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The arithmetic of certain del Pezzo surfaces and K3 surfaces

par NGUYEN Ngoc Dong Quan

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Résumé. Nous construisons des surfaces de del Pezzo de degré 4 violant le principe de Hasse expliqué par l'obstruction de Brauer-Manin. En utilisant ces surfaces de del Pezzo de degré 4, nous montrons qu'il y a des familles algébriques de surfaces $K3$ violant le principe de Hasse expliqué par l'obstruction de Brauer-Manin. Divers exemples sont donnés.

Abstract. We construct del Pezzo surfaces of degree 4 violating the Hasse principle explained by the Brauer-Manin obstruction. Using these del Pezzo surfaces, we show that there are algebraic families of $K3$ surfaces violating the Hasse principle explained by the Brauer-Manin obstruction. Various examples are given.

1. Introduction

Let k be a global field and $\mathbb{A}_k$ be the adèle ring of k. Let $\mathcal{V}$ be a smooth geometrically irreducible variety defined over k and let $\mathrm{Br}(\mathcal{V})$ be the Brauer group of $\mathcal{V}$. It is well-known that

$$\mathcal{V}(k) \subseteq \mathcal{V}(\mathbb{A}_k)^{\mathrm{Br}} \subseteq \mathcal{V}(\mathbb{A}_k).$$

Here,

$$\mathcal{V}(\mathbb{A}_k)^{\mathrm{Br}} = \left\{ (P_v) \in \mathcal{V}(\mathbb{A}_k) \text{ such that } \sum_v \mathrm{inv}_v \mathcal{A}(P_v) = 0 \text{ for all } \mathcal{A} \in \mathrm{Br}(\mathcal{V}) \right\}$$

and for each valuation v and each Azumaya algebra $\mathcal{A}$ in $\mathrm{Br}(\mathcal{V})$, $\mathrm{inv}_v : \mathrm{Br}(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z}$ is the local invariant map from class field theory and $\mathcal{A}(P_v)$ is defined as follows. A point $P_v \in \mathcal{V}(k_v)$ gives a map $\mathrm{Spec}(k_v) \longrightarrow \mathcal{V}$, and hence induces a pullback map $\mathrm{Br}(\mathcal{V}) \longrightarrow \mathrm{Br}(k_v)$; we write $\mathcal{A}(P_v)$ for the image of $\mathcal{A}$ under this map.

We say that $\mathcal{V}$ satisfies *the Hasse principle* if the following is true

$$\mathcal{V}(k) \neq \emptyset \Leftrightarrow \mathcal{V}(k_v) \neq \emptyset \ \forall v.$$

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If $\mathcal{V}(k) = \emptyset$ and $\mathcal{V}(\mathbb{A}_k) \neq \emptyset$, we say that $\mathcal{V}$ is *a counter-example to the Hasse principle*. Further, if we also have $\mathcal{V}(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$, we say that $\mathcal{V}$ is *a counter-example to the Hasse principle explained by the Brauer-Manin obstruction.*

In 1921, Hasse proved that smooth quadric hypersurfaces of arbitrary dimension satisfy the Hasse principle. The first counterexamples of genus one curves to the Hasse principle were discovered by Lind ([9]) in 1940 and independently by Reichardt ([13]).

We are concerned in this paper with constructing del Pezzo surfaces of degree 4 violating the Hasse principle explained by the Brauer-Manin obstruction; and it then follows that there exist algebraic families of $K3$ surfaces violating the Hasse principle explained by the Brauer-Manin obstruction. More precisely, we shall prove the following

Theorem 1.1. *Let p be a prime such that $p = 64k^2 + 40k + 5$ for $k \in \mathbb{Z}_{\geq 0}$. Let $\mathcal{X} \subset \mathbb{P}^4_{\mathbb{Q}}$ be the del Pezzo surface defined by*

$$uv = x^2 - py^2 \tag{1.1}$$

$$(u + (8k+1)v)(u + (8k+2)v) = x^2 - pz^2 \tag{1.2}$$

Then, $\mathcal{X}$ is a counterexample to the Hasse principle explained by the Brauer-Manin obstruction.

and

Theorem 1.2. *Let p be a prime such that $p = 64k^2 + 40k + 5$ for $k \in \mathbb{Z}_{\geq 0}$. Let $(\Gamma, \Lambda, \Sigma) \in \mathbb{Q}^3$ be a point on the conic $\mathcal{Q} \in \mathbb{P}^2_{\mathbb{Q}}$ defined by*

$$\mathcal{Q} : (p^2 - (8k+1)^2)X^2 - Y^2 + Z^2 = 0, \tag{1.3}$$

such that $\Gamma \neq 0$.

Let $\Omega \in \mathbb{Q}$ be a rational number such that the triple $(\lambda, \mu, \nu) \in \mathbb{Q}^3$ defined by

$$\begin{cases} \lambda := \Gamma^2 \\ \mu := \Omega^2 - \Gamma^2 \\ \nu := \Lambda^2 - \Omega^2 - (p^2 - 1)\Gamma^2 \end{cases}, \tag{1.4}$$

satisfies the following.

(A1) $\lambda\mu\nu \neq 0$.

(A2) $C_1C_2C_3C_4C_5 \neq 0$ *where*

$$C_1 := \nu^2 - 4p^2\lambda\mu,$$
$$C_2 := (8k+1)^2(8k+2)^2\lambda^2 + 16k(8k+1)\lambda\mu + \mu^2,$$
$$C_3 := p(8k+1)^2\lambda + p\mu + (8k+1)\nu,$$
$$C_4 := p(8k+2)^2\lambda + p\mu + (8k+2)\nu,$$
$$C_5 := (8k+1)\nu^2 + p(8k+1)(8k+2)\lambda\nu + p\mu\nu + p^2\lambda\mu.$$

Let $\mathcal{K} \subset \mathbb{P}^5_{\mathbb{Q}}$ be the $K3$ surface defined by

$$\mathcal{K} : \begin{cases} u^2 & = xy + pz^2 \\ u^2 - pv^2 & = (x + (8k+1)y)(x + (8k+2)y) \\ w^2 & = \lambda x^2 + \mu y^2 + \nu z^2 \end{cases} . \tag{1.5}$$

Then, $\mathcal{K}$ is a violation of the Hasse principle explained by the Brauer-Manin obstruction.

We shall prove Theorem 1.1 in Section 2. In Section 3, we shall prove Theorem 1.2 and as a corollary, we shall show that there are algebraic families of $K3$ surfaces violating the Hasse principle explained by the Brauer-Manin obstruction.

Remark 1.3. Let $k = 0$ in Theorem 1.1. Then, $p = 5$. Let $\mathcal{X}_5$ be the del Pezzo surface of degree 4 defined by

$$\mathcal{X}_5 : \begin{cases} uv = x^2 - 5y^2 \\ (u+v)(u+2v) = x^2 - 5z^2 \end{cases} .$$

Then, by Theorem 1.1, $\mathcal{X}_5$ violates the Hasse principle explained by the Brauer-Manin obstruction. This is the well-known Birch and Swinnerton-Dyer del Pezzo surface (see [1]).

Remark 1.4. It is easy to check that the point $(\Gamma_0, \Lambda_0, \Sigma_0) = (1, p, 8k+1)$ lies on the conic $\mathcal{Q}$ with $\Gamma_0 = 1 \neq 0$.

In Section 3, we shall show that for a given point $(\Gamma, \Lambda, \Sigma)$ on the conic $\mathcal{Q}$ with $\Gamma \neq 0$, there are infinitely many polynomials $\Omega(\mathbf{T}) \in \mathbb{Q}[\mathbf{T}]$ such that the quadruple $(\Gamma, \Lambda, \Sigma, \Omega(\mathbf{T}))$ satisfies $A1$ and $A2$ for any $\mathbf{T} \in \mathbb{Q}$. This is the key fact that we shall use to construct algebraic families of $K3$ surfaces violating the Hasse principle explained by the Brauer-Manin obstruction.

2. The Hasse principle for certain degree 4 del Pezzo surfaces

Recall that the Kronecker symbol (still denoted by $\left(\dfrac{a}{b}\right)$) is the extension of the Jacobi symbol to $(\mathbb{Z} \setminus \{0\})^2$ by defining $\left(\dfrac{a}{-1}\right) = \mathrm{sign}(a)$, $\left(\dfrac{a}{2}\right) =$

$\left(\dfrac{2}{a}\right)$ for a odd and $\left(\dfrac{a}{2}\right) = 0$ for a even and extending by multiplicativity (see [3, p.36]).

We begin by proving the main lemmas that we need in the proof of Theorem 1.1.

Lemma 2.1. *Let p be a prime such that $p = 64k^2 + 40k + 5$ for $k \in \mathbb{Z}_{\geq 0}$. Then, 2 and $8k+3$ are quadratic non-residues in $\mathbb{F}_p^\times$, and $4k+1$ is a quadratic residue in $\mathbb{F}_p^\times$.*

Proof. We see from Theorem 2.2.9 ([3, p.38]) that

$$\left(\frac{4k+1}{p}\right) = \left(\frac{4k+1}{(16k+6)(4k+1)-1}\right) = \left(\frac{4k+1}{-1}\right) = \operatorname{sign}(4k+1) = 1.$$

Since $p = 64k^2 + 40k + 5 \equiv 5 \pmod 8$, 2 is a quadratic non-residue in $\mathbb{F}_p^\times$. Further, we see that $2(4k+1)(8k+3) \equiv 64k^2 + 40k + 6 \equiv 1 \pmod p$. Thus, since 2 is a quadratic non-residue in $\mathbb{F}_p^\times$ and $(4k+1)$ is a quadratic residue modulo p, it follows that $8k+3$ is a quadratic non-residue in $\mathbb{F}_p^\times$, proving our contention. $\square$

Lemma 2.2. *Suppose the same assumptions and notations as in Theorem 1.1. Let $\mathbb{Q}(\mathcal{X})$ be the function field of $\mathcal{X}$ and let $\mathcal{A}$ be the class of the quaternion algebra $\left(p, \dfrac{u}{u+(8k+1)v}\right)$. Then, $\mathcal{A}$ is an Azumaya algebra of $\mathcal{X}$, that is, $\mathcal{A}$ belongs to the subgroup $Br(\mathcal{X})$ of $Br(\mathbb{Q}(\mathcal{X}))$. Further, the quaternion algebras $\mathcal{A}$, $\mathcal{B} = \left(p, \dfrac{u}{u+(8k+2)v}\right)$, $\mathcal{C} = \left(p, \dfrac{v}{u+(8k+1)v}\right)$ and $\mathcal{D} = \left(p, \dfrac{v}{u+(8k+2)v}\right)$ all represent the same class in $Br(\mathbb{Q}(\mathcal{X}))$.*

Proof. Let $K = \mathbb{Q}(\sqrt{p})$. We consider the following two conics defined over K and lying on $\mathcal{X}$

$$\Gamma_1 : u = 0, \quad x - \sqrt{p}y = 0, \quad (8k+1)(8k+2)v^2 = x^2 - pz^2$$

and

$$\Gamma_2 : u + (8k+1)v = 0, \quad x - \sqrt{p}z = 0, \quad (8k+1)v^2 + x^2 - py^2 = 0.$$

Let σ be the generator of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$. Then, it follows that $\Gamma_1 + \sigma\Gamma_1$ is the section of the surface $\mathcal{X}$ by the hyperplane $u = 0$, and similarly, $\Gamma_2 + \sigma\Gamma_2$ is the section of $\mathcal{X}$ by the hyperplane $u + (8k+1)v = 0$. Hence, we deduce that

$$\operatorname{div}\left(\frac{u}{u+(8k+1)v}\right) = \Gamma_1 + \sigma\Gamma_1 - \Gamma_2 - \sigma\Gamma_2.$$

Thus, it follows from Lemma 1 in [16] (or from Proposition 2.2.3 in [5]) that $\mathcal{A}$ is an Azumaya algebra of $\mathcal{X}$.

The last contention follows immediately from the defining equations of $\mathcal{X}$. $\square$

Proof of Theorem 1.1. One can verify that $\mathcal{X}$ is smooth if $p \neq 0$. Hence, $\mathcal{X}$ is del Pezzo surface of degree 4. Now we show that $\mathcal{X}$ is everywhere locally solvable.

One can check that the points

$$\begin{aligned} Q_1 &:= (u : v : x : y : z) = (-p : 0 : 0 : 0 : \sqrt{-p}), \\ Q_2 &:= (-(8k+1)\sqrt{p} : \sqrt{p} : (8k+2)\sqrt{p} : \sqrt{p} : 8k+2), \\ Q_3 &:= (1 : 1 : 1 : 0 : \sqrt{-1}), \end{aligned}$$

lie on $\mathcal{X}$. For an odd prime $l \neq p$, at least one of p, $-p$ and -1 is a square in $\mathbb{Q}_l^\times$. Hence, in any event, at least one of the points Q_1, Q_2, Q_3 lies on $\mathcal{X}(\mathbb{Q}_l)$ for odd prime $l \neq p$. Further, since

$$p = 64k^2 + 40k + 5 \equiv 1 \pmod 4,$$

-1 is a square in $\mathbb{Q}_p$. Therefore, the point Q_3 also belongs to $\mathcal{X}(\mathbb{Q}_p)$.

Suppose that $l = \infty$. Then, p is a square in $\mathbb{R} = \mathbb{Q}_\infty$. Thus, the point Q_2 lies on $\mathcal{X}$.

Suppose that $l = 2$. Then, since

$$p(-p + 8k + 2) = -4096k^4 - 4608k^3 - 1792k^2 - 280k - 15 \equiv 1 \pmod 8,$$

the point $Q_4 := (-p^2 : p : 0 : p : (8k+2)\sqrt{p(-p+8k+2)})$ belongs to $\mathcal{X}(\mathbb{Q}_2)$. Therefore, $\mathcal{X}$ is locally solvable at 2.

Hence, in any event, $\mathcal{X}$ is everywhere locally solvable.

Let $\mathbb{Q}(\mathcal{X})$ be the function field of $\mathcal{X}$ and let $\mathcal{A}$ be the class of the quaternion algebra $\left(p, \dfrac{u}{u + (8k+1)v}\right)$. Then, by Lemma 2.2, we know that $\mathcal{A}$ is an Azumaya algebra of $\mathcal{X}$.

For each prime l (including $l = \infty$), let $P_l := (u : v : x : y : z)$ be a point in $\mathcal{X}(\mathbb{Q}_l)$ such that it is represented by integral coordinates with at least one unit among them. Let $\mathcal{A}(P_l) \in \mathrm{Br}(\mathbb{Q}_l)$ be the evaluation of $\mathcal{A}$ at P_l and let $\mathrm{inv}_l : \mathrm{Br}(\mathbb{Q}_l) \longrightarrow \mathbb{Q}/\mathbb{Z}$ be the invariant map from class field theory as introduced in the Introduction. We shall prove that for any $P_l \in \mathcal{X}(\mathbb{Q}_l)$,

$$\mathrm{inv}_l(\mathcal{A}(P_l)) = \begin{cases} 0 & \text{if } l \neq p \\ 1/2 & \text{if } l = p \end{cases}.$$

Suppose that $l = \infty$. Then, p is positive and hence, a square in $\mathbb{R} = \mathbb{Q}_\infty$. Hence, the Hilbert symbol $\left(p, \dfrac{u}{u + (8k+1)v}\right)_\infty = 1$ at points P_∞ for which u and $u + (8k+1)v$ are non-zero. Since the map $P \mapsto \mathrm{inv}_\infty(\mathcal{A}(P))$ is continuous on $\mathcal{X}(\mathbb{R})$, it implies that $\mathrm{inv}_\infty(\mathcal{A}(P_\infty)) = 0$ for any point $P_\infty \in \mathcal{X}(\mathbb{R})$.

Suppose that l is an odd prime such that p is a square in $\mathbb{Q}_l^\times$ and $l \neq p$. Then, repeating in the same manner as in the case when $l = \infty$, we deduce that $\mathrm{inv}_l(\mathcal{A}(P_l)) = 0$.

Suppose that l is an odd prime such that p is not a square in $\mathbb{Q}_l^\times$ and $l \neq p$. Then, at least one of u, v is non-zero modulo l; otherwise, it follows from (1.1) and (1.2) that $x^2 - py^2 \equiv 0 \pmod{l}$ and $x^2 - pz^2 \equiv 0 \pmod{l}$. We see that x, y are non-zero modulo l; otherwise $u = v = x = y = z = 0$ modulo l, contradiction. Hence, $p \equiv (x/y)^2 \pmod{l}$ and thus, p is a square in $\mathbb{Q}_l$, contradiction. Similarly, at least one of $(u + (8k+1)v)$ and $(u + (8k+2)v)$ is non-zero modulo l. Hence, at least one of the numbers $\dfrac{u}{u + (8k+1)v}$, $\dfrac{u}{u + (8k+2)v}$, $\dfrac{v}{u + (8k+1)v}$ and $\dfrac{v}{u + (8k+2)v}$ is an l-adic unit in $\mathbb{Z}_l^\times$, say U. Hence, the Hilbert symbol $(p, U)_l = 1$. Therefore, $\mathrm{inv}_l(\mathcal{A}(P_l)) = 0$.

Suppose that $l = 2$. We shall prove that at least one of u, v is odd. Assume the contrary, that is, $u = 2u_1$ and $v = 2v_1$ for some u_1, $v_1 \in \mathbb{Z}_2$. Then, since $p \equiv 5 \pmod{8}$, we deduce that $x^2 - y^2 \equiv 0 \pmod{4}$. Note that x, y and z must be odd; otherwise, for example, assume that x is even. Then, it follows from equations (1.1) and (1.2) that y and z must be even as well, contradiction.

Now, since x, y are odd, it implies that $x^2 - y^2 \equiv 0 \pmod{8}$. Hence, modulo 8 equation (1.1), we deduce that $4u_1v_1 \equiv x^2 - 5y^2 \pmod{8}$. Hence, $4(u_1v_1 + y^2) \equiv 0 \pmod{8}$. Thus, $(u_1v_1 + y^2)$ is even. Hence, u_1 and v_1 are odd.

Similarly, we also have that $x^2 - z^2 \equiv 0 \pmod{8}$. Thus, modulo 8 equation (1.2), it follows that $4(u_1 + (8k+1)v_1)(u_1 + (8k+2)v_1) \equiv x^2 - 5z^2 \pmod{8}$. Hence, it implies that $4[(u_1 + (8k+1)v_1)(u_1 + (8k+2)v_1) + z^2] \equiv 0 \pmod{8}$. Since z is odd, $(u_1 + (8k+1)v_1)(u_1 + (8k+2)v_1)$ is odd, a contradiction since u_1 and v_1 are odd. Thus, at least one of u and v is odd and hence it implies that at least one of $(u + (8k+1)v)$ and $(u + (8k+2)v)$ is odd. So, at least one of the numbers $\dfrac{u}{u + (8k+1)v}$, $\dfrac{u}{u + (8k+2)v}$, $\dfrac{v}{u + (8k+1)v}$ and $\dfrac{v}{u + (8k+2)v}$ is a 2-adic unit in $\mathbb{Z}_2^\times$, say U. Hence, since $p \equiv 1 \pmod{4}$, the Hilbert symbol $(p, U)_2 = 1$. Therefore, $\mathrm{inv}_2(\mathcal{A}(P_2)) = 0$.

Suppose that $l = p$. Then, reducing the defining equations of $\mathcal{X}$ modulo p, we deduce that

$$\begin{cases} uv \equiv x^2 \pmod{p} \\ (u + (8k+1)v)(u + (8k+2)v) \equiv x^2 \pmod{p} \end{cases}.$$

It follows from the last two congruences that $u^2 + (16k+3)uv + (8k+1)(8k+2)v^2 \equiv uv \pmod{p}$. Hence, we deduce that

$$(u + (8k+1)v)^2 \equiv (-8k-1)v^2 \pmod{p}.$$

We know that $v \not\equiv 0 \pmod p$; otherwise, $u \equiv 0 \pmod p$ and $x \equiv 0 \pmod p$. Reducing equations (1.1) and (1.2) modulo p^2, we deduce that $py^2 \equiv 0 \pmod{p^2}$ and $pz^2 \equiv 0 \pmod{p^2}$; so, $y \equiv 0 \pmod p$ and $z \equiv 0 \pmod p$, contradiction. Hence, it implies that $\left(\dfrac{v}{u+(8k+1)v}\right)^2 \equiv \dfrac{1}{-8k-1} \pmod p$. Since $p = 64k^2 + 40k + 5$, it follows that $p = 4(4k+1)^2 + 8k + 1$ and hence,

$$\frac{1}{-8k-1} = \frac{1}{4(4k+1)^2} \pmod p.$$

Hence, $\dfrac{v}{u+(8k+1)v} \equiv \pm\dfrac{1}{2(4k+1)} \equiv \pm(8k+3) \pmod p$. By Lemma 2.1, we deduce that the local Hilbert symbol

$$\left(p, \frac{v}{u+(8k+1)v}\right)_p = \left(\frac{\pm(8k+3)}{p}\right) = \left(\frac{\pm 1}{p}\right)\left(\frac{8k+3}{p}\right) = -1.$$

Hence, $\mathrm{inv}_p(\mathcal{A}(P_p)) = 1/2$.

Therefore, $\sum_l \mathrm{inv}_l(\mathcal{A}(P_l)) = 1/2$ for any $(P_l)_l \in \mathcal{X}(\mathbb{A}_{\mathbb{Q}})$ and hence, $\mathcal{X}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \emptyset$, proving our contention. □

2.1. Examples of del Pezzo surfaces violating the Hasse principle. We recall the following conjecture.

Conjecture 2.3. *(see [2]) (Bouniakowsky's conjecture) Let $P(x) \in \mathbb{Z}[x]$ be an irreducible polynomial and let $N = \gcd\{P(n) : n \in \mathbb{Z}_{>0}\}$. Then, there are infinitely many positive integers n such that $\dfrac{|P(n)|}{N}$ is a prime.*

We see from the Bouniakowsky conjecture that there should be infinitely many primes p such that

$$p = 64k^2 + 40k + 5,$$

for a positive integer k. Hence, there should be infinitely many del Pezzo surfaces of degree 4 defined as in Theorem 1.1 violating the Hasse principle explained by the Brauer-Manin obstruction.

In Table 2.1 below, we give a list of the first few values of p in Theorem 1.1.

Example 2.4. Let $(k, p) = (1, 109)$ and let $\mathcal{X}_{109}$ be the del Pezzo surface defined by

$$\mathcal{X}_{109} : \begin{cases} uv = x^2 - 109y^2 \\ (u+9v)(u+10v) = x^2 - 109z^2 \end{cases}.$$

Then, by Theorem 1.1, $\mathcal{X}_{109}$ is a counterexample to the Hasse principle explained by the Brauer-Manin obstruction.

k	0	1	3	6	8	14	16
p	5	109	701	2549	4421	13109	17029
k	17	19	23	27	32	36	37
p	19181	23869	34781	47741	66821	84389	89101
k	38	39	41	44	47	48	52
p	93941	98909	109229	125669	143261	149381	175141
k	56	59	61	63	66	72	74
p	202949	225149	240589	256541	281429	334661	353429
k	76	77	88	93	94	98	99
p	372709	382541	499141	557261	569269	618581	631229
k	111	113	116	118	124	129	131
p	792989	821741	865829	895861	989029	1070189	1103549

TABLE 2.1. Degree 4 del Pezzo surfaces violating the Hasse principle.

3. The Hasse principle for $K3$ surfaces

In this section, we shall prove Theorem 1.2 and hence, applying Theorem 1.2, we shall construct algebraic families of $K3$ surfaces violating the Hasse principle explained by the Brauer-Manin obstruction.

We state the following well-known lemma that we shall need in the proof of Theorem 1.2.

Lemma 3.1. *(See* [6, Lemma 4.8]*) Let k be a number field and let $\mathcal{V}_1$ and $\mathcal{V}_2$ be (proper) k-varieties. Assume that there is a k-morphism $\alpha : \mathcal{V}_1 \to \mathcal{V}_2$ and that $\mathcal{V}_2(\mathbb{A}_k)^{Br} = \emptyset$. Then,*

$$\mathcal{V}_1(\mathbb{A}_k)^{Br} = \emptyset.$$

The next lemma shows that given a point $(\Gamma, \Lambda, \Sigma) \in \mathcal{Q}(\mathbb{Q})$ with $\Gamma \neq 0$, there are infinitely many polynomials $\Omega(\mathbf{T}) \in \mathbb{Q}(\mathbf{T})$ such that the quadruple $(\Gamma, \Lambda, \Sigma, \Omega(\mathbf{T}))$ satisfies $A1$ and $A2$ for any $\mathbf{T} \in \mathbb{Q}$. Hence, by Theorem 1.2, this implies that there are algebraic families of $K3$ surfaces violating the Hasse principle explained by the Brauer-Manin obstruction.

Lemma 3.2. *Let p be a prime such that $p = 64k^2 + 40k + 5$ for $k \in \mathbf{Z}_{\geq 0}$. Let $(\Gamma, \Lambda, \Sigma) \in \mathbb{Q}^3$ be a point on the conic $\mathbf{Q}$ defined as in Theorem 1.2 with $\Gamma \neq 0$. Then, there are infinitely many polynomials $\Omega(\mathbf{T}) \in \mathbb{Q}[\mathbf{T}]$ such that the quadruple $(\Gamma, \Lambda, \Sigma, \Omega(\mathbf{T}))$ satisfies $A1$ and $A2$ in Theorem 1.2 for any $\mathbf{T} \in \mathbb{Q}$.*

Proof. Assume that Ω is a rational number such that one of C_i is zero for $1 \leq i \leq 5$. Then, we see that Ω is a rational root of one of the following polynomials

$$
\begin{aligned}
F_1(\mathbf{T}) &= \mathbf{T}^4 - 2((p^2+1)\Gamma^2 + \Lambda^2)\mathbf{T}^2 + (p^2+1)^2\Gamma^4 + 2(1-p^2)\Gamma^2\Lambda^2 + \Lambda^4,\\
F_2(\mathbf{T}) &= \mathbf{T}^4 + (128k^2 + 16k - 2)\Gamma^2\mathbf{T}^2\\
&\quad + (4096k^4 + 3072k^3 + 704k^2 + 80k + 5)\Gamma^4,\\
F_3(\mathbf{T}) &= (8k+2)^2\mathbf{T}^2\\
&\quad + (-32768k^5 - 40960k^4 - 19456k^3 - 4480k^2 - 512k - 24)\Gamma^2\\
&\quad + (8k+1)\Lambda^2,\\
F_4(\mathbf{T}) &= (p - 8k - 2)\mathbf{T}^2\\
&\quad + (-32768k^5 - 45056k^4 - 23552k^3 - 5888k^2 - 712k - 33)\Gamma^2\\
&\quad + (8k+2)\Lambda^2,
\end{aligned}
$$

and

$$
\begin{aligned}
F_5(\mathbf{T}) &= -(8k+2)^2\mathbf{T}^4\\
&\quad + \Big((-64k^2p + 16kp^2 - 24kp - 16k - p^3 + 3p^2 - 2)\Gamma^2\\
&\qquad\quad + (-16k + p - 2)\Lambda^2\Big)\,\mathbf{T}^2\\
&\quad + (-64k^2p^3 + 64k^2p + 8kp^4 - 24kp^3 - 16kp^2 + 24kp + 8k\\
&\qquad\quad + p^4 - p^3 - 3p^2 + p + 1)\Gamma^4\\
&\quad + \Big(64k^2p - 16kp^2 + 24kp + 16k - 2p^2 + p + 2\Big)\,\Gamma^2\Lambda^2\\
&\quad + (8k+1)\,\Lambda^4.
\end{aligned}
$$

Similarly, if Ω is a rational number such that $\lambda\mu\nu = 0$, then Ω is a rational root of the degree 4 polynomial defined by

$$H(\mathbf{T}) = \Gamma^2(\mathbf{T}^2 - \Gamma^2)(\mathbf{T}^2 - \Lambda^2 + (p^2-1)\Gamma^2).$$

We define

$$\mathbf{A} := \{z \in \mathbb{Q} : G(z) = 0 \text{ or } H(z) = 0\},$$

where

$$G(\mathbf{T}) = \prod_{i=1}^{5} F_i(\mathbf{T}).$$

Then, $\mathbf{A}$ is nonempty since $H(\mathbf{T})$ has at least two rational roots $\mathbf{T} = \pm\Gamma$. Further, since we see that $\deg(G(\mathbf{T})) = 16$ and $\deg(H(\mathbf{T})) = 4$, it follows that the cardinality of $\mathbf{A}$ is finite.

We define

$$m_0 := \max\{|z| : z \in \mathbf{A}\}.$$

Then, one can check that the polynomial $\Omega(\mathbf{T}) \in \mathbb{Q}[\mathbf{T}]$ defined by

$$\Omega(\mathbf{T}) = \mathbf{T}^{2n} + m,$$

for $m \geq m_0 + 1$ and $n \geq 1$, does not take any values in $\mathbf{A}$ for any $\mathbf{T} \in \mathbb{Q}$. Thus, the quadruple $(\Gamma, \Lambda, \Sigma, \Omega(\mathbf{T}))$ satisfies $A1$ and $A2$ for any $\mathbf{T} \in \mathbb{Q}$. $\square$

Before proceeding to prove Theorem 1.2, we shall prove that the surface $\mathcal{K}$ defined in Theorem 1.2 is a $K3$ surface.

Lemma 3.3. *Suppose the same notations and assumptions as in Theorem 1.2. Then, $\mathcal{K}$ is smooth, that is, $\mathcal{K}$ is a $K3$ surface.*

Remark 3.4. The proof below shows that $\mathcal{K}$ is smooth if and only if $C_1C_2C_3C_4C_5 \neq 0$ where C_i, $1 \leq i \leq 5$ were defined in Theorem 1.2.

Proof. One can prove the smoothness of $\mathcal{K}$ using the Jacobian criterion. This approach is elementary but tedious. We present below a more elegant proof using the geometric properties of the situation which was kindly provided by the referee.

We know that the surface $\mathcal{K}$ is a double cover of the del Pezzo surface $\mathcal{X}$ defined in Theorem 1.1, ramified along the curve $\mathcal{C} \subset \mathcal{X}$ cut out by $\lambda x^2 + \mu y^2 + \nu z^2 = 0$. It is known that $\mathcal{X}$ is smooth; hence, to prove that $\mathcal{K}$ is smooth, it suffices to show that $\mathcal{C}$ is smooth. We see from the defining equations of $\mathcal{X}$ that $\mathcal{C}$ is a double cover of the curve $\mathcal{D} \subset \mathbb{P}^3$ defined by

$$\begin{cases} u^2 & = xy - \dfrac{p(\lambda x^2 + \mu y^2)}{\nu} \\ u^2 - pv^2 & = (x + (8k+1)y)(x + (8k+2)y) \end{cases}$$

with ramification locus $\mathcal{L} \subset \mathcal{D}$ defined by $\lambda x^2 + \mu y^2 = 0$. Recall that an intersection of two quadrics $\mathcal{Q}_1(\underline{x}) = \mathcal{Q}_2(\underline{x}) = 0$ is smooth if and only if the homogeneous polynomial $\det(s\mathcal{Q}_2(\underline{x}) + t\mathcal{Q}_2(\underline{x})) \in \mathbb{Q}[s,t]$ has no multiple root (see [14]). Using this fact, one can check that $\mathcal{D}$ is smooth if and only if $C_1C_3C_4C_5 \neq 0$ and that $\mathcal{L}$ is smooth as soon as $C_2 \neq 0$. Hence, $\mathcal{K}$ is smooth as soon as all C_i's are nonzero. $\square$

3.1. Proof of Theorem 1.2.

Proof. We prove that $\mathcal{K}$ is everywhere locally solvable. We consider the following cases.

Case I. l is an odd prime such that -1 is a square in $\mathbb{Q}_l^\times$. In particular, p is among these primes.

Let $Q_1 := (x : y : z : u : v : w) = (1 : 1 : 0 : 1 : \sqrt{-1} : \Omega)$. Since $\lambda + \mu = \Omega^2$ and $p = 64k^2 + 40k + 5$, one can check that Q_1 lies on $\mathcal{K}$. Since, -1 is a square in $\mathbb{Q}_l^\times$, it follows that $Q_1 \in \mathcal{K}(\mathbb{Q}_l)$.

Case II. l is an odd prime such that p is a square in $\mathbb{Q}_l^\times$.

Let $Q_2 := (-(8k+1)\sqrt{p} : \sqrt{p} : \sqrt{p} : (8k+2)\sqrt{p} : (8k+2) : \sqrt{p}\Sigma)$. Since $(\Gamma, \Lambda, \Sigma)$ lies on the conic $\mathcal{Q}$, it follows from (1.3) that Q_2 lies on $\mathcal{K}$. Furthermore, since $\sqrt{p} \in \mathbb{Q}_l^\times$, it follows that $Q_2 \in \mathcal{K}(\mathbb{Q}_l)$.

Case III. l is an odd prime such that $-p$ is a square in $\mathbb{Q}_l^\times$.

Let $Q_3 := (-p : 0 : 0 : 0 : \sqrt{-p} : p\Gamma)$. Then, since $\lambda = \Gamma^2$, one sees that Q_3 lies on $\mathcal{K}$. Since $\sqrt{-p} \in \mathbb{Q}_l^\times$, it follows that $Q_3 \in \mathcal{K}(\mathbb{Q}_l)$.

Case IV. $l = 2$.

Let $Q_4 := (-p^2 : p : p : 0 : (8k+2)\sqrt{p(8k+2-p)} : p\Lambda)$. Since

$$p = (8k+2)^2 + (8k+1),$$

and

$$\nu = \Lambda^2 - \Omega^2 - (p^2-1)\Gamma^2,$$

one can check that Q_4 lies on $\mathcal{K}$. Furthermore, we know that

$$p(8k+2-p) = -4096k^4 - 4608k^3 - 1792k^2 - 280k - 15 \equiv 1 \pmod{8}.$$

Hence, $p(8k+2-p)$ is a square in $\mathbb{Q}_2^\times$. Thus, $Q_4 \in \mathcal{K}(\mathbb{Q}_2)$.

Therefore, in any event, $\mathcal{K}$ is everywhere locally solvable.

Now we show that $\mathcal{K}(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} = \emptyset$. Indeed, on letting

$$x = U,\ y = V,\ z = Y,\ u = X,\ v = Z,\ w = W,$$

one sees that $\mathcal{K}$ lies on the del Pezzo surface $\mathcal{X}$ in Theorem 1.1. Hence, there exists a morphism

$$\phi : \mathcal{K} \longrightarrow \mathcal{X}.$$

Thus, since $\mathcal{X}(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} = \emptyset$, it follows from Lemma 3.1 that $\mathcal{K}(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} = \emptyset$, proving our contention. $\square$

3.2. Algebraic families of $K3$ surfaces violating the Hasse principle. In this subsection, we shall apply Theorem 1.2 to explicitly construct algebraic families of $K3$ surfaces violating the Hasse principle explained by the Brauer-Manin obstruction.

Corollary 3.5. *Let p be a prime such that $p = 64k^2 + 40k + 5$ for some integer $k \in \mathbb{Z}_{\geq 0}$. Let m, n be integers such that $m \geq 2$ and $n \geq 1$. Let $\mathcal{K}_p \subset \mathbb{P}^5_\mathbb{Q}$ be the $K3$ surface defined by*

$$\mathcal{K}_p : \begin{cases} u^2 & = xy + pz^2 \\ u^2 - pv^2 & = (x + (8k+1)y)(x + (8k+2)y) \\ w^2 & = x^2 + (\mathbf{T}^{4n} + 2m\mathbf{T}^{2n} + m^2 - 1)y^2 \\ & \quad -(\mathbf{T}^{4n} + 2m\mathbf{T}^{2n} + m^2 - 1)z^2 \end{cases} \tag{3.1}$$

for $\mathbf{T} \in \mathbb{Q}$*. Then,* $\mathcal{K}_p$ *violates the Hasse principle explained by the Brauer-Manin obstruction.*

Proof. Throughout the proof, we shall use the same notations as in Lemma 3.2.

The conic $\mathcal{Q}$ in Theorem 1.2 defined by

$$\mathcal{Q} : (p^2 - (8k+1)^2)X^2 - Y^2 + Z^2 = 0,$$

has a point $(\Gamma, \Lambda, \Sigma) = (1, p, 8k+1)$ with $\Gamma = 1 \neq 0$. Then, the polynomials $F_i(\mathbf{T})$ in the proof of Lemma 3.2 are of the form.

$$F_1(\mathbf{T}) = (\mathbf{T}^2 - 1)(\mathbf{T}^2 - (16384k^4 + 20480k^3 + 8960k^2 + 1600k + 101)),$$

$$F_2(\mathbf{T}) = \mathbf{T}^4 + (128k^2 + 16k - 2)\mathbf{T}^2 + 4096k^4 + 3072k^3 + 704k^2 + 80k + 5,$$

$$F_3(\mathbf{T}) = (64k^2 + 32k + 4)\mathbf{T}^2 + 4096k^4 + 3584k^3 + 960k^2 + 88k + 1,$$

$$F_4(\mathbf{T}) = (64k^2 + 32k + 3)\mathbf{T}^2 + 4096k^4 + 4608k^3 + 1792k^2 + 288k + 17,$$

$$F_5(\mathbf{T}) = -(\mathbf{T}^2 - 1)(4(4k+1)^2\mathbf{T}^2 - (1024k^3 + 896k^2 + 232k + 19)).$$

Since $k \geq 0$, one sees that $F_3(\mathbf{T}) \geq 1$ and $F_4(\mathbf{T}) \geq 17$ for all $\mathbf{T} \in \mathbb{Q}$. Hence, F_3 and F_4 do not have any rational roots. We know that the discriminant of F_2 is

$$\Delta_{F_2} = -8192k^3 - 3072k^2 - 384k - 16 \leq -16 < 0.$$

Hence, it follows that F_2 does not have any rational roots. We contend that F_1 has exactly two rational roots ± 1. Indeed, we see that

$$A = 16384k^4 + 20480k^3 + 8960k^2 + 1600k + 101 \equiv 5 \pmod 8;$$

hence, it implies that A is not a square in $\mathbb{Q}_2^\times$. In particular, this implies that A is not a perfect square in $\mathbb{Q}$, which shows that the set of rational roots of F_1 has exactly two elements ± 1. Similarly, we can see that

$$B = 1024k^3 + 896k^2 + 232k + 19 \equiv 3 \pmod 8,$$

which implies that B is not a square in $\mathbb{Q}_2^\times$. Hence, B is not a square in $\mathbb{Q}$. Thus, F_5 has exactly two rational roots ± 1.

Therefore, the polynomial $G(\mathbf{T}) = \prod_{i=1}^5 F_i(\mathbf{T})$ has exactly two rational roots ± 1.

Now, the polynomial $H(\mathbf{T})$ in the proof of Lemma 3.2 is of the form

$$H(\mathbf{T}) = (\mathbf{T}^2 - 1)^2.$$

Hence, $H(\mathbf{T})$ has exactly two rational roots ± 1. Therefore, we deduce that

$$\mathbf{A} := \{z \in \mathbb{Q} : G(z) = 0 \text{ or } H(z) = 0\} = \{\pm 1\}.$$

For each $m \geq 2$ and $n \geq 1$, we define the polynomial $\Omega(\mathbf{T}) \in \mathbb{Q}[\mathbf{T}]$ as follows.

$$\Omega(\mathbf{T}) = \mathbf{T}^{2n} + m.$$

Then, one can easily check that $\Omega(\mathbf{T})$ does not take values ± 1 for any $\mathbf{T} \in \mathbb{Q}$.

We define

$$\begin{aligned}
\lambda &= \Gamma^2 = 1, \\
\mu &= \Omega(\mathbf{T})^2 - \Gamma^2 = \mathbf{T}^{4n} + 2m\mathbf{T}^{2n} + m^2 - 1, \\
\nu &= \Lambda^2 - \Omega(\mathbf{T})^2 - (p^2-1)\Gamma^2 = -(\mathbf{T}^{4n} + 2m\mathbf{T}^{2n} + m^2 - 1).
\end{aligned}$$

Then, since $\Omega(\mathbf{T})$ does not take values ± 1 for any $\mathbf{T} \in \mathbb{Q}$, repeating in the same arguments as in the proof of Lemma 3.2, we deduce that the quadruple $(\lambda, \mu, \nu, \Omega(\mathbf{T}))$ satisfies $A1$ and $A2$ for any $\mathbf{T} \in \mathbb{Q}$. Hence, by Theorem 1.2, $\mathcal{K}_p$ violates the Hasse principle explained by the Brauer-Manin obstruction. □

In Table 3.1 below, we tabulate algebraic families $\mathcal{K}_p$ of $K3$ surfaces violating the Hasse principle explained by the Brauer-Manin obstruction for the first few values of p in Corollary 3.5 with $m = 2$ and $n = 1$.

k	p	The defining equations of $\mathcal{K}_p$
0	5	$\mathcal{K}_5 : \begin{cases} u^2 = xy + 5z^2 \\ u^2 - 5v^2 = (x+y)(x+2y) \\ w^2 = x^2 + (\mathbf{T}^4 + 4\mathbf{T}^2 + 3)y^2 - (\mathbf{T}^4 + 4\mathbf{T}^2 + 3)z^2 \end{cases}$
1	109	$\mathcal{K}_{109} : \begin{cases} u^2 = xy + 109z^2 \\ u^2 - 109v^2 = (x+9y)(x+10y) \\ w^2 = x^2 + (\mathbf{T}^4 + 4\mathbf{T}^2 + 3)y^2 - (\mathbf{T}^4 + 4\mathbf{T}^2 + 3)z^2 \end{cases}$
3	701	$\mathcal{K}_{701} : \begin{cases} u^2 = xy + 701z^2 \\ u^2 - 701v^2 = (x+25y)(x+26y) \\ w^2 = x^2 + (\mathbf{T}^4 + 4\mathbf{T}^2 + 3)y^2 - (\mathbf{T}^4 + 4\mathbf{T}^2 + 3)z^2 \end{cases}$

TABLE 3.1. Algebraic families of $K3$ surfaces $\mathcal{K}_p$ violating the Hasse principle.

Remark 3.6. The following remark is suggested by the referee.

It is possible that there should be infinitely many algebraic families of $K3$ surfaces violating the Hasse principle arising from Corollary 3.5. To prove this, one needs to verify whether these families are nonisotrivial or if they are isotrivial, can one show that at least they are nonconstant. The author did not pursue this question since Corollary 3.5 is sufficient for our purposes.

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