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par Evgeniy ZORIN

Résumé. Nous établissons un nouveau *lemme de multiplicité* pour les solutions d'un système différentiel généralisant les relations différentielles classiques de Ramanujan. Ce résultat peut être utile pour l'étude des propriétés arithmétiques des valeurs de la fonction zêta de Riemann aux entiers positifs impairs (Nesterenko, 2011).

Abstract. We establish a new *multiplicity lemma* for solutions of a differential system extending Ramanujan's classical differential relations. This result can be useful in the study of arithmetic properties of values of Riemann zeta function at odd positive integers (Nesterenko, 2011).

1. Introduction

In what follows we denote by $\sigma_k(n)$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$ the sum of kth powers of divisors of n:

$$\sigma_k(n) := \sum_{d|n} d^k.$$

In this paper we consider the following sets of functions. First of all, Fourier series of the Eisenstein functions

$$E_{2k}(z) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) z^n, \quad k \in \mathbb{N}, \tag{1.1}$$

where B_{2k} are Bernoulli numbers. Also we consider

$$g_{u,v}(z) := \sum_{n=1}^{\infty} n^u \sigma_{-v}(n) z^n, \quad u, v \in \mathbb{N}, \quad 0 \leq u < v, \quad v \text{ is odd.} \tag{1.2}$$

We recall the following well-known fact: functions E_2, E_4 and E_6 are algebraically independent over $\mathbb{C}(z)$ and all the other functions E_{2k}, $k \geq 4$ belong to $\mathbb{Q}[E_4, E_6]$ (see for instance [8]). More precisely, for all $k \geq 4$ there exists a polynomial $A_k \in \mathbb{C}[X, Y]$ such that

$$E_{2k}(z) = A_k(E_4(z), E_6(z))$$

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and these polynomials $A_k(X,Y)$, $k \geq 4$ contain only monomials M of bidegrees $(\deg_X M, \deg_Y M)$ satisfying $2\deg_X M + 3\deg_Y M = k$.

In [2] Yu.Nesterenko proved the following theorem:

Theorem 1.1 (Nesterenko). *For any complex q, $0 < |q| < 1$, the set*

$$q, E_2(q), E_4(q), E_6(q)$$

contains at least three algebraically independent over $\mathbb{Q}$ numbers.

In particular, this result allowed him to establish the algebraic independence of π, $\mathbf{e}^{\pi}$ and $\Gamma\left(\frac{1}{4}\right)$. Indeed, using modular properties of Eisenstein series one can easily calculate

$$E_2(\mathbf{e}^{-2\pi}) = \frac{3}{\pi}, \quad E_4(\mathbf{e}^{-2\pi}) = 3\frac{\Gamma(\frac{1}{4})^8}{(2\pi)^6}, \quad E_6(\mathbf{e}^{-2\pi}) = 0$$

(see Chapter 1 of [4]). Hence applying Theorem 1.1 with $q = \mathbf{e}^{-2\pi}$ one obtains algebraic independence of π, $\mathbf{e}^{\pi}$ and $\Gamma\left(\frac{1}{4}\right)$.

Later Yu.Nesterenko [3] showed that considering a bigger set of functions, notably introducing functions (1.2), we can capture other extremely interesting cases, for instance one has $\zeta(4n+3) \in \mathbb{Q}(E_2(\mathbf{e}^{-2\pi}), g_{0,4n+3}(\mathbf{e}^{-2\pi}))$ $(n \in \mathbb{N})$. So extending Theorem 1.1 for a set of functions like

$$E_2, E_4, E_6, g_{u,v}, \quad 0 \leq u < v, \quad v \equiv 3 \mod 4$$

one could potentially illuminate the question of transcendence or irrationality of ζ at odd positive points. See [3] for some more discussion.

An important stage in the proof of Theorem 1.1 is the following *multiplicity lemma*:

Theorem 1.2 (Nesterenko). *For any polynomial $A(z, X_1, X_2, X_3) \in \mathbb{C}[z, X_1, X_2, X_3]$, $A \neq 0$, the inequality holds*

$$\mathrm{ord}_{z=0} A(z, P(z), Q(z), R(z)) < c(\deg_z A + 1)(\deg_{\underline{X}} A + 1)^3,$$

where c is an absolute constant.

We invite the interested reader to consult Chapters 3 and 10 in [4] as well as [6, 7, 9] concerning the use of multiplicity estimates in algebraic independence proofs.

In this paper we adopt the method from [2] and [4, Chapter 10] to establish (for any fixed odd $m \geq 3$) a multiplicity lemma for the whole set of functions

$$(1.3) \qquad E_2(z), E_4(z), E_6(z), g_{u,v}(z), \ v \text{ is odd}, \ 0 \leq u < v \leq m,$$

see Theorem 2.1 below. Note that *algebraic independence* over $\mathbb{C}(z)$ of the set of functions (1.3) was already established in 2010 by P. Kozlov ([1], for a statement see also [3], page 2), we use this result in the proof of Theorem 2.1.

Proofs of Theorems 1.1 and 1.2 are based on the fact that functions E_2, E_4, E_6 satisfy the following system of differential equations:

(1.4)
$$\delta E_2 = \frac{1}{12}\left(E_2^2 - E_4\right), \delta E_4 = \frac{1}{3}\left(E_2E_4 - E_6\right), \delta E_6 = \frac{1}{2}\left(E_2E_6 - E_4^2\right),$$

where $\delta := z\frac{d}{dz}$.

The set of functions (1.3) satisfy the following extended system of differential equations [3]. We keep the three equations (1.4) and for any odd $v \geq 3$

$$\begin{aligned} \delta g_{u,v}(z) &= g_{u+1,v}(z), \quad 0 \leq u < v-1, \\ \delta g_{v-1,v}(z) &= B_{v+1}\frac{A_{v+1}(E_4(z), E_6(z)) - 1}{2v+2}; \end{aligned} \tag{1.5}$$

in the case $v = 1$ we have the equation

$$\delta g_{0,1}(z) = \frac{1}{24}\left(1 - E_2(z)\right). \tag{1.6}$$

2. Multiplicity Lemma

Let $m \in \mathbb{N}$ be a fixed positive odd integer. We introduce the following notation:

$$R := \mathbb{C}[z, X_1, X_2, X_3, Y_{0,1}, Y_{0,3}, Y_{1,3}, Y_{2,3}, \ldots, Y_{m-1,m}]. \tag{2.1}$$

Theorem 2.1. *Let $m \geq 1$ be an odd integer. There exists a constant C depending on m only such that for all non-zero $P \in R$*

$$\begin{aligned} &\mathrm{ord}_{z=0} P(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), g_{0,3}(z), \ldots, g_{0,m}(z), \ldots, g_{m-1,m}(z)) \\ &\leq C\left(\deg_z P + 1\right)\left(\deg_{\underline{X},\underline{Y}} P + 1\right)^{\left(\frac{m+1}{2}\right)^2 + 3}, \end{aligned} \tag{2.2}$$

where $\deg_{\underline{X},\underline{Y}} P$ denotes the total degree of P in the variables X_1, X_2, X_3, $Y_{0,1}, \ldots, Y_{m-1,m}$, i.e. all the variables appearing in the definition (2.1) *of R but z.*

Remark 2.1. *The exponent $\left(\frac{m+1}{2}\right)^2 + 3$ in the r.h.s. of* (2.2) *equals the number of functions different than z in the l.h.s. of* (2.2) *(and also the transcendence degree of R over $\mathbb{C}(z)$). It is an easy exercise from linear algebra that for any given set $f_1, \ldots, f_l$ of functions analytic at $z = 0$ and any $N \in \mathbb{N}$, $N \geq 1$, one can construct a polynomial $P_{N,\underline{f}} \in \mathbb{C}[z, X_1, \ldots, X_l]$ of degree $\deg P \geq N$ and satisfying*

$$\mathrm{ord}_{z=0} P_{N,\underline{f}}(z, f_1(z), \ldots, f_l(z)) \geq C_0\left(\deg_z P_{N,\underline{f}} + 1\right)\left(\deg_{\underline{X}} P_{N,\underline{f}} + 1\right)^l,$$

where $C_0 = \frac{1}{(l+1)!}$.

Hence Theorem 2.1 provides multiplicity estimate for the set of functions (1.3) *with the optimal exponent.*

In the sequel we denote

$$\begin{aligned} D_0 &:= z\frac{d}{dz} + \frac{1}{12}\left(X_1^2 - X_2\right)\frac{d}{dX_1} + \frac{1}{3}\left(X_1X_2 - X_3\right)\frac{d}{dX_2} \\ &\quad + \frac{1}{2}\left(X_1X_3 - X_2^2\right)\frac{d}{dX_3}, \\ D_1 &:= \frac{1}{24}\left(1 - X_1\right)\frac{d}{dY_{0,1}}, \\ D_v &:= \sum_{k=0}^{v-2} Y_{k+1,v}\frac{d}{dY_{k,v}} + B_{v+1}\frac{A_{v+1}(X_2,X_3)-1}{2v+2}\frac{d}{dY_{v-1,v}}, \end{aligned}$$

$v = 3, 5, \ldots, m$, and

$$D := D_0 + \sum_{k=0}^{(m-1)/2} D_{2k+1}. \tag{2.3}$$

The differential operator D satisfies

$$\begin{aligned} &DP(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)) \\ &\qquad = z\frac{d}{dz}P(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)). \end{aligned} \tag{2.4}$$

We deduce Theorem 2.1 using Nesterenko's conditional Multiplicity Lemma (see Theorem 2.2 below). This result deals with differential system of the following type:

$$f_i'(z) = \frac{A_i(z,\underline{f})}{A_0(z,\underline{f})}, \quad i = 1, \ldots, n, \tag{2.5}$$

where $A_i(z, X_1, \ldots, X_n) \in \mathbb{C}[z, X_1, \ldots, X_n]$ for $i = 0, ..., n$ (we suppose that A_0 is a non-zero polynomial).

Remark 2.2. *It is easy to see that the system* (1.4)∪(1.5)∪(1.6) *is of the type* (2.5).

One associates to the system (2.5) the differential operator

$$D_A = A_0(z, X_1, \ldots, X_n)\frac{\partial}{\partial z} + \sum_{i=1}^{n} A_i(z, X_1, \ldots, X_n)\frac{\partial}{\partial X_i}. \tag{2.6}$$

In our case (i.e. the case of the system (2.5)) this formula gives exactly the differential operator D as defined in (2.3).

Theorem 2.2 (Nesterenko, see Theorem 1.1, Chapter 10 [4]). *Suppose that functions*

$$\underline{f} = (f_1(z), \ldots, f_n(z)) \in \mathbb{C}[[z]]^n$$

are analytic at the point $z = 0$ and form a solution of the system (2.5). If there exists a constant K_0 such that every D_A-stable prime ideal $\mathcal{P} \subset \mathbb{C}[X_1', X_1, \ldots, X_n]$, $\mathcal{P} \neq (0)$, satisfies

$$\min_{P \in \mathcal{P}} \mathrm{ord}_{z=0} P(z, \underline{f}) \leq K_0, \tag{2.7}$$

then there exists a constant $K_1 > 0$ such that for any polynomial $P \in \mathbb{C}[X_1', X_1, \ldots, X_n]$, $P \neq 0$, the following inequality holds

$$\mathrm{ord}_{z=0}(P(z, \underline{f})) \leq K_1(\deg_{\underline{X}'} P + 1)(\deg_{\underline{X}} P + 1)^n. \tag{2.8}$$

To deduce Theorem 2.1 from Theorem 2.2 it is sufficient to prove Proposition 2.1 here below.

Proposition 2.1. *If $\mathcal{P}$ is a non-zero prime ideal of*

$$R = \mathbb{C}[z, X_1, X_2, X_3, Y_{0,1}, \ldots, Y_{m-1,m}]$$

with $D\mathcal{P} \subset \mathcal{P}$, then either $z \in \mathcal{P}$ or $\Delta = X_2^3 - X_3^2 \in \mathcal{P}$.

Proof of Theorem 2.1 modulo Proposition 2.1. If we have the result announced in Proposition 2.1, then any prime D-stable ideal $\mathcal{P}$ contains the polynomial

$$\Theta := z\Delta = z\left(X_2^3 - X_3^2\right). \tag{2.9}$$

In this case we have obviously

$$\begin{aligned} &\min_{P \in \mathcal{P}} \mathrm{ord}_{z=0} P(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)) \\ &\qquad \leq \mathrm{ord}_{z=0} \Theta(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)) = 2, \end{aligned} \tag{2.10}$$

as $\Theta(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \ldots, g_{m-1,m}(z)) = z\Delta(E_2(z), E_4(z))$ by definition (2.9) and $\mathrm{ord}_{z=0}\Delta(E_2(z), E_4(z)) = 1$. The upper bound (2.10) assures the hypothesis (2.7) of Theorem 2.2, and applying this theorem we obtain the upper bound (2.8), that is (2.2) in our case. □

To prove Proposition 2.1, we describe at first principal D-stable ideals of R.

Lemma 2.1. *There exist only two non-zero D-invariant principal prime ideals of R, namely, the ideals generated by z and Δ.*

Proof. Suppose that $A \in R$ is any irreducible polynomial with the property that $A | DA$. Thus

$$DA = AB, \quad B \in R. \tag{2.11}$$

We readily verify with the definition of D (see (2.3)) that $\deg_{\underline{Y}} DA \leq \deg_{\underline{Y}} A$ and $\deg_z DA \leq \deg_z A$, hence (2.11) implies $\deg_{\underline{Y}} B = \deg_z B = 0$, i.e. $B \in \mathbb{C}[X_1, X_2, X_3]$.

For any $F \in R$ we define the *weight* of F as $\phi : R \to \mathbb{N}$,

$$\phi(F) := \deg_t F(z, tX_1, t^2X_2, t^3X_3, t^{m+1}\underline{Y}).$$

Then ϕ satisfies the following properties:

(1) For any $F \in R$

$$\phi(DF) \leq \phi(F) + 1.$$

(2) For any $F, G \in R$

$$\phi(FG) = \phi(F) + \phi(G).$$

These properties together with (2.11) imply

$$\phi(A) + \phi(B) = \phi(DA) \leq \phi(A) + 1,$$

hence $\phi(B) \leq 1$. Thus $B \in \mathbb{C}[X_1]$ and $\deg B \leq 1$, i.e. $B = aX_1 + b$, $a, b \in \mathbb{C}$. So (2.11) in fact means

$$DA = (aX_1 + b)\,A, \tag{2.12}$$

where $a, b \in \mathbb{C}$.

We claim that $b \in \mathbb{N}$ (and later we prove also that $a \in \mathbb{Z}$). To prove this we consider one more weight $\phi_2 : R \to \mathbb{Z}$ (in fact we shall use this weight only for monomials from R). For any $F \in R$, we denote

$$\phi_2(F) := \deg_t F(z, tX_1, t^2X_2, t^3X_3, t^{-1}Y_{0,1}, \ldots, t^{-m}Y_{0,m}, \\ t^{-m+1}Y_{1,m}, \ldots, t^{-1}Y_{m-1,m})$$

(i.e. we assign to the variable $Y_{u,v}$ the weight $\phi_2(Y_{u,v}) := u - v$).

Let $M \in R$ be a monomial. We readily verify that all the monomials of

$$\frac{A_{v+1}(X_2, X_3) - 1}{2v + 2} \frac{d}{dY_{v-1,v}} M$$

have the weight ϕ_2 strictly bigger than $\phi_2(M)$. The same is true for

$$\frac{1}{24}\,(1 - X_1)\,\frac{d}{dY_{0,1}} M,$$

as well as for

$$Y_{k+1,v} \frac{d}{dY_{k,v}} M \quad (0 \leq k < v - 1)$$

and

$$\left(\frac{1}{12}\left(X_1^2 - X_2\right)\frac{d}{dX_1} + \frac{1}{3}\,(X_1X_2 - X_3)\,\frac{d}{dX_2} + \frac{1}{2}\left(X_1X_3 - X_2^2\right)\frac{d}{dX_3}\right) M.$$

The monomial $z\frac{d}{dz}M$ either is 0 or satisfy $\phi_2(z\frac{d}{dz}M) = \phi_2(M)$.

Thus remembering the definition of D, see (2.3), we readily verify that every non-zero monomial appearing in DM has the weight ϕ_2 not less than $\phi_2(M)$ and the only monomial in DM that eventually has the weight ϕ_2 equal to $\phi_2(M)$ is $z\frac{d}{dz}M$.

Let C be the sum of monomials of A with minimal weight ϕ_2. Then in the r.h.s. of (2.12) the sum of monomials of minimal weight is equal to bC (all the other monomials of $(aX_1+b)A$ obviously have a bigger value of weight ϕ_2).

Applying to the l.h.s. of (2.12) (that is DA) the above considerations we conclude

$$z\frac{d}{dz}C = bC. \tag{2.13}$$

Comparing the coefficients on the both sides of (2.13) we find $b=\deg_z C$, in particular $b\in\mathbb{N}$.

We proceed to the proof $a\in\mathbb{Z}$. Substituting $X_1=E_2(z)$, $X_2=E_4(z)$, $X_3=E_6(z)$, $Y_{u,v}=g_{u,v}(z)$, $0\le u<v\le m$ in (2.12) we obtain

$$\begin{aligned}(aE_2(z)+b)\,&A\left(z,E_2(z),E_4(z),E_6(z),g_{0,1}(z),\dots,g_{m-1,m}(z)\right)\\ &= DA\left(z,E_2(z),E_4(z),E_6(z),g_{0,1}(z),\dots,g_{m-1,m}(z)\right).\end{aligned} \tag{2.14}$$

We consider the Taylor expansion (at the point $z=0$) of

$$A\left(z,E_2(z),E_4(z),E_6(z),g_{0,1}(z),\dots,g_{m-1,m}(z)\right). \tag{2.15}$$

It is a function analytic at $z=0$ because all the functions E_2, E_4, E_6, $g_{0,1}$, $\dots$, $g_{m-1,m}$ are. Let M denote the order at $z=0$ of the function (2.15), that is

$$\begin{aligned}A&\left(z,E_2(z),E_4(z),E_6(z),g_{0,1}(z),\dots,g_{m-1,m}(z)\right)\\ &= cz^M + (\text{terms of order } > M),\end{aligned}$$

where $c\ne 0$. Using (2.4) we obtain

$$\begin{aligned}DA&\left(z,E_2(z),E_4(z),E_6(z),g_{0,1}(z),\dots,g_{m-1,m}(z)\right)\\ &= cMz^M + (\text{terms of order } > M).\end{aligned}$$

In view of (1.1) we have $E_2(z)=1+$terms of order >1. We readily deduce from (2.14)

$$(a+b)cz^M + (\text{terms of order } > M) = cMz^M + (\text{terms of order } > M).$$

Comparing coefficients with z^M on both sides in this expression (and simplifying out c) we readily deduce $a+b=M$. We have already established $b\in\mathbb{N}$. Obviously, $M\in\mathbb{N}$ (as it is a degree in a Taylor series). We conclude $a\in\mathbb{Z}$.

So we have established that coefficients a,b involved in (2.12) are in fact integers.

Note that

$$D(\Delta^{-a}z^{-b}) = (-aX_1 - b)\,\Delta^{-a}z^{-b}. \tag{2.16}$$

We denote

$$\begin{aligned} S\left(z, E_2, E_4, E_6, g_{0,1}, \dots, g_{m-1,m}\right) &:= \\ & A\left(z, E_2, E_4, E_6, g_{0,1}, \dots, g_{m-1,m}\right) \times \Delta^{-a}z^{-b}. \end{aligned} \tag{2.17}$$

Applying the differential operator D to the r.h.s. of (2.17) and using (2.12), (2.16) we obtain

$$DS = 0.$$

Using (2.4) on the latter equality we conclude

$$\frac{d}{dz}S(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \dots, g_{m-1,m}(z)) = 0,$$

hence

$$S(z, E_2(z), E_4(z), E_6(z), g_{0,1}(z), \dots, g_{m-1,m}(z)) \in \mathbb{C}.$$

Recall that functions $z, E_2, E_4, E_6, g_{0,1}, \dots, g_{m-1,m}$ are all algebraically independent over $\mathbb{C}$, see [3] page 2. For this reason we deduce $S[z, X_1, X_2, X_3, \underline{Y}] \in \mathbb{C}$ and thereby

$$A = \lambda \Delta^a z^b$$

with $\lambda \in \mathbb{C}$. If we suppose that A is irreducible, we obtain immediately that either $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$. □

Proof of Proposition 2.1. We consider the following nested sequence of rings

$$\begin{aligned} \mathbb{C}[z, \underline{X}] &\subset \mathbb{C}[z, \underline{X}, Y_{0,1}] \\ &\subset \mathbb{C}[z, \underline{X}, Y_{0,1}, Y_{2,3}] \\ &\subset \mathbb{C}[z, \underline{X}, Y_{0,1}, Y_{1,3}, Y_{2,3}] \\ &\subset \mathbb{C}[z, \underline{X}, Y_{0,1}, Y_{0,3}, Y_{1,3}, Y_{2,3}] \\ &\quad\vdots \\ &\subset \mathbb{C}[z, \underline{X}, Y_{0,1}, \dots, Y_{m-3,m-2}] \\ &\subset \mathbb{C}[z, \underline{X}, Y_{0,1}, \dots, Y_{m-3,m-2}, Y_{m-1,m}] \\ &\subset \mathbb{C}[z, \underline{X}, Y_{0,1}, \dots, Y_{m-3,m-2}, Y_{m-2,m}, Y_{m-1,m}] \\ &\quad\vdots \\ &\subset \mathbb{C}[z, \underline{X}, Y_{0,m}, \dots, Y_{m-1,m}] = R. \end{aligned} \tag{2.18}$$

We readily verify with the definition of D that every term R_i appearing in the chain (2.18) satisfies $DR_i \subset R_i$.

Let $\mathcal{P} \subset R$ be a prime ideal of R satisfying $D\mathcal{P} \subset \mathcal{P}$. If $\mathcal{P} \cap \mathbb{C}[z, \underline{X}] \neq \{0\}$, it contains a polynomial $z\Delta$ as shown in [5][Theorem 1.4]. So the claim of

Proposition 2.1 is proved in this case. We suppose henceforth $\mathcal{P} \cap \mathbb{C}[z, \underline{X}] = \{0\}$.

As we suppose $\mathcal{P} \neq \{0\}$ and $\mathcal{P} \cap \mathbb{C}[z, \underline{X}] = \{0\}$, we find in the chain (2.18) at some step an extension of rings $R_i \subset R_{i+1}$ satisfying $\mathcal{P} \cap R_i = \{0\}$ and $\mathcal{P} \cap R_{i+1} \neq \{0\}$. In this case the ideal (of the ring R_{i+1}) $\mathcal{P} \cap R_{i+1}$ is a principal one, because we add exactly one variable at each step in the chain (2.18), i.e. $\text{tr.deg.}_{R_i} R_{i+1} = 1$. Hence $\mathcal{P} \cap R_{i+1}$ is a D-stable principal ideal (of the ring R_{i+1}, and also this ideal generates a principal D-stable ideal of the ring R, because D-stability of a principal ideal means exactly the condition $Q|DQ$ on a generator of the ideal). We deduce with Lemma 2.1 that $z\Delta \in \mathcal{P} \cap R_{i+1} \subset \mathcal{P}$, Q.E.D. □

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