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On the number of representations in the Waring-Goldbach problem with a prime variable in an arithmetic progression

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Résumé. Nous démontrons un théorème de type Bombieri-Vinogradov sur le nombre de représentations d'un entier N sous la forme $N = p_1^g + p_2^g + \cdots + p_s^g$ avec $p_1, p_2, \cdots, p_s$ des nombres premiers et $p_1 \equiv l \pmod{k}$, sous une hypothèse convenable $s = s(g)$ pour chaque entier $g \geq 2$.

Abstract. We prove a Bombieri-Vinogradov type theorem for the number of representations of an integer N in the form $N = p_1^g + p_2^g + \ldots + p_s^g$ with $p_1, p_2, \ldots, p_s$ prime numbers such that $p_1 \equiv l \pmod{k}$, under suitable hypothesis on $s = s(g)$ for every integer $g \geq 2$.

1. Introduction

The problem of representing an integer N as the sum of gth powers of primes $p_1, \ldots, p_s$ with the smallest possible number $s = s(g)$ of variables for any integer $g \geq 1$, i.e.

$$N = p_1^g + p_2^g + \ldots + p_s^g, \tag{1.1}$$

is known as the Waring-Goldbach problem. It is a hybrid of the famous Goldbach conjecture (the case $g = 1$) and the Waring problem, which concerns how gth powers of integers, whether prime or not, may generate additively all integers with the least number of summands. An integer N is *admissible* for (1.1), if it satisfies some sort of congruence condition, which is certainly necessary. Indeed, for example every odd prime p satisfies $p^2 \equiv 1 \pmod{8}$, which implies that any $N \not\equiv s \pmod{8}$ cannot be the sum of s squares of odd primes (for the general case see the statement of BVTWG below). In [11], Ch. 8, we find the definition of $H(g)$, the least integer s such that every sufficiently large admissible N can be represented in the form (1.1). The early investigations of Vinogradov [21],[22] and Hua [10] have provided the basic specimens for the testing and development of the Hardy-Littlewood method which yielded the following upper bound:

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$$H(g) \leq \begin{cases} 2^g + 1 & \text{if } 1 \leq g \leq 10, \\ 2g^2(2\log g + \log\log g + 5/2) & \text{if } g > 10. \end{cases}$$

Subsequently, several authors have studied the equation (1.1) under some further restrictions on the prime variables such as

$$p_i \equiv l_i \ (\mathrm{mod}\ k_i),\ (k_i, l_i) = 1,\ i \in \{1, \ldots, s\}, \tag{1.2}$$

where $(k, l) = 1$ means that k and l are relatively prime (here and in what follows, $(k_1, \ldots, k_t)$ denotes the greatest common divisor of the integers $k_1, \ldots, k_t$). Particular attention has focused on the weighted number of solutions of (1.1) under the restriction (1.2) given by

$$I(N, M, g, s, \mathbf{k}, \mathbf{l}) := \sum_{\substack{p_1^g + \ldots + p_s^g = N \\ p_i \equiv l_i \ (\mathrm{mod}\ k_i),\ p_i \leq M \\ i=1,\ldots,s}} \prod_{i=1}^{s} \log p_i,$$

where the sum is over the s-dimensional vectors $\langle p_1, \ldots, p_s \rangle$ satisfying the assigned conditions and $\mathbf{k} := \langle k_1, \ldots, k_s \rangle$, $\mathbf{l} := \langle l_1, \ldots, l_s \rangle$.

Let us consider the following problems associated to the equation (1.1):

$(1.2)^*$, that is (1.1) under (1.2), where $k_i = k$, $\forall i \in \{1, \ldots, s\}$;

$(1.2)^{**}$, that is (1.1) under (1.2), where $k_1 = k$ and $k_i = 1$, $\forall i \in \{2, \ldots, s\}$;

(symbols followed by * and ** will refer to $(1.2)^*$ and $(1.2)^{**}$, respectively).

As one could expect, the main efforts are devoted to solve such problems in the most famous and prototypal case $g = 1$. The early results to be mentioned are those of Zulauf [26], [27] and Ayoub [1], who proved independently Vinogradov's three primes theorem (i.e. $H(1) \leq 3$) under $(1.2)^*$ for every sufficiently large $N \equiv l_1 + l_2 + l_3 \ (\mathrm{mod}\ k)$ with $(k, l_i) = 1$, $(i = 1, 2, 3)$, and uniformly for all $k \leq L^D$, where $L := \log N$ and $D > 0$ is a constant. In particular, Zulauf's result yields an asymptotic formula for every sufficiently large admissible N,

$$I(N, N, 1, 3, \mathbf{k}, \mathbf{l})^* = I(N, N, 1, 3, k, \langle l_1, l_2, l_3 \rangle)^* = \mathcal{M}T + o(N^2 L^{-A}),$$

which holds with the expected main term $\mathcal{M}T$ and for an arbitrary constant $A > 0$ uniformly for all $k \leq L^D$. Such a rather severe range of uniformity for the moduli k's is essentially that of the prime number theorem for arithmetic progressions, namely the Siegel-Walfisz theorem, which plays a crucial role in the application of the Hardy-Littlewood method for additive problems involving prime numbers. However, a well-known partial extension of the Siegel-Walfisz formula is provided by the Bombieri-Vinogradov theorem ([18], Theorem 15.1). In this direction, some authors

([2],[5],[7],[8],[9],[14],[15],[16],[19],[25]) have established asymptotic formulae for $I(N,N,1,3,\mathbf{k},\mathbf{l}), I(N,N,1,3,k,\mathbf{l})^*$ or $I(N,N,1,3,k,l)^{**}$ going beyond Zulauf's bound for k, though most of these formulae hold uniformly for *almost all* moduli up to a certain power of N and sometimes at the cost of possibly a few exceptions of classes $\mathbf{l}$. Usually such results are obtained via a so-called *Bombieri-Vinogradov type theorem for the Waring-Goldbach problem*, which we state in a general form as:

BVTWG. *Let g and s positive integers. If $p^\theta | g$ and $p^{\theta+1} \nmid g$, we define*

$$\gamma = \gamma(g,p) := \begin{cases} \theta+2, & \text{if } p=2,\ 2|g, \\ \theta+1, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \eta = \eta(g) := \prod_{(p-1)|g} p^\gamma.$$

Assuming that N is a sufficiently large integer with $N \equiv s \pmod{\eta}$, for every constant $A>0$ there exists $B=B(A)>0$ such that

$$\sum_{\substack{\mathbf{k} \\ k_i \in [1,K_i]}} \max_{\mathbf{l}\in\mathcal{B}(N,\mathbf{k})} \max_{M\le N} \left| I(N,M,g,s,\mathbf{k},\mathbf{l}) - \mathcal{MT} \right| \ll N^{s/g-1}L^{-A},$$

where $K_i \le N^{1/(2g)}/L^B$ $(i=1,\dots,s)$, $\mathcal{MT}$ is the expected main term and

$$\mathcal{B}(N,\mathbf{k}) = \mathcal{B}(N,\mathbf{k},g,s) := \{\mathbf{l}\colon 1\le l_i \le k_i, (k_i,l_i)=1, (k_1,\dots,k_s) | N - \sum_{i=1}^{s} l_i^g\}.$$

As far as we know, a few results are available in the literature for the nonlinear case $g \ge 2$. Among them we recall [23], where it is proved the solvability of $(1.2)^*$ when $g=2$, $s=5$ for *all* the moduli $k \le N^\delta$ with an effective constant $\delta > 0$, though no asymptotic formula for $I(N,N,2,5,k,\mathbf{l})^*$ is provided.

In the present paper we consider the problem $(1.2)^{**}$ for any $g \ge 2$ and establish a BVTWG** for $I_k(N) := I(N,N,g,s,k,l)^{**}$ with $k \le N^{1/2g}L^{-B}$, under suitable hypothesis on $s = s(g)$. More precisely, let $W(N,g,2t)$ be the number of solutions $\langle x_1,\dots,x_{2t}\rangle$ with $1 \le x_i \le N^{1/g}$ of the Diophantine equation $x_1^g + x_2^g + \dots + x_t^g = x_{t+1}^g + \dots + x_{2t}^g$. We have the following result.

Theorem. *Let $g,t \ge 2$ be integers and $v = v(g,t) \ge 0$ a real number such that $W(N,g,2t) \ll N^{2t/g-1}L^v$ for every sufficiently large $N \equiv 2t+1 := s \pmod{\eta}$. For every constant $A>0$ there exists $B=B(A)>0$ such that*

$$\sum_{k \le N^{1/(2g)}/L^B} \max_{\substack{1\le l\le k \\ (l,k)=1}} \left| I_k(N) - \mathcal{M}(N)\mathfrak{S}_k(N)\varphi(k)^{-1} \right| \ll N^{s/g-1}L^{-A}, \tag{1.3}$$

where $\mathcal{M}(N) := g^{-s} \sum_{\substack{1\le m_1,\dots,m_s\le N \\ m_1+\dots+m_s=N}} (m_1\dots m_s)^{1/g-1}$, $\varphi(k) := \sum_{\substack{1\le l\le k \\ (l,k)=1}} 1$ and $\mathfrak{S}_k(N)$ is the singular series defined in (3.11).

We remark that $\mathcal{M}(N) \asymp N^{s/g-1}$ ([20], Theorem 2.3) and $\mathfrak{S}_k(N)$ is uniformly bounded in N (see (3.10), (3.11) below). The evaluation of $W(N, g, 2t)$ is a deep matter within the classical theory of the Waring problem. Hua's Lemma ([11], Theorem 4) yields the inequality $W(N, g, 2t) \ll N^{2t/g-1}L^v$ with a certain $v > 0$, whenever $2t \geq 2^g$ for every $g \geq 2$. However, nowadays one can take lower values of t and $v = 0$ when $g \geq 6$, namely for $6 \leq g \leq 8$ if $2t \geq 2^{g-3}7$ (see [4]) and for any $g \geq 9$ if $2t \geq g^2(\log g + \log\log g + \mathcal{O}(1))$ (see [6]). Besides, Wooley [24] has recently announced a further strong improvements on the estimate of $W(N, g, 2t)$ when $g \geq 7$. Hence, at the moment our Theorem has the following immediate consequence.

Corollary. *For every constant $A > 0$ there exists $B = B(A) > 0$ such that (1.3) holds for every sufficiently large integer $N \equiv s \pmod{\eta}$ with*

$$s \geq \begin{cases} 2^g + 1 & \textit{if } 2 \leq g \leq 5, \\ 2^{g-3}7 + 1 & \textit{if } 6 \leq g \leq 8, \\ g^2(\log g + \log\log g + \mathcal{O}(1)) & \textit{if } g \geq 9. \end{cases}$$

The proof of the Theorem is an application of the Hardy-Littlewood circle method, where we generalize the treatment of the major arcs terms, via the Bombieri-Vinogradov theorem, applied in [12]. In order to evaluate the minor arcs contribution we employ the strategy of Halupczok [7], because the method of [12] allows to establish only a weaker result where l is required to be a fixed integer (in an unpublished paper [13] we have considered (1.3) without the maximum).

2. Notation and outline of the proof

Among the definitions already given in the previous section we recall that (m, n) denotes the greatest common divisor of m and n. Since (x, y) is also the open interval with real endpoints x, y, the meaning will be evident from the context. On the other side, $[m, n]$ will denote the least common multiple of m and n. For simplicity we often write $m \equiv n\,(k)$ instead of $m \equiv n \pmod{k}$ and set $\sum_{a=1}^{q}{}^{*} := \sum_{\substack{a=1\\(a,q)=1}}^{q}$, $e(x) := \exp(2\pi i x)$. The number of the divisors of n is $d(n)$, μ denotes Möbius' function and $*$ is the usual convolution product of arithmetic functions. The letter p, with or without subscript, is devoted to prime numbers. We will appeal to the well-known inequalities $\varphi(n) \gg n(\log\log\, 10n)^{-1}$, $\sum_{n \leq x} 1/n \ll \log x$ and $\sum_{n \leq x} d(n) \ll x \log x$ without further references. Moreover, we will adopt the following convention concerning the positive real numbers ε and c. Whenever ε appears in a statement, either implicitly or explicitly, we assert that for each $\varepsilon > 0$, the statement holds for sufficiently large values of the main parameter. Notice that the "value" of ε may consequently change from statement to statement, and hence also the dependence of implicit constants on ε. For example, by adopting this convention for c as well, we write $(\log x)\, e^{-c\sqrt{\log x}} \ll e^{-c\sqrt{\log x}}$. The implicit constants in $\ll$ and $\mathcal{O}$ symbols might depend only on g, t, A.

Let V, Y, K be real numbers such that $V \geq 5(A + 2t + v + 2) + 2$, $Y \geq 2^{6g}(V + 1)$ and $0 < K \leq P^{1/2} L^{-B}$ with $P := N^{1/g}$ and $B := A + 4Y + 6$. Then, we set $Q := L^Y$, $\tau := NQ^{-1}$,

$$\mathcal{E} := \sum_{k \leq K} \max_{\substack{1 \leq l \leq k \\ (l,k)=1}} \left| I_k(N) - \mathcal{M}(N) \mathfrak{S}_k(N) \varphi(k)^{-1} \right|,$$

$$S_k(\alpha) := \sum_{\substack{p \leq P \\ p \equiv l\ (k)}} e(\alpha p^g) \log p, \; S(\alpha) := S_1(\alpha)\,, \; M(\alpha) := \frac{1}{g} \sum_{m=1}^{N} m^{1/g-1} e(\alpha m).$$

If one defines the union of the major arcs as

$E_1 := \bigcup\limits_{q \leq Q} \bigcup\limits_{\substack{a=0 \\ (a,q)=1}}^{q-1} \Big(\frac{a}{q} - \frac{1}{q\tau}\,, \frac{a}{q} + \frac{1}{q\tau} \Big)$ and the minor arcs as

$E_2 := \Big(-\frac{1}{\tau}\,, 1 - \frac{1}{\tau} \Big) \setminus E_1$, then one has

$$I_k(N) = \int_{-1/\tau}^{1-1/\tau} S_k(\alpha)\, S^{s-1}(\alpha)\, e(-N\alpha)\, d\alpha := I_k^{(1)}(N) \; + \; I_k^{(2)}(N),$$

where $I_k^{(i)}(N) := \displaystyle\int_{E_i} S_k(\alpha)\, S^{s-1}(\alpha)\, e(-N\alpha)\, d\alpha\,, \quad i = 1, 2.$

Since $\mathcal{E} \leq \mathcal{E}_1 + \mathcal{E}_2$ with

$$\mathcal{E}_1 = \mathcal{E}_1(g) := \sum_{k \leq K} \max_{\substack{1 \leq l \leq k \\ (l,k)=1}} \left| I_k^{(1)}(N) - \mathcal{M}(N) \mathfrak{S}_k(N) \varphi(k)^{-1} \right|,$$

$$\mathcal{E}_2 = \mathcal{E}_2(g) := \sum_{k \leq K} \max_{\substack{1 \leq l \leq k \\ (l,k)=1}} \left| I_k^{(2)}(N) \right|,$$

the Theorem will follow from the inequalities

$$\mathcal{E}_1 \ll P^{s-g} L^{-A}, \tag{2.1}$$

$$\mathcal{E}_2 \ll P^{s-g} L^{-A}. \tag{2.2}$$

3. Major arcs: the estimate of $\mathcal{E}_1$

In this section we prove (2.1) by finding an asymptotic formula with an error term which is small on average for

$$I_k^{(1)}(N) = \sum_{q \leq Q} \sideset{}{^*}\sum_{a=1}^{q} I_k(a, q), \tag{3.1}$$

where

$$I_k(a,q) := \int_{-1/q\tau}^{1/q\tau} S_k\Big(\frac{a}{q}+\alpha\Big)\, S\Big(\frac{a}{q}+\alpha\Big)^{s-1} e\Big(-N\Big(\frac{a}{q}+\alpha\Big)\Big)\, d\alpha, \tag{3.2}$$

with integers a, q and real numbers α satisfying

$$q \le Q, \qquad (a,q)=1, \qquad |\alpha| \le (q\tau)^{-1} = (qN)^{-1}Q = q^{-1}P^{-g}Q. \tag{3.3}$$

Thus, we write

$$S_k\Big(\frac{a}{q}+\alpha\Big) = \sum_{\substack{m=1 \\ m\equiv l \bmod (k,q)}}^{q}{}^{*} e\Big(\frac{am^g}{q}\Big) T(\alpha) \,+\, \mathcal{O}(\log q) \tag{3.4}$$

with $T(\alpha) := \sum\limits_{\substack{p\le P \\ p\equiv l\ (k) \\ p\equiv m\ (q)}} e(\alpha p^g)\log p = \sum\limits_{\substack{p\le P \\ p\equiv f\ ([k,q])}} e(\alpha p^g)\log p$, where the integer $f = f(l,k,m,q)$ is such that $(f,[k,q])=1$ and the congruence $x\equiv f \bmod [k,q]$ is equivalent to the system $x\equiv l\ (k)$, $x\equiv m\ (q)$. Indeed, from $m\equiv l \bmod (k,q)$ it follows that $m-l = t(k,q) = tw_1k + tw_2q$ for some integers t, w_1, w_2. This reveals that $f := m - tw_2q = l + tw_1k$ is the unique simultaneous solution mod $[k,q]$ of $x\equiv l\ (k)$ and $x\equiv m\ (q)$. Further, $(k,l)=(m,q)=1$ implies $(f,[k,q])=1$. Now let us denote

$$\Delta(z,h) := \max_{y\le z}\ \max_{\substack{l \\ (l,h)=1}} \Big| \sum_{\substack{p\le y \\ p\equiv l\ \ (h)}} \log p - \frac{y}{\varphi(h)} \Big| \tag{3.5}$$

and apply partial summation to get

$$\begin{aligned} T(\alpha) &= -\int_0^P \frac{d}{dy}e(\alpha y^g) \sum_{\substack{p\le y \\ p\equiv f\ ([k,q])}} \log p \ dy + e(\alpha N) \sum_{\substack{p\le P \\ p\equiv f\ ([k,q])}} \log p \\ &= -\int_0^P \Big(\frac{y}{\varphi[k,q]} + \mathcal{O}\Big(\Delta(P,[k,q])\Big)\Big)\frac{d}{dy}e(\alpha y^g)\, dy \\ &\qquad + \Big(\frac{P}{\varphi[k,q]} + \mathcal{O}\Big(\Delta(P,[k,q])\Big)\Big) e(\alpha N) \\ &= \frac{1}{\varphi[k,q]}\Big(P\, e(\alpha N) - \int_0^P y\,\Big(\frac{d}{dy}e(\alpha y^g)\Big)\, dy\Big) \\ &\qquad + \mathcal{O}\Big(\int_0^P \Delta(P,[k,q])\, |\alpha| y^{g-1} dy\Big) \ + \mathcal{O}(\Delta(P,[k,q])). \end{aligned}$$

Integration by parts and the well-known formula (see [20], Ch.2) $\int_0^P e(\alpha y^g)\, dy = M(\alpha) + \mathcal{O}(1+N|\alpha|)$ together with (3.3) lead to

$$T(\alpha) = \frac{M(\alpha)}{\varphi[k,q]} + \mathcal{O}\Big(\Big(1+\frac{N}{q\tau}\Big)\, \Delta(P,[k,q])\Big).$$

Now we substitute the latter in (3.4) and note that (3.3) implies

$$S_k\Big(\frac{a}{q}+\alpha\Big) = \frac{c_k(a,q)}{\varphi[k,q]}\,M(\alpha) + \mathcal{O}(Q\,\Delta(P,[k,q])), \tag{3.6}$$

where $c_k(a,q) := \sum^{q}_{\substack{m=1\\ m\equiv l \bmod (k,q)}}{}^{*}\, e\Big(\frac{am^g}{q}\Big)$. For $c(a,q) := c_1(a,q)$ one might follow the proof of Lemma 7.15 in [11] to obtain

$$S\Big(\frac{a}{q}+\alpha\Big) = \frac{c(a,q)}{\varphi(q)}\,M(\alpha) + \mathcal{O}\Big(Pe^{-c\sqrt{L}}\Big). \tag{3.7}$$

Formulae (3.3), (3.6), (3.7) and the trivial bound $S_k(\beta) \ll PLk^{-1}$ imply

$$\begin{aligned} S_k\Big(\frac{a}{q}+\alpha\Big)S\Big(\frac{a}{q}+\alpha\Big)^{s-1} e\Big(-N\Big(\frac{a}{q}+\alpha\Big)\Big) \\ = \frac{c(a,q)^{s-1}}{\varphi(q)^{s-1}}\,\frac{c_k(a,q)}{\varphi[k,q]}\,e\Big(-N\frac{a}{q}\Big)\,M(\alpha)^s\,e(-N\alpha) \\ + \mathcal{O}\Big(P^s k^{-1}\,e^{-c\sqrt{L}}\Big) + \mathcal{O}(P^{s-1}Q\,\Delta(P,[k,q])). \end{aligned}$$

Therefore, (3.2) becomes

$$\begin{aligned} I_k(a,q) = \frac{c(a,q)^{s-1}}{\varphi(q)^{s-1}}\,\frac{c_k(a,q)}{\varphi[k,q]}\,e\Big(-N\frac{a}{q}\Big)\int_{-1/q\tau}^{1/q\tau} M(\alpha)^s\,e(-N\alpha)\,d\alpha \\ + \mathcal{O}\Big(P^{s-g}k^{-1}\,e^{-c\sqrt{L}}\Big) + \mathcal{O}(q^{-1}P^{s-g-1}\,Q^2\,\Delta(P,[k,q])). \end{aligned}$$

Consequently, if we set $b_k(q) := \sum_{a=1}^{q}{}^{*}\, c_k(a,q)c(a,q)^{s-1}\,e(-Na/q)$, one has

$$\begin{aligned} \sum_{a=1}^{q}{}^{*} I_k(a,q) = \frac{b_k(q)}{\varphi[k,q]\varphi(q)^{s-1}}\int_{-1/q\tau}^{1/q\tau} M(\alpha)^s\,e(-N\alpha)\,d\alpha \\ + \mathcal{O}(P^{s-g}k^{-1}e^{-c\sqrt{L}}) + \mathcal{O}(P^{s-g-1}Q^2\Delta(P,[k,q])). \end{aligned} \tag{3.8}$$

Since it is well-known that (see [20], Ch.2)

$$\int_{-1/q\tau}^{1/q\tau} M(\alpha)^s\,e(-N\alpha)\,d\alpha = \mathcal{M}(N) + \mathcal{O}((q\tau)^{s/g-1}),$$

then (3.8) makes (3.1) into

$$I_k^{(1)}(N) = \mathcal{M}(N) \sum_{q \le Q} \frac{b_k(q)}{\varphi[k,q]\varphi(q)^{s-1}} + \mathcal{O}(P^{s-g} k^{-1}\, e^{-c\sqrt{L}}) \\ + \mathcal{O}\Big(\tau^{s/g-1} \sum_{q \le Q} \frac{|b_k(q)| q^{s/g-1}}{\varphi[k,q]\varphi(q)^{s-1}}\Big) \\ + \mathcal{O}(P^{s-g-1} Q^2 \sum_{q \le Q} \Delta(P, [k,q])). \tag{3.9}$$

At the end of the section it will be shown that

$$b_k(q) \ll q^{s/2+1+\varepsilon}. \tag{3.10}$$

This allows to deduce the absolute convergence of the *singular* series

$$\mathfrak{S}_k(N) := \sum_{q=1}^{+\infty} \frac{b_k(q)\varphi(k,q)}{\varphi(q)^s}. \tag{3.11}$$

Moreover, by writing $\varphi[k,q] = \varphi(k)\varphi(q)/\varphi(k,q)$ in (3.9), since $P^{s-g} \ll \mathcal{M}(N) \ll P^{s-g}$ ([20], Theorem 2.3), from (3.10) we get

$$I_k^{(1)}(N) = \mathcal{M}(N) \frac{\mathfrak{S}_k(N)}{\varphi(k)} + \mathcal{O}\Big(L \frac{P^{s-g}}{k} \sum_{q > Q} q^{-\frac{s}{2}+1+\varepsilon}(k,q)\Big) \\ + \mathcal{O}\Big(\frac{P^{s-g}}{k}\, e^{-c\sqrt{L}}\Big) + \mathcal{O}\Big(\frac{\tau^{s/g-1} L}{k} \sum_{q \le Q} q^{-s\frac{g-2}{2g}+\varepsilon}(k,q)\Big) \\ + \mathcal{O}(P^{s-g-1}\, Q^2 \sum_{q \le Q} \Delta(P, [k,q])).$$

Since $g \ge 2$ and $s \ge 5$, then

$$\mathcal{E}_1 \ll P^{s-g} L\, \Sigma_1 + \tau^{s/g-1}\, L \Sigma_2 \; + P^{s-g-1}\, Q^2\, \Sigma_3 + P^{s-g}\, e^{-c\sqrt{L}}, \tag{3.12}$$

with

$$\Sigma_1 := \sum_{k \le K} \sum_{q > Q} \frac{(k,q)}{k q^{3/2-\varepsilon}}, \qquad \Sigma_2 := \sum_{k \le K} \sum_{q \le Q} \frac{(k,q)}{k} q^{\varepsilon},$$

$$\Sigma_3 := \sum_{k \le K} \sum_{q \le Q} \Delta(P, [k,q]).$$

Let us estimate Σ_1. We have

(3.13)
$$\begin{aligned}\Sigma_1 &= \sum_{d\le Q} d \sum_{k\le K} \sum_{\substack{q>Q\\(k,q)=d}} \frac{1}{kq^{3/2-\varepsilon}} + \sum_{Q<d\le K} d \sum_{k\le K} \sum_{\substack{q>Q\\(k,q)=d}} \frac{1}{kq^{3/2-\varepsilon}}\\ &\ll \sum_{d\le Q} \frac{d^\varepsilon}{d^{3/2}} \sum_{k\le K/d} \frac{1}{k} \sum_{q>Q/d} \frac{q^\varepsilon}{q^{3/2}} + \sum_{Q<d\le K} \frac{d^\varepsilon}{d^{3/2}} \sum_{k\le K/d} \frac{1}{k} \sum_{q=1}^{\infty} \frac{q^\varepsilon}{q^{3/2}}\\ &\ll L \sum_{d\le Q} \frac{1}{d^{3/2-\varepsilon}} \sum_{q>Q/d} \frac{1}{q^{3/2-\varepsilon}} + L \sum_{Q<d} \frac{1}{d^{3/2-\varepsilon}} \ll \frac{L^2}{Q^{1/2-\varepsilon}}.\end{aligned}$$

While for the sum Σ_2 we get

(3.14)
$$\begin{aligned}\Sigma_2 &\ll \sum_{d\le Q} d \sum_{k\le K} \frac{1}{k} \sum_{\substack{q\le Q\\(k,q)=d}} q^\varepsilon \ll \sum_{d\le Q} d^\varepsilon \sum_{k\le K/d} \frac{1}{k} \sum_{q\le Q/d} q^\varepsilon\\ &\ll Q^{1+\varepsilon} L \sum_{d\le Q} \frac{1}{d} \ll Q^{1+\varepsilon} L^2.\end{aligned}$$

Finally, we estimate Σ_3 by writing

(3.15)
$$\Sigma_3 = \sum_{h\le QK} \omega(h)\,\Delta(P,h) \quad \text{with} \quad \omega(h) := \sum_{k\le K} \sum_{\substack{q\le Q\\ [k,q]=h}} 1.$$

Since $\omega(h) = \sum_{d\le Q} \sum_{q\le Q} \sum_{\substack{k\le K\\ [k,q]=h,\,(k,q)=d}} 1 = \sum_{d\le Q} \sum_{q\le Q/d} \sum_{\substack{k\le K/d\\ kqd=h}} 1 \le \sum_{d\le Q} \sum_{q\le Q/d} 1 \ll QL$,
by applying the Bombieri-Vinogradov theorem ([18], Theorem 15.1), from (3.5), (3.15) and the definitions of K and Q, one gets

(3.16)
$$\Sigma_3 \ll PQ^2L^{6-B}.$$

Hence, the inequality (2.1) follows from (3.12), (3.13), (3.14), (3.16) and the definitions of B, Q and τ.

It remains to prove (3.10). First let us show that $b_k(q) = b_k(q,g,l,N)$ is a multiplicative function of q. At this aim, we write $q = q_1q_2$ with $(q_1,q_2) = 1$ and define $k_i := (k,q_i)$ for every $k \le K$ and $i = 1,2$. Consequently, one has $(k,q) = (k,q_1q_2) = k_1k_2$ and there exist integers $a_1, a_2, m_1, m_2, n_1, n_2$ such that $a = a_2q_1 + a_1q_2$, $m = m_2q_1 + m_1q_2$ and $n = n_2q_1 + n_1q_2$. Thus,

$$\begin{aligned} b_k(q) = & \sum_{a=1}^{q}{}^{*} e\Big(-\frac{aN}{q}\Big) \sum_{\substack{m=1 \\ m\equiv l \bmod (k,q)}}^{q}{}^{*} e\Big(\frac{am^g}{q}\Big)\Big(\sum_{n=1}^{q}{}^{*} e\Big(\frac{an^g}{q}\Big)\Big)^{s-1} \\ = & \sum_{a_1=1}^{q_1}{}^{*}\sum_{a_2=1}^{q_2}{}^{*} \varsigma(a_1,a_2,-N) \sum_{\substack{m=1 \\ m\equiv l\,(k_1k_2)}}^{q_1q_2}{}^{*} \varsigma(a_1,a_2,m^g)\Big(\sum_{n=1}^{q_1q_2}{}^{*} \varsigma(a_1,a_2,n^g)\Big)^{s-1}, \end{aligned}$$

where we denote $\varsigma(a_1,a_2,h) := e\Big(\frac{(a_2q_1+a_1q_2)h}{q_1q_2}\Big)$ for every integer h.

Note that $m^g \equiv m_2^g q_1^g + m_1^g q_2^g$, $n^g \equiv n_2^g q_1^g + n_1^g q_2^g \bmod (q_1q_2)$. Moreover, it is easy to see that $(k,l)=1$ implies the equivalence of $m_2q_1+m_1q_2 \equiv l\ (k_1k_2)$ with the congruences $m_1q_2 \equiv l\ (k_1)$, $m_2q_1 \equiv l\ (k_2)$. Hence, one has

$$\begin{aligned} \sum_{\substack{m=1 \\ m\equiv l\,(k_1k_2)}}^{q_1q_2}{}^{*} \varsigma(a_1,a_2,m^g) = & \sum_{\substack{m=1 \\ m\equiv l\,(k_1k_2)}}^{q_1q_2}{}^{*} e\Big(\frac{(a_2q_1+a_1q_2)m^g}{q_1q_2}\Big) \\ = & \sum_{\substack{m_1=1 \\ m_2q_1+m_1q_2\equiv l\ (k_1k_2)}}^{q_1}{}^{*} \sum_{m_2=1}^{q_2}{}^{*} e\Big(\frac{a_1m_1^gq_2^g}{q_1}\Big)e\Big(\frac{a_2m_2^gq_1^g}{q_2}\Big) \\ = & \sum_{\substack{m_1=1 \\ m_1q_2\equiv l\ (k_1)}}^{q_1}{}^{*} e\Big(\frac{a_1m_1^gq_2^g}{q_1}\Big) \sum_{\substack{m_2=1 \\ m_2q_1\equiv l\ (k_2)}}^{q_2}{}^{*} e\Big(\frac{a_2m_2^gq_1^g}{q_2}\Big) \\ = & \sum_{\substack{m_1=1 \\ m_1\equiv l\ (k_1)}}^{q_1}{}^{*} e\Big(\frac{a_1m_1^g}{q_1}\Big) \sum_{\substack{m_2=1 \\ m_2\equiv l\ (k_2)}}^{q_2}{}^{*} e\Big(\frac{a_2m_2^g}{q_2}\Big). \end{aligned}$$

Analogously,

$$\sum_{n=1}^{q_1q_2}{}^{*} \varsigma(a_1,a_2,n^g) = \sum_{n_1=1}^{q_1}{}^{*} e\Big(\frac{a_1n_1^g}{q_1}\Big) \sum_{n_2=1}^{q_2}{}^{*} e\Big(\frac{a_2n_2^g}{q_2}\Big).$$

Thus, $b_k(q) = b_k(q_1)b_k(q_2)$, i.e. $b_k(q)$ is a multiplicative function of q.

Now, let us suppose that every prime divisor of q_2 divides g, while $(q_1,g)=1$. Since by Lemma 8.3 of [11] and by the multiplicativity of $b_k(q)$ one has $b_k(q)=0$ unless $q_2 \ll 1$ and q_1 is squarefree, then (3.10) is proved whenever one shows that

$$b_k(p) = \sum_{a=1}^{p}{}^{*} c_k(a,p)c(a,p)^{s-1}e(-Na/p) \ll p^{s/2+1}$$

for each prime p. At this aim, note that $|c_k(a,p)|=1$ if $p|k$, and $c_k(a,p) = c(a,p)$ otherwise. Since Lemma 4.3 of [20] implies $c(a,p) \ll p^{1/2}$, then we conclude that $b_k(p) \ll (p-1)p^{s/2} \le p^{s/2+1}$, as required.

We remark that in [12] for $g = 2$ and $s = 5$ the stronger bound $b_k(q) \ll q^{3+\varepsilon}$ is proved.

4. Minor arcs: the estimate of $\mathcal{E}_2$

By following the method in [7] we write

$$\mathcal{E}_2 \le \sum_{r=1}^{D} \sum_{K_r<k\le 2K_r} \max_{\substack{l\\(l,k)=1}} |I_k^{(2)}(N)| \le L \sum_{r=1}^{D} \sum_{K_r<k\le 2K_r} \max_{\substack{l\\(l,k)=1}} \sum_{\substack{p\le P\\ p\equiv l\,(k)}} |J(N-p^g)|,$$

where $D := [\log_2 K] \ll L$, $K_r := K/2^r$ and

$$J(m) = J(m,s,E_2) := \int_{E_2} S(\alpha)^{s-1} e(-m\alpha)\, d\alpha.$$

Since by hypothesis one has $s-1 = 2t$ and

$$\int_0^1 |\sum_{m\le P} e(\alpha m^g)|^{2t} d\alpha = W(N,g,2t) \ll P^{2t-g} L^v,$$

then we will apply the bound,

$$J(m) \le \int_0^1 |S(\alpha)|^{2t}\, d\alpha \le L^{2t} W(N,g,2t) \ll L^{2t+v} P^{2t-g}. \tag{4.1}$$

Therefore, we get

$$\sum_{\substack{p\le P\\ p\equiv l\,(k)}} |J(N-p^g)| \ll L^{2t+v} P^{2t-g} X(P;k,l) + L^{-A-2} P^{2t-g} \pi(P;k,l),$$

where $X(P;k,l) := \#\{p \le P:\ p \equiv l\,(k),\ |J(N-p^g)| > P^{2t-g}/L^{A+2}\}$ and $\pi(P;k,l) := \#\{p\le P:\ p\equiv l\,(k)\}$ as usual.

The Cauchy-Schwarz inequality and the trivial bound $\pi(P;k,l) \ll P/k$ imply

$$\begin{aligned} \mathcal{E}_{2,r} &:= \sum_{K_r<k\le 2K_r} \max_{\substack{l\\(l,k)=1}} \sum_{\substack{p\le P\\ p\equiv l\,(k)}} |J(N-p^g)| \\ &\ll L^{2t+v} P^{2t-g} \Big(K_r \sum_{K_r<k\le 2K_r} \max_{\substack{l\\(l,k)=1}} X(P;k,l)^2 \Big)^{1/2} \\ &\quad + L^{-A-2} P^{2t-g+1}. \end{aligned} \tag{4.2}$$

Since $\mathcal{E}_2 \le L \sum_{r=1}^{D} \mathcal{E}_{2,r}$, then (2.2) follows whenever one proves that even the first summand on the right hand side of (4.2) is $\ll L^{-A-2} P^{2t-g+1}$.

Considering the term in brackets, the contribution of the k's such that $d(k) > L^C$ with $C := 2A + 4t + 2v + 5$ fits this request because it is

$$\leq K_r \sum_{\substack{K_r<k\leq 2K_r \\ d(k)>L^C}} \max_{\substack{l \\ (l,k)=1}} \pi(P;k,l)^2 \ll \frac{P^2}{K_r} \sum_{\substack{K_r<k\leq 2K_r \\ d(k)>L^C}} 1 < \frac{P^2}{K_r L^C} \sum_{k\leq 2K_r} d(k) \ll \frac{P^2}{L^{C-1}}.$$

Let us prove that the same estimate holds for the remaining k's, i.e.

$$\mathcal{D}_r^{1/2} := \Big(K_r \sum_{\substack{K_r<k\leq 2K_r \\ d(k)\leq L^C}} \max_{\substack{l \\ (l,k)=1}} X(P;k,l)^2 \Big)^{1/2} \ll P L^{-A-2t-v-2}.$$

At this aim, we consider the arithmetic function $\xi_l(k) := kX(P;k,l)$ with its Möbius inverse $f_l := \mu * \xi_l$ and write

$$\mathcal{D}_r < \sum_{\substack{K_r<k\leq 2K_r \\ d(k)\leq L^C}} \frac{1}{k} \max_{\substack{l \\ (l,k)=1}} \xi_l(k)^2 = \sum_{\substack{K_r<k\leq 2K_r \\ d(k)\leq L^C}} \frac{1}{k} \max_{\substack{l \\ (l,k)=1}} \big(\sum_{d|k} f_l(d)\big)^2.$$

Again by the Cauchy-Schwarz inequality we have

$$\mathcal{D}_r < \sum_{\substack{K_r<k\leq 2K_r \\ d(k)\leq L^C}} \frac{d(k)}{k} \sum_{d|k} \max_{\substack{l \\ (l,k)=1}} f_l(d)^2 \leq L^C \sum_{K_r<k\leq 2K_r} \frac{1}{k} \sum_{d|k} \max_{\substack{l \\ (l,k)=1}} f_l(d)^2.$$

Since $X(P;k,l+r) = X(P;k,l)$ for any $r \equiv 0\,(k)$, then $f_l(d)$ is d-periodic with respect to l for every $d|k$. Consequently, $\max_{\substack{0\leq l<k \\ (l,k)=1}} f_l(d)^2 = \max_{\substack{0\leq l<d \\ (l,d)=1}} f_l(d)^2$. Moreover, one may easily verify that (see also [17], equation 10)

$$\sum_{0\leq l<d} f_l(d)^2 = d \sum_{\substack{0\leq l<d \\ (l,d)=1}} \Big| \sum_{m\leq P} c(m)e(\alpha m) \Big|^2 := d \sum_{\substack{0\leq l<d \\ (l,d)=1}} |C(\alpha)|^2, \text{ say,}$$

where $c(m)$ is the characteristic function of the set $\mathcal{X}$ of prime numbers $p \leq P$ such that $|J(N-p^g)| > P^{2t-g}/L^{A+2}$. Thus, we obtain

$$\begin{aligned}
\mathcal{D}_r &< L^C \sum_{K_r<k\leq 2K_r} \frac{1}{k} \sum_{d|k} \sum_{0\leq l<d} f_l(d)^2 \\
&\leq L^C \sum_{d\leq 2K_r} \sum_{\substack{0\leq l<d \\ (l,d)=1}} |C(\alpha)|^2 \sum_{\substack{K_r<k\leq 2K_r \\ k\equiv 0\,(d)}} \frac{d}{k} \\
&= L^C \sum_{d\leq 2K_r} \sum_{\substack{0\leq l<d \\ (l,d)=1}} |C(\alpha)|^2 \sum_{K_r/d<k/d\leq 2K_r/d} \frac{1}{k/d} \\
&\ll L^{C+1} \sum_{d\leq 2K_r} \sum_{\substack{0\leq l<d \\ (l,d)=1}} |C(\alpha)|^2.
\end{aligned}$$

The large sieve inequality (see [18]) and the hypothesis on K imply

$$\mathcal{D}_r \ll L^{C+1}(P+K_r^2)\sum_{m\le P}|c(m)|^2 \le L^{C+1}P\#\mathcal{X} \quad \text{for every } r\le D.$$

Now we observe that

$$\#\mathcal{X} < \frac{L^{A+2}}{P^{2t-g}}\sum_{p\le P}|J(N-p^g)| = \frac{L^{A+2}}{P^{2t-g}}\int_{E_2} S(\alpha)^{s-1}\tilde{S}(\alpha)e(-N\alpha)\,d\alpha,$$

where $\tilde{S}(\alpha) := \sum_{p\le P}(a_p \log p)e(p^g\alpha)$ for some unimodular numbers a_p.

By considering the underlying Diophantine equation and recalling that $s-1=2t$, plainly the integral on the right is

$$\ll \sup_{\alpha\in E_2}|S(\alpha)|\int_0^1 |S(\alpha)^{s-2}\tilde{S}(\alpha)|\,d\alpha \ll L^{2t}W(N,g,2t)\sup_{\alpha\in E_2}|S(\alpha)|.$$

Thus, Vinogradov's estimate, $\sup_{\alpha\in E_2}|S(\alpha)| \ll PL^{-V}$, together with the definitions of V, Q, τ, E_2 (see [11], Theorem 10) and (4.1) imply that

$$\#\mathcal{X} < \frac{P^{g-2t+1}}{L^{V-A-2t-2}}W(N,g,2t) \ll \frac{P}{L^{V-A-2t-v-2}} \le \frac{P}{L^{4A+8t+4v+10}}\ .$$

Since $C := 2A+4t+2v+5$, then we conclude

$$\mathcal{D}_r^{1/2} \ll PL^{C/2-2A-4t-2v-9/2} = PL^{-A-2t-v-2},$$

as it is required. The Theorem is completely proved.

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