\}$ respectively. Then

$$\tilde{e}_1 = C_1(e_1) = v = C_2(e_1) = \hat{e}_1.$$

For $i > 1$ the equality $\tilde{e}_i = \hat{e}_i$ follows from the uniqueness of the choice of the coefficients in Equation (2.1). Hence bases $\{\tilde{e}_i\}$ and $\{\hat{e}_i\}$ coincide. Therefore, $C_1 = C_2$. □

2.3. Existence and finiteness of ς-reduced Hessenberg matrices.

Theorem 2.1. *For any matrix $A \in \mathrm{SL}(n, \mathbb{Z})$ whose characteristic polynomial is irreducible over $\mathbb{Q}$ the set $\mathrm{RPH}(A)$ is nonempty and finite.*

For the proof of this theorem we use the following general proposition.

Proposition 2.2. *Any Hessenberg matrix A with $\varsigma(A) > 0$ is uniquely defined by its Hessenberg type A^{red} and the characteristic polynomial.*

Proof. Suppose that for a matrix $A = (a_{i,j})$ we know its characteristic polynomial

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0.$$

and its Hessenberg type A^{red}. From A^{red} we know all columns of A except for the last one. Direct calculations show that for any k the coefficient c_k is a polynomial in the variables $a_{i,j}$ not depending on $a_{1,n}, \ldots, a_{k,n}$. The unique monomial in c_k containing $a_{k+1,n}$ is

$$\Big(\prod_{j=k+1}^{n-1} a_{j+1,j}\Big) a_{k+1,n}.$$

Since $\varsigma(A) \neq 0$, the product in brackets is nonzero. Hence $a_{k+1,n}$ is a function of c_k and $a_{i,j}$ where $1 \leq i \leq n$ and $1 \leq j < n$, and the last column is uniquely defined. □

Example. Hessenberg complexity together with the characteristic polynomial is not sufficient to distinguish all the integer conjugacy classes. The two matrices

$$\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 3 & 8 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 2 & 5 \\ 1 & 1 & 2 \\ 0 & 3 & 7 \end{pmatrix}$$

are not integer conjugate. However their Hessenberg complexity equals 3, and they have the same characteristic polynomial.

Proof of Theorem 2.1. By Proposition 2.1 we have $\mathrm{PH}(A) \neq \emptyset$. Since $\varsigma(\mathrm{PH}(A)) \subset \mathbb{Z}_+$, there exists $\tilde{A} \in \mathrm{PH}(A)$ with minimal complexity ς, so $\tilde{A} \in \mathrm{RPH}(A)$.

It follows that $\mathrm{RPH}(A) \neq \emptyset$. All integer conjugate matrices have the same characteristic polynomial. Hence by Proposition 2.2 there exists at most one matrix $\overline{A} \in \mathrm{RPH}(A)$ of any given Hessenberg type. By definition all matrices of $\mathrm{RPH}(A)$ have the same Hessenberg complexity (say, c). The number of Hessenberg types whose Hessenberg complexity equals c is finite. Therefore, $\mathrm{RPH}(A)$ is finite. □

2.4. Families of Hessenberg matrices with given Hessenberg type. Consider an arbitrary Hessenberg type $\Omega = A^{\mathrm{red}}$ with integer coefficients $(a_{i,j})$. Denote by $H(\Omega)$ the set of all Hessenberg matrices in $\mathrm{SL}(n, \mathbb{Z})$ of Hessenberg type Ω.

Denote by $M_k(\Omega)$ the matrix obtained from the zero matrix by replacing the last column by $v_k := (a_{k,1}, \ldots, a_{k,k+1}, 0, \ldots, 0)^t$. Consider the simplex $\sigma(\Omega) = \mathrm{conv}(O, v_1, \ldots, v_{n-1})$.

Definition. The *integer volume* of a simplex σ with integer vertices is the index of the sublattice generated by its edges in the lattice $\langle\sigma\rangle \cap \mathbb{Z}^n$, denote it by $\mathrm{IV}(\sigma)$.

Proposition 2.3.
i). The set $H(\Omega)$ is not empty if and only if $\mathrm{IV}(\sigma(\Omega)) = 1$.
ii). If $A_0 \in H(\Omega)$, then $H(\Omega) = A_0 + \langle M_1(\Omega), \ldots, M_{n-1}(\Omega)\rangle_{\mathbb{Z}}$.

The proof of Theorem 2.3 is based on Lemma 2.1.

Definition. Consider a k-dimensional subspace π satisfying $\mathrm{rk}\,\pi \cap \mathbb{Z}^d = k$ and an integer vector $v \notin \pi$. The *integer distance* from v to π is the index of the sublattice $\langle v, \pi\rangle_{\mathbb{Z}}$ in the lattice $\langle v, \pi\rangle_{\mathbb{R}} \cap \mathbb{Z}^n$. We denote it by $\mathrm{ID}(v, \pi)$.

Lemma 2.1. *Consider a Hessenberg matrix A of type Ω, let its last column be an integer vector v. Then $A \in \mathrm{SL}(n, \mathbb{Z})$ if and only if the following conditions hold:*

$$\mathrm{IV}(\sigma(\Omega)) = 1 \qquad \textit{and} \qquad \mathrm{ID}(v, \langle\sigma(\Omega)\rangle) = 1.$$

Proof. If $A \in \mathrm{SL}(n,\mathbb{Z})$, then A preserves all integer volumes and integer distances. Let $S_e^{n-1} := \mathrm{conv}(O, e_1, \ldots, e_{n-1})$. Since $\mathrm{IV}(S_e^{n-1}) = 1$, we have

$$\mathrm{IV}(\sigma(\Omega)) = \mathrm{IV}(A(S_e^{n-1})) = 1.$$

In addition we have $A(e_n) = v$. Hence

$$\mathrm{ID}(v, \langle \sigma(\Omega) \rangle) = \mathrm{ID}(e_n, \langle e_1, \ldots, e_{n-1} \rangle) = 1.$$

Conversely, it is easy to see that the two conditions stated here imply that the A-image of $\mathbb{Z}^n$ is a sublattice of index 1 in $\mathbb{Z}^n$, i.e., $A \in \mathrm{SL}(n,\mathbb{Z})$. □

Proof of Theorem 2.3. (*i*). Suppose that $\mathrm{IV}(\sigma(\Omega)) = 1$. Then choose v at unit integer distance to the plane spanned by $\sigma(\Omega)$. Then by Lemma 2.1 $H(\Omega) \neq \emptyset$. Conversely if $H(\Omega) \cap \mathrm{SL}(n,\mathbb{Z}) \neq \emptyset$, then by Lemma 2.1 we have $\mathrm{IV}(\sigma(\Omega)) = 1$.

Statement (*ii*) is straightforward, since the determinant of the matrix is additive with respect to the operation of addition of vectors in the last column. □

Example. The Hessenberg matrices of type $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$ form the family

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} + \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right\rangle_{\mathbb{Z}}.$$

3. Complete geometric invariants of conjugacy classes

In this section we introduce a complete geometric invariant of integer conjugacy classes proposed by F. Klein in [9] and further extended by G. Voronoi [13]. For additional information see [2, 3, 4].

3.1. Continued fractions in the sense of Klein-Voronoi.

3.1.1. ***General definitions.*** Assume that $A \in \mathrm{GL}(n,\mathbb{R})$ has n distinct eigenvalues. Suppose that the real eigenvalues of A are $r_1, \ldots, r_k$ and the complex conjugate eigenvalues are $c_1, \overline{c}_1, \ldots, c_l, \overline{c}_l$, where $k+2l = n$. Denote by $L_{\mathbb{R}}(A)$ the space spanned by real eigenvectors.

Definition. Put

$$T_A = \{B \in \mathrm{GL}(n,\mathbb{R}) \mid AB = BA, \mathrm{spec}(B) \subset S^1, B|_{L_{\mathbb{R}}(A)} = \mathrm{id}|_{L_{\mathbb{R}}(A)}\},$$

where $\mathrm{spec}(B)$ is the spectrum of B, S^1 is the complex unit circle. Actually T_A is an abelian group with matrix multiplication as the group operation. For $v \in \mathbb{R}^n$ we denote its orbit $\{B(v) \mid B \in T_A\}$ by $T_A(v)$.

It is clear that the orbit $T_A(v)$ is homeomorphic to a torus of dimension not greater than l. In case v does not lie in any invariant plane of A, we have $\dim T_A(v) = l$.

Example. If A is totally real then we have $T_A(v) = \{v\}$. If A has only one pair of complex eigenvalues then a general orbit $T_A(v)$ is an ellipse around the $(n-2)$-dimensional invariant subspace corresponding to the real eigenvalues.

Let g_i be a real eigenvector with eigenvalue r_i for $i = 1, \dots, k$; let g_{k+2j-1} and g_{k+2j} be vectors of the real and the imaginary parts of some complex eigenvector with eigenvalue c_j for $j = 1, \dots, l$. Let

$$\pi = \langle g_1, \dots, g_k, g_{k+1}, g_{k+3}, \dots, g_{k+2l-1} \rangle_{\mathbb{R}},$$

and let π_+ be the cone of π whose points have nonnegative last l coordinates in the basis $(g_1, \dots, g_k, g_{k+1}, g_{k+3}, \dots, g_{k+2l-1})$. Note that for any v the orbit $T_A(v)$ intersects the cone π_+ in a unique point.

Consider the arrangement of all k real invariant hyperplanes of A, which are of the form $\langle g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_n \rangle_{\mathbb{R}}$. By $C_1(A), \dots, C_{2^k}(A)$ we denote the connected components of their complement in $\mathbb{R}^n$.

Definition. The set

$$S_i(A) := \bigcup_{p \in \partial\left(\operatorname{conv}\left(\{q \in \pi_+ | T_A(q) \cap C_i(A) \cap \mathbb{Z}^n \neq \emptyset,\ q \neq 0\}\right)\right)} T_A(p)$$

is called the *sail* of a cone C_i. The *Klein-Voronoi continued fraction* of A as the union of all sails:

$$\textsc{kvcf}(A) = \bigcup\nolimits_{i=1}^{2^k} S_i(A).$$

For an arbitrary m-dimensional plane $\tilde{\pi} \subset \pi$, the set

$$\bigcup_{p \in S_i(A) \cap \tilde{\pi}} T_A(p)$$

is called an *m-dimensional orbit-face* if the set $S_i(A) \cap \tilde{\pi}$ is homeomorphic to the m-dimensional ball. Integer points of the sail are said to be *vertices* of this sail.

3.1.2. *Algebraic continued fractions.* Consider $A \in \mathrm{GL}(n, \mathbb{Z})$ whose characteristic polynomial is irreducible over $\mathbb{Q}$. Suppose that it has k real roots $r_1, \dots, r_k$ and $2l$ complex roots $c_1, \bar{c}_1, \dots, c_l, \bar{c}_l$, where $k + 2l = n$.

Definition. The group of all elements of $\mathrm{GL}(n, \mathbb{Z})$ commuting with A is called the *Dirichlet group* of A and is denoted by $\Xi(A)$.

The group $\Xi(A)$ takes $\textsc{kvcf}(A)$ to itself and permutes the sails. By the Dirichlet unit theorem, $\Xi(A) \simeq \mathbb{Z}^{k+l-1} \oplus G$, where G is a finite abelian group. From the definition it follows that $\textsc{kvcf}(A)/\Xi(A)$ is homeomorphic to one or several copies of the $(n-1)$-dimensional torus.

A *fundamental domain* of $\textsc{kvcf}(A)$ is a collection of its orbit-faces, one from each equivalence class of $\textsc{kvcf}(A)/\Xi(A)$.

3.2. Geometric complete invariants of Dirichlet groups.

Theorem 3.1. *Assume that $A, B \in \mathrm{GL}(n,\mathbb{Z})$ have characteristic polynomials which are irreducible over $\mathbb{Q}$. Then $\Xi(A) = \Xi(B)$ if and only if $\mathrm{KVCF}(A) = \mathrm{KVCF}(B)$.*

Remark. If the characteristic polynomial of a matrix is irreducible over $\mathbb{Q}$, then all its eigenvalues are distinct, so all matrices of Theorem 3.1 possess a Klein-Voronoi continued fraction.

Proof. If $\Xi(A) = \Xi(B)$, then A and B commute. Hence they have the same eigenvectors (since they do not have multiple eigenvalues). Therefore, by definition $\mathrm{KVCF}(A) = \mathrm{KVCF}(B)$.

Conversely, assume that $\mathrm{KVCF}(A) = \mathrm{KVCF}(B)$. We assume that A has real eigenvectors $g_1, \ldots, g_k$ (corresponding to eigenvalues r_i), and also some complex conjugate eigenvectors $g_{k+2j-1} \pm \sqrt{-1} g_{k+2j}$ (corresponding to conjugate complex eigenvalues $c_j, \overline{c}_j$), for $j = 1, \ldots, l$. We denote coordinates with respect to the basis $\{g_i\}$ by $x_1, \ldots, x_k, y_1, z_1, \ldots y_l, z_l$ and define the function Φ_A by

$$\Phi_A(x_1, \ldots, z_l) = \prod_{i=1}^{k} x_i \prod_{j=1}^{l} (y_j^2 + z_j^2)$$

Similarly define Φ_B for B. Note that A preserves the form Φ_A up to a multiplicative scalar: a simple calculation shows that for any $v \in \mathbb{R}^n$ it holds $\Phi_A(A(v)) = \det(A)\Phi_A(v)$. The same is true for B and Φ_B.

Denote by $D_\rho(0)$ the ball of radius ρ centered at the origin. The set $\mathrm{KVCF}(A) = \mathrm{KVCF}(B)$ asymptotically coincides with the set $\Phi_A = 0$ (and $\Phi_B = 0$ respectively) at infinity, i.e., a for any ε there exists $N > 0$ such that for any point $p \in \mathrm{KVCF}(A) \setminus D_\rho(0)$ there exist a point $q \in \{\Phi_A = 0\} \setminus D_\rho(0)$ satisfying $|p - q| < \varepsilon$ and vice versa. Therefore, $\{\Phi_A = 0\} = \{\Phi_B = 0\}$, and hence the matrices A and B have the same invariant subspaces. In particular, their one-dimensional real eigenspaces corresponding to real eigenvectors and two-dimensional eigenspaces (we denote them by π_1, …, π_l) defined by pairs of complex conjugate roots coincide. Hence in order to prove that A and B commute it is enough to prove that they commute for the vectors of the invariant planes π_1, …, π_l.

Let us show that A and B restricted to π_i $(i = 1, \ldots, l)$ commute. Consider $v \in \mathrm{KVCF}(A)$, it is clear from Definition 3.1.1 that

$$T_A(v) = \mathrm{KVCF}(A) \cap (v + \langle \pi_1, \ldots, \pi_l \rangle) = \mathrm{KVCF}(B) \cap (v + \langle \pi_1, \ldots, \pi_l \rangle) = T_B(v).$$

Consider an arbitrary $C \in T_A$. Since all eigenvalues of C have unit modulus and C is diagonalizable in the eigenbasis of A, C preserves Φ_A up to a scalar $\det(C) = \pm 1$. Therefore, $|\Phi_A|$ is constant on $T_A(v)$. By the same reason $|\Phi_B|$ is constant on $T_A(v) = T_B(v)$. Therefore, by linearity, $\Phi_A = c \cdot \Phi_B$ for some constant $c \neq 0$. Hence, A preserves Φ_B up to a multiplicative scalar.

Consider now the plane π_j for some $1 \le j \le l$ and take coordinates (y_j, z_j). It is clear that if a linear operator preserves $\Phi_B(v)$ up to a multiplicative scalar then its restriction to the plane π_j preserves the level sets of the form

$$y_j^2 + z_j^2.$$

There are two types of matrices which do this:

$$\lambda \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \quad \text{and} \quad \lambda \begin{pmatrix} -\cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

where $\alpha, \lambda \in \mathbb{R}$ are arbitrary. The matrices of the second family have two real eigenvalues in π_j, which by the above is not the case for B. Therefore, both A and B are from the first family. All matrices of the first family commute. Hence A commutes with B in the planes π_j for $1 \le j \le l$.

Therefore A and B are simultaneously diagonalizable in a certain complex basis, and in consequence they commute. Since A and B are irreducible and commute, $\Xi(A) = \Xi(B)$. □

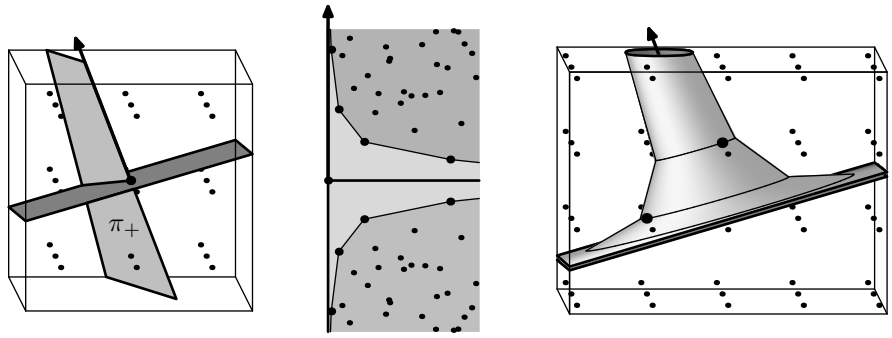
3.3. Geometric invariants of conjugacy classes. On the one hand, from Theorem 3.1 it follows that $\textsc{kvcf}(A)$ uniquely identifies $\Xi(A)$. On the other hand, the matrix A acts on $\textsc{kvcf}(A)$; we denote this transformation by P_A. It is clear that distinct matrices of $\Xi(A)$ define nonequivalent shifts. So the matrix $A \in \mathrm{GL}(n, \mathbb{Z})$ is uniquely identified with a pair $(\textsc{kvcf}(A), P_A)$. The group $\mathrm{GL}(n, \mathbb{Z})$ naturally acts on pairs $(\textsc{kvcf}(A), P_A)$ by left multiplication on the first factor and by conjugation on the second factor. Using these notions, Theorem 3.1 takes the following form.

Theorem 3.2. (On complete geometric invariants.) *Two matrices $A_1, A_2 \in \mathrm{GL}(n, \mathbb{Z})$ whose characteristic polynomial are irreducible over the field $\mathbb{Q}$, are integer conjugate if and only if the pairs $(\textsc{kvcf}(A_1), P_{A_1})$ and $(\textsc{kvcf}(A_2), P_{A_2})$ are in the same $\mathrm{GL}(n, \mathbb{Z})$-orbit.* □

Remark. An important consequence of this theorem is that all geometric $\mathrm{GL}(n, \mathbb{Z})$-invariants of $\textsc{kvcf}(A)$ — like integer distances and integer volumes of certain integer point configurations of $\textsc{kvcf}(A)$ — are invariants of the conjugacy class of A. Fundamental domains of Klein continued fractions and their invariants in a totally-real three dimensional case ($n = k = 3, l = 0$) were studied e.g. in [10, 11, 6]. Currently not much is known about the construction of Klein-Voronoi continued fractions in the non-totally-real case.

3.4. Particular example. Let us study a linear operator A defined by the Hessenberg matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$



a) *b)* *c)*

FIGURE 3.1. For the matrix A studied in Subsection 3.4 we show its cone π_+ in (a), the set $\text{KVCF}(A) \cap \pi_+$ in (b), and a sail of $\text{KVCF}(A)$ in (c).

It has one real and two complex conjugate eigenvalues. Therefore, the corresponding cone π_+ is a two-dimensional half-plane. In Figure 3.1a the halfplane π_+ is colored in light gray and the invariant plane corresponding to the pair of complex eigenvectors is in dark gray. The vector shown in Figure 3.1a with endpoint at the origin is an eigenvector of A.

In Figure 3.1b we show the cone π_+. The invariant plane separates π_+ onto two parts. The dots on π_+ correspond to points v whose orbits contain integer points (i.e. $T_A(v) \cap \mathbb{Z}^3 \neq \emptyset$). We take the boundaries of convex hulls in each of the two parts. Actually, the boundary of first convex hull is taken to the boundary of the second by $-\text{id}$.

Finally, in Figure 3.1c we show one of two sails. Three orbit-vertices shown in the figure correspond to the vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$: the large dark points $(0,1,0)$ and $(0,0,1)$ are visible on the corresponding orbit-vertices.

The group $\Xi(A)$ is homeomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with generators A and $-\text{id}$. Since $A(1,0,0) = (0,1,0)$, a fundamental domain of $\text{KVCF}(A)$ contains one orbit-vertex and one vertex edge. For instance, we choose the orbit-vertex $T_A(1,0,0)$ and the orbit-edge corresponding to the "tube" connecting orbit-vectors $T_A(1,0,0)$ and $T_A(0,1,0)$.

Future work. In the forthcoming paper [7] we focus on the ς-reducibility properties in the set of Hessenberg matrices of $\text{SL}(3,\mathbb{Z})$. We apply the techniques of Klein-Voronoi continued fractions to the first interesting open case here: the case of matrices having one real and two complex conjugate eigenvalues. We show that in this case Hessenberg matrices almost always distinguish corresponding conjugacy classes.

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Oleg Karpenkov
TU Graz
24, Kopernikusgasse
8010, Graz, Austria
E-mail: karpenkov@tugraz.at
URL: http://www.geometrie.tugraz.at/karpenkov/