JOURNAL
de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 25, n^o 1 (2013), p. 43-57.

<http://jtnb.cedram.org/item?id=JTNB_2013__25_1_43_0>

Journal de Théorie des Nombres de Bordeaux **25** (2013), 43-57

Algebraic independence of the generating functions of Stern's sequence and of its twist

par PETER BUNDSCHUH et KEIJO VÄÄNÄNEN

RÉSUMÉ. Très récemment, la fonction génératrice $A(z)$ de la suite $(a_n)_{n\geq 0}$ de Stern, définie par $a_0 := 0, a_1 := 1$, et $a_{2n} := a_n, a_{2n+1} := a_n + a_{n+1}$ pour tout entier $n > 0$, a été considérée du point de vue arithmétique. Coons [8] a montré la transcendance de $A(\alpha)$ pour tout α algébrique avec $0 < |\alpha| < 1$, et ce résultat fut généralisé dans [6] de sorte que, pour les mêmes α, les nombres $A(\alpha), A'(\alpha), A''(\alpha), \ldots$ sont algébriquement indépendants. À peu près au même temps, Bacher [4] a étudié la version tordue (b_n) de la suite de Stern, définie par $b_0 := 0, b_1 := 1$, et $b_{2n} := -b_n, b_{2n+1} := -(b_n + b_{n+1})$ pour tout $n > 0$.

Les objectifs principaux du présent travail sont d'établir les analogues sur la fonction génératrice $B(z)$ de (b_n) des résultats arithmétiques mentionnés plus haut concernant $A(z)$, de démontrer l'indépendance algébrique de $A(z), B(z)$ sur le corps $\mathbb{C}(z)$, d'utiliser ce fait pour en déduire que, pour tout nombre complexe α avec $0 < |\alpha| < 1$, le degré de transcendance du corps $\mathbb{Q}(\alpha, A(\alpha), B(\alpha))$ sur $\mathbb{Q}$ est au moins 2, et de fournir des majorations assez bonnes pour l'exposant d'irrationalité de $A(r/s)$ et de $B(r/s)$, où r, s sont des entiers avec $0 < |r| < s$ et $(\log|r|)/(\log s)$ suffisamment petit.

ABSTRACT. Very recently, the generating function $A(z)$ of the Stern sequence $(a_n)_{n\geq 0}$, defined by $a_0 := 0, a_1 := 1$, and $a_{2n} := a_n, a_{2n+1} := a_n + a_{n+1}$ for any integer $n > 0$, has been considered from the arithmetical point of view. Coons [8] proved the transcendence of $A(\alpha)$ for every algebraic α with $0 < |\alpha| < 1$, and this result was generalized in [6] to the effect that, for the same α's, all numbers $A(\alpha), A'(\alpha), A''(\alpha), \ldots$ are algebraically independent. At about the same time, Bacher [4] studied the twisted version (b_n) of Stern's sequence, defined by $b_0 := 0, b_1 := 1$, and $b_{2n} := -b_n, b_{2n+1} := -(b_n + b_{n+1})$ for any $n > 0$.

The aim of our paper is to show the analogs on the generating function $B(z)$ of (b_n) of the above-mentioned arithmetical results on $A(z)$, to prove the algebraic independence of $A(z), B(z)$ over the field $\mathbb{C}(z)$, to use this fact to conclude that, for any

Manuscrit reçu le 22 septembre 2012.

complex α with $0 < |\alpha| < 1$, the transcendence degree of the field $\mathbb{Q}(\alpha, A(\alpha), B(\alpha))$ over $\mathbb{Q}$ is at least 2, and to provide rather good upper bounds for the irrationality exponent of $A(r/s)$ and $B(r/s)$ for integers r, s with $0 < |r| < s$ and sufficiently small $(\log |r|)/(\log s)$.

1. Introduction and results

Stern's sequence has a long history and seemingly appeared first in print in 1858 [15]. Recall that this sequence $(a_n)_{n=0,1,\ldots}$ is defined by $a_0 := 0, a_1 := 1$ and, for $n \in \mathbb{N} := \{1, 2, \ldots\}$, by

$$a_{2n} := a_n \quad \text{and} \quad a_{2n+1} := a_n + a_{n+1}.$$

In a recent paper, Coons [8] discussed arithmetic and related analytic questions concerning the generating function[1]

$$A(z) := \sum_{n=0}^{\infty} a_{n+1} z^n$$

of Stern's sequence. More precisely, he proved that $A(z)$ is transcendental over the rational function field $\mathbb{C}(z)$ [8, Theorem 2.2] and used this to show the transcendence (over $\mathbb{Q}$) of $A(\alpha)$ at every $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$ [8, Theorem 2.4]. Here $\overline{\mathbb{Q}}$ denotes the field of all complex algebraic numbers, and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

In a very recent article, the first author [6] fairly generalized both of these Coons' results by proving the algebraic independence over $\mathbb{Q}$ of the numbers $A(\alpha), A'(\alpha), A''(\alpha), \ldots$ for every α as before. Analytically, the basic ingredient of this proof is the fact that the function $A(z)$ is hypertranscendental. Remember that an analytic function is called *hypertranscendental* if it satisfies no algebraic differential equation, that is, no finite collection of derivatives of the function is algebraically dependent over $\mathbb{C}(z)$.

It should be pointed out that two more proofs of Coons' result on the transcendence of $A(\alpha)$ are included in [6]: The first one, in Remark 3, depends, as Coons', on Mahler's classical result, Theorem 2.1 below, whereas the second one, in Sec. 5, is based on a consequence of Schmidt's Subspace Theorem (see Corvaja and Zannier [9, Corollary 1]).

Let us finally notice that $A(z)$ satisfies the Mahler-type functional equation[2]

$$A(z) = p(z)A(z^2) \quad \text{with} \quad p(z) := 1 + z + z^2 \tag{1.1}$$

[1] The usual definition of a generating function differs from this by a factor z on the right-hand side. But this is unimportant for the arithmetical questions to be considered here.

[2] In [6], from p. 365 on, the following $p(z)$ was denoted by $P(z)$.

which plays a central role in all proofs in Secs. 2, 3, and 5 of [6]. It is easily deduced from the above recursive definition of the sequence (a_n), and makes evident the product representation

$$A(z) = \prod_{k=0}^{\infty} p(z^{2^k}), \tag{1.2}$$

whence $A(z)$ does not vanish on $\mathbb{D}$.

In a recent preprint, Bacher [4] (see also Allouche [2]) introduced the twisted version of Stern's sequence by putting $b_0 := 0, b_1 := 1$ and, for $n \in \mathbb{N}$,

$$b_{2n} := -b_n \quad \text{and} \quad b_{2n+1} := -(b_n + b_{n+1}). \tag{1.3}$$

This definition shows $b_n \in \mathbb{Z}$ for any $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, b_{2^k} = (-1)^k$ for any $k \in \mathbb{N}_0$, and $|b_n| \leq n$ for every $n \in \mathbb{N}_0$. On denoting the generating function of the sequence (b_n) by

$$B(z) := \sum_{n=0}^{\infty} b_{n+1} z^n, \tag{1.4}$$

this is a non-terminating power series with integral coefficients having convergence radius 1. Thus, by a result of Carlson [7], $B(z)$ either defines a rational function or cannot be analytically continued beyond the unit circle (hence is transcendental).

At the beginning of Sec. 2 below, we will prove that $B(z)$ satisfies the following Mahler-type functional equation

$$B(z) = 2 - p(z)B(z^2) \tag{1.5}$$

with $p(z)$ as in (1.1). This fact not only will enable us to rule out the beforementioned first alternative but also to prove the following analogue of the transcendence of $A(\alpha)$, again using Mahler's Theorem 2.1.

Theorem 1.1. *For every $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$, the number $B(\alpha)$ is transcendental.*

Remark. It would be interesting to see a proof of this statement via Schmidt's Subspace Theorem along similar lines as used in Sec. 5 of [6].

Our next result is the B-analogue of Theorems 1 and 2 from [6].

Theorem 1.2. *The function $B(z)$ is hypertranscendental. Moreover, for every $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$, the numbers $B(\alpha), B'(\alpha), B''(\alpha), \ldots$ are algebraically independent.*

Our main concern in Secs. 3 and 4 will be to study the algebraic independence of A and B, first from the 'functional', and then from the 'numerical' point of view. More precisely, we will be able to first establish the following.

Theorem 1.3. *The functions $A(z)$ and $B(z)$ are algebraically independent over $\mathbb{C}(z)$.*

This result for which we will give an elementary proof is the main analytic ingredient in showing the subsequent common strengthening of Coons' Theorem 2.4 and of our preceding Theorem 1.1.

Theorem 1.4. *For any non-zero $\alpha \in \mathbb{D}$, the transcendence degree of $\mathbb{Q}(\alpha, A(\alpha), B(\alpha))$ over $\mathbb{Q}$ is at least 2. In particular, for every $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$, the numbers $A(\alpha)$ and $B(\alpha)$ are algebraically independent.*

Remark. In view of Theorem 1.2, its A-analogue [6, Theorem 2] and of Theorem 1.4 one may ask for the algebraic independence of all numbers $A(\alpha), B(\alpha), A'(\alpha), B'(\alpha), \ldots$ at non-zero algebraic points α in the unit disk.

Sec. 5 will be devoted to some quantitative questions in the present realm. We first give an algebraic independence measure of $A(\alpha)$ and $B(\alpha)$ with algebraic α.

Theorem 1.5. *Let $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$. For any $H, s \in \mathbb{N}$ and for any polynomial $P \in \mathbb{Z}[x_1, x_2] \setminus \{0\}$ whose total degree does not exceed s and whose coefficients are not greater than H in absolute value, the inequality*

$$|P(A(\alpha), B(\alpha))| > \exp(-\mu s^2(\log H + s^2 \log(s+1))),$$

holds, where μ is a positive constant depending only on α and the functions A and B.

This theorem immediately implies that the irrationality exponent of $A(\alpha)$ or $B(\alpha)$ with real algebraic α and $0 < |\alpha| < 1$, is finite. Recall here that the irrationality exponent $\mu(\xi)$ of a real irrational number ξ is defined to be the infimum of the real numbers μ such that the inequality

$$\left|\xi - \frac{p}{q}\right| \leq q^{-\mu}$$

has only finitely many solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$. For certain rational α's, we can get a more precise and rather sharp result by using the ideas of the works [1] and [5]. Namely the following result holds.

Theorem 1.6. *If $r, s \in \mathbb{Z} \setminus \{0\}, s \geq 2$ and $\gamma := (\log |r|)/(\log s) < 1$, then the estimates*

$$\mu(A(r/s)) \leq \frac{375(1-\gamma)}{32(4-9\gamma)}, \qquad if\ \gamma < \frac{4}{9},$$

$$\mu(B(r/s)) \leq \frac{32(1-\gamma)}{3(3-7\gamma)}, \qquad if\ \gamma < \frac{3}{7},$$

hold. In particular, one has $\mu(A(1/s)) \leq \frac{375}{128} = 2,929\ldots$, and $\mu(B(1/s)) \leq \frac{32}{9} = 3,555\ldots$.

2. Functional equation, non-continuability, and proof of Theorems 1.1 and 1.2

To prove equation (1.5), we start from (1.4) and use (1.3).

$$\begin{aligned} B(z)-1 &= \sum_{j=1}^{\infty} b_{2j} z^{2j-1} + \sum_{j=1}^{\infty} b_{2j+1} z^{2j} \\ &= -\sum_{j=1}^{\infty} b_j z^{2j-1} - \sum_{j=1}^{\infty} (b_j + b_{j+1}) z^{2j} \\ &= -z \sum_{j=1}^{\infty} b_j z^{2(j-1)} - z^2 \sum_{j=1}^{\infty} b_j z^{2(j-1)} - \Big(\sum_{j=1}^{\infty} b_j z^{2(j-1)} - 1 \Big). \end{aligned}$$

By (1.4), the last three sums equal $B(z^2)$, and this leads to (1.5).

Next we assume that B is a rational function which means that there are coprime $u, v \in \mathbb{C}[z] \setminus \{0\}$ such that $B(z) = u(z)/v(z)$. Then (1.5) can be equivalently written as

$$u(z)v(z^2) = 2v(z)v(z^2) - p(z)u(z^2)v(z)$$

implying the divisibility condition

$$v(z^2) \mid (1+z+z^2)v(z) \quad (\text{in } \mathbb{C}[z]) \tag{2.1}$$

since $u(z^2), v(z^2)$ are also coprime. Now (2.1) implies $\deg v \le 2$, where $\deg v = 0$ can be immediately excluded since, by (1.5), B cannot be a polynomial. Next, in the case $\deg v = 1$, hence $v(z) = z + c$ (w.l.o.g.) with some $c \in \mathbb{C}^\times$, (2.1) would read as

$$\frac{(1+z+z^2)(z+c)}{z^2+c} = z + 1.$$

This implies $c \neq -1$, and inserting $z = 1$ leads to a contradiction. In the remaining case $\deg v = 2$, degree considerations show

$$(1+z+z^2)v(z) = d\, v(z^2) \tag{2.2}$$

with some $d \in \mathbb{C}^\times$, in fact with $d = 1$. But then, by (2.2), $v(1) = 0, v(\zeta) = v(\overline{\zeta}) = 0$ for $\zeta := e^{2\pi i/3}$, and these are too many distinct zeros for a degree 2 polynomial.

Proof of Theorem 1.1. To directly prepare this proof, we next quote the one-dimensional version of a transcendence criterion going back to Mahler [11] (see also [14, Theorem 1.2]). Let K be an algebraic number field and O_K its ring of integers. Assume that $f \in K[[z]]$ has convergence radius $r > 0$ and satisfies a functional equation

$$f(z^t) = \frac{g_0(z) + \ldots + g_m(z) f(z)^m}{h_0(z) + \ldots + h_m(z) f(z)^m} \tag{2.3}$$

with $t \in \mathbb{N} \setminus \{1\}, m \in \{1, ..., t-1\}, g_\mu, h_\mu \in O_K[z]$ $(\mu = 0, ..., m)$, $(g_m, h_m) \neq (0,0)$. If $\Delta(z)$ denotes the resultant of the two polynomials

$$g_0(z) + ... + g_m(z)X^m \quad \text{and} \quad h_0(z) + ... + h_m(z)X^m$$

with respect to the indeterminate X, then the following holds.

Theorem 2.1 (Mahler's theorem)**.** *Assume that* K, f, Δ *are as before, and that* f *is transcendental over* $K(z)$*. If* $\alpha \in \overline{\mathbb{Q}}^\times$ *satisfies* $|\alpha| < \min(1, r)$ *and* $\Delta(\alpha^{t^j}) \neq 0$ *for any* $j \in \mathbb{N}_0$*, then* $f(\alpha)$ *is transcendental.*

To finish the proof of Theorem 1.1, we apply Theorem 2.1 to $f = B$ which has been recognized as transcendental over $\mathbb{C}(z)$. We may take $K = \mathbb{Q}, r = 1$, and, since (2.3) reads here

$$B(z^2) = \frac{2 - B(z)}{p(z)}\,, \tag{2.4}$$

$t = 2, m = 1, g_0(z) = 2, g_1(z) = -1, h_0(z) = p(z), h_1(z) = 0$, and therefore

$$\Delta(z) = \det \begin{pmatrix} -1 & 2 \\ 0 & p(z) \end{pmatrix} = -p(z).$$

Since $0 \notin p(\mathbb{D})$, our proof is complete. □

Proof of Theorem 1.2. The proof of the hypertranscendence of $B(z)$ is essentially based on Theorem 3 from Nishioka's paper [12] which is deduced from a necessary condition for the existence of differentially algebraic solutions of certain types of functional equations.

Theorem 2.2 ([12, Theorem 3])**.** *Let* C *be a field of characteristic* 0*, and suppose that* $f \in C[[z]]$ *has the following two properties:*

(i) For suitable $m \in \mathbb{N}_0$*, the series* $f, Df, \dots, D^m f$ *are algebraically dependent over* $C(z)$*, where* D *denotes the differential operator* $z\frac{d}{dz}$*.*

(ii) For suitable $t \in \mathbb{N} \setminus \{1\}, f$ *satisfies the functional equation*

$$f(z^t) = u(z)f(z) + v(z), \tag{2.5}$$

where $u, v \in C(z), u \neq 0$*. If* $u(z) = s_M z^M + ...$ *with* $M \in \mathbb{Z}, s_M \in C^\times$ *define* $Q := [M/(t-1)]$*.*

Then there exists some $w \in C(z)$ *satisfying*

$$w(z^t) = u(z)w(z) + v(z)$$

or

$$w(z^t) = u(z)w(z) + v(z) - \gamma\frac{u_1(z)z^{Qt}}{u_2(z)},$$

where $u_1(z) = u(z)/(s_M z^M), u_2 \in C(z) \setminus \{0\}$ *fulfils the condition* $u_2(z^t) = u_2(z)/u_1(z)$*, and* $\gamma \in C$ *is the constant term in the* z*-expansion of the quotient* $v(z)u_2(z)/(u_1(z)z^{Qt})$ *in case* $s_M = 1$ *and* $M = Q(t-1)$*, but* $\gamma = 0$ *otherwise.*

Towards a contradiction to the first statement of our Theorem 1.2, we assume that there is an $m \in \mathbb{N}_0$ such that the functions $B, B', ..., B^{(m)}$ are algebraically dependent over $\mathbb{C}(z)$. As it is easily seen by induction, the equation

$$D^{\mu}B = \sum_{\lambda=1}^{\mu} c_{\lambda\mu} z^{\lambda} B^{(\lambda)}$$

holds for any $\mu \in \mathbb{N}$ with explicit $c_{1\mu}, ..., c_{\mu\mu} \in \mathbb{N}, c_{\mu\mu} = 1$. Therefore, our above assumption on the derivatives of B is equivalent to the algebraic dependence of $B, DB, ..., D^m B$ over $\mathbb{C}(z)$, whence condition (i) of Theorem 2.2 is satisfied.

By (2.4), condition (2.5) is satisfied for $f = B$ if we take $t = 2, u(z) = -1/p(z), v(z) = 2/p(z)$. Since our $u(z) = -1 + z + ...$ near the origin, we have $M = 0, s_M = -1, Q = 0$, and thus $\gamma = 0$ in Theorem 2.2. Therefore there is a $w \in \mathbb{C}(z)$ satisfying $w(z^2) = (2 - w(z))/p(z)$, and this contradicts what we have proved at the beginning of the present section.

The proof of the second, i.e., the arithmetic part of Theorem 1.2, depends essentially on the following inhomogeneous generalization of Nishioka's original result in [13, Corollary 2] quoted in [6].

Theorem 2.3 ([14, Theorem 4.2.1])**.** *Let K denote an algebraic number field, and let $t \in \mathbb{N} \setminus \{1\}$. Suppose that $f_1, ..., f_m \in K[[z]]$ converge in some disk $U \subset \mathbb{D}$ about the origin, where they satisfy the matrix functional equation*

$${}^{\tau}(f_1(z^t), ..., f_m(z^t)) = \mathcal{A}(z) \cdot {}^{\tau}(f_1(z), ..., f_m(z)) + {}^{\tau}(b_1(z), ..., b_m(z))$$

with $\mathcal{A} \in \mathrm{Mat}_{m\times m}(K(z)), \tau$ indicating the matrix transpose, and $b_1, ..., b_m \in K(z)$. If $\alpha \in \overline{\mathbb{Q}}^{\times} \cap U$ is such that none of the α^{t^j} ($j \in \mathbb{N}_0$) is a pole of $b_1, ..., b_m$ and the entries of $\mathcal{A}$, then the following inequality holds

$$\mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}(f_1(\alpha), ..., f_m(\alpha)) \geq \mathrm{trdeg}_{K(z)}K(z)(f_1(z), ..., f_m(z)).$$

With this tool, the proof of Theorem 1.2 can be easily completed, very much parallel to the one of Theorem 2 in [6]. Therefore we leave the details to the reader. □

3. Proof of Theorem 1.3

To establish this result, we first recall that, by (1.1) and (1.5), the functions A and B satisfy the following system of functional equations

$$A(z^2) = \frac{A(z)}{p(z)}, \quad B(z^2) = \frac{2 - B(z)}{p(z)}. \tag{3.1}$$

Assume now that $A(z)$ and $B(z)$ are algebraically dependent over $\mathbb{C}(z)$. Then there exists some $P \in \mathbb{C}[z, x, y] \setminus \{0\}$ depending on x *and* on y (remember that both of $A(z)$ and $B(z)$ are transcendental over $\mathbb{C}(z)$) such that

$$P(z,A(z),B(z))=0 \tag{3.2}$$

holds identically in $\mathbb{D}$. Assuming further that $k:=\deg_y P\in\mathbb{N}$ is minimal, we write

$$P(z,x,y)=\sum_{j=0}^{k}P_j(z,x)y^j \qquad (\text{with } P_k(z,x)\neq 0). \tag{3.3}$$

From this we obtain, by (3.1) and (3.2),

$$\begin{aligned}0=P(z^2,A(z^2),B(z^2))&=\sum_{j=0}^{k}P_j\Big(z^2,\frac{A(z)}{p(z)}\Big)\Big(\frac{2-B(z)}{p(z)}\Big)^j\\&=\sum_{i=0}^{k}(-B(z))^i\sum_{j=i}^{k}\binom{j}{i}2^{j-i}P_j\Big(z^2,\frac{A(z)}{p(z)}\Big)p(z)^{-j}.\end{aligned}$$

Multiplying this equation by $p(z)^{k+\ell}$, where $\ell:=\max_{0\le j\le k}\deg_x P_j(z,x)$, the last double sum becomes a polynomial in $z,A(z),B(z)$ suggesting us to define

$$\begin{aligned}Q(z,x,y)&:=\sum_{j=0}^{k}Q_j(z,x)y^j,\\Q_j(z,x)&:=(-1)^j\sum_{i=j}^{k}\binom{i}{j}2^{i-j}p(z)^{k+\ell-i}P_i\Big(z^2,\frac{x}{p(z)}\Big)\end{aligned}$$

whence $Q_k(z,x)=(-1)^k p(z)^{\ell}P_k(z^2,x/p(z))\neq 0$ and $Q(z,A(z),B(z))=0$ identically in $\mathbb{D}$, by our construction. Thus

$$\begin{aligned}R(z,x,y)&:=P_k(z,x)Q(z,x,y)-Q_k(z,x)P(z,x,y)\\&=\sum_{j=0}^{k-1}\Big(P_k(z,x)Q_j(z,x)-P_j(z,x)Q_k(z,x)\Big)y^j\in\mathbb{C}[z,x,y]\end{aligned}$$

has $\deg_y R<k$ and satisfies $R(z,A(z),B(z))=0$ in $\mathbb{D}$. By our above minimality condition on k, the coefficients of all y^j $(j=0,...,k-1)$ in R must vanish, in particular, the one of y^{k-1}. This leads after some minor computation to

(3.4)

$$P_{k-1}(z,x)P_k\Big(z^2,\frac{x}{p(z)}\Big)+P_k(z,x)\Big(p(z)P_{k-1}\Big(z^2,\frac{x}{p(z)}\Big)+2kP_k\Big(z^2,\frac{x}{p(z)}\Big)\Big)=0.$$

Note that (3.4) implies $P_{k-1}\neq 0$ (since $P_k\neq 0$). Denoting $s_j:=\deg_x P_j$ for $j\in\{k-1,k\}$ we infer from (3.4) that $s_k>s_{k-1}$ cannot hold. To rule out the possibility $s_k < s_{k-1}$ ($\Leftrightarrow s_{k-1} - 1 - s_k \geq 0$), we write for the same j's

$$P_j(z,x) = \sum_{\sigma=0}^{s_j} p_{j,\sigma}(z)x^\sigma \tag{3.5}$$

and consider on the left-hand side of (3.4) the factor of $x^{s_{k-1}+s_k}$, namely

$$p_{k-1,s_{k-1}}(z)p_{k,s_k}(z^2)p(z)^{-s_k} + p_{k,s_k}(z)p_{k-1,s_{k-1}}(z^2)p(z)^{1-s_{k-1}}$$

which has to vanish. This condition is equivalent to

$$p(z)^{s_{k-1}-1-s_k}p_{k-1,s_{k-1}}(z)p_{k,s_k}(z^2) + p_{k,s_k}(z)p_{k-1,s_{k-1}}(z^2) = 0. \tag{3.6}$$

Denoting by λ_j ($\neq 0$) the leading coefficient of $p_{j,s_j}(z)$ for $j \in \{k-1, k\}$, the leading coefficient of the polynomial on the left-hand side of (3.6) equals $\kappa\lambda_k\lambda_{k-1}$ with $\kappa \in \{1,2\}$, whence equation (3.6) cannot hold. Thus $s_{k-1} = s_k =: s$ must be valid.

We next insert (3.5) in the left-hand side of (3.4) and determine the coefficient of x^{2s} which equals

$$p_{k-1,s}(z)p_{k,s}(z^2) + p(z)p_{k,s}(z)p_{k-1,s}(z^2) + 2kp_{k,s}(z)p_{k,s}(z^2), \tag{3.7}$$

and this expression must vanish identically in z. Writing $d_j := \deg_z p_{j,s}$ for $j \in \{k-1, k\}$ the degrees of the three summands in (3.7) are

$$d_{k-1} + 2d_k, \;\; 2 + 2d_{k-1} + d_k, \;\; 3d_k, \tag{3.8}$$

respectively. It is easily checked that these degrees are distinct if and only if $d_k - d_{k-1} \notin \{0,1,2\}$. If $d_k - d_{k-1}$ is 0 or 2, then precisely the middle or the last term in (3.8), resp., is the largest one. Thus, there remains only one case to be excluded, namely $d_k - d_{k-1} = 1$, where the second and third term in (3.8) are equal, whereas the first one is smaller. With $d := d_k = d_{k-1} + 1$ we write

$$p_{k,s}(z) = a_1z^d + a_2z^{d-1} + \dots, \quad p_{k-1,s}(z) = b_1z^{d-1} + \dots \quad (\text{with } a_1b_1 \neq 0). \tag{3.9}$$

Inserting this in (3.7), this polynomial expansion starts with

$$a_1(b_1 + 2ka_1)z^{3d} + \big(2a_1b_1 + a_2(b_1 + 2ka_1)\big)z^{3d-1} + \dots$$

implying $b_1 + 2ka_1 = 0$ (since $a_1 \neq 0$) and then $2a_1b_1 = 0$. This contradicts the condition $a_1b_1 \neq 0$ in (3.9). Hence our initial assumption on A, B was incorrect and Theorem 3 is proved.

Remark. Whereas we presented above an elementary and self-contained proof, it should be noticed that our Theorem 3 could also be deduced from Proposition 3 in Kubota's very general paper [10] dealing with solutions of systems of multidimensional Mahler-type functional equations fairly generalizing our system (3.1). For the convenience of the reader, we briefly indicate the necessary specializations: For L, M and Ω, take $\mathbb{C}(z)$, the quotient field of $\mathbb{C}[[z]]$, and $(2) \in \mathrm{Mat}_{1\times 1}(\mathbb{N}_0)$, respectively; as system (22) in [10], take our (3.1), whence $f_1 = B, f_2 = A$ hence $m = 2, k = 1$.

4. Proof of Theorem 1.4

To this purpose, we want to apply Theorem 2.3 to $f_1 = A, f_2 = B$; hence we may take $K = \mathbb{Q}, t = 2, U = \mathbb{D}$. By (3.1), the column vector ${}^{\tau}(A, B)$ satisfies a matrix functional equation as supposed in Theorem 2.3 with

$$\mathcal{A}(z) = \frac{1}{p(z)} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ b_1(z) = 0,\ b_2(z) = \frac{2}{p(z)}\,.$$

Notice next that

$$\mathrm{trdeg}_{\mathbb{Q}(z)}\mathbb{Q}(z)(A(z), B(z)) = \mathrm{trdeg}_{\mathbb{C}(z)}\mathbb{C}(z)(A(z), B(z)) = 2,$$

the second equality holding by our Theorem 1.3; the first comes from a standard argument in transcendence theory (compare, e.g., the remarks in [14, pp. 6, 89]) taking $A, B \in \mathbb{Q}[[z]]$ into account. Thus, according to Theorem 2.3, we have $\mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}(A(\alpha), B(\alpha)) \geq 2$, in fact $= 2$, for any $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$. So far, the algebraic points α in Theorem 1.4.

The case of transcendental points $\alpha \in \mathbb{D}$ was considered by Amou [3]. We first note that his Theorem 1 with m-dimensional matrix functional equation for ${}^{\tau}(f_1, ..., f_m)$ similar to the one in Theorem 2.3 is not enough to conclude. But his Theorem 3 to be quoted below will work. Note that therein the hypotheses are much stronger compared to those in Theorem 2.3: Namely, the matrix $\mathcal{A}(z)$ must be diagonal, $\mathcal{A}(z)^{-1}$ and $\mathcal{A}(z)^{-1} \cdot {}^{\tau}(b_1(z), ..., b_m(z))$ must have polynomial entries, and m must be equal to 2. More precisely, Amou's result can be quoted as follows.

Theorem 4.1 ([3, Theorem 3])**.** *Let K denote an algebraic number field, and let $t \in \mathbb{N} \setminus \{1\}$. Suppose that $f_1, f_2 \in K[[z]]$ converge in $\mathbb{D}$, are algebraically independent over $K(z)$, and satisfy the functional equations*

$$f_\mu(z) = a_\mu(z) f_\mu(z^t) + b_\mu(z) \qquad (\mu = 1, 2), \tag{4.1}$$

where $a_\mu, b_\mu \in K[z]$. Then, for any transcendental $\alpha \in \mathbb{D}$, the following inequality holds

$$\mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}(\alpha, f_1(\alpha), f_2(\alpha)) \geq 2.$$

Applying Theorem 4.1 as before to $f_1 = A, f_2 = B$ we note that (3.1) can be equivalently written as

$$A(z) = p(z)A(z^2), \quad B(z) = -p(z)B(z^2) + 2$$

and this is of type (4.1), completing our proof.

5. Proof of Theorems 1.5 and 1.6

First we note that the proof of Theorem 1.5 follows immediately from the following special case of [14, Theorem 4.4.2].

Theorem 5.1. *Assume that the functions f_1, f_2 satisfy the assumptions of Theorem 4.1 and that $\alpha \in \overline{\mathbb{Q}}^{\times} \cap \mathbb{D}$ has the property $a_1(\alpha^{t^k})a_2(\alpha^{t^k}) \neq 0$ for any $k \in \mathbb{N}_0$ Then, for any $H, s \in \mathbb{N}$ and for any polynomial $P \in \mathbb{Z}[x_1, x_2] \setminus \{0\}$ whose total degree does not exceed s and whose coefficients are not greater than H in absolute value, the inequality*

$$|P(f_1(\alpha), f_2(\alpha))| > \exp(-\mu s^2(\log H + s^2 \log(s+1))),$$

holds, where μ is a positive constant depending only on α and the functions f_1 and f_2.

To prove Theorem 1.6, we use Padé approximations of A and B similarly to the procedure in [1] and [5] for the generating function of the Thue-Morse sequence. In our case, we do not have an analogous non-vanishing result but we compute several low degree Padé approximants and iterate these by the functional equation of A or B. This leads to a rather dense sequence of good rational approximations for $A(r/s)$ and $B(r/s)$. These approximations together with the following approximation lemma from [1] can finally be used to prove Theorem 1.6.

Lemma 5.1. *Let ξ, δ, ρ and θ be real numbers such that $0 < \delta \leq \rho$ and $\theta \geq 1$. Assume that there exist a sequence $(p_n/q_n)_{n\geq 1}$ of rational numbers and some positive constants C_0, C_1 and C_2 such that the inequalities*

$$q_n < q_{n+1} \leq C_0 q_n^{\theta} \quad \textit{and} \quad \frac{C_1}{q_n^{1+\rho}} \leq \left|\xi - \frac{p_n}{q_n}\right| \leq \frac{C_2}{q_n^{1+\delta}}.$$

hold. Then the estimate

$$\mu(\xi) \leq (1+\rho)\theta/\delta.$$

is valid. If, moreover, p_n and q_n are coprime for all sufficiently large n, then one has $\theta \geq \delta$ and

$$\mu(\xi) \leq \max(1+\rho, 1+\theta/\delta).$$

Proof of Theorem 1.6. To consider $A(r/s)$, we start by computing, simply by solving systems of linear homogeneous equations, Padé approximants[3] $[3/5]_A$, $[4/6]_A$, $[5/7]_A$ and $[6/8]_A$, which are given by the equations

$$Q_{i,0}(z)A(z) - P_{i,0}(z) = R_{i,0}(z), \qquad i = 1,2,3,4, \tag{5.1}$$

with the polynomials

$$Q_{1,0}(z) = 2 - z - z^2 - z^3 - 2z^4 + 4z^5, \;\; P_{1,0}(z) = 2 + z + 2z^2 - 2z^3,$$

[3]For this usual notation, compare, e.g. [5].

$$Q_{2,0}(z) = 3-2z+z^2+z^3-5z^4+2z^5-z^6,\ \ P_{2,0}(z) = 3+z+5z^2+z^3+5z^4,$$
$$Q_{3,0}(z)=1+z-z^2+z^3-z^4-2z^5+z^6-z^7, P_{3,0}(z)=1+2z+2z^2+3z^3+2z^4+3z^5,$$
$$Q_{4,0}(z)=1-2z-z^2+2z^3-3z^4+2z^5+2z^6-2z^7+2z^8,$$
$$P_{4,0}(z)=1-z-z^2-2z^3-2z^4-2z^5-4z^6,$$

and the corresponding remainder terms

$$R_{1,0}(z) = 6z^9 + \mathrm{O}(z^{10}),\ R_{2,0}(z) = 2z^{11} + \mathrm{O}(z^{12}),$$
$$R_{3,0}(z) = -2z^{13} + \mathrm{O}(z^{14}),\ R_{4,0}(z) = 4z^{15} + \mathrm{O}(z^{16}).$$

We now apply the functional equation (1.1) of $A(z)$ to (5.1) and obtain a sequence of approximations. Namely, assuming

$$Q_{i,n}(z)A(z) - P_{i,n}(z) = R_{i,n}(z), \tag{5.2}$$

we get

$$Q_{i,n}(z^2)p(z)A(z^2) - p(z)P_{i,n}(z^2) = p(z)R_{i,n}(z^2)$$

or

$$Q_{i,n+1}(z)A(z) - P_{i,n+1}(z) = R_{i,n+1}(z),$$

where

$$Q_{i,n+1}(z) = Q_{i,n}(z^2),\ P_{i,n+1}(z) = p(z)P_{i,n}(z^2),\ R_{i,n+1}(z) = p(z)R_{i,n}(z^2).$$

Starting from (5.1) and making this step repeatedly we get a sequence of equations (5.2) with

$$Q_{i,n}(z) = Q_{i,0}(z^{2^n}),\ P_{i,n}(z) = P_{i,0}(z^{2^n})\prod_{k=0}^{n-1} p(z^{2^k}),$$
$$R_{i,n}(z) = R_{i,0}(z^{2^n})\prod_{k=0}^{n-1} p(z^{2^k}).$$

Here deg $Q_{i,n} = (4+i)2^n$, deg $P_{i,n} \le (4+i)2^n - 2$, and $\mathrm{ord}_0 R_{i,n} = (7+2i)2^n$. Therefore we do not have Padé approximants if $n \ge 1$, because the vanishing orders at the origin are not sufficiently large. However, also these approximations are useful.

We now have

$$\frac{1}{2} < Q_{i,n}(r/s) < 4$$

for all $n \ge c_0$, where c_0 (as $c_1, c_2, \dots$ later) is an effectively computable positive constant independent of n. By denoting

$$q_{i,n} = s^{(4+i)2^n} Q_{i,n}(r/s),$$

we obtain four sequences of positive integers $q_{i,n}$ such that

$$c_1 s^{(4+i)2^n} \le q_{i,n} \le c_2 s^{(4+i)2^n} \tag{5.3}$$

for all $n \geq c_3$. Further, by (5.2),

$$A(r/s) - \frac{p_{i,n}}{q_{i,n}} = r_{i,n}$$

where

$$p_{i,n} = s^{(4+i)2^n} P_{i,n}(r/s), \quad r_{i,n} = \frac{s^{(4+i)2^n}}{q_{i,n}} R_{i,n}(r/s).$$

Note here that all $p_{i,n}$ are integers. Further, since $a_n \leq n$, the values of the terms $\mathrm{O}(z^{(8+4i)2^n})$ in $R_{i,0}(z^{2^n})$ at the point $z = r/s$ are bounded by $c_4|r/s|^{(8+2i)2^n}$ and

$$c_5 \leq \Big| \prod_{k=0}^{n-1} p((r/s)^{2^k}) \Big| \leq c_6$$

for all $n \geq c_7$. This implies (recall $|r| = s^\gamma$ with $0 \leq \gamma < 1$) that

$$\frac{c_8}{q_{i,n}^{(8+2i-1)(1-\gamma)/(4+i)}} \leq |r_{i,n}| \leq \frac{c_9}{q_{i,n}^{(8+2i-1)(1-\gamma)/(4+i)}} \tag{5.4}$$

for all $n \geq c_{10}$. Starting from $n \geq c_{10}$ we now build a sequence of positive integers $q_{1,n}, q_{2,n}, q_{3,n}, q_{4,n}, q_{1,n+1}, q_{2,n+1}, \ldots$, and denote it by (q_n). Let (p_n) be the corresponding sequence of $p_{i,n}$. By (5.3) and (5.4) we then have, for all $n \geq c_{11}$,

$$q_n < q_{n+1} \leq c_{12} q_n^{5/4}, \qquad \frac{c_{13}}{q_n^{15(1-\gamma)/8}} \leq \left| A(r/s) - \frac{p_n}{q_n} \right| \leq \frac{c_{14}}{q_n^{9(1-\gamma)/5}}. \tag{5.5}$$

We can now apply Lemma 5.1 and get immediately the proof of the claim on $A(r/s)$.

To begin our consideration of $B(r/s)$, we give $[2/2]_B$, $[7/5]_B$ and $[10/10]_B$, namely

$$Q_{i,0}(z)B(z) - P_{i,0}(z) = R_{i,0}(z), \qquad i = 1, 2, 3, \tag{5.6}$$

where now the polynomials read

$$Q_{1,0}(z) = 1 - z + z^2, \;\; P_{1,0}(z) = 1 - 2z + 2z^2,$$

$$Q_{2,0}(z) = 1 + z - z^2 - z^3 + z^4 + z^5, \;\; P_{2,0}(z) = 1 - 2z^2 + z^3 + 4z^4 - 4z^6 - 2z^7,$$

$$Q_{3,0}(z) = 1 - z + z^3 - 2z^4 - z^5 + 3z^6 - z^8 - z^9 + z^{10},$$

$$P_{3,0}(z) = 1 - 2z + z^2 + 2z^3 - 3z^4 + 4z^6 - 4z^7 - 5z^8 + 2z^9 + 6z^{10},$$

and the remainder terms

$$R_{1,0}(z) = -2z^8 + \mathrm{O}(z^{10}), \; R_{2,0}(z) = 2z^{16} + \mathrm{O}(z^{17}), \; R_{3,0}(z) = -4z^{21} + \mathrm{O}(z^{22}).$$

Analogously to the above consideration, we now apply the functional equation (1.5) of B and get the approximations

$$Q_{i,n}(z)B(z) - P_{i,n}(z) = R_{i,n}(z) \tag{5.7}$$

where

$$Q_{i,n}(z) = Q_{i,0}(z^{2^n}),$$

$$P_{i,n}(z) = (-1)^n P_{i,0}(z^{2^n}) \prod_{k=0}^{n-1} p(z^{2^k}) + 2Q_{i,0}(z^{2^n}) \sum_{k=0}^{n-1} (-1)^k \prod_{j=0}^{k-1} p(z^{2^j}),$$

$$R_{i,n}(z) = (-1)^n R_{i,0}(z^{2^n}) \prod_{k=0}^{n-1} p(z^{2^k}).$$

Here $\deg Q_{1,n} = 2 \cdot 2^n, \deg P_{1,n} \leq 4 \cdot 2^n - 2, \mathrm{ord}_0 R_{1,n} = 8 \cdot 2^n, \deg Q_{2,n} = 5 \cdot 2^n, \deg P_{2,n} \leq 9 \cdot 2^n - 2, \mathrm{ord}_0 R_{2,n} = 16 \cdot 2^n, \deg Q_{3,n} = 10 \cdot 2^n, \deg P_{3,n} \leq 12 \cdot 2^n - 2$ and $\mathrm{ord}_0 R_{3,n} = 21 \cdot 2^n$. Thus only the cases $i = 1, 2$ give Padé approximants for all $n \geq 0$, because the vanishing order in the case $i = 3$ is not sufficiently large for $n \geq 1$.

By these approximants, we then get sequences

$$q_{i,n} = s^{d_i \cdot 2^n - 2} Q_{i,n}(r/s), \;\; p_{i,n} = s^{d_i \cdot 2^n - 2} P_{i,n}(r/s), \;\; r_{i,n} = \frac{s^{d_i \cdot 2^n - 2}}{q_{i,n}} R_{i,n}(r/s),$$

where $d_1 = 4$, $d_2 = 9$, $d_3 = 12$. Here $q_{i,n}$ and $p_{i,n}$ are integers satisfying, for all $n \geq c_{15}$,

$$c_{16} s^{d_i \cdot 2^n} \leq q_{i,n} \leq c_{17} s^{d_i \cdot 2^n}, \qquad \frac{c_{18}}{q_{i,n}^{f_i(1-\gamma)}} \leq \left| B(r/s) - \frac{p_{i,n}}{q_{i,n}} \right| \leq \frac{c_{19}}{q_{i,n}^{f_i(1-\gamma)}}$$

where $f_1 = 2, f_2 = 16/9, f_3 = 7/4$. Now the sequence $q_{1,n+1}, q_{2,n}, q_{3,n}, q_{1,n+2}, q_{2,n+1}, q_{3,n+1}, \ldots$, to be denoted again by (q_n), satisfies the conditions

$$0 < q_n < q_{n+1} \leq c_{20} q_n^{4/3}, \qquad \frac{c_{21}}{q_n^{2(1-\gamma)}} \leq \left| B(r/s) - \frac{p_n}{q_n} \right| \leq \frac{c_{22}}{q_n^{7(1-\gamma)/4}}$$

for all $n \geq c_{23}$. Here the integers p_n are the corresponding $p_{i,n}$. Therefore we can again apply Lemma 5.1 to get the proof of our theorem for $B(r/s)$. □

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