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Résumé. Nous démontrons que, pour toute unité ϵ dans un corps de nombres réel K de degré $n+1$, il existe seulement un nombre fini de n-uples dans K^n qui ont un développement purement périodique par l'algorithme de Jacobi-Perron. Ce résultat généralise le cas des fractions continues pour $n=1$. Pour $n=2$ nous donnons un algorithme qui permet de calculer explicitement tous ces couples.

Abstract. We prove that, for any unit ϵ in a real number field K of degree $n+1$, there exits only a finite number of n-tuples in K^n which have a purely periodic expansion by the Jacobi-Perron algorithm. This generalizes the case of continued fractions for $n=1$. For $n=2$ we give an explicit algorithm to compute all these pairs.

1. Introduction

One of the generalizations of the continued fraction algorithm to higher dimensions is the Jacobi-Perron Algorithm (JPA). Its main interest lies in the great simplicity of its definition. When it is used for rational numbers, it gives a very simple algorithm to get the gcd of n integers. It also quickly gives an integer matrix with determinant ± 1 whose first column is given. It gives rational simultaneous approximations to a set $(\alpha_1, \dots, \alpha_n)$ of real numbers $\mathbb{Q}$ linearly independent with 1. These approximations are only best approximations when $n=2$ and $(1, \alpha_1, \alpha_2)$ is a basis of a real cubic field with complex conjugates and when the JPA expansion is ultimately periodic. The authors [2] gave an algorithm, using only integers, to compute the AJP expansion when the numbers α_i are algebraic numbers. The quality of the approximations have been studied in general by J. Lagarias [10]. For more efficients algorithms see H. Cohen [7].

It is well-known that the continued fraction expansion of a real quadratic number α is always ultimately periodic. Moreover α has a purely periodic expansion if and only if $\alpha > 1$ and its conjugate $\overline{\alpha}$ satisfies $-1 < \overline{\alpha} < 0$ (we say that α is reduced). If l denotes the period length and $\left(\frac{p_i}{q_i}\right)$ $(i \geq 0)$ the

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sequence of convergents of α then we have :

$$\begin{pmatrix} p_l & p_{l-1} \\ q_l & q_{l-1} \end{pmatrix} \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \epsilon \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \tag{1.1}$$

where $\epsilon = q_l\alpha + q_{l-1}$ is a unit of $\mathbb{Q}(\alpha)$.

This property is also valid for purely periodic JPA expansions. It is called the Hasse-Bernstein theorem [3].

Moreover, in the quadratic case, we have the following property (P) : if $\epsilon > 1$ is a unit of a real quadratic field K, then there exists only a finite number of reduced elements in K whose continued fraction expansion is associated with ϵ by (1.1).

A natural question of O. Perron [13] was : let $(\alpha_i)_{1\leq i\leq n}$ be n elements of a real number field K of degree $(n+1)$ such that $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ be $\mathbb{Q}$-linearly independent. Under which algebraic conditions these numbers have a periodic JPA expansion? Now, this question is not solved, even for degree 3.

Several authors have presented a few classes of periodic JPA expansions. C. Levesque and G. Rhin [11] introduced a pair (α_1, α_2) of cubic numbers depending on a parameter. Their JPA expansion is purely periodic with period length tending to infinity with the parameter. The first author [1] proved that this JPA expansion and the Voronoï expansion are closely connected and that the unit obtained is fundamental in the number field generated by α_2. Leon Bernstein [4] studied the JPA expansion of the pair $(\sqrt[3]{m}, \sqrt[3]{m^2})$ for some integers m with $1 < m < 1000$. In [5] he presented the JPA expansion of the same type when $m = D^3 + 6D$ with $D = 2K$ and $K \geq 2$. The period length is equal to 8 and gives the square of the fundamental unit of the field $\mathbb{Q}(\sqrt[3]{m})$. In 1984, E. Dubois and R. Paysant-Le Roux [8] proved that, in all real number fields of degree $(n+1)$, there exist n numbers whose JPA expansion is periodic.

In this paper we use another point of view. Given a unit in a real number field of degree $(n+1)$, can we compute all n-tuples which have a purely periodic JPA expansion associated with this unit? We prove the following result : Let K be a real number field of degree $(n+1)$ and ϵ a unit of K, there exists only a finite number of elements in K^n whose JPA expansion is purely periodic and associated with ϵ. We give a bound depending on the trace of the unit ϵ and a method to get all these expansions. So, we obtain the generalization of the property (P) to any real number field.

Moreover in the case of real cubic fields ($n = 2$) we give an explicit algorithm to find all the periodic expansions related to a fixed unit.

In Section 2, we give the definition of the Jacobi-Perron algorithm and the Hasse-Bernstein theorem. Our theorem is proved in Section 3. Section 4 is devoted to the case of real cubic fields and the explicit algorithm is given in Section 5. In Section 6 we present some numerical results which show that, surprisingly, there is a lot of such expansions. For instance, in the real cubic field with discriminant 621, we give a unit with trace equal to 435 which gives 2758 pairs (α_1, α_2) which have a purely periodic AJP expansion associated with this unit. We also give some improvements of Bernstein's results.

2. Preliminaries : JPA and Hasse-Bernstein theorem.

Definition 2.1. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a vector in $\mathbb{R}^n$ $(n \geq 1)$.The **Jacobi-Perron Algorithm (JPA)** expansion [13] of α is given by the two sequences :

- $(a^{(\nu)})_{\nu \geq 0}$ in $\mathbb{Z}^n$ where $a^{(\nu)} = (a_1^{(\nu)}, a_2^{(\nu)}, \dots, a_n^{(\nu)})$;
- $(\alpha^{(\nu)})_{\nu \geq 0}$ in $\mathbb{R}^n$ where $\alpha^{(\nu)} = (\alpha_1^{(\nu)}, \alpha_2^{(\nu)}, \dots, \alpha_n^{(\nu)})$

defined by :

$$(2.1) \qquad \begin{cases} \alpha^{(0)} = \alpha \; ; \\ \text{for } \nu \geq 0 \;\; a_i^{(\nu)} = [\alpha_i^{(\nu)}] \;\; \text{for } 1 \leq i \leq n \; ; \\ \text{if } \alpha_1^{(\nu)} \neq a_1^{(\nu)} \;\; \alpha_n^{(\nu+1)} = \frac{1}{\alpha_1^{(\nu)} - a_1^{(\nu)}} \; ; \\ \alpha_i^{(\nu+1)} = \frac{\alpha_{i+1}^{(\nu)} - a_{i+1}^{(\nu)}}{\alpha_1^{(\nu)} - a_1^{(\nu)}} \text{ for } 1 \leq i < n. \end{cases}$$

where [x] is the integer part of x. We define $a_0^{(\nu)} = 1$ and $\alpha_0^{(\nu)} = 1$ for all ν.

Remark. In this case, the integers $a_i^{(\nu)}$, $\nu \geq 0$, $0 \leq i \leq n$, satisfy the following **Perron Conditions**

$$(2.2) \qquad (a_n^{(\nu)}, a_{n-1}^{(\nu+1)}, \dots, a_{n-i}^{(\nu+i)}) \geq (a_i^{(\nu)}, a_{i-1}^{(\nu+1)}, \dots, a_1^{(\nu+i-1)}, a_0^{(\nu+i)})$$

in lexicographical order.

Definition 2.2. We define the sequence $A^{(\nu)} = (A_0^{(\nu)}, A_1^{(\nu)}, A_2^{(\nu)}, \dots, A_n^{(\nu)})$ of vectors in $\mathbb{Z}^n$ by :

$$(2.3) \qquad \begin{cases} \text{for } 0 \leq i \leq n \text{ and } 0 \leq j \leq n, \quad A_i^{(j)} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ else} \end{cases} \\ \text{for } \nu \geq 0 \text{ and } 0 \leq i \leq n, \\ A_i^{(\nu+n+1)} = A_i^{(\nu)} + a_1^{(\nu)} A_i^{(\nu+1)} + \dots + a_n^{(\nu)} A_i^{(\nu+n)}. \end{cases}$$

So, we have the following formulas :

(1) $\alpha_i = \dfrac{\sum_{j=0}^{n} A_i^{(\nu+j)} \alpha_j^{(\nu)}}{\sum_{j=0}^{n} A_0^{(\nu+)} \alpha_j^{(\nu)}}$ for all $\nu \geq 0$ and $1 \leq i \leq n$.

(2) by writing

$$\mathcal{A}_\nu = \begin{pmatrix} a_n^{(\nu)} & 1 & 0 & \dots & 0 \\ a_{n-1}^{(\nu)} & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_1^{(\nu)} & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix} \tag{2.4}$$

$$\text{we have } \mathcal{A}_0\mathcal{A}_1\dots\mathcal{A}_{\nu-1} = \begin{pmatrix} A_n^{(\nu+n)} & \dots & \dots & A_n^{(\nu)} \\ \vdots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \vdots \\ A_0^{(\nu+n)} & \dots & \dots & A_0^{(\nu)} \end{pmatrix}.$$

We set $\mathcal{A}^{(l)} = \mathcal{A}_0\mathcal{A}_1\dots\mathcal{A}_{l-1}$.

Definition 2.3. The JPA expansion is **periodic** if there exist two integers $k \geq 0$ and $l > 0$ such that $a_i^{(k+\nu)} = a_i^{(k+\nu+l)}$ for all $\nu \geq 0$ and $0 < i \leq n$. l is called the period length.

If k and l are the smallest integers which verify this equality then k is the preperiod length and l is the primitive period length. If $k = 0$ the expansion is **purely periodic** .

Remark. If the JPA expansion of $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is purely periodic with period length l, then

$$\epsilon = A_0^{(l)} + \alpha_1 A_0^{(l+1)} + \dots + \alpha_n A_0^{(l+n)} = \prod_{\nu=0}^{l-r} \alpha_n^{(\nu)}.$$

is a unit of $K = \mathbb{Q}[\alpha_1, \dots, \alpha_n]$. This is the Hasse-Bernstein theorem [3] : in this case we have

$$\mathcal{A}^{(l)} \begin{pmatrix} \alpha_n \\ \vdots \\ \alpha_1 \\ 1 \end{pmatrix} = \epsilon \begin{pmatrix} \alpha_n \\ \vdots \\ \alpha_1 \\ 1 \end{pmatrix} \tag{2.5}$$

We say that this JPA expansion is associated with the unit ϵ.

Definition 2.4. We say that a matrix $\mathcal{A}$ is a **JPA matrix** of length $l > 0$ if there is a finite sequence of integers $(a_i^{(\nu)})$, $0 \leq \nu \leq l-1$, $0 < i \leq n$ such that $\mathcal{A} = \mathcal{A}_0\mathcal{A}_1\dots\mathcal{A}_{l-1}$ where the matrices $\mathcal{A}_i$ are defined by (2.4) and the integers $a_i^{(\nu)}$ satisfy the Perron conditions. We say that each $\mathcal{A}_i$ is an **elementary JPA matrix**.

Definition 2.5. We say that a matrix $\mathcal{A}$ is a **JPA period matrix** if $\mathcal{A}$ is a **JPA matrix** of length $l > 0$ and if the infinite sequence of integers $(a_i^{(\nu)})$, $0 \le \nu \le l-1$, $0 < i \le n$ defined by $a_i^{(kl+\nu)} = a_i^{(\nu)}$, for $0 \le \nu \le l-1$, $0 < i \le n$ and $k \ge 1$ satisfies the Perron conditions.

3. The main theorem.

Theorem 3.1. *Let K be a real number field of degree $(n+1)$ and $\epsilon > 1$ a unit of K. There exists only a finite number of elements in K^n whose JPA expansion is purely periodic and associated with ϵ.*

Proof. Let Z_K be the ring of algebraic integers of K, $(1, \omega_1, \ldots, \omega_n)$ an integral basis of $\mathbb{Z}_K$ and A the matrix of the multiplication by ϵ defined by

$$A \begin{pmatrix} \omega_n \\ \vdots \\ \omega_1 \\ 1 \end{pmatrix} = \epsilon \begin{pmatrix} \omega_n \\ \vdots \\ \omega_1 \\ 1 \end{pmatrix} \tag{3.1}$$

We search a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in K^n such that the JPA expansion of α is purely periodic and associated with ϵ, that is to say that we search the period matrices $\mathcal{A}$ such that

$$\mathcal{A} \begin{pmatrix} \alpha_n \\ \vdots \\ \alpha_1 \\ 1 \end{pmatrix} = \epsilon \begin{pmatrix} \alpha_n \\ \vdots \\ \alpha_1 \\ 1 \end{pmatrix}. \tag{3.2}$$

Remark. The matrices A^t and $\mathcal{A}^t$ are the matrices of the multiplication by ϵ in two different basis of the $\mathbb{Q}$-vector space K.

We suppose now that $\mathcal{A}$ is a JPA period matrix of length l satisfying (3.2).

With the previous notations we have the following lemmas :

Lemma 3.1. *For $0 \le i \le n$ and $0 \le k \le n$ we have*

$$A_i^{(l+k)} \le \max(A_i^{(l+n)}, 1).$$

Proof. If $0 \le \nu \le n$ then $A_i^{(\nu)} = \begin{cases} 1 \text{ if } i = \nu \\ 0 \text{ otherwise} \end{cases}$. We have $A_i^{(\nu+n+1)} = A_i^{(\nu)} + a_1^{(\nu)} A_i^{(\nu+1)} + \ldots + a_n^{(\nu)} A_i^{(\nu+n)}$, for $\nu \ge 0$ and $0 \le i \le n$. As $a_n^{(\nu)} \ge 1$ and $a_i^{(\nu)} \ge 0$ then $A_i^{(\nu+n+1)} \ge A_i^{(\nu+n)}$. These statements prove the lemma. $\square$

Lemma 3.2. *For $0 \le i \le n$ we have $A_i^{(l+n)} \le (n+1) A_n^{(l+n)}$.*

Proof. For $1 \le i \le n$, we have

$$\alpha_i = \frac{\sum_{j=0}^n A_i^{(l+j)}\alpha_j}{\sum_{j=0}^n A_0^{(l+j)}\alpha_j},$$

therefore

$$\frac{\alpha_n}{\alpha_i} = \frac{\sum_{j=0}^n A_n^{(l+j)}\alpha_j}{\sum_{j=0}^n A_i^{(l+j)}\alpha_j} \ge 1.$$

Then $\sum_{j=0}^n A_i^{(l+j)}\alpha_j \le \sum_{j=0}^n A_n^{(l+j)}\alpha_j$. For all $0 \le j \le n$ we have $\alpha_j \le \alpha_n$ and $A_n^{(l+j)} \le A_n^{(l+n)}$, therefore for all $0 \le i \le n$ we have $A_i^{(l+n)}\alpha_n \le \sum_{j=0}^n A_i^{(l+j)}\alpha_j \le (n+1)A_n^{(l+n)}\alpha_n$ and $A_i^{(l+n)} \le (n+1)A_n^{(l+n)}$. □

According to the relations (3.1) and (3.2) we have $\mathrm{trace}(A) = \mathrm{trace}(\mathcal{A})$. Since these matrices have positive elements then $A_n^{(l+n)} \le \mathrm{trace}(\mathcal{A})$. So, if we denote $\mathcal{A} = (m_{ij})_{1\le i,j\le n+1}$ then $m_{ij} \le (n+1)\mathrm{trace}(A)$. Therefore, there exists only a finite number of matrices $\mathcal{A}$ satisfying (3.2). □

4. Real cubic fields (n=2).

Let K be a real number field of degree 3 and $\epsilon > 1$ a unit of K. We search the JPA period matrices $\mathcal{A}$ of length l associated with ϵ. It means that we search the matrices

$$\mathcal{A}^{(l)} = \begin{pmatrix} A_2^{(l+2)} & A_2^{(l+1)} & A_2^{(l)} \\ A_1^{(l+2)} & A_1^{(l+1)} & A_1^{(l)} \\ A_0^{(l+2)} & A_0^{(l+1)} & A_0^{(l)} \end{pmatrix}$$

$$= \mathcal{A}_0\mathcal{A}_1\ldots\mathcal{A}_{l-1} = \begin{pmatrix} a_2^{(0)} & 1 & 0 \\ a_1^{(0)} & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdots \cdot \begin{pmatrix} a_2^{(l-1)} & 1 & 0 \\ a_1^{(l-1)} & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

where the integers $a_i^{(\nu)}$ ($i \in \{1,2\}$, $0 \le \nu \le l-1$) satisfy the following **periodicity conditions** (including the Perron conditions (2.2)) : $a_2^{(\nu)} \ge 1$, $0 \le a_1^{(\nu)} \le a_2^{(\nu)}$ and if $a_2^{(\nu)} = a_1^{(\nu)}$ then $\nu < l-1$ and $a_1^{(\nu+1)} \ge 1$ or $\nu = l-1$ and $a_1^{(0)} \ge 1$.

We have the following lemma.

Lemma 4.1. *Let $\mathcal{A}$ be a JPA matrix with $\mathcal{A} = \mathcal{A}_0\mathcal{A}_1\ldots\mathcal{A}_{m-1}$ where $m \ge 1$, then we have*

1) $A_0^{(m+2)} \ge 1$ *and the sequences* $A_j^{(\nu+2)}$ $(0 \le \nu \le m)$ *are non decreasing for $j = 0$ and $j = 2$.*
2) *For* $0 \le j, \nu \le 2$, $A_j^{(m+\nu)} \le A_2^{(m+2)}$;
3) $\frac{\mathrm{trace}(\mathcal{A})}{3} \le A_2^{(m+2)} \le \mathrm{trace}(\mathcal{A})$.

Proof. The results of the lemma are clearly true for $m = 1$ i.e. in the case where $\mathcal{A}$ is equal to $\mathcal{A}_0$.

1) We have $A_0^{(3)} = 1$, $A_0^{(4)} = a_2^{(1)} \geq 1$, $A_0^{(5)} = a_2^{(2)}a_2^{(1)} + a_1^{(2)} \geq a_2^{(2)}a_2^{(1)} \geq a_2^{(1)}$. So, by Definition 2.2 the proof follows by induction for $\nu \geq 6$. For the second property use $A_2^{(2)} = 1$ and $A_2^{(\nu+3)} \geq A_2^{(\nu+2)}$ for $\nu \geq 0$ by Definition 2.2.
2) The result is clearly true for $m = 1$. We have
$$A_2^{(2)} = 1, A_2^{(1)} = A_2^{(0)} = 0$$
and
$$A_2^{(4)} - A_1^{(4)} = a_2^{(1)}\left(a_2^{(0)} - a_1^{(0)}\right) + a_1^{(1)} - 1.$$
So, knowing that the sequence $A_2^{(\nu+3)}$ is non decreasing, as
$$\begin{aligned} A_2^{(\nu+3)} - A_i^{(\nu+3)} = a_2^{(\nu)}\left(A_2^{(\nu+2)} - A_i^{(\nu+2)}\right) + a_1^{(\nu)}\left(A_2^{(\nu+1)} - A_i^{(\nu+1)}\right) \\ + \left(A_2^{(\nu)} - A_i^{(\nu)}\right) \end{aligned}$$
for $\nu \geq 2$ and $0 \leq i \leq 1$, we prove the second result by induction.
3) Use the trace formula and the previous result.

$\square$

Proposition 4.1. *Let a, b, c be three fixed integers such that $a \geq b \geq 0$ and $a \geq c \geq 1$. There exist at most 4 JPA matrices $\mathcal{A}$ with first column* $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Proof. We will use the following notation
$$\mathcal{A}_\nu = \begin{pmatrix} a_2^{(\nu)} & 1 & 0 \\ a_1^{(\nu)} & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = (a_2^{(\nu)}, a_1^{(\nu)}).$$
It is clear that, if $\mathcal{A}$ satisfies the periodicity conditions, then for all j $(0 \leq j \leq l-2)$, $\mathcal{A}_j^{-1}\mathcal{A}_{j-1}^{-1}\ldots\mathcal{A}_0^{-1}\mathcal{A}$ is a JPA matrix satisfying the conditions of lemma 4.1.

• First we search $(a_2^{(0)}, a_1^{(0)})$: since the first column of $\mathcal{A}_0^{-1}\mathcal{A}$ is $\begin{pmatrix} c \\ a - a_2^{(0)}c \\ b - a_1^{(0)}c \end{pmatrix}$, we have :

- if $\mathcal{A}_0^{-1}\mathcal{A} = I_3$ then $\mathcal{A} = \mathcal{A}_0 = (a, b)$ and $l = 1$.
- else, by lemma 4.1, we have

$$\begin{cases} c \geq a - a_2^{(0)}c \geq 0 \\ c \geq b - a_1^{(0)}c \geq 1 \end{cases} \tag{4.1}$$

We note $a = d_2c + r_2$, $b = d_1c + r_1$ with $0 \le r_2 < c$ and $0 \le r_1 < c$ and study the four different cases :
Case 1 : if $r_1r_2 \neq 0$. In this case we have $\mathcal{A}_0 = (d_2, d_1)$.
Case 2 : if $r_1 = 0$ and $r_2 \neq 0$. In this case there is at most a unique solution (if $d_1 \neq 0$) : $\mathcal{A}_0 = (d_2, d_1 - 1)$.
Case 3 : if $r_1 \neq 0$ and $r_2 = 0$. In this case there are at most 2 solutions : $\mathcal{A}_0 = (d_2, d_1)$ and $\mathcal{A}_0 = (d_2 - 1, d_1)$.
Case 4 : if $r_1 = 0$ and $r_2 = 0$. Here $c = 1$, because c divides the determinant which is equal to 1. There are at most 4 solutions : $\mathcal{A}_0 = (a, b)$, $(a, b-1)$,$(a-1, b)$ and $(a-1, b-1)$.

We let the reader verify that in all cases there are at most 4 possible JPA matrices. For example, in the last case if $a > b \ge 1$ the 4 solutions are :

$$\begin{aligned}
\mathcal{A} &= (a, b) \\
\mathcal{A} &= (a, b-1)(1, 0) \\
\mathcal{A} &= (a-1, b-1)(1, 1) \\
\mathcal{A} &= (a-1, b-1)(1, 0)(1, 0).
\end{aligned}$$

• Then we iterate this process and it will stop when, for an integer l, $\mathcal{A}_{l-1}^{-1} \dots \mathcal{A}_0^{-1}\mathcal{A} = I_3$ and then $\mathcal{A}_0\mathcal{A}_1 \dots \mathcal{A}_{l-1}$ is a JPA matrix with length l. At each step we have to verify the Perron conditions, particularly at step $(l-1)$, that is if $a_2^{(l-1)} = a_1^{(l-1)}$ then $a_1^{(0)}$ must be positive, then we get a period matrix. □

5. Algorithm in the case n=2.

Let K be a real number field of degree 3 such that $K = \mathbb{Q}(\alpha)$ where α is a real root of a polynomial $Q = X^3 - a_2X^2 - a_1X - a_0 \in \mathbb{Z}[X]$. Let $\epsilon > 1$ in $\mathbb{Z}[\alpha]$ be a unit of K (given, for example, by the Voronoï algorithm or by Pari [12]). Let A be the matrix defined by the relation

$$A \begin{pmatrix} \alpha^2 \\ \alpha \\ 1 \end{pmatrix} = \epsilon \begin{pmatrix} \alpha^2 \\ \alpha \\ 1 \end{pmatrix}$$

1. We compute all vectors (a, b, c) such that $\frac{\operatorname{trace}(A)}{3} \le a \le \operatorname{trace}(A)$, $a \ge b \ge 0$, $a \ge c \ge 1$ and $\gcd(a, b, c) = 1$.
2. We apply proposition 4.2 to each vector (a, b, c) and obtain all period matrices $\mathcal{A}$ with first column $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.
3. For each matrix $\mathcal{A}$ found, we verify that $P_A(\mathcal{A}) = 0$ where P_A denotes the characteristic polynomial of A. Indeed, there exists

(α_2, α_1) satisfying the system

$$\mathcal{A}\begin{pmatrix}\alpha_2\\ \alpha_1\\ 1\end{pmatrix} = \epsilon\begin{pmatrix}\alpha_2\\ \alpha_1\\ 1\end{pmatrix} \tag{5.1}$$

if and only if ϵ is an eigenvalue of $\mathcal{A}$ that is to say $P_A(\mathcal{A}) = 0$.

4. For each remaining matrix $\mathcal{A}$, we solve the system (5.1) as follows : we denote $\mathcal{A} = (a_{ij})$ and $\mathcal{A}^{-1} = (d_{ij})$ and so the system (5.1) will be :

$$\begin{cases}(a_{11}-\epsilon)\alpha_2 + a_{12}\alpha_1 + a_{13} = 0\\ a_{21}\alpha_2 + (a_{22}-\epsilon)\alpha_1 + a_{23} = 0\\ a_{31}\alpha_2 + a_{32}\alpha_1 + (a_{33}-\epsilon) = 0\end{cases} \tag{5.2}$$

or

$$\begin{cases}\alpha_2 = -\frac{a_{32}}{a_{31}}\alpha_1 + \frac{\epsilon - a_{33}}{a_{31}}\\ (\frac{d_{31}}{a_{31}} + \epsilon)\alpha_1 = \frac{a_{21}}{a_{31}}\epsilon + \frac{d_{21}}{a_{31}}.\end{cases} \tag{5.3}$$

If $\eta = \frac{d_{31}}{a_{31}} + \epsilon$ and $P_{\mathcal{A}} = X^3 - t_1X^2 + t_2X - 1$ is the characteristic polynomial of $\mathcal{A}$ then $\eta^{-1} = -\frac{1}{P_{\mathcal{A}}(\frac{d_{31}}{a_{31}})}\left(\epsilon^2 - t_1\epsilon + t_2 - \frac{d_{31}}{a_{31}}\epsilon + \frac{d_{31}^2}{a_{31}^2} + \frac{d_{31}}{a_{31}}t_1\right)$ since $P_{\mathcal{A}}(\epsilon) = 0$.

We have then to replace η^{-1} in (5.3) to get some formulas of the type :

$$\alpha_1 = \frac{p + q\epsilon + r\epsilon^2}{m}; \qquad \alpha_2 = \frac{s + t\epsilon + u\epsilon^2}{m}$$

where p, q, r, s, t, u, m are integers.

The numerical results point out that these integers have common divisors. In order to use smaller integers, we will use the following lemma which allows us to eliminate these common factors.

Lemma 5.1. *Let $A = (a_{ij})_{1\le i,j\le 3}$ be a matrix with coefficients in any field and $D = (d_{ij})$ be the comatrix of A. If we denote :*

Δ *the determinant of A,*

$P_A = X^3 - t_1X^2 + t_2X - \Delta$ *the characteristic polynomial of A,*

$d_0 = a_{31}d_{22} - a_{32}d_{21}$,

$\Delta_1 = d_{31}^3 + t_1d_{31}^2a_{31} + t_2d_{31}a_{31}^2 + a_{31}^3\Delta$,

$\Delta_2 = a_{21}a_{31}^2\Delta + d_{21}(d_{31}^2 + d_{31}t_1a_{31} + t_2a_{31}^2)$,

$d_3 = a_{31}d_{21} - a_{21}d_{31}$,

$d_4 = a_{31}d_{32} - a_{32}d_{31}$,

then we have $\Delta_1 = d_3d_4$ and $\Delta_2 = d_3d_0$.

Proof. It is easy to verify that we have the following formulas :

$$\begin{aligned}
t_1 &= a_{11} + (a_{22} + a_{33});\\
t_2 &= ((a_{22} + a_{33})a_{11} + (-a_{21}a_{12} + (-a_{31}a_{13} + (a_{33}a_{22} - a_{32}a_{23}))));\\
\Delta_1 &= (a_{32}^2 a_{31} a_{21}^2 + (-a_{32}a_{31}^2 a_{22} + a_{33}a_{32}a_{31}^2)a_{21} - a_{32}a_{31}^3 a_{23})a_{11}\\
&\quad + ((-a_{32}a_{31}^2 a_{21}^2 + (a_{31}^3 a_{22} - a_{33}a_{31}^3)a_{21} + a_{31}^4 a_{23})a_{12}\\
&\quad + (a_{32}^3 a_{21}^3 + (-2a_{32}^2 a_{31} a_{22} + a_{33}a_{32}^2 a_{31})a_{21}^2\\
&\quad + (a_{32}a_{31}^2 a_{22}^2 - a_{33}a_{32}a_{31}^2 a_{22} - a_{32}^2 a_{31}^2 a_{23})a_{21} + a_{32}a_{31}^3 a_{23}a_{22}));\\
\Delta_2 &= (-a_{33}a_{32}a_{31}a_{21}^2 + (a_{33}a_{31}^2 a_{22} - a_{33}^2 a_{31}^2)a_{21} + a_{33}a_{31}^3 a_{23})a_{11}\\
&\quad + ((a_{32}a_{31}^2 a_{21}^2 + (-a_{31}^3 a_{22} + a_{33}a_{31}^3)a_{21} - a_{31}^4 a_{23})a_{13}\\
&\quad + (-a_{33}a_{32}^2 a_{21}^3 + (a_{33}a_{32}a_{31}a_{22} + (a_{32}^2 a_{31}a_{23} - a_{33}^2 a_{32}a_{31}))a_{21}^2\\
&\quad + (-a_{32}a_{31}^2 a_{23}a_{22} + 2a_{33}a_{32}a_{31}^2 a_{23})a_{21} - a_{32}a_{31}^3 a_{23}^2)),
\end{aligned}$$

and then to verify that we have $\Delta_1 = d_3 d_4$ and $\Delta_2 = d_3 d_0$. □

We denote $A^2 = (b_{ij})$, $e_0 = a_{31}d_{12} - a_{32}d_{11}$ and with the previous lemma we obtain :

$$\alpha_1 = \frac{d_0 - b_{11}\epsilon + a_{31}\epsilon^2}{d_4}; \qquad \alpha_2 = \frac{e_0 + b_{32}\epsilon - a_{32}\epsilon^2}{d_4}.$$

Then we obtain formulas of the type ;

$$\alpha_1 = \frac{P + Q\epsilon + R\epsilon^2}{M}; \qquad \alpha_2 = \frac{S + T\epsilon + U\epsilon^2}{M}$$

where P, Q, R, S, T, U, M are integers which are smaller than the previous integers p, q, r, s, t, u, m.

6. Numerical results

We present here four examples of computations. Some other expansions may be found on the web site [15]. We take, in a real cubic field of any signature, a positive unit $\epsilon > 1$ whose trace and norm are positive. E. Dubois, A. Farhane and R. Paysant-Le Roux [9] have proved that ϵ needs to be a Pisot number.

1. Let K be the real cubic field of discriminant -324 generated by the positive root α of the polynomial $x^3 - 3\,x^2 - 2$ and $\epsilon = 5\,\alpha^2 + \alpha + 3$ be the fundamental unit with trace$(\epsilon) = 57$.

There are 163 purely periodic expansions related to ϵ. For each AJP period matrix which is given by a periodic expansion of primitive length l there are $l - 1$ other period matrices deduced from this periodic expansion by the action of the cyclic permutation of order l. We say that we get a period class of length l. Here we have one class of length 1, 9 of length 2, 6 of length 4 and 24 of length 5.

2. Let K be the totally real cubic field of discriminant 621 generated by the positive root $\alpha > 9$ of the polynomial $x^3 - 9\ x^2 - 9\ x - 2$ and $\epsilon = 4\,\alpha^2 + 4\,\alpha + 1$ with $\mathrm{trace}(\alpha) = 435$. Then we get 2758 periodic expansions. The largest period length is 10. For example, if we take

$$\alpha_1{}^{(0)} = \frac{232\,\alpha^2 - 2168\,\alpha - 1308}{47}$$

and

$$\alpha_2{}^{(0)} = \frac{-206\,\alpha^2 + 1938\,\alpha + 1129}{47},$$

then we have a purely periodic expansion of length 10 :

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 6 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \end{pmatrix}$$

and the period matrix is

$$\begin{pmatrix} 215 & 192 & 182 \\ 104 & 93 & 28 \\ 150 & 134 & 127 \end{pmatrix}.$$

3. Let K be the real cubic field of discriminant -108 generated by the positive root α of the polynomial $x^3 - 3x^2 - 3x - 1$ and $\epsilon = \alpha$. There are 3 periodic expansions related to α : one period class of length 2 and one of length 1. If we take $\epsilon = \alpha^2$ then $\mathrm{trace}(\epsilon) = 15$ and there are 25 periodic expansions : one class of length 4, 6 classes of length 3 and the three other expansions are deduced from those related to α by doubling the period.

4. The case of the fields $\mathbb{Q}(\sqrt[3]{m})$. We have the following lemma :

Lemma 6.1. *Let $a \geq 1$ and b be two integers such that $a \geq b \geq 0$ and $\alpha > a$ the largest real root of the polynomial*

$$X^3 - aX^2 - bX - 1.$$

Then $(\alpha(\alpha - a), \alpha)$ has a purely periodic JPA expansion with period length equal to 1*.*

Let $a \geq 4$ be an integer and α the greatest real root of the polynomial

$$X^3 - aX^2 + 3X - 1.$$

Then $(a\alpha - \alpha^2 - 2, \frac{\alpha + \alpha(\alpha - a) + 1}{a - 3})$ has a purely periodic JPA expansion with period length equal to 3*.*

Proof. Left to the reader. For the second case, use the fact that the product of the two elements of the pair is equal to 1. □

Let $\omega = \sqrt[3]{m}$. For $m = 4$, the fundamental unit of the ring $\mathbb{Z}[\omega]$ is equal to $5 + 3\omega + 2\omega^2$. Its minimal polynomial is

$$X^3 - 15X^2 + 3X - 1.$$

The lemma gives a pair with period length equal to 3. But there are 25 pairs with period lengths 2, 3 or 4.

We remark that, nowadays, it is easier to get a fundamental unit with Pari. So we may easily treat the case $m = 17$.

For $m = 17$ the period length of the JPA expansion of (ω, ω^2) is equal to 61. The fundamental unit is the real root of the polynomial

$$X^3 - 972X^2 + 54X - 1.$$

There are 5637 pairs whose JPA expansion is purely periodic associated with this unit.

For the case where $m = D^3 + 6D$ with $D = 2K$ and $K \geq 2$, Bernstein obtained by his JPA expansion the square of the fundamental unit of $\mathbb{Q}(\omega)$.The fundamental unit, see Bernstein [6] and Stender [14], is the real root of the polynomial

$$X^3 - (12K^4 + 18K^2 + 3)X^2 + 3X - 1.$$

Then the lemma gives a pair with period length equal to 3 (instead of 8 in Bernstein's paper). For $K = 2,\ m = 267$, we get 838 pairs with period length from 2 to 9. For $K = 3$ we get 6040 pairs.

Remark. It seems that, for a fixed unit ϵ, the number of purely periodic expansions related to ϵ^k grows as k tends to infinity.

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[15] `http://www.math.univ-metz.fr/~rhin`

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