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On classifying Laguerre polynomials which have Galois group the alternating group

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Résumé. Nous démontrons que le discriminant du polynôme de Laguerre généralisé $L_n^{(\alpha)}(x)$, pour un couple (n,α) d'entiers avec $n \geq 1$, n'est le carré d'un entier non nul que si (n,α) fait partie d'une trentaine d'ensembles explicites et infinis ou si (n,α) fait partie d'un ensemble supplémentaire qui est fini. Donc nous obtenons de nouvelles informations concernant la réalisation du groupe alterné A_n comme groupe de Galois du polynôme $L_n^{\alpha}(x)$ sur les nombres rationnels $\mathbb{Q}$. Par exemple, nous établissons que pour tous les entiers positifs n avec $n \equiv 2 \pmod 4$ (avec un nombre fini de cas exceptionnels), la seule valeur d'α pour laquelle le groupe de Galois est le groupe alterné A_n est le cas où $\alpha = n$.

Abstract. We show that the discriminant of the generalized Laguerre polynomial $L_n^{(\alpha)}(x)$ is a non-zero square for some integer pair (n,α), with $n \geq 1$, if and only if (n,α) belongs to one of 30 explicitly given infinite sets of pairs or to an additional finite set of pairs. As a consequence, we obtain new information on when the Galois group of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$ is the alternating group A_n. For example, we establish that for all but finitely many positive integers $n \equiv 2 \pmod 4$, the only α for which the Galois group of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$ is A_n is $\alpha = n$.

1. Introduction

In this paper, we investigate the Galois group associated with the generalized Laguerre polynomial $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$, where

$$L_n^{(\alpha)}(x) = \sum_{j=0}^{n} \frac{(n+\alpha)(n-1+\alpha)\cdots(j+1+\alpha)}{(n-j)!j!}(-x)^j.$$

Here, n is a positive integer (the degree of $L_n^{(\alpha)}(x)$) and we restrict ourselves to the case that α is an integer. In the late 1920's and early 1930's, I. Schur

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[20, 21, 22] established the irreducibility of these polynomials in the case $\alpha \in \{0, 1, -n-1\}$ and then obtained the following results associated with their Galois groups over the rationals:

- $L_n^{(0)}(x)$ has Galois group S_n for each n.
- $L_n^{(1)}(x)$ has Galois group S_n for each even n with $n+1$ not a square.
- $L_n^{(1)}(x)$ has Galois group A_n for each odd n and each even n with $n+1$ a square.
- $L_n^{(-n-1)}(x)$ has Galois group S_n for each $n \not\equiv 0 \pmod 4$.
- $L_n^{(-n-1)}(x)$ has Galois group A_n for each $n \equiv 0 \pmod 4$.

Previously, D. Hilbert [16] had established the existence of polynomials of any degree n over the rationals with Galois groups S_n and A_n, and Schur [22] commented that his work gave concrete examples of such polynomials except in the case of the Galois group being A_n with $n \equiv 2 \pmod 4$. Since then polynomials over the rationals of degree $n \equiv 2 \pmod 4$ and Galois group A_n have been obtained and are part of the history associated with what is now known as the Inverse Galois Theory Problem (see [19]). In 1989, R. Gow [9] showed nevertheless that the gap in Schur's work, that of finding Laguerre polynomials with Galois group A_n with $n \equiv 2 \pmod 4$, could be filled if one could show that $L_n^{(\alpha)}(x)$ is irreducible in the case that $\alpha = n$. Recently, T. Kidd, O. Trifonov and the second author [6] (see also [7]) made use of work by M. Bennett, O. Trifonov and the second author [1] to establish that, for $n \equiv 2 \pmod 4$ and $n > 2$, the polynomial $L_n^{(\alpha)}(x)$ is irreducible. Together with the reducibility of $L_2^{(2)}(x)$ and the work of Schur and Gow, this confirms then that for each positive integer n, there is an integer α for which the Galois group associated with $L_n^{(\alpha)}(x)$ over the rationals is A_n.

The present paper began with an interest in determining whether this gap in Schur's work could have been completed in a different way, more precisely by choosing some $\alpha \neq n$ to obtain examples of $L_n^{(\alpha)}(x)$ having Galois group A_n in the case $n \equiv 2 \pmod 4$. Based on early investigations, it became increasingly clear that a different choice for α would be hard if not impossible to come by.

Conjecture 1.1. *For every positive integer $n \equiv 2 \pmod 4$ with $n > 2$, the only integer α for which A_n is the Galois group associated with $L_n^{(\alpha)}(x)$ over the rationals is $\alpha = n$.*

One goal of ours here is to come close to establishing the above conjecture by showing the following.

Theorem 1.1. *For all but finitely many positive integers $n \equiv 2 \pmod 4$, the Galois group associated with $L_n^{(\alpha)}(x)$ over the rationals is A_n for some integer α if and only if $\alpha = n$.*

We will establish this result by showing that for all but finitely many positive integers $n \equiv 2 \pmod 4$, the discriminant of $L_n^{(\alpha)}(x)$, where $\alpha \in \mathbb{Z}$, is a square if and only if $\alpha = n$. Our methods allow us to conclude that the exceptional set of positive integers n satisfying $n \equiv 2 \pmod 4$ and the Galois group associated with $L_n^{(\alpha)}(x)$ over the rationals is A_n for some $\alpha \neq n$ is effectively computable. Furthermore, for any of these finitely many exceptional $n \equiv 2 \pmod 4$ with $n > 2$, there are finitely many effectively computable $\alpha \in \mathbb{Z}$ such that the Galois group associated with $L_n^{(\alpha)}(x)$ over the rationals is A_n. As suggested by the conjecture above, we believe that there are no exceptional $n > 2$.

As noted earlier, Schur [22] showed that $L_n^{(1)}(x)$ has Galois group A_n over $\mathbb{Q}$ in the case that n is odd. It is worth noting that our results imply that for all but finitely many positive integers $n \equiv 3 \pmod 4$, if $L_n^{(\alpha)}(x)$ has Galois group A_n over $\mathbb{Q}$, then either $\alpha = 1$ or both α is twice the square of an even number and $n = \alpha - 1$. Here, we know of just one exceptional pair where $n \equiv 3 \pmod 4$ and $L_n^{(\alpha)}(x)$ has Galois group A_n, namely $(n, \alpha) = (7, 26)$.

We will establish a considerably more precise result concerning the set of integer pairs (n, α) with $n \geq 1$ for which $L_n^{(\alpha)}(x)$ has Galois group A_n over $\mathbb{Q}$. First, we note that $L_2^{(-2)}(x) = x^2/2$ has discriminant 0 and trivially has Galois group A_2. One can easily check that this is the only case of a quadratic $L_n^{(\alpha)}(x)$ with a double root. As a consequence, the remaining pairs (n, α) for which $L_n^{(\alpha)}(x)$ has Galois group A_n over $\mathbb{Q}$ will satisfy that the discriminant $\mathrm{Discr}\,(n, \alpha)$ of the monic polynomial $(-1)^n n! L_n^{(\alpha)}(x)$ is a non-zero square. Of importance to us is an observation of Schur's [22] that this discriminant satisfies the nice identity

$$\mathrm{Discr}\,(n, \alpha) = \prod_{j=2}^{n} j^j (\alpha + j)^{j-1}. \tag{1.1}$$

Our goal will then be to classify rather precisely the pairs (n, α) for which $\mathrm{Discr}\,(n, \alpha)$ is a non-zero square. This classification is rather complicated, so we postpone elaborating on the details of this classification until our next section. We will also give there a precise result for the Galois groups in the case of $\alpha \in \{0, 1, 2, \ldots, 10\}$ and each positive integer n.

Before proceeding, we note that there have been a variety of other recent results concerning the irreducibility and Galois structure of $L_n^{(\alpha)}(x)$ (see the corollaries in the next section for some related results). T.-Y. Lam and the second author [5] showed that if α is a rational number that is not a negative integer, then $L_n^{(\alpha)}(x)$ is irreducible for n sufficiently large. F. Hajir [13] extended this result showing that for α a rational number that is not a negative integer and n sufficiently large depending on α, the Galois group of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$ is A_n if $\mathrm{Discr}(n,\alpha)$ is a square and S_n otherwise. In [14], Hajir and Wong showed, among other results, that if $n \ge 6$ is fixed, then for all but finitely many values of α, $L_n^{(\alpha)}(x)$ is irreducible and the Galois group of $L_n^{(\alpha)}(x)$ is S_n. Hajir [11, 12, 13] and E. A. Sell [23] have investigated the irreducibility and Galois groups associated with $L_n^{(\alpha)}(x)$ for $\alpha = -n-r$ where r is a positive integer. In particular, Hajir [12] showed that for each r and each n sufficiently large depending on r, the polynomial $L_n^{(-n-r)}(x)$ is irreducible and has associated Galois group containing A_n; furthermore, in the case $1 \le r \le 9$, he established this same result but for all $n \ge 1$. Hajir [12] also observed that the Bessel polynomials, which were determined to be irreducible by O. Trifonov and the second author in [8], are the case $\alpha = -2n-1$ of the Laguerre polynomials. As a consequence of work of Grosswald [10], $L_n^{(-2n-1)}(x)$ has associated Galois group S_n. We also comment that there is recent work of S. Laishram and T. N. Shorey [17] and T. N. Shorey and R. Tijdeman [24] associated with the factorization of Laguerre polynomials.

As a final remark to this introduction, we mention a related problem of interest to us. In addition to the resolution of the conjectures in this paper, it would be nice to know something about the set of integer pairs (n,α), with $n \ge 1$, for which the Galois group associated with $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$ is different from both S_n and A_n. In particular, is the set of such pairs finite?

2. More precise statements of the main results

We begin by clarifying notation we will use throughout this paper. We let $\square$ denote the square of an unspecified non-zero integer and $\boxplus$ denote the square of an unspecified non-zero rational number. We use $Q(n)$ to denote the squarefree part of a natural number n; in other words, $Q(n)$ is n divided by the largest square factor of n. One may also view $Q(n)$ as the product of the distinct primes q for which $q^e \| n$ with e odd. Let $P(n)$ denote the largest prime factor of a natural number n. If $c + d\sqrt{t} = (a + b\sqrt{t})^u$, where c, d, a, b and t are rational numbers with $\sqrt{t}$ irrational and u is a nonnegative integer, then we refer to c as the rational part of $(a + b\sqrt{t})^u$.

Our goal is to describe rather precisely the set $\mathcal{A}$ of pairs (n,α), where n and α are integers with $n \ge 1$, for which $\mathrm{Discr}(n,\alpha)$ is a non-zero square. Thus, the Galois group associated with $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$ is A_n only if $(n,\alpha) \in \mathcal{A}$ or, as noted earlier, $(n,\alpha) = (2,-2)$. We view the pairs (n,α) in $\mathcal{A}$ as being in two sets, say $\mathcal{A}_0$ and $\mathcal{A}_\infty$. We are interested in describing explicitly the second of these sets, $\mathcal{A}_\infty$. The set $\mathcal{A}_0$ will be a finite set of exceptional pairs that we do not attempt to make explicit here, though the methods here are effective and would "in theory" allow for these pairs to be computed. Of significance to us is that $\mathcal{A}_0$ is finite, so these contribute at most a finite number of pairs (n,α) for which the Galois group associated with $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$ is A_n. The set $\mathcal{A}_\infty$, on the other hand, is a union of infinite sets of pairs (n,α) for which $\mathrm{Discr}\,(n,\alpha)$ is a non-zero square. This collection of infinite sets of pairs (n,α) are of two types. One type consists of pairs (n,α) where at least one of the variables n or α ranges over a positive proportion of the integers. The second type is a thinner set where one of the variables n or α ranges over a set of integers having asymptotic density 0 and the other variable is either fixed or explicitly described in terms of the first variable. As will be clear from our description of these sets, as X tends to infinity, the number of pairs (n,α) with $1 \le n \le X$ and $|\alpha| \le X$ of the first type is asymptotic to C_1X for some constant C_1 and the number of such pairs of the second type is asymptotic to $C_2\sqrt{X}$ for some constant C_2. Given that the total number of pairs (n,α) with $1 \le n \le X$ and $|\alpha| \le X$ is asymptotic to $2X^2$, there are very few cases where the Galois group of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$ is A_n. On the other hand, as noted at the end of the introduction, there may only be finitely many cases where the Galois group of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$ is not A_n or S_n.

The infinite sets (n,α) of the first type in $\mathcal{A}_\infty$ are the (n,α) satisfying one of the following.

(i) $n = 1$ and α arbitrary
(ii) $\alpha = n$ and $n \equiv 0 \pmod 2$
(iii) $\alpha = 1$ and n is odd or $n+1$ is an odd square
(iv) $\alpha = -1$ and n is even or n is an odd square
(v) $\alpha = -n-1$ and $n \equiv 0 \pmod 4$
(vi) $\alpha = -n-2$ and $n \equiv 1 \pmod 4$
(vii) $\alpha = -2n-2$ and $n \equiv 0 \pmod 4$

Observe that only two of these classes involve integers $n \equiv 2 \pmod 4$, namely (ii) and (iv). As we will see, (iv) leads to $L_n^{(\alpha)}(x)$ having Galois group A_n over $\mathbb{Q}$ only in the case $n = 2$. For all other $n \equiv 2 \pmod 4$ indicated above for which A_n is the Galois group of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$, we necessarily have $\alpha = n$.

The infinite sets (n,α) of the second type in $\mathcal{A}_\infty$ are the (n,α) satisfying one of the following.

(viii) $\alpha = 3$ and $n+2$ is the rational part of $(2+\sqrt{3})^{2k+1}$ for some $k \in \mathbb{Z}^+$
(ix) $\alpha = 3$, $n \equiv 1 \pmod{24}$ and $(n+2)/3$ is a square
(x) $\alpha = 5$ and $n+3$ is the rational part of $(4+\sqrt{15})^{2k+1}$ for some $k \in \mathbb{Z}^+$

(xi) $\alpha = n - 6$ and $(2\alpha + 7)/3$ is the rational part of $(1 + \sqrt{2})^{4k}$ for some $k \in \mathbb{Z}^+$
(xii) $\alpha = n - 1$ is even and n is a square
(xiii) $n = 2$ and $\alpha + 2$ is a square
(xiv) $n = 3$, $\alpha \equiv 1 \pmod 3$ and $(\alpha + 2)/3$ is a square
(xv) $n = 4$ and $\alpha + 3$ is the rational part of $(2 + \sqrt{3})^k$ for some integer $k \geq 3$
(xvi) $n = 5$ and $\alpha + 3$ is the rational part of $(4 + \sqrt{15})^k$ for some integer $k \geq 2$
(xvii) $\alpha = n+2$ and $n+1$ is the square of the rational part of $(1+\sqrt{2})^{2k+1}$ for some $k \in \mathbb{Z}^+$
(xviii) $\alpha = n + 1$ and $n + 1$ is twice a square
(xix) $\alpha = n + 3$ is an even square
(xx) $\alpha = -n - 3$, $n \equiv 0 \pmod 4$ and $n + 1$ is a square
(xxi) $\alpha = -n - 4$, $n \equiv 1 \pmod{24}$ and $(n + 2)/3$ is a square
(xxii) $\alpha = -n - 5$ and $n + 2$ is the rational part of $(2 + \sqrt{3})^{2k+1}$ for some $k \in \mathbb{Z}^+$
(xxiii) $\alpha = -n - 6$ and $n + 3$ is the rational part of $(4 + \sqrt{15})^{2k+1}$ for some $k \in \mathbb{Z}^+$
(xxiv) $\alpha = -2n + 4$ and $(2n - 5)/3$ is the rational part of $(1 + \sqrt{2})^{4k}$ for some $k \in \mathbb{Z}^+$
(xxv) $\alpha = -2n$ and n is an odd square
(xxvi) $n = 4$ and $-\alpha - 3$ is the rational part of $(2 + \sqrt{3})^k$ for some integer $k \geq 3$
(xxvii) $n = 5$ and $-\alpha - 3$ is the rational part of $(4 + \sqrt{15})^k$ for some integer $k \geq 2$
(xxviii) $\alpha = -2n - 4$ and $n + 1$ is the square of the rational part of $(1 + \sqrt{2})^{2k+1}$ for some $k \in \mathbb{Z}^+$
(xxix) $\alpha = -2n - 2$ and $(n + 1)/2$ is the square of an odd number
(xxx) $\alpha = -2n - 4$ and $n + 3$ is an even square

One checks that in every case above besides (ii) and (iv) already noted and (xiii) where $n = 2$, we have $n \not\equiv 2 \pmod 4$. Thus, for $n > 2$, the only pairs (n, α) with $n \equiv 2 \pmod 4$ listed above and for which the Galois group of $L_n^{(\alpha)}(x)$ is A_n satisfy $\alpha = n$. Hence, Theorem 1.1 follows once we have established the following result.

Theorem 2.1. *There is a finite set $\mathcal{A}_0$ of pairs (n, α) such that Discr(n, α) is a non-zero square if and only if (n, α) satisfies one of the properties (i)-(xxx) or $(n, \alpha) \in \mathcal{A}_0$.*

Based on computations for $|\alpha| \leq 1000$ and $n \leq 1000$, we make the following conjecture that implies Conjecture 1.1.

Conjecture 2.1. *The only integer pair in $\mathcal{A}_0$ is $(7, 26)$.*

As commented on earlier, the elements of $\mathcal{A}_0$ are effectively computable. We are optimistic that the methods here will eventually lead to a resolution of Conjecture 2.1 and, hence, Conjecture 1.1.

Let $\Delta = \Delta(n, \alpha)$ denote the part of $\operatorname{Discr}(n, \alpha)$ that is free of the obvious *square* factors in (1.1). In other words, we set

$$\Delta = \Delta(n,\alpha) = \prod_{1 \le 2k+1 \le n} (2k+1) \prod_{1 \le 2k \le n} (\alpha + 2k). \tag{2.1}$$

We are therefore interested in establishing Theorem 2.1 with $\operatorname{Discr}(n, \alpha)$ there replaced by $\Delta(n, \alpha)$, though we observe an exception to this here. If one of the square factors eliminated from $\operatorname{Discr}(n, \alpha)$ to form $\Delta(n, \alpha)$ is 0, then it is possible for $\operatorname{Discr}(n, \alpha) = 0$ and $\Delta(n, \alpha) \neq 0$. We are not interested in such pairs for the purposes of establishing Theorem 2.1. This situation occurs precisely when $-n \le \alpha < -1$ and α is odd. In this context, we have the identity

$$L_n^{(\alpha)}(x) = \frac{(n+\alpha)!}{n!}(-x)^{-\alpha} L_{n+\alpha}^{(-\alpha)}(x) \qquad \text{for } -n \le \alpha \le -1.$$

Thus, the Galois group of $L_n^{(\alpha)}(x)$ for $\alpha \in \{-1, -2, \ldots, -n\}$ is the same as the Galois group of $L_{n+\alpha}^{(-\alpha)}(x)$.

To determine the pairs for which $\Delta(n, \alpha)$ is a non-zero square, we will want information about Diophantine equations of the form

$$N(N+d)(N+2d)\cdots(N+(k-1)d) = b\,y^2,$$

where b, d, N, k and y are integers. Our conclusion in Theorem 2.1 relies on a result in [4].

Theorem 2.2 (Filaseta, Laishram, Saradha)**.** *For a fixed integer $d > 0$, and fixed real numbers $\epsilon > 0$ and $C \ge d$, the equation*

$$N(N+d)(N+2d)\cdots(N+(k-1)d) = by^2$$

has finitely many solutions in positive integers N, k, b and y with

$$\gcd(N,d) = 1, \quad k \ge 3, \quad N \ge (C - d + \epsilon d)k \quad \text{and} \quad P(b) \le Ck.$$

We observe here that Theorem 2.1 can be used in combination with the following result of Hajir [12] to obtain more explicit information about the Galois group of $L_n^{(\alpha)}(x)$ in the case $\alpha = -n - r$, where $1 \le r \le 9$. As noted in the introduction, Hajir [12] established the following.

Theorem 2.3 (Hajir)**.** *If $1 \le r \le 9$, then for all $n \ge 1$, we have $L_n^{(-n-r)}(x)$ is irreducible and has Galois group containing A_n.*

As a consequence, for $1 \le r \le 9$, the Galois group of $L_n^{(-n-r)}(x)$ is A_n if and only if $\Delta(n, -n-r) = \square$. Otherwise, the Galois group is S_n. Theorem

2.1 implies that such $\Delta(n,\alpha) = \square$ if and only if (n,α) belongs to some finite set of pairs or one of the following hold:

- $\alpha = -n-1$ and $n \equiv 0 \pmod 4$.
- $\alpha = -n-2$ and $n \equiv 1 \pmod 4$.
- $\alpha = -n-3$, $n \equiv 0 \pmod 4$ and $n+1$ is a square.
- $\alpha = -n-4$, $n \equiv 1 \pmod 4$ and $n+2$ is 3 times an odd square
- $\alpha = -n-5$, $n \equiv 0 \pmod 4$ and $n+2$ is the rational part of $(2+\sqrt{3})^{2k+1}$, where $k \geq 1$ is an integer.
- $\alpha = -n-6$, $n \equiv 1 \pmod 4$ and $n+3$ is the rational part of $(4+\sqrt{15})^{2k+1}$, where $k \geq 1$ is an integer.

There are known elements in the set of finite pairs associated with $\alpha = -n-r$ with $1 \leq r \leq 9$ coming from items listed for $\mathcal{A}_\infty$. Specifically, we have the pairs $(4,-10)$ (so $\alpha = -n-6$) arising from (vii) and $(9,-18)$ (so $\alpha = -n-9$) arising from (xxv), along with trivial pairs of the form $(1,-1-r)$ for $r \in \{1,2,\ldots,9\}$ from (i) which are not in the sets listed above.

In the way of indicating more about the Galois groups associated with $L_n^{(\alpha)}(x)$, we end the paper with a proof of the following result, related to Theorem 2.3 and the comments above (see (4.1) and (4.2) later in this paper).

Theorem 2.4. *Let n and α be integers with $n \geq 1$ and $0 \leq \alpha \leq 10$. Then the Galois group of $L_n^{(\alpha)}(x)$ is A_n if it is listed in Table 1 below. Otherwise, the Galois group is S_n or $(n,\alpha) \in \{(4,5),(6,3)\}$. The polynomial $L_4^{(5)}(x)$ factors as a linear polynomial times an irreducible cubic over $\mathbb{Q}$, and the Galois group of $L_4^{(5)}(x)$ is S_3. The polynomial $L_6^{(3)}(x)$ is irreducible over $\mathbb{Q}$, and its Galois group is the projective general linear group $PGL_2(5)$ of order* 120 *(isomorphic to S_5) generated by the permutations* $(12)(34)(56)$ *and* $(1,2,3,4,6)$ *in S_6.*

The proof of Theorem 2.4 will make use of Newton polygons, which we briefly describe here. Let p be a prime, and s and r be integers relatively prime to p. If m is a non-zero rational number and a is an integer such that $m = p^a(s/r)$, we define $\nu(m) = \nu_p(m) = a$. We define $\nu(0) = +\infty$. Consider $f(x) = \sum_{j=0}^n a_j x^j \in \mathbb{Q}[x]$ with $a_n a_0 \neq 0$,. Let S be the set of points in the extended plane given by

$$S = \{(0,\nu(a_n)),(1,\nu(a_{n-1})),(2,\nu(a_{n-2})),\ldots,(n-1,\nu(a_1)),(n,\nu(a_0))\}.$$

Consider the lower edges along the convex hull of these points. The left-most endpoint is $(0,\nu(a_n))$, and the right-most endpoint is $(n,\nu(a_0))$. The endpoints of all the edges belong to S, and the slopes of the edges increase

α	condition(s) for Galois group to be A_n
0	$n = 1$
1	$n + 1$ is a square or n is odd
2	$n \in \{1, 2\}$
3	$3n + 6$ is a square and n is odd or $n + 2$ is the rational part of $(2 + \sqrt{3})^{2k+1}$ with $k \in \mathbb{Z}^+$
4	$n \in \{1, 4\}$
5	$n + 3$ is the rational part of $(4 + \sqrt{15})^{2k-1}$ with $k \in \mathbb{Z}^+$
6	$n \in \{1, 6\}$
7	$n \in \{1, 2\}$
8	$n \in \{1, 7, 8, 9\}$
9	$n \in \{1\}$
10	$n \in \{1, 3, 10\}$

Table 1

from left to right. The polygonal path formed by these edges is called the Newton polygon of $f(x)$ with respect to the prime p.

Theorem 2.4 will rely heavily on the following result from [12].

Theorem 2.5 (Hajir)**.** *Suppose $f(x) \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$ and of degree n and that a prime in $(n/2, n-2)$ divides the denominator of a slope of an edge of a Newton polygon of $f(x)$. Then the Galois group of $f(x)$ over $\mathbb{Q}$ is A_n or S_n depending on whether the discriminant of $f(x)$ is a square or not, respectively.*

In addition, we will make use of the following result from [3].

Theorem 2.6 (Filaseta, Finch, Leidy)**.** *Let n and α be integers with $n \geq 1$ and $0 \leq \alpha \leq 10$. Then $L_n^{(\alpha)}(x)$ is irreducible unless (n, α) is one of the pairs $(2, 2)$, $(4, 5)$ and $(2, 7)$.*

Before leaving this section, we give a few corollaries. The first of these follows immediately from Theorem 2.1. The others can be viewed as improvements of results of Hajir in [13] and [12], respectively, mentioned in the introduction; however, we emphasize that these corollaries are simply a direct consequence of Hajir's work and the first corollary below in the case of Corollary 2.2 and of Hajir's work and Theorem 2.1 in the case of Corollary 2.3. The first corollary also has connections to the result of Hajir and Wong [14] mentioned in the introduction.

Corollary 2.1. *There is an absolute constant C (independent of n) such that each of the following holds:*

- *If n is a fixed positive integer ≥ 6, then there are $\leq C$ integers α for which Discr(n, α) is a non-zero square.*
- *If α is a fixed integer not in the set $\{-1, 1, 3, 5\}$, then there are $\leq C$ positive integers n for which Discr(n, α) is a non-zero square.*

Corollary 2.2. *Fix a nonnegative integer $\alpha \notin \{1, 3, 5\}$. Then for all but finitely many positive integers n, the Galois group associated with $L_n^{(\alpha)}(x)$ over the rationals is S_n.*

Corollary 2.3. *Fix a positive integer $r > 6$. Then for all but finitely many positive integers n, the Galois group associated with $L_n^{(-n-r)}(x)$ over the rationals is S_n.*

In the next two sections, we investigate in detail the solutions to the Diophantine equation $\Delta(n, \alpha) = \square$, treating the cases $\alpha > 0$ and $\alpha < 0$ separately. Note that the case $\alpha = 0$ is covered by the work of I. Schur [22] mentioned earlier; consequently, $\Delta(n, 0) = \square$ if and only if $n = 1$, which is included in (i) of our list of elements in $\mathcal{A}_\infty$. To help prepare for the proof of Theorem 2.4, we show as we proceed that there are no elements of $\mathcal{A}_0$ in the case $1 \leq \alpha \leq 10$. In the final section, we give a proof of Theorem 2.4.

3. Solutions to $\Delta(n, \alpha) = \square$, for $\alpha > 0$

We consider two possibilities depending on whether $1 \leq \alpha \leq n$ or $\alpha > n$.

3.1. The discriminant for $1 \leq \alpha \leq n$. We write $n = \alpha + w$ and look at various cases for different parities of α and w.

Case 1: α and w are both odd.

Here, we are interested in

$$\Delta = 3 \cdot 5 \cdots (\alpha + w - 1) \cdot (\alpha + 2)(\alpha + 4) \cdots (\alpha + \alpha + w) = \square.$$

Observe that the (possibly empty) product

$$(\alpha + 2)(\alpha + 4) \cdots (\alpha + w - 1)$$

appears in both $3 \cdot 5 \cdots (\alpha + w - 1)$ and $(\alpha + 2)(\alpha + 4) \cdots (\alpha + \alpha + w)$. Thus, $\Delta = \square$ if and only if

$$3 \cdot 5 \cdots \alpha \cdot (\alpha + w + 1) \cdots (\alpha + \alpha + w) = \square.$$

Recalling the notation given at the beginning of the second section, the above is equivalent to

$$(\alpha + w + 1)(\alpha + w + 3) \cdots (\alpha + w + \alpha) = b\square, \tag{3.1}$$

where $b = Q(3 \cdot 5 \cdots \alpha)$ and, consequently, $P(b) \le \alpha$. The left-hand side of (3.1) above is a product of $k = (\alpha+1)/2$ consecutive odd integers. Taking $d = 2$, $C = 2$ and $\epsilon = 0.5$, we find that

$$\alpha + w + 1 > (\alpha+1)/2 = (C - d + \epsilon d)\, k \quad \text{and} \quad P(b) < \alpha + 1 = Ck.$$

For $\alpha \ge 5$, we appeal to Theorem 2.2 which implies (3.1) has finitely many solutions in $n = \alpha + w$ and $k = (\alpha+1)/2$ and, hence, in α and n. These finitely many pairs (n, α) are part of the set $\mathcal{A}_0$. We observe here that some of our arguments (in this case with $\alpha \ge 7$), the role of Theorem 2.2 can be replaced by more explicit results when $C \le 2$ (see [18]).

Now, we consider the cases where $1 \le \alpha \le 3$, where α is odd and w is odd. For $\alpha = 1$, we have $w = n - 1$ and $b = 1$ in (3.1) so that

$$n + 1 = \square.$$

Given the conditions of this case, we have here that $n+1$ is an odd square. This corresponds to part of (iii) in the list of elements of $\mathcal{A}_\infty$.

For $\alpha = 3$, we obtain from (3.1) that

$$(n+1)(n+3) = 3\,\square.$$

Here, n is even. Rewriting the above, we see that the n for which $\Delta(n,3) = \square$ in this case correspond to even positive integers n satisfying the Pell equation

$$(n+2)^2 - 3y^2 = 1.$$

The last equation implies that $n + 2$ is the rational part of $(2 + \sqrt{3})^j$ for some nonnegative integer j. On the other hand, one checks that the rational part of $(2+\sqrt{3})^j$ is an even integer if and only if j is odd. We deduce that $\Delta(n,3)$ is a square if and only if $n+2$ is the rational part of $(2+\sqrt{3})^{2k+1}$ for some positive integer k. This leads then to the elements of $\mathcal{A}_\infty$ listed in (viii).

As noted at the end of the last section, we wish to clarify that there are no elements of $\mathcal{A}_0$ for $1 \le \alpha \le 10$. We are interested in odd α in this case and have already addressed $\alpha \in \{1,3\}$. For each remaining $\alpha \in \{5,7,9\}$, the solutions to (3.1) correspond to integral points on a curve. We give some details as to how we can determine explicitly these points in this case. We rely on our details here as being illustrative of how we address future arguments showing that there are no elements of $\mathcal{A}_0$ with $1 \le \alpha \le 10$.

For $\alpha = 5$, equation (3.1) becomes

$$(n+1)(n+3)(n+5) = 15z^2,$$

for some integer z. Multiplying both sides by 15^3 and setting $x = 15n$ and $y = 15^2 z$, we deduce

$$y^2 = (x+15)(x+45)(x+75) = x^3 + 135x^2 + 5175x + 50625.$$

Sage (an open-source mathematics software system) indicates that there are 11 integral points on this curve. The ones for which x is a positive integral multiple of 15 are $(105, \pm 1800)$. This corresponds to $n = 7$. However, given that α and w are odd in the case under consideration, we have n is even. So $n = 7$ is not an admissible n for this case.

For $\alpha = 7$, equation (3.1) becomes

$$(n+1)(n+3)(n+5)(n+7) = 105 z_1^2,$$

for some integer z_1. One can check that this is an elliptic curve of rank 2 and, hence, has infinitely many rational points on it; for example, $n = 607/23$ provides a non-trivial example. To find the positive integer solutions in n (and z_1) that are of interest to us, we proceed as follows. We consider each divisor d of 105. Noting that n is even, the above equation implies that for one of these d, we must have

$$(n+1)(n+3)(n+5) = d z_2^2,$$

for some integer z_2. Multiplying both sides by d^3 and setting $x = dn$ and $y = d^2 z_2$, we deduce

$$y^2 = (x+d)(x+3d)(x+5d).$$

We are then interested in points (x, y) on one of these elliptic curves with x and y positive integers divisible by d. Simple arguments can allow one to reduce the number of d one needs to consider, but regardless Sage can be used to determine all such points (x, y). There are a few such points. For example, for $d = 105$, one gets the unusually large solution $x = 352695$ and $y = 209739600$. But this leads to $n = 3359$ which is odd and also is such that the product $(n+1)(n+3)(n+5)(n+7)$ is not 105 times a square. The only positive integers n leading to $(n+1)(n+3)(n+5)(n+7)$ being 105 times a square are $n = 2$ and $n = 9$. The former leads to $w < 0$ and the latter has n odd, neither related to the case under consideration.

Taking $\alpha = 9$ in (3.1) leads to

$$(n+1)(n+3)(n+5)(n+7)(n+9) = 105 z^2,$$

for some integer z. There are completely elementary approaches to this equation, taking advantage of the fact that two of the five factors on the left necessarily will be squares. On the other hand, we also get here that $(n+1)(n+3)(n+5) = d z_2^2$ for some integer z_2 and some divisor d of 105. We are led then to the same elliptic curves that we considered for $\alpha = 7$ above. One verifies here that there are no solutions with n a positive even integer > 9. (We note, however, that $n = 11$ leads to an odd solution to this Diophantine equation.)

Case 2: α is odd and w is even.

In this case, we obtain

$$\Delta = 3 \cdot 5 \cdots (\alpha + w) \cdot (\alpha + 2)(\alpha + 4) \cdots (\alpha + \alpha + w - 1) = \square.$$

The factor $(\alpha + 2)(\alpha + 4) \cdots (\alpha + w)$ appears twice in the products above, and we deduce that

$$3 \cdot 5 \cdots \alpha \cdot (\alpha + w + 2) \cdots (\alpha + \alpha + w - 1) = \square,$$

which is equivalent to

$$(\alpha + w + 2) \cdots (\alpha + w + \alpha - 1) = b\,\square, \tag{3.2}$$

where $b = Q(3 \cdot 5 \cdots \alpha)$ and, consequently, $P(b) \leq \alpha$. The left-hand side of (3.2) above is a block of $k = (\alpha - 1)/2$ consecutive odd integers. We take $d = 2$, $C = 3$ and $\epsilon = 0.5$. Then

$$\alpha + w + 2 > \alpha - 1 = (C - d + \epsilon d)k.$$

For $\alpha \geq 3$, we have $\alpha \leq 3(\alpha - 1)/2$ so that

$$P(b) \leq 3(\alpha - 1)/2 = Ck.$$

For $\alpha \geq 7$, we apply Theorem 2.2 to deduce that (3.2) has finitely many solutions in $n = \alpha + w$ and $k = (\alpha - 1)/2$ and, hence, in α and n.

If $\alpha = 1$, then $\Delta = (3 \cdot 5 \cdots n)^2$. Here, n is odd. Thus, this gives part of (iii) in our list of elements in $\mathcal{A}_\infty$.

For $\alpha = 3$, we have

$$\Delta(n, 3) = 3 \cdot 5 \cdots n \cdot 5 \cdot 7 \cdots (n + 2) = \square.$$

The latter corresponds to $n + 2$ being 3 times an odd square. Since odd squares are 1 modulo 8, this leads to (ix) in our list of elements in $\mathcal{A}_\infty$.

For $\alpha = 5$, we have

$$\Delta(n, 5) = 3 \cdot 5 \cdots n \cdot 7 \cdot 9 \cdots (n + 4) = \square.$$

We deduce in this case that $(n + 2)(n + 4) = 15\,\square$. Such n arise from solutions of the Pell equation

$$(n + 3)^2 - 15y^2 = 1.$$

We are interested in odd $n \geq \alpha$ above. One checks such n correspond to $n + 3$ being the rational part of $(4 + \sqrt{15})^{2k+1}$, for some positive integer k, giving (x) in our list of elements in $\mathcal{A}_\infty$.

To verify that there are no elements of $\mathcal{A}_0$ for $1 \leq \alpha \leq 10$ in this case, we still need to consider $\alpha = 7$ and $\alpha = 9$. These lead to the equations

$$(n + 2)(n + 4)(n + 6) = 105z^2$$

and

$$(n + 2)(n + 4)(n + 6)(n + 8) = 105z^2,$$

respectively, which correspond to equations considered in the previous case. The positive integers n leading to solutions to the first are 1, 8, 26 and 3358.

The positive integers n leading to solutions to the second are 1 and 10. We require both $n \geq \alpha$ and n odd in this case, so there are no elements of $\mathcal{A}_0$ arising from this case with $1 \leq \alpha \leq 10$.

Case 3: α and w are both even.

For this case, we have

$$\Delta = 3 \cdot 5 \cdots (\alpha + w - 1) \cdot (\alpha + 2)(\alpha + 4) \cdots (\alpha + \alpha + w) = \square.$$

Observe that

$$3 \cdot 5 \cdots (\alpha + w - 1) = 3 \cdot 5 \cdots (\alpha - 1) \cdot (\alpha + 1) \cdots (\alpha + w - 1) \tag{3.3}$$

and

$$\begin{aligned} &(\alpha + 2)(\alpha + 4) \cdots (\alpha + \alpha + w) \\ &\quad = \left(\frac{\alpha}{2} + 1\right)\left(\frac{\alpha}{2} + 2\right) \cdots \alpha \cdot (\alpha + 1) \cdots \left(\alpha + \frac{w}{2}\right) \cdot 2^{(\alpha+w)/2}. \end{aligned} \tag{3.4}$$

We focus on the product of the numbers $\alpha + j$ where j is odd and $1 \leq j \leq w/2$ which is a factor in both (3.3) and (3.4). From

$$\begin{aligned} 3 \cdot 5 \cdots (\alpha - 1) &= \frac{\alpha!}{(\alpha/2)!\, 2^{\alpha/2}} \\ \text{and} \quad \left(\frac{\alpha}{2} + 1\right)\left(\frac{\alpha}{2} + 2\right) \cdots \alpha &= \frac{\alpha!}{(\alpha/2)!}, \end{aligned} \tag{3.5}$$

we see that $\Delta = \square$ if and only if

$$\begin{aligned} &2^{w/2} \left(\frac{\alpha!}{(\alpha/2)!}\right)^2 \prod_{\substack{1 \leq j \leq w/2 \\ j \text{ odd}}} (\alpha + j)^2 \\ &\qquad \times \prod_{\substack{w/2 < u \leq w-1 \\ u \text{ odd}}} (\alpha + u) \prod_{\substack{1 \leq v \leq w/2 \\ v \text{ even}}} (\alpha + v) = \square. \end{aligned} \tag{3.6}$$

We consider a few different subcases and address the α satisfying $1 \leq \alpha \leq 10$ afterwards.

First, we suppose that $w \equiv 0 \pmod 4$ and $w \leq 4(\alpha + 1)$. Then (3.6) is equivalent to

$$(\alpha+2)(\alpha+4) \cdots \left(\alpha+\frac{w}{2}\right) \cdot \left(\alpha+\frac{w}{2}+1\right)\left(\alpha+\frac{w}{2}+3\right) \cdots (\alpha+w-1) = \square. \tag{3.7}$$

We denote the squarefree part of $\alpha + t$ by a_t. In other words, a_t is the squarefree positive integer such that $\alpha + t = a_t b_t^2$ for some integer b_t. We restrict ourselves to $t \in T$, where

$$T = \{2, 4, \ldots, w/2\} \cup \{(w/2) + 1, (w/2) + 3, \ldots, w - 1\}.$$

Observe that (3.7) implies that any prime p that divides a_t for some $t \in T$ must also divide a_s for some $s \in T$ with $s \neq t$. Since in this case p divides the difference $(\alpha+t)-(\alpha+s) = t-s$, we deduce that $p \leq w-3$. Thus, each prime divisor of each a_t with $t \in T$ is $\leq w-3$. In other words, $P(a_t) \leq w-3$ for all $t \in T$. It follows that (3.7) has a solution if and only if

$$\left(\alpha + \frac{w}{2} + 1\right)\left(\alpha + \frac{w}{2} + 3\right) \cdots (\alpha + w - 1) = b\,\square \tag{3.8}$$

has a solution, where b is the squarefree part of $a_2 a_4 \cdots a_{w/2}$. In particular, $P(b) \leq w-3$. The left-hand side of (3.8) is a block of $k = w/4$ consecutive odd integers. Recalling $w \leq 4(\alpha+1)$, and setting $d = 2$, $C = 4$ and $\epsilon = 0.5$, we find that

$$\alpha + \frac{w}{2} + 1 \geq 3w/4 = (C - d + \epsilon d)k \quad \text{and} \quad P(b) < w = Ck.$$

By Theorem 2.2, for $w \geq 12$, (3.8) has finitely many solutions in $\alpha + (w/2)$ and k and, hence, in α and n.

We consider the $w \equiv 0 \pmod 4$ with $0 \leq w < 12$. If $w = 0$, then the left-hand side of (3.6) is clearly a square. Hence, in this case, where $n = \alpha$ with α even, we have that $\Delta(n,\alpha)$ is a square. This is (ii) in our list of elements in $\mathcal{A}_\infty$.

For $w = 4$, equation (3.7) becomes $(\alpha+2)(\alpha+3) = \square$. A product of two consecutive positive integers cannot be a square. Therefore, here, $\Delta(n,\alpha)$ cannot be a square.

For $w = 8$, equation (3.7) becomes

$$(\alpha+2)(\alpha+4)(\alpha+5)(\alpha+7) = \square. \tag{3.9}$$

This describes an elliptic curve which we analyze as follows. Denoting the right-hand side above by z^2 and dividing both sides of the equation by $(\alpha+2)^4$, we obtain

$$\begin{aligned}
\left(\frac{z}{(\alpha+2)^2}\right)^2 &= \frac{(\alpha+4)(\alpha+5)(\alpha+7)}{(\alpha+2)^3} \\
&= \frac{30 + 31(\alpha+2) + 10(\alpha+2)^2 + (\alpha+2)^3}{(\alpha+2)^3} \\
&= 30\left(\frac{1}{\alpha+2}\right)^3 + 31\left(\frac{1}{\alpha+2}\right)^2 + 10\left(\frac{1}{\alpha+2}\right) + 1.
\end{aligned}$$

Multiplying by 900 on both sides and setting $y = 30z/(\alpha+2)^2$ and $x = 30/(\alpha+2)$, we deduce

$$y^2 = x^3 + 31x^2 + 300x + 900.$$

We use Sage to obtain information about the group of rational points on this elliptic curve. It has rank 0 so that all of its rational points are torsion. There are 8 torsion points. Four of them, including the point at infinity, correspond to $\alpha \in \{-2, -4, -5, -7\}$ in (3.9). Two others are $(x, y) = (0, \pm 30)$ which, given that $x = 30/(\alpha + 2)$, do not lead to solutions to (3.9). The other two points are $(x, y) = (-12, \pm 6)$ which correspond to $\alpha = -9/2$ in (3.9). Since we are interested in integers $\alpha \geq 1$, there are no pairs (n, α) in this case for which $\Delta(n, \alpha)$ is a square.

Now, we consider the subcase that $w \equiv 0 \pmod 4$ and $w > 4(\alpha+1)$. We note for later purposes the argument here does not require $w \equiv 0 \pmod 4$ and, therefore, also applies to w satisfying $w \equiv 2 \pmod 4$ and $w > 4(\alpha+1)$. To handle $w > 4(\alpha+1)$, we make use of the following explicit gap estimate for primes found in [15].

Lemma 3.1 (Harborth and Kemnitz)**.** *For every $N \geq 48683$, the interval $(N, 1.001N]$ contains a prime.*

Since

$$n = \alpha + w > 5\alpha + 4 > 1001\alpha/999,$$

we deduce that $n > 1.001(n+\alpha)/2$. Thus, by Lemma 3.1, we may conclude that there is a prime in the interval $\big((n + \alpha)/2, n\big]$, whenever $(n + \alpha)/2 \geq 48683$. We show that, in this case, $\Delta(n, \alpha)$ cannot be a square. First of all, we observe that a prime p in $\big((n + \alpha)/2, n\big]$ is odd for $(n + \alpha)/2 \geq 48683$. From (2.1), we see that p divides $\Delta(n, \alpha)$. Also, $2p > n + \alpha$, which implies that exactly one factor of p appears in the prime factorization of $\Delta(n, \alpha)$. Thus, $\Delta(n, \alpha)$ cannot be a square for $(n + \alpha)/2 \geq 48683$. Note that there are only finitely many pairs (n, α) not considered in this case. In other words, there are finitely many positive integers n and α satisfying $(n + \alpha)/2 < 48683$.

In the subcase that $w \equiv 2 \pmod 4$, we can deduce in a similar manner to the above that $\Delta(n, \alpha) = \Box$ only if

$$\Big(\alpha + \frac{w}{2} + 2\Big)\Big(\alpha + \frac{w}{2} + 4\Big) \cdots \Big(\alpha + w - 1\Big) = b\,\Box, \tag{3.10}$$

where b is a squarefree integer satisfying $P(b) \leq w - 3$. On the left-hand side of (3.10), we have a block of $k = (w - 2)/4$ consecutive odd integers. We consider two possibilities here depending on whether $w \leq 4\alpha + 14$ or $w > 4\alpha + 14$.

For $w \equiv 2 \pmod 4$ and $w \leq 4\alpha + 14$, we take $d = 2$, $C = 4$ and $\epsilon = 0.5$ in Theorem 2.2. Since

$$\alpha + \frac{w}{2} + 2 \geq 3(w - 2)/4 = (C - d + \epsilon d)k \quad \text{and} \quad P(b) < w - 2 = Ck,$$

we obtain that for $w \geq 14$, (3.10) has finitely many solutions in $\alpha + (w/2)$ and k and, hence, in α and n.

For $w = 2$, the left-hand side of (3.6) is 2 times a square, leading to an impossibility (since the right-hand side of (3.6) is a square).

For $w = 6$, from (3.6), we see that $\Delta(n, \alpha)$ is a square if and only if

$$(\alpha + 2)(\alpha + 5) = 2\,\square.$$

Multiplying both sides of this equation by 4 and rewriting, we obtain that $u = 2\alpha + 7$ corresponds to a solution in u and v to the Diophantine equation

$$u^2 - 2v^2 = 9.$$

Since 2 is not a square modulo 3, this Diophantine equation only has solutions with both u and v divisible by 3. In particular, we must have $\alpha \equiv 1 \pmod 3$ and $x = (2\alpha + 7)/3$ corresponds to a solution in x and y to the Pell equation

$$x^2 - 2y^2 = 1.$$

The condition α is even implies that $x = (2\alpha + 7)/3$ is 1 modulo 4. We deduce in this case that $\alpha = (3x - 7)/2$ where x is the rational part of $(1 + \sqrt{2})^{4k}$ for some positive integer k. This is (xi) in our list of elements in $\mathcal{A}_\infty$.

For $w = 10$, we deduce from (3.6) that $\Delta(n, \alpha)$ is a square if and only if

$$(\alpha + 2)(\alpha + 4)(\alpha + 7)(\alpha + 9) = 2\,\square.$$

Denoting the right-hand side above by $2z^2$ and dividing both sides of the equation by $(\alpha + 2)^4$, we obtain

$$\begin{aligned} 2\left(\frac{z}{(\alpha+2)^2}\right)^2 &= \frac{(\alpha+4)(\alpha+7)(\alpha+9)}{(\alpha+2)^3} \\ &= \frac{70 + 59(\alpha+2) + 14(\alpha+2)^2 + (\alpha+2)^3}{(\alpha+2)^3} \\ &= 70\left(\frac{1}{\alpha+2}\right)^3 + 59\left(\frac{1}{\alpha+2}\right)^2 + 14\left(\frac{1}{\alpha+2}\right) + 1. \end{aligned}$$

Multiplying by $2^5 \cdot 35^2$ on both sides and setting

$$y = \frac{2^3 \cdot 35 \cdot z}{(\alpha+2)^2} \quad \text{and} \quad x = \frac{2^2 \cdot 35}{\alpha + 2},$$

we deduce

$$y^2 = x^3 + 118x^2 + 3920x + 39200.$$

Using Sage, we deduce that the group of rational points on this elliptic curve has rank 0 so that all of its rational points are torsion. There are 4 torsion points, corresponding to $\alpha \in \{-2, -4, -5, -7\}$. Since we are in a case where $\alpha \geq 1$, there are no pairs (n, α) here for which $\Delta(n, \alpha)$ is a square.

Observe that if $w \equiv 2 \pmod 4$ and $w > 4\alpha + 14$, then $w > 4(\alpha + 1)$. When we gave our argument for the subcase where $w \equiv 0 \pmod 4$ and $w > 4(\alpha + 1)$, we already noted that for $w \equiv 2 \pmod 4$ and $w > 4(\alpha + 1)$, we have $\Delta(n,\alpha)$ is not a square except possibly for finitely many pairs (n,α).

For the purposes of Theorem 2.4, we want to check what pairs of positive integers (n,α) satisfy α and w are both even, $1 \le \alpha \le 10$ and $\Delta(n,\alpha)$ is a square. Our arguments here will not depend on the the parity of w and, hence, will apply equally well in Case 4 below. For $(n+\alpha)/2 \ge 48683$, observe that

$$w = n - \alpha = n + \alpha - 2\alpha \ge 2 \cdot 48683 - 20 > 44 \ge 4(\alpha+1),$$

which allows us to appeal to the argument involving Lemma 3.1 to see that there is a prime in $\big((n+\alpha)/2, n\big]$ and that such a prime necessarily divides $\Delta(n,\alpha)$ to exactly the first power. Thus, $\Delta(n,\alpha)$ is not a square for $(n+\alpha)/2 \ge 48683$. For the finitely many cases involving $(n+\alpha)/2 < 48683$, the computations are simple enough, but we observe that they can be simplified further by checking directly that, for $1 \le \alpha \le 10$, there is a prime p in the interval $\big((n+\alpha)/2, n\big]$ for all $n \ge 17$. Hence, as before, we have that p divides $\Delta(n,\alpha)$ to exactly the first power, so $\Delta(n,\alpha)$ is not a square. We are left with considering integer pairs (n,α) with $1 \le \alpha \le 10$ and $1 \le n \le 16$, where we can restrict to even α though this is not necessary at this point. One checks directly that the pairs (n,α) that arise with $\Delta(n,\alpha)$ a square are listed in Table 1, and each of these corresponds to a pair in $\mathcal{A}_\infty$. Thus, no pairs (n,α) in $\mathcal{A}_0$ occur here with $1 \le \alpha \le 10$.

Case 4: α is even and w is odd.

The argument here has similarities to Case 3. In this case, we are interested in

$$\Delta = 3 \cdot 5 \cdots (\alpha + w)(\alpha+2)(\alpha+4)\cdots(\alpha+\alpha+w-1) = \square.$$

This is equivalent to

$$\begin{aligned} &3 \cdot 5 \cdots (\alpha-1)(\alpha+1)\cdots(\alpha+w) \\ &\qquad \times \left(\frac{\alpha}{2}+1\right)\left(\frac{\alpha}{2}+2\right)\cdots \alpha(\alpha+1)\cdots\left(\alpha+\frac{w-1}{2}\right)\cdot 2^{(\alpha+w-1)/2} = \square. \end{aligned}$$

We make use of (3.5) to deduce here that $\Delta = \square$ if and only if

$$\begin{aligned} &2^{(w-1)/2}\left(\frac{\alpha!}{(\alpha/2)!}\right)^2 \prod_{\substack{1\le i\le (w-1)/2\\ i \text{ odd}}} (\alpha+i)^2 \\ &\qquad \times \prod_{\substack{(w-1)/2<j\le w\\ j \text{ odd}}} (\alpha+j) \prod_{\substack{1\le l\le (w-1)/2\\ l \text{ even}}} (\alpha+l) = \square. \end{aligned} \tag{3.11}$$

We consider different subcases.

First, we consider the subcase that $w \equiv 1 \pmod 4$ and $w \leq 4\alpha - 7$. Then (3.11) is equivalent to

$$(3.12) \qquad \begin{aligned} &(\alpha+2)(\alpha+4)\cdots\left(\alpha+\frac{w-1}{2}\right) \\ &\quad\times\left(\alpha+\frac{w-1}{2}+1\right)\left(\alpha+\frac{w-1}{2}+3\right)\cdots(\alpha+w)=\square. \end{aligned}$$

As before, we denote the squarefree part of $\alpha + t$ by a_t. We consider here $t \in T$, where

$$T = \{2, 4, \ldots, (w-1)/2\} \cup \{((w-1)/2)+1, ((w-1)/2)+3, \ldots, w\}.$$

Equation (3.12) implies that each prime that divides some a_t for $t \in T$ must also divide a_s for some $s \in T$ with $s \neq t$. Since each factor $\alpha + j$ appearing on the left-hand side of (3.12) is in the interval $[\alpha+2, \alpha+w]$, we deduce that each prime divisor of an a_t is $\leq w-2$. In other words, $P(a_t) \leq w-2$ for all $t \in T$. Thus, (3.12) has a solution if and only if

$$(3.13) \qquad \left(\alpha+\frac{w-1}{2}+1\right)\left(\alpha+\frac{w-1}{2}+3\right)\cdots(\alpha+w)=b\square$$

has a solution, where b is the squarefree part of $a_2 a_4 \cdots a_{(w-1)/2}$. Note that $P(b) \leq w-2$. The left-hand side of (3.13) is a block of $k = (w+3)/4$ consecutive odd integers. Recall that $w \leq 4\alpha - 7$. Setting $d = 2$, $C = 4$ and $\epsilon = 0.5$, we find that

$$\alpha + \frac{w-1}{2} + 1 \geq 3(w+3)/4 = (C - d + \epsilon d)k \quad \text{and} \quad P(b) < w+3 = Ck.$$

Theorem 2.2 implies now that for $w \geq 9$, equation (3.13) has finitely many solutions in $\alpha + (w+1)/2$ and k and, hence, in α and n.

In the case where $w = 1$, we deduce from (3.11) that

$$n\left(\frac{\alpha!}{(\alpha/2)!}\right)^2 = (\alpha+1)\left(\frac{\alpha!}{(\alpha/2)!}\right)^2 = \square.$$

For $w = 1$, we deduce that $\Delta = \square$ if and only if n is an odd square, leading to (xii) in our list of elements in $\mathcal{A}_\infty$.

When $w = 5$, equation (3.12) becomes

$$(3.14) \qquad (\alpha+2)(\alpha+3)(\alpha+5) = \square.$$

Since solutions to (3.14) correspond to integral points on an elliptic curve, we deduce there are finitely many α satisfying (3.14) and, hence, there are finitely many α such that $\Delta(n, \alpha) = \Delta(\alpha+5, \alpha)$ is a square. In fact, Sage indicates that the elliptic curve has rank 0 and the torsion group associated with the elliptic curve over the rationals consists of the points $(-2, 0)$, $(-3, 0)$, $(-5, 0)$ and the point at infinity. Hence, there are no α for which $\Delta(\alpha+5, \alpha)$ is a square in this case.

For the subcase that $w \equiv 1 \pmod 4$ and $w > 4\alpha - 7$, we make use of Lemma 3.1 as before. This argument works equally well with $w \equiv 3 \pmod 4$, which we will make use of momentarily. Since α is even in the current case under consideration, we have $\alpha \geq 2$. We deduce that $n > 5\alpha - 7 > 1001\alpha/999$, so $n > 1.001(n+\alpha)/2$. By Lemma 3.1, we conclude that there is a prime in the interval $\big((n+\alpha)/2, n\big]$ provided $(n+\alpha)/2 \geq 48683$. As before, Δ cannot be a square if $(n+\alpha)/2 \geq 48683$, and there are finitely pairs (n, α) with $(n+\alpha)/2 \leq 48683$.

In the subcase that $w \equiv 3 \pmod 4$, we can deduce similarly as above that $\Delta(n, \alpha) = \square$ if and only if

$$\left(\alpha + \frac{w-1}{2} + 2\right)\left(\alpha + \frac{w-1}{2} + 4\right) \cdots (\alpha + w) = b\,\square, \tag{3.15}$$

where b is the squarefree part of $2a_2a_4 \cdots a_{(w-3)/2}$ and $P(b) \leq w - 2$. Here, we note that, despite the presence of the factor 2 in this product defining b, if b is even, then there are no solutions to (3.15) since the left-hand side is odd. On the other hand, the factor 2 is needed here as we need the squarefree part of $2a_2a_4 \cdots a_{(w-3)/2}$ to be odd in order for (3.15) to imply $\Delta(n, \alpha) = \square$.

The left-hand side of (3.15) is a block of $k = (w+1)/4$ consecutive odd integers. We consider two possibilities here depending on whether $w \leq 4\alpha + 3$ or $w > 4\alpha + 3$.

For the subcase that $w \equiv 3 \pmod 4$ and $w \leq 4\alpha + 3$, we take $d = 2$, $C = 4, \epsilon = 0.5$, $k = (w+1)/4$ and $w \leq 4\alpha + 3$ in Theorem 2.2. Since

$$\alpha + \frac{w-1}{2} + 2 \geq 3(w+1)/4 = (C - d + \epsilon d)k \quad \text{and} \quad P(b) < w + 1 = Ck,$$

we obtain that for $w \geq 11$, (3.15) has finitely many solutions in $\alpha + (w+3)/2$ and k and, hence, in α and n.

When $w = 3$, equation (3.11) implies $\alpha + 3 = 2\,\square$. Since $n = \alpha + w = \alpha + 3$ and n is odd, this is impossible. Thus, there are no solutions with $w = 3$.

When $w = 7$, equation (3.11) implies

$$(\alpha + 2)(\alpha + 5)(\alpha + 7) = 2\,\square. \tag{3.16}$$

Equation (3.16) is an elliptic curve, and we deduce that there are finitely many integers α satisfying (3.16). In fact, Sage indicates here that the group of rational points on this elliptic curve has rank 0 and the only torsion points correspond to $\alpha \in \{-2, -5, -7\}$ and the point at infinity. Thus, $\Delta(n, \alpha) = \Delta(\alpha + 7, \alpha)$ is never a non-zero square in this case.

For the subcase that $w \equiv 3 \pmod 4$ and $w > 4\alpha + 3$, we observe that $w > 4\alpha - 7$ and recall the argument given for $w \equiv 1 \pmod 4$ and $w > 4\alpha - 7$. As noted there, Δ is a square for at most finitely many (n, α).

Recall that, at the end of Case 3, we already checked the pairs (n, α) for which α is even, $1 \le \alpha \le 10$ and $\Delta(n, \alpha)$ is a square. We showed that there are no such pairs (n, α) in $\mathcal{A}_0$.

3.2. The discriminant for $\boldsymbol{\alpha > n}$. If $n = 1$, then $L_n^{(\alpha)}(x)$ is a linear polynomial and the Galois group is trivially A_1. In this case $\Delta = 1$, corresponding to (i) in our list of elements in $\mathcal{A}_\infty$. We turn first then to considering $2 \le n \le 5$. These cases can be handled directly using (2.1). If $n = 2$, then $\Delta = \square$ if and only if

$$\alpha + 2 = \square.$$

If $n = 3$, then $\Delta = \square$ if and only if

$$\alpha + 2 = 3\,\square.$$

For $n = 4$, we have $\Delta = \square$ if and only if

$$(\alpha + 2)(\alpha + 4) = 3\,\square.$$

This corresponds to solutions in $x = \alpha + 3$ and y to the Pell equation

$$x^2 - 3y^2 = 1.$$

Thus, in this case, $\alpha + 3$ is the rational part of $(2 + \sqrt{3})^k$. Given $\alpha > n$, we want k here to be an integer ≥ 3. For $n = 5$, we likewise get $\Delta = \square$ if and only if

$$(\alpha + 2)(\alpha + 4) = 15\,\square.$$

This corresponds to solutions in $x = \alpha + 3$ and y to the Pell equation

$$x^2 - 15y^2 = 1.$$

Hence, in this case, $\alpha + 3$ is the rational part of $(4 + \sqrt{15})^k$, where k is an integer ≥ 2. The above corresponds to items (xiii)-(xvi) in our list of elements in $\mathcal{A}_\infty$.

We are left now with pairs (n, α) with $\alpha > n \ge 6$. We consider two cases depending on the parity of α. Before continuing, we observe that for $1 \le \alpha \le 10$ as in Theorem 2.4, the condition $n < \alpha$ allows us quickly to determine the pairs (n, α) explicitly for which $\Delta(n, \alpha)$ is a square. Besides the pairs with $n = 1$ and the pairs $(2, 7)$ and $(3, 10)$ which occur above, there is only the one additional pair $(7, 8)$, from (xviii) in our list of elements in $\mathcal{A}_\infty$, that follows from a direct computation.

Case 1: α is odd.

For the moment, suppose that n is even. Then

$$\Delta = 3 \cdot 5 \cdots (n-1)(\alpha + 2)(\alpha + 4) \cdots (\alpha + n) = \square$$

if and only if

$$(\alpha+2)(\alpha+4)\cdots(\alpha+n)=b\,\square, \tag{3.17}$$

where b is the squarefree part of $3\cdot 5\cdots(n-1)$. The left-hand side of the above equation is a product of $k=n/2$ consecutive odd positive integers. Setting $d=2$, $C=2$ and $\epsilon=0.5$, we obtain

$$\alpha+2>n/2=(C-d+\epsilon d)k \quad \text{and} \quad P(b)\le n-1<n=Ck.$$

Since $n\ge 6$, Theorem 2.2 implies in this case that there are finitely many solutions to (3.17) in $\alpha+2$ and $k=n/2$ and, hence, in α and n.

If instead n is odd and necessarily ≥ 7, we apply a very similar argument. In this case,

$$\Delta=3\cdot 5\cdots n\,(\alpha+2)(\alpha+4)\cdots(\alpha+n-1)=\square,$$

leading to

$$(\alpha+2)(\alpha+4)\cdots(\alpha+n-1)=b\,\square,$$

where b is the squarefree part of $3\cdot 5\cdots n$. Taking $k=(n-1)/2$, $d=2$, $C=3$ and $\epsilon=0.5$ in Theorem 2.2 then implies as before that there are finitely many pairs (n,α) in this case for which $\Delta(n,\alpha)=\square$.

Case 2: α is even.

We define

$$N=\begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd}\end{cases} \quad \text{and} \quad M=\begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd.}\end{cases}$$

Then

$$\Delta=3\cdot 5\cdots N(\alpha+2)(\alpha+4)\cdots(\alpha+M).$$

Observe that $n\ge 6$ implies $M\ge 6$. We look at two subcases.

Subcase (i). $M<4\alpha/7$.

Here, $\Delta=\square$ if and only if

$$\left(\frac{\alpha}{2}+1\right)\left(\frac{\alpha}{2}+2\right)\cdots\left(\frac{\alpha}{2}+\frac{M}{2}\right)=b\,\square, \tag{3.18}$$

where b is the squarefree part of $3\cdot 5\cdots N$ or the squarefree part of $2\cdot 3\cdot 5\cdots N$ depending on whether $M/2$ is even or odd, respectively. The left-hand side of the last equation above is a block of $k=M/2\ge 3$ consecutive integers. Taking $d=1$, $C=2.5$ and $\epsilon=0.25$, we find that

$$\frac{\alpha}{2}+1>7M/8=(C-d+d\epsilon)k \quad \text{and} \quad P(b)\le N\le Ck.$$

From Theorem 2.2, we obtain (3.18) has finitely many solutions in $\alpha/2$ and k and, hence, in M (and, likewise, n) and α.

Subcase (ii). $M \geq 4\alpha/7$.

We write $\alpha = M + w$, $m = M/2$ and $u = w/2$. Since α and M are even, w is as well. Since $\alpha > n \geq M \geq 6$, we deduce that m and u are positive integers. In this subcase, $w \leq 3M/4$ and $u \leq 3m/4$. From

$$\Delta = 3 \cdot 5 \cdots N \cdot (\alpha+2)(\alpha+4)\cdots(\alpha+M) = \square,$$

we deduce that

$$3 \cdot 5 \cdots N \cdot 2^m (m+u+1)(m+u+2)\cdots(m+u+m) = \square.$$

We also observe that

$$\Delta(2m, 2m) = 3 \cdot 5 \cdots (2m-1) 2^m (m+1)(m+2)\cdots(m+m),$$

which leads to

$$\Delta(2m, 2m) = \left(\frac{(2m)!}{m!}\right)^2 \quad \text{for } m \geq 1.$$

Set $A = 1$ if n is even and $A = n$ if n is odd. Define

$$\Lambda = \frac{\Delta}{\Delta(2m,2m)} = \frac{A \cdot (m+u+1)(m+u+2)\cdots(m+u+m)}{(m+1)(m+2)\cdots(m+m)}.$$

After canceling factors, we see that we can rewrite Λ as

$$\Lambda = \frac{A \cdot (2m+1)(2m+2)\cdots(2m+u)}{(m+1)(m+2)\cdots(m+u)}.$$

We rewrite each factor in the denominator of the form $m + j$ with $j \leq u/2$ as $(2m+2j)/2$ and cancel the factors of the form $2m + 2j$ with $j \leq u/2$ to obtain

$$\Lambda = \begin{cases} \dfrac{2^{u/2} A \cdot (2m+1)(2m+3)\cdots(2m+u-1)}{(m+\frac{u}{2}+1)(m+\frac{u}{2}+2)\cdots(m+u)} & \text{if } 2 \mid u \\[2ex] \dfrac{2^{(u-1)/2} A \cdot (2m+1)(2m+3)\cdots(2m+u)}{(m+\frac{u-1}{2}+1)(m+\frac{u-1}{2}+2)\cdots(m+u)} & \text{if } 2 \nmid u. \end{cases} \tag{3.19}$$

Observe that $\Delta = \square$ is equivalent to $\Lambda = \boxplus$ (the square of a rational number).

We consider the case that $\Lambda = \boxplus$. If n is even, we recall that $A = 1$. For n odd, we have $A = n = M + 1 = 2m + 1$. Any prime that divides two of the numbers among

$$2m+1, 2m+3, \ldots, 2m+u-1, m+\frac{u}{2}+1, m+\frac{u}{2}+2, \ldots, m+u$$

must be $\leq 2u - 1$. Similarly, any prime that divides two numbers among

$$2m+1, 2m+3, \ldots, 2m+u, m+\frac{u-1}{2}+1, m+\frac{u-1}{2}+2, \ldots, m+u$$

must be $\leq 2u-1$. We deduce that if p is an odd prime $> 2u-1$ dividing one of the denominators appearing in the expressions for Λ in (3.19), then p exactly divides the denominator to an even power. In other words, we have

$$\Big(m+\frac{u}{2}+1\Big)\Big(m+\frac{u}{2}+2\Big)\cdots(m+u)=b\,\square \tag{3.20}$$

or

$$\Big(m+\frac{u-1}{2}+1\Big)\Big(m+\frac{u-1}{2}+2\Big)\cdots(m+u)=b\,\square, \tag{3.21}$$

where $P(b) \leq \max\{2, 2u-1\}$. Observe that the left-hand sides of (3.20) and (3.21) are products of $k = u/2$ and $k = (u+1)/2$ consecutive integers, respectively. We take $d = 1$, $C = 4$ and $\epsilon = 1/3$. Noting that $m \geq 4u/3$ we have

$$m+\frac{u}{2}+1 > \frac{5u}{3} = (C-d+\epsilon d)k,$$

if u is even. Also,

$$m+\frac{u-1}{2}+1 \geq \frac{5(u+1)}{3} = (C-d+\epsilon d)k$$

if u is odd and $u \geq 7$. Furthermore, we have $P(b) \leq 2u-1 < Ck$ in either of these cases. Theorem 2.2 implies that, for $u \geq 6$, equations (3.20) and (3.21) have finitely many solutions in m and u, giving rise to only finitely many pairs (n, α) in this case.

We are left with considering the cases where $u \leq 5$ and $n \geq 6$. If $u = 1$ and n is even, we obtain that $\Lambda = \boxplus$ if and only if

$$\frac{2m+1}{m+1} = \boxplus.$$

Since $m+1$ and $2m+1$ are relatively prime, we deduce that the above equation is equivalent to each of $m+1$ and $2m+1$ being a square. Setting $m+1 = y^2$ and $2m+1 = x^2$, we obtain the Pell equation

$$x^2 - 2y^2 = -1.$$

We deduce in this case that $\alpha = n+2$ where $n+1$ is the square of the rational part of $(1+\sqrt{2})^{2k+1}$ for some positive integer k. This is (xvii) in our list of elements in $\mathcal{A}_\infty$.

If $u = 1$ and n is odd, we similarly obtain that $\Lambda = \boxplus$ if and only if $m+1$ is a square. Here, $\alpha = n+1$ and $n = 2m+1$. We deduce that $\Delta(n,\alpha) = \Delta(n, n+1)$ is a square with n odd if and only if $n+1$ is twice a square, giving (xviii) in our list of elements in $\mathcal{A}_\infty$.

If $u = 2$ and n is even, we obtain that $\Lambda = \boxplus$ if and only if

$$\frac{2(2m+1)}{m+2} = \boxplus.$$

Necessarily, $2|(m+2)$ and $\gcd(m+2, 2m+1) \in \{1, 3\}$. Thus, we must either have both $(m+2)/2$ and $2m+1$ equal to a $\square$ or both equal to 3 times a $\square$. In the former case, $2m+4$ and $2m+1$ are squares differing by 3; in the latter case, $(2m+4)/3$ and $(2m+1)/3$ are consecutive positive integers that are squares. Since m is a positive integer, neither of these can happen.

If $u = 2$ and n is odd, we obtain that $\Lambda = \boxplus$ if and only if

$$\frac{2}{m+2} = \boxplus.$$

This occurs if and only if $m+2$ is twice a square. Since n is odd, we have $n = 2m+1$, and we deduce that $\Lambda = \boxplus$ if and only if $\alpha = n+3$ is an even square. This gives (xix) in our list of elements in $\mathcal{A}_\infty$.

If $u = 3$ and n is even, we obtain that $\Lambda = \boxplus$ if and only if

$$\frac{2(2m+1)(2m+3)}{(m+2)(m+3)} = \boxplus.$$

This equation implies that any prime appearing to an odd power in the prime factorization of $2m+1$ must be ≤ 5. If $u = 3$ and n is odd, we obtain that $\Lambda = \boxplus$ if and only if

$$\frac{2(2m+3)}{(m+2)(m+3)} = \boxplus.$$

Thus, in either case with $u = 3$, we deduce that

$$(m+2)(m+3)(2m+3) = b\,\square,$$

where b is a squarefree positive integer and $P(b) \le 5$. Since there are only finitely many possible integral values of m satisfying this elliptic curve equation for each such b, we are done in the case $u = 3$. In other words, there are only finitely many pairs (n, α) in this case, corresponding to $\alpha = n+5$ or $\alpha = n+6$ with α even, for which $\Delta(n, \alpha)$ is a square.

The cases $u = 4$ and $u = 5$ can be handled in a similar manner to the case $u = 3$, allowing us to conclude that there are at most finitely many (n, α) in these cases, corresponding to $\alpha = n+j$ with α even and $j \in \{7, 8, 9, 10\}$, where $\Delta(n, \alpha)$ is a square. In each of these cases, we can deduce that

$$(m+3)(m+4)(2m+3) = b\,\square,$$

where b is a squarefree positive integer and $P(b) \le 7$. We omit further details.

4. Solutions to $\Delta(n, \alpha) = \square$, for $\alpha < 0$

For $n = 1$ here, we still have $\Delta(n, \alpha) = 1$ corresponding to (i) in our list of elements in $\mathcal{A}_\infty$. We restrict ourselves then to $n > 1$. Set $\beta = -\alpha$. Thus, in this section, we consider the equation $\Delta(n, -\beta) = \square$, where β and n are positive integers with $n \ge 2$.

We begin by observing that if $\beta \in \{1, 2, \ldots, n\}$, then $L_n^{(-\beta)}(x)$ has the factor x. In this case, the Galois group associated with $L_n^{(-\beta)}(x)$ is contained in S_{n-1} and cannot be A_n unless $n = 2$ and $L_n^{(-\beta)}(x)$ is reducible. Directly using

$$2 \cdot L_2^{(-\beta)}(x) = x^2 + 2(\beta - 2)x + (\beta - 1)(\beta - 2),$$

one checks that $L_2^{(-\beta)}(x)$ is reducible for $\beta = 1$ and 2. Thus, the Galois group associated with $L_n^{(-\beta)}(x)$ is A_n for $\beta \in \{1, 2, \ldots, n\}$ if only if $(n, -\beta) \in \{(2, -1), (2, -2)\}$.

Despite the above, we have slightly more work to do in the case $\beta \in \{1, 2, \ldots, n\}$ since it is possible for $\Delta(n, -\beta) = \square$ when the Galois group is not A_n. However, if $n \geq \beta \geq 2$, then one checks that 0 is a root with multiplicity at least β so that $\operatorname{Discr}(n, -\beta) = 0$ (see the discussion after (2.1)). Since we are interested in non-zero square values of $\operatorname{Discr}(n, -\beta)$ for Theorem 2.1, we are left with considering $\Delta(n, -1)$ in the case that $\beta \leq n$. From (2.1), we see that

$$\Delta(n, -1) = \begin{cases} \displaystyle\prod_{1 \leq 2k+1 \leq n} (2k+1)^2 & \text{if } n \text{ is even} \\ n \cdot \displaystyle\prod_{1 \leq 2k+1 \leq n-2} (2k+1)^2 & \text{if } n \text{ is odd.} \end{cases}$$

Thus, $\Delta(n, -1)$ is a non-zero square if and only if either n is even or n is an odd square. This corresponds to (iv) in our list of elements in $\mathcal{A}_\infty$.

Now, we consider $\beta > n$. Since

$$\Delta(n, -\beta) = \prod_{1 \leq 2k+1 \leq n} (2k+1) \prod_{2 \leq 2k \leq n} (-\beta + 2k),$$

we obtain for even n that

$$\begin{aligned} \Delta(n, -\beta) &= (-1)^{n/2} \prod_{1 \leq 2k+1 \leq n} (2k+1) \prod_{2 \leq 2k \leq n} (\beta - 2k) \\ &= (-1)^{n/2} \prod_{1 \leq 2k+1 \leq n} (2k+1) \prod_{2 \leq 2k \leq n} (\beta - n - 2 + 2k) \\ &= (-1)^{n/2} \Delta(n, \beta - n - 2). \end{aligned} \tag{4.1}$$

Similarly, for odd n, one obtains

$$\Delta(n, -\beta) = (-1)^{(n-1)/2} \Delta(n, \beta - n - 1). \tag{4.2}$$

We want to relate these to the cases we have already studied of $L_n^{(\alpha)}(x)$ with $\alpha \geq 0$. But we need to consider the possibility that $\beta - n - 2$ or $\beta - n - 1$ above is < 0. Given we have $\beta > n$, we examine next what happens if n is even and $\beta = n + 1$. From (4.1), we have $\Delta(n, -n-1) = (-1)^{n/2} \Delta(n, -1)$. From (2.1), we see that $\Delta(n, -1) > 0$. In order for $\Delta(n, -n-1)$ to be a non-zero square, we necessarily need then that $n \equiv 0 \pmod 4$. Our above analysis for $\Delta(n,-1)$ implies that $\Delta(n,-1)$ is a non-zero square if n is even. We deduce in this case that $\Delta(n,-n-1)$ is a non-zero square if and only if $n \equiv 0 \pmod 4$. This gives (v) in our list of elements in $\mathcal{A}_\infty$.

The goal now is simply to combine information about $\Delta(n,\alpha)$ already obtained for $\alpha > 0$ with (4.1) and (4.2) to obtain information about $\Delta(n,-\beta)$. The following implications indicate how the items (vi), (vii), and (xx)-(xxx) in our list of elements in $\mathcal{A}_\infty$ were obtained. We omit the details as they are straightforward. We note, however, that (4.1) and (4.2) imply that $\Delta(n,-\beta)$ can be a non-zero square only if $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$, respectively.

$$\begin{array}{rclrclrcl}
\text{(iii)} & \Longrightarrow & \text{(vi)} & \text{(x)} & \Longrightarrow & \text{(xxiii)} & \text{(xvi)} & \Longrightarrow & \text{(xxvii)} \\
\text{(ii)} & \Longrightarrow & \text{(vii)} & \text{(xi)} & \Longrightarrow & \text{(xxiv)} & \text{(xvii)} & \Longrightarrow & \text{(xxviii)} \\
\text{(iii)} & \Longrightarrow & \text{(xx)} & \text{(xii)} & \Longrightarrow & \text{(xxv)} & \text{(xviii)} & \Longrightarrow & \text{(xxix)} \\
\text{(ix)} & \Longrightarrow & \text{(xxi)} & \text{(xv)} & \Longrightarrow & \text{(xxvi)} & \text{(xix)} & \Longrightarrow & \text{(xxx)} \\
\text{(viii)} & \Longrightarrow & \text{(xxii)} & & & & & &
\end{array}$$

5. The proof of Theorem 2.4

We begin with a simple consequence of Theorem 2.5.

Lemma 5.1. *Let α and n be nonnegative integers with $n \geq \max\{\alpha, 1\}$ for which $L_n^{(\alpha)}(x)$ is irreducible over $\mathbb{Q}$. If there exists a prime in the interval $((n+\alpha)/2, n-2)$, then the Galois group of $L_n^{(\alpha)}(x)$ is A_n or S_n depending, respectively, on whether $\Delta(n,\alpha)$ is a square or not.*

Proof. Observe that $L_n^{(\alpha)}(x) = \sum_{j=0}^{n} a_j x^j$, where

$$a_j = (-1)^j \frac{(n+\alpha)(n-1+\alpha)\cdots(j+1+\alpha)}{(n-j)!j!}.$$

We ignore possible cancellation of factors and refer to the numerator of a_j as $(-1)^j(n+\alpha)(n-1+\alpha)\cdots(j+1+\alpha)$ and the denominator of a_j as $(n-j)!j!$. Suppose that there exists a prime p in the interval $((n+\alpha)/2, n-2)$. Then $p^2 \geq 2p > n+\alpha$. Since $n \geq \alpha$, we also have $p > (n+\alpha)/2 \geq \alpha$. In particular, $p \geq \alpha + 1$. It follows that p divides the numerator of a_0 once and the denominator of a_0 once. In other words, $\nu_p(a_0) = 0$. For, $0 < j \leq p-1-\alpha$, we also have that p divides the numerator of a_j once and the denominator of a_j at most once. For $p-1-\alpha < j < p$, we have $p - \alpha \leq j$ so that

$$n - j \leq n + \alpha - p < p.$$

Thus, p does not divide the denominator for such j. Hence, $\nu_p(a_j) \geq 0$ for $0 < j < p$. Observe that

$$n/2 < p < j + 1 + \alpha \leq n + \alpha < 2p \quad \text{for } p \leq j \leq n-1.$$

We deduce that $\nu_p(a_j) = -1$ for $p \le j \le n$. The Newton polygon of $L_n^{(\alpha)}(x)$ with respect to p therefore consists of two edges, one with slope 0 having endpoints $(0, -1)$ and $(n-p, -1)$ and the other with slope $1/p$ having endpoints $(n-p, -1)$ and $(n, 0)$. Theorem 2.5 now implies that the Galois group of $L_n^{(\alpha)}(x)$ contains A_n. $\square$

We now give the proof of Theorem 2.4. Using Maple (mathematics software system), we verified that there is a prime in the interval $(N, 1.5N]$ for each $9 \le N \le 48683$. By Lemma 3.1, we easily have the same result for $N > 48683$. Set $N = (n+\alpha)/2$. Observe that

$$1.5N = \frac{3(n+\alpha)}{4} < n-2$$

as long as $n > 3\alpha + 8$. For Theorem 2.4, we are interested in $0 \le \alpha \le 10$. Hence, there is a prime in $((n+\alpha)/2, n-2)$ provided $n > 38$ for any such α. One checks directly that the interval $((n+\alpha)/2, n-2)$ contains a prime for all pairs (n, α) with $20 \le n \le 38$ and $0 \le \alpha \le 10$. Given Theorem 2.6 and Lemma 5.1, we need only determine those pairs (n, α) for which $\Delta(n, \alpha)$ is a square for $n \ge 20$ and $0 \le \alpha \le 10$ and to compute each of the Galois groups of $L_n^{(\alpha)}(x)$ for $1 \le n \le 19$ and $0 \le \alpha \le 10$. The former is done by referring directly to Theorem 2.1 and to the analysis for $1 \le \alpha \le 10$ given in the previous sections. The latter computations for $1 \le n \le 19$ were done using Magma (mathematics software system), which can determine the Galois groups for any irreducible polynomial with degree less than or equal to 19. Theorem 2.4 follows.

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