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The period-index problem in WC-groups IV: a local transition theorem

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Résumé. Soit K un corps de valuation discrète complet avec corps résiduel parfait k. En supposant des bornes supérieures pour la relation entre l'indice et la période pour des groupes de Weil-Châtelet sur k, nous déduisons des bornes supérieures correspondantes pour la relation entre l'indice et la période pour des groupes de Weil-Châtelet sur K. À une constante dépendant seulement de la dimension d'un torseur près, nous retrouvons des théorèmes de Lichtenbaum et Milne dans un contexte "sans dualité". Nos techniques utilisent les *modèles LLR* des torseurs sous des variétés abeliennes avec bonne réduction et une généralisation de l'obstruction période-indice à la cohomologie plate. Dans un appendice, nous considérons des sujets apparentés relevant de l'arithmétique du corps.

Abstract. Let K be a complete discretely valued field with perfect residue field k. Assuming upper bounds on the relation between period and index for WC-groups over k, we deduce corresponding upper bounds on the relation between period and index for WC-groups over K. Up to a constant depending only on the dimension of the torsor, we recover theorems of Lichtenbaum and Milne in a "duality free" context. Our techniques include the use of *LLR models* of torsors under abelian varieties with good reduction and a generalization of the period-index obstruction map to flat cohomology. In an appendix, we consider some related issues of a field-arithmetic nature.

Introduction

0.1. Notation and Terminology.

For a field K, we let K^{sep} denote a separable closure of K and $\overline{K}$ an algebraic closure of K. We write $\mathfrak{g}_K$ for $\mathrm{Gal}(K^{\mathrm{sep}}/K)$.

By a CDVF, we mean a field which is complete with respect to a discrete valuation.

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If $X_{/K}$ is an integral variety, let X^{reg} denote its regular locus and $I(X)$ the index of X, i.e., the gcd of all degrees of closed points on X.

For M a $\mathfrak{g}_K$-module and $\eta \in H^i(K,M)$ a Galois cohomology class, we denote by $P(\eta)$ and $I(\eta)$ the **period** and **index** of η (c.f. [WCII, §2]). Especially, if $A_{/K}$ is an abelian variety, then $H^1(K,A)$ is canonically isomorphic to the **Weil-Châtelet group** of A, which parameterizes **torsors** (X,μ) under A. Under this correspondence, we have $I(\eta) = I(X)$.

Let $G_{/K}$ be an algebraic group scheme. Then G gives rise to a sheaf of groups on the flat (fppf) site of $\operatorname{Spec} K$. We put $H^0(K,G) = G(K)$, and we denote by $H^1(K,G)$ the first flat cohomology, a pointed set. If G is commutative, for all $i \geq 0$ we have flat cohomology groups $H^i(K,G)$. Recall that flat and étale (= Galois, here) cohomology coincide when $G_{/K}$ is a smooth, commutative group scheme [Mil, Thm. 3.9], and we shall not be considering the étale cohomology of non-smooth group schemes, so we do not distinguish notationally between flat and Galois cohomology. We trust that no confusion will arise.

Recall that a principal polarization on an abelian variety $A_{/K}$ is a K-rational element λ of the Néron-Severi group $NS(A)$ such that the corresponding homomorphism $\varphi_\lambda : A \to A^\vee$ has the property that $(\varphi_\lambda)_{/K^{\mathrm{sep}}} = \varphi_{\mathcal{L}}$ for some ample line bundle $\mathcal{L} \in \operatorname{Pic}(A_{/\overline{K}})$. We say that a polarization λ is **strong** if the line bundle $\mathcal{L}$ can be chosen to be K-rational. The coboundary map in cohomology of the short exact sequence of $\mathfrak{g}_K$-modules

$$0 \to \mathbf{Pic}^0(A)(K^{\mathrm{sep}}) \to \mathbf{Pic}(A)(K^{\mathrm{sep}}) \to NS(A)(K^{\mathrm{sep}}) \to 0,$$

yields a homomorphism $\Phi_{\mathrm{PS}} : H^0(NS(A)) \to H^1(K,A^\vee)$ such that $\lambda \in NS(A)(K)$ is strong iff $\Phi_{\mathrm{PS}}(\lambda) = 0$. We recall from [PoSt, §4] that

$$\Phi_{\mathrm{PS}}(H^0(K,NS(A))) \subseteq H^1(K,A^\vee)[2].$$

It will then follow from Theorem 12 that any polarization can be made strong by passing to a field extension of degree at most $2^{2\dim A}$. Every polarization on an elliptic curve (E,O) is represented by the K-rational divisor $n[O]$ for some $n \in \mathbb{Z}^+$, hence is strong.

0.2. The Main Theorem.

Main Theorem.

Let K be a complete discretely valued field with perfect residue field k.
a) Suppose that there exists $i \in \mathbb{N}$ and a function $c : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that: for all abelian varieties $A_{/k}$ and all torsors $\eta \in H^1(k,A)$,

$$I(\eta) \leq c(\dim A)P(\eta)^i.$$

Then there exists a function $C : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that for all finite extensions L/K, all principally polarizable abelian varieties $A_{/L}$ and all torsors $\eta \in$

$H^1(L, A)$,

$$I(\eta) \leq C(\dim A) P(\eta)^{\dim A + i}.$$

b) Suppose that $\operatorname{char}(k) = 0$*, that there exists* $i \in \mathbb{N}$ *and a function* $c : \mathbb{Z}^+ \to \mathbb{Z}^+$ *such that: for all finite extensions* l/k*, all nontrivial abelian varieties* $A_{/l}$ *and all torsors* $\eta \in H^1(l, A)$*, we have*

$$I(\eta) \leq c(\dim A) P(\eta)^{\dim A + i - 1}.$$

Then there exists a function $C : \mathbb{Z}^+ \to \mathbb{Z}^+$ *such that for all finite extensions* L/K*, all principally polarizable abelian varieties* $A_{/L}$ *and all torsors* $\eta \in H^1(L, A)$,

$$I(\eta) \leq C(\dim A) P(\eta)^{\dim A + i}.$$

0.3. Outline of the paper.

The theorem as stated above is admittedly rather technical. So we believe that most (if not all) readers will benefit from a discussion which places it in a larger context. We do so at some length in §1, beginning in §1.1 by recalling some prior instances of "transition theorems" in Galois cohomology and field arithmetic. Throughout the rest of §1 we repeatedly give examples and remarks to show that many of the complications in the statement of the Main Theorem are necessary.

In §2 we introduce a new technical tool, the period-index obstruction map Δ in flat cohomology, which allows us to work with torsors with period divisible by the characteristic of the ground field, a case that was disallowed in our previous work on the subject. As a first indication that our formalism is a fruitful one, we derive a foundational result (Theorem 12) bounding the index of any torsor under an abelian variety A in terms of the period and $\dim A$, which had up until now only been established in special cases.

The philosophy of the present work is to analyze the local period-index problem from geometric perspective, more precisely in terms of the geometry of regular R-models of torsors. That we can proceed in this way is thanks in large part to two recent deep results on regular models due to Liu-Lorenzini-Raynaud and Gabber-Liu-Lorenzini. We begin §3 with careful statements of these results (the latter of which is, as of this writing, not yet publicly available). The rest of the section is devoted to a proof of the Main Theorem as well as a more precise result (Theorem 7) for iterated Laurent series fields over the complex numbers.

We have also included an Appendix, §4, which considers relations between the property of a field that all torsors under abelian varieties have rational points and some other, better known, properties of a field-arithmetic nature.

0.4. Acknowledgments.

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1. Motivating the Main Theorem

1.1. Transition theorems.

Recall the C_i property of fields: a polynomial with coefficients in K which is homogeneous polynomial of degree n in more than n^i variables has a nontrivial zero. A field C_0 if and only if it is algebraically closed [FMV, Lemma 3.2]. Moreover:

Theorem 1. *a) (Chevalley) A finite field is C_1.*
b) (Tsen) If K is C_i, then so is any algebraic extension L/K.
c) (Tsen) If K is C_i and L/K has transcendence degree j, then L is C_{i+j}.
d) (Lang) A CDVF with algebraically closed residue field is C_1.
e) (Greenberg) If k is C_i, then $k((t))$ is C_{i+1}.

Proof. See [Ch], [Ts], [La52], [Gr67]. □

Part d) can be rephrased as: if K is a CDVF with C_0-residue field, then K is C_1. This suggests that Greenberg's theorem might be generalized to the statement that a CDVF K with C_i residue field is C_{i+1}. In particular, E. Artin conjectured that p-adic fields are C_2. But this turned out to be false: $\mathbb{Q}_p$ is not C_i for any i.

More recently other numerical invariants of fields with properties analogous to those of Theorem 1 have been considered. Especially, in [CG] J.-P. Serre defined, for a prime number p, the **p-cohomological dimension** $\mathrm{cd}_p(K)$ of a field K as well as the **cohomological dimension** $\mathrm{cd}(K) = \sup_p \mathrm{cd}_p(K)$. The relation $\mathrm{cd}_p(k) \leq i$ satisfies all the properties of Theorem 1. Moreover, a C_1 field satisfies $\mathrm{cd}_p(k) \leq 1$ for all p and even the stronger property mentioned above: if K is a CDVF with residue field k satisfying $\mathrm{cd}_p(k) \leq i$, then $\mathrm{cd}_p(K) \leq i+1$ [CG, Prop. II.12].

In some cases there are relations between the cohomological dimension and rational points on certain K-varieties. For a perfect field K, $\mathrm{cd}(K) \leq 1$ is equivalent to the condition that any torsor under a connected linear group has a K-rational point. For a perfect field K, Serre conjectures that $\mathrm{cd}(K) \leq 2$ implies that every torsor under a simply connected, semisimple linear group has a K-rational point, which is now known to be true in (at least) many cases. Finally, Voevodsky's work gives an interpretation of $\mathrm{cd}_2(K) \leq i$ in terms of the u-invariant of quadratic forms.

There is a closely related problem on Brauer groups. Say a field K is **Br(d)** if for all finite extensions L/K and all $P \in \mathbb{Z}^+$, every class in $\mathrm{Br}(L)[P]$ has a splitting field of degree dividing P^d. A perfect field K is Br(0) – that is, $\mathrm{Br}(L) = 0$ for all finite L/K – iff $\mathrm{cd}(K) \leq 1$. Much of the content of class field theory is encoded in the statement that local and global fields are Br(1). It is not hard to show that if K is CDVF with perfect Br(i) residue field, then K is Br(i+1). On the other hand, one has the following "folk conjecture", which can be traced back (in the form of a question) to J.-L. Colliot-Thélène.

Conjecture 2. *If a field K is Br(d), then the rational function field $K(t)$ is Br(d+1).*

Indeed, some of the most exciting recent work in the field has been the verification of this conjecture in certain special cases: when $K = k(C)$ is the function field of a curve over (i) a p-adic field (Saltman [Sa]), (ii) an algebraically closed field (de Jong [deJ]), (iii) the function field of a curve over a finite field (Lieblich [Lie]). (In some cases, one must restrict to classes of period prime to p.) See also the recent paper [HKS] for more information on such results.

1.2. Property WC(i).

A field K is **WC(i)** if for every abelian variety $A_{/K}$ and every $\eta \in H^1(K, A)[P]$, the index of η divides P^i. In particular, a field is WC(0) iff every torsor under an abelian variety $A_{/K}$ has a K-rational point.

Remark 1.2.1: Let L/K be a finite field extension, $A_{/L}$ an abelian variety, and $B = \mathrm{Res}_{L/K}$ be the Weil restriction of B from L down to K. Shapiro's Lemma gives a canonical isomorphism $H^1(L, A) = H^1(K, B)$. It follows that the property WC(i) is automatically inherited by all algebraic field extensions.

Example 1.2.2: A PAC field is WC(0). In particular, separably closed $\Longrightarrow$ WC(0).[1]

Theorem 3. *(Lang,* [La56]*) A finite field is WC(0).*

Geyer and Jarden have constructed many non-PAC WC(0) fields [GeJa].

Example 1.2.3: The field $k = \mathbb{R}$ is not WC(0): e.g. the smooth model of

$$y^2 = -(x^4 + 1)$$

is a genus one curve without an $\mathbb{R}$-point. On the other hand, since $\mathfrak{g}_{\mathbb{R}} = \mathbb{Z}/2\mathbb{Z}$, every Galois cohomology group $H^i(\mathbb{R}, M)$ with $i > 0$ is 2-torsion. In particular, any nontrivial torsor under a real abelian variety has period equals index equals 2, so $\mathbb{R}$ is certainly WC(1).

[1] For more about the PAC property, see the Appendix.

In view of the local transition theorems for properties C_i and Br(i), one might guess that if K is a complete discretely valued field with WC(i) residue field k, then K is itself WC(i+1), at least in equal characteristic. However, this is generally very far from being the case, and is not even true in the (most favorable) case in which k is algebraically closed of characteristic 0.

For $g \in \mathbb{Z}^+$, let us write $\mathbb{C}_g$ for the iterated Laurent series field $\mathbb{C}((t_1))\cdots((t_g))$.[2] In particular $\mathbb{C}_1 = \mathbb{C}((t))$. The field $\mathbb{C}_g$ has cohomological dimension g at every prime p, has property C_g but not C_{g-1}, and has property Br(g-1). The absolute Galois group of $\mathbb{C}_g$ is isomorphic to $\hat{\mathbb{Z}}^g = (\varprojlim \mathbb{Z}/n\mathbb{Z})^g$.

The following two classical results limit the WC(i) properties of these fields:

Proposition 4. *(Lang-Tate* [LaTa, p. 678]*) Let $g, n \in \mathbb{Z}^+$ with $n > 1$.*
a) Let $E_{/\mathbb{C}_g}$ be any elliptic curve with $j(E) \in \mathbb{C}$.
Then there exists a torsor $\eta \in H^1(\mathbb{C}_g, E^g)$ with $P(\eta) = n,\ I(\eta) = n^g$.
b) Let $E_{/\mathbb{C}_{2g}}$ be any elliptic curve with $j(E) \in \mathbb{C}$.
Then there exists a torsor $\eta \in H^1(\mathbb{C}_{2g}, E^g)$ with $P(\eta) = n,\ I(\eta) = n^{2g}$.

Theorem 5. *(Shafarevich* [Ш]*) For every $n > 1$, there exists an abelian variety $A_{/\mathbb{C}((t))}$ and a torsor $\eta \in H^1(\mathbb{C}((t)), A)$ such that $P(\eta) = n$, $I(\eta) > n$.*

So $\mathbb{C}_1 = \mathbb{C}((t))$ is *not* WC(1).

1.3. Property Almost WC(i).

We do not know whether $\mathbb{C}((t))$ is WC(2). However, it is at least "very close":

Theorem 6. *There exists a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that: for any $g \in \mathbb{Z}^+$, any g-dimensional abelian variety $A_{/\mathbb{C}((t))}$ and any torsor $\eta \in H^1(\mathbb{C}((t)), A)$, we have*

$$I(\eta) \leq f(g) P(\eta)^2.$$

This motivates the following moderate loosening of the WC(i) property:

A field k is **almost WC(i)** if there exists a function $f(g)$ such that: for all finite extensions l/k, all abelian varieties $A_{/l}$ and all $\eta \in H^1(l, A)$, we have

$$I(\eta) \leq f(\dim A) P(\eta)^i.$$

Thus Theorem 6 asserts that $\mathbb{C}_1 = \mathbb{C}((t))$ is almost WC(2). Moreover, Proposition 4 shows that $\mathbb{C}((t))$ is *not* almost WC(0), whereas Theorem 5 does not rule out the possibility that $\mathbb{C}((t))$ is almost WC(1).

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[2] Here we are, of course, using $\mathbb{C}$ to denote the field of complex numbers, although any algebraically closed field of characteristic 0 would serve as well.

More generally, we shall prove:

Theorem 7. *For all $i \in \mathbb{Z}^+$, the field $\mathbb{C}_i$ is almost WC(i+1).*

Theorems 6 and 7 are almost immediate consequences of the proof of the Main Theorem, so we give their proofs at the end of §3.

In the other direction, Proposition 4 shows that $\mathbb{C}_i$ is not almost WC(i-1).

1.4. Property (almost) weakly (pp) WC(i).

The situation is much different for fields of mixed characteristic.

Proposition 8. *Let $K/\mathbb{Q}_p$ be an algebraic extension such that K has finite degree over its maximal unramified subextension. Then K is not WC(g) for any $g \in \mathbb{N}$.*

Proof. Let $g \in \mathbb{N}$. In [WCII, §3.2] we construct a finite field extension K_g/K, an elliptic curve $E_{/K_g}$ and a torsor $\eta \in H^1(K_g, E^g)$ with $P(\eta) = p$, $I(\eta) = p^{g+1}$. Thus the field K_g is not WC(g). The conclusion follows by applying Remark 1.2.1. $\square$

Nevertheless there are certainly nontrivial results on the period-index problem in WC-groups over p-adic fields, beginning with the celebrated theorem of Lichtenbaum that period equals index for genus one curves over a p-adic field. One of the main results of [WCII] is the following generalization:

Theorem 9. *Let $A_{/K}$ be a principally polarized abelian variety over a p-adic field. For $n \in \mathbb{Z}^+$, let $\eta \in H^1(K, A)[n]$. Assume that* at least *one of the following holds:*
(i) $\dim A = 1$.
(ii) n is odd.
(iii) There is a $\mathfrak{g}_K$-module isomorphism $A[n] \cong H \oplus H^*$, where H, H^* are Cartier dual $\mathfrak{g}_K$-modules, each isotropic for the Weil pairing.
Then $I(\eta) \leq (g!)\ P(\eta)^g$.

Proof. This is [WCII, Thm. 2]. $\square$

If K is a sufficiently large p-adic field, then the proof of Proposition 8 constructs torsors η under abelian varieties $A_{/K}$ of dimension g, period p and index p^g. Thus the bound of Theorem 9 is sharp up to a multiplicative constant.

This motivates the following definitions:

We say that K is **weakly WC(i)** (resp. **weakly pp WC(i)**) if: for all finite extensions L/K, all nontrivial abelian varieties $A_{/L}$ (resp. all nontrivial principally polarized abelian varieties $A_{/L}$) and all classes $\eta \in H^1(L, A)$, we have

$$I(\eta) \leq P(\eta)^{\dim A + i - 1}.$$

In particular, in any weakly pp WC(1) field k, the analogue of Lichtenbaum's theorem holds: period equals index for all genus one curves over all finite extensions of k.

Finally, we say that k is **almost weakly WC(i)** (resp. **almost weakly pp WC(i)**) if there exists a function $C(g)$ such that for all finite extensions l/k, all nontrivial abelian varieties $A_{/L}$ (resp. all nontrivial pp abelian varieties $A_{/L}$) and all classes $\eta \in H^1(L, A)$ we have

$$I(\eta) \leq C(g)P^{\dim A+i-1}.$$

With this terminology, Theorem 9 *nearly* asserts that a p-adic field is almost weakly pp WC(1), the drawback being the requirement of at least one of the additional hypotheses (i) through (iii). This drawback is removed by our Main Theorem.

1.5. Restatement of the Main Theorem.

Let us now restate our main result using the language of the previous sections.

Main Theorem. *Let K be a CDVF with almost weakly WC(i) residue field k.*
a) If k is moreover almost WC(i), then K is almost weakly pp WC(i+1).
b) If $\mathrm{char}(k) = 0$*, then K is almost weakly pp WC(i+1).*

1.6. Examples and remarks.

Remark 1.6.1: The hypothesis on the perfection of k is essential. Indeed, in [LLR, Remark 9.4] the authors construct a CDVF field K whose residue field is imperfect but separably closed (and hence WC(0) – c.f. the Appendix) of characteristic $p > 0$ and, for all integers $0 < r \leq n$, a genus one curve over K of period p^n and index p^{n+r}. Thus K is *not* almost weakly pp WC(1).

Example 1.6.2: Since $\mathbb{R}$ is almost WC(0), our Main Theorem shows that $K = \mathbb{R}((t))$ is almost WC(1). Other hand, K is *not* WC(1): there exists a genus one curve $C_{/K}$ of period 2 and index 4 [WCIII].

We saw above that a p-adic field has WC(0) residue field and is not almost WC(i) for any i. Here is a similar example in equicharacteristic 0:

Example 1.6.3: Let k be a Hilbertian PAC field of characteristic 0 [FA, Thm. 18.10.3]. Then k is WC(0), but for all $P > 1$, $\mathrm{Hom}(\mathfrak{g}_k, \mathbb{Z}/P\mathbb{Z})$ is infinite. So for every $g \in \mathbb{Z}^+$ and $P > 1$, there exists a Tate elliptic curve $E_{/K}$ and a torsor $X \in H^1(K, E^g)$ with period P and index P^g.

By a famous theorem of Lang, any finite field k is WC(0) [La56].[3] Thus the Main Theorem asserts that any **local field** – i.e., a finite extension of

[3] See the appendix for further discussion and a connection to an earlier result of F.K. Schmidt.

$\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$ – is weakly almost pp WC(1). As mentioned above, this should be compared with Theorem 9: in this case the Main Theorem applies to all torsors under abelian varieties at the cost of a larger function $C(g)$.

Especially, let us take $g = 1$ and compare with the theorems of Lichtenbaum and Milne. Our proof yields $I \mid 192P$ for genus one curves over a field K with perfect WC(0)-residue field. In particular $(6, P) = 1 \implies P = I$. Working a little more carefully and adding the hypothesis that $\mathfrak{g}_k$ is procyclic, one finds easily that $I \leq 16P$ and $I \mid 48P$. This leaves open the question of whether the full Lichtenbaum theorem extends to our setting or whether the use of Tate duality is essential. We hope to return to this point in a future work.

Another noteworthy feature of our Main Theorem is that, in contrast to the work of [Li68], [WCII], [WCII], torsors of period divisible by $\mathrm{char}(K)$ are *not* ruled out. However, Milne proved that period equals index for genus one curves over $\mathbb{F}_q((t))$ by establishing an analogue of Tate local duality in flat cohomology. Again, we recover Milne's theorem up to a constant in a general setting to which Tate duality seems inapplicable. Our new technique is to generalize the period-index obstruction map $\Delta : H^1(K, A[P]) \to \mathrm{Br}(K)$ to the setting of flat cohomology on $\operatorname{Spec} K$. Fortunately for us, we need only quite formal properties of Δ whose proofs are direct analogues of the corresponding ones in the étale case. In fact we also have work in progress on the explicit computation of Δ in terms of **symbol algebras** as is done in [WCI], [WCII] when $A[P]$ is an étale group scheme whose corresponding Galois representation has sufficiently small image: [WCV].

2. The period-index obstruction map in flat cohomology

Let K be an arbitrary field, $A_{/K}$ an abelian variety of dimension g and $P \in \mathbb{Z}^+$. Then the morphism $[P] : A \to A$ is an isogeny: in particular, on $\overline{K}$-points we have a short exact sequence

$$0 \to A[P](\overline{K}) \to A(\overline{K}) \stackrel{[P]}{\to} A(\overline{K}) \to 0. \tag{2.1}$$

We recall the following basic fact [Mum, §18]:

Proposition 10. *The following are equivalent:*
(i) The isogeny $[P] : A \to A$ is separable.
(ii) The finite K-group scheme $A[P]$ is étale.
(iii) The characteristic of k does not *divide P.*

When the equivalent conditions of Proposition 10 hold, $[P] : A \to A$ is an étale covering, so the fiber over any point $Q \in A(K^{\mathrm{sep}})$ is a finite étale algebra. It follows that the sequence

$$0 \to A[P](K^{\mathrm{sep}}) \to A(K^{\mathrm{sep}}) \stackrel{[P]}{\to} A(K^{\mathrm{sep}}) \to 0 \tag{2.2}$$

is exact.

2.1. Perfect fields of positive characteristic.

Suppose that K is a perfect field of characteristic $p > 0$, and let $P = p^k$, $k \in \mathbb{Z}^+$. In this case (2.1) and (2.2) are one and the same, so that (2.2) is exact. Letting $A[P]^\circ$ denote the maximal étale quotient of $A[P]$, we may reinterpret (2.2) as a short exact sequence of abelian sheaves on the small étale site of $\operatorname{Spec} K$

$$0 \to A[P]^\circ \to A \xrightarrow{[P]} A \to 0,$$

and taking étale = Galois cohomology we get

$$0 \to A(K)/PA(K) \to H^1(K, A[P]^\circ) \to H^1(K, A)[P] \to 0.$$

Example 2.1.1: Let $E_{/K}$ be a supersingular elliptic curve. Then $E[P]^\circ = 0$, so that every torsor under E has a K-rational point.

Returning to the general case, let $\eta \in H^1(K, A)[P]$ be any torsor under A, and choose any lift of η to $\xi \in H^1(K, A[P]^\circ)$. By [WCII, Prop. 12], ξ can be split by an extension of degree at most $\# A[P]^\circ$ and has index dividing $\# A[P]^\circ$. But $\# A[P]^\circ \mid P^g$, with equality iff A is *ordinary*. Since ξ splits $\implies$ η splits, we have the following result.

Proposition 11. *Let K be a perfect field of positive characteristic p, let P be a power of p and let $\eta \in H^1(K, A)[P]$. Then $I(\eta) \mid P(\eta)^g$.*

Remark 2.1.2: A similar argument applies to the "usual" Kummer sequence

$$1 \to \mu_p(\overline{K}) \to \mathbb{G}_m(\overline{K}) \xrightarrow{[p]} \mathbb{G}_m(\overline{K}) \to 1,$$

to give that

$$\operatorname{Br}(K)[p] = H^2(K, (\mu_p)^\circ) = H^2(K, 0) = 0.$$

2.2. The Kummer sequence in flat cohomology.

We return to the case of general K and P. In this case we have a short exact sequence of commutative algebraic K-group schemes

$$0 \to A[P] \to A \xrightarrow{[P]} A \to 0.$$

Viewing this as a short exact sequence of abelian sheaves on the flat site of K and taking flat cohomology we get

$$0 \to A(K)/PA(K) \to H^1(K, A[P]) \to H^1(K, A)[P] \to 0.$$

As an application of this formalism, we shall prove:

Theorem 12. *Let K be a field, $A_{/K}$ a g-dimensional abelian variety, and $\eta \in H^1(K, A)$ a torsor. Then $I(\eta) \mid P(\eta)^{2g}$.*

Remark 2.2.1: Special cases of this result are due to Lang-Tate [LaTa, p. 678], Lichtenbaum [Li70, Thm. 8] and Harase [Ha, Thm. 4].

Proof. We will show that any class $\xi \in H^1(K, A[P])$ splits over a field of degree at most P^{2g}. By the surjectivity of $H^1(K, A[P]) \to H^1(K, A)[P]$, this implies that any $\eta \in H^1(K, A)[P]$ has a splitting field of degree at most P^{2g}. By an easy primary decomposition argument as in [WCII, Prop. 12], we conclude that $I(\eta) \mid P^{2g}$.

To establish this, consider the K-group scheme $A[P]$: it is the group scheme of all automorphisms φ of A such that $[P] = [P] \circ \varphi$. Therefore, by the principle of descent, the flat cohomology group $H^1(K, A[P])$ classifies $(\overline{K}/K)$-twisted forms of $[P] : A \to A$, i.e., morphisms $q : X \to A$ fitting into a diagram

$$\begin{array}{ccc} X_{/\overline{K}} & \xrightarrow{\sim} & A_{/\overline{K}} \\ q\big\downarrow & & [P]\big\downarrow \\ A_{/\overline{K}} & \xrightarrow{1} & A_{/\overline{K}} \end{array}$$

In particular, $X_{/K}$ is a torsor under A and the map $q : X \to A$ has degree equal to the degree of $[P] : A \to A$, namely P^{2g}. Thus $q^*([O])$ is an effective K-rational zero-cycle on X of degree P^{2g}, so yields a closed point Q on X such that $[K(Q) : K] \le P^{2g}$. Over the field extension $K(Q)$, the morphism $q : (X, Q) \to (A, O)$ and is therefore isomorphic to $[n]_{/K(Q)} : A \to A$. It follows that $K(Q)$ is a splitting field for ξ, completing the proof. $\square$

Remark 2.2.2: In the earlier literature on the subject, together with the index one finds the **separable index**, the least positive degree of a K-rational divisor with support in K^{sep}. Our proof of Theorem 12 does *not* give an upper bound for the separable index. However, the recent preprint [LiuGa] shows that the index is equal to the separable index for smooth varieties over any field. In particular this applies to torsors under abelian varieties and gives that the separable index divides the $(2g)$th power of the period.

Corollary 13. *Let $(A, \lambda)_{/K}$ be a principally polarized abelian variety. Then there exists a field extension L/K of degree at most $2^{2\dim A}$ such that $(A, \lambda)_{/L}$ is* strongly *principally polarized.*

Proof. As recalled in §0.1, the obstruction to λ being strong is an element $\Phi_{\mathrm{PS}}(\lambda) \in H^1(K, A)[2]$. Now apply Theorem 12. $\square$

2.3. The period-index obstruction in flat cohomology.

Let $(A, \lambda)_{/K}$ be a strongly principally polarized abelian variety over an arbitrary field, and let $P \ge 2$ be an integer. Let $\mathcal{L}$ be the Pth multiple of the principal polarization. Then D. Mumford has defined a **theta group**

$$0 \to \mathbb{G}_m \to \mathcal{G}_{\mathcal{L}} \to A[P] \to 0, \tag{2.3}$$

in which $\mathbb{G}_m$ is the exact center [Mum, § 23]. When $\operatorname{char}(k) \nmid P$, (2.3) can be viewed as a sequence of sheaves of groups on the étale site of K. Because $\mathbb{G}_m$ is central in $\mathcal{G}_{\mathcal{L}}$, there is a connecting map in nonabelian Galois cohomology

$$\Delta : H^1(K, A[P]) \to H^2(K, \mathbb{G}_m).$$

The map Δ was first studied by C.H. O'Neil in the case of $\dim A = 1$ [O'N]. She had the fundamental insight that Δ is intimately related to the period-index problem, and accordingly she named Δ the **period-index obstruction map**.

Now we define the period-index obstruction map in arbitrary characteristic. To do this, it suffices to identify (2.3) as a central, short exact sequence of sheaves of groups on the flat site of $\operatorname{Spec} K$. We then get a connecting homomorphism

$$\Delta : H^1(K, A[P]) \to H^2(K, \mathbb{G}_m) = H^2(K, \mathbb{G}_m) = \operatorname{Br}(K).$$

We recall our conventions: whenever we write $H^i(K, G)$, it is understood that this is cohomology on the flat site of $\operatorname{Spec} K$. If $G_{/K}$ is smooth, this reduces to Galois cohomology. Since $A[P]$ is *not* smooth if $\operatorname{char}(k)$ divides P, $H^1(K, A[P])$ *cannot* be identified with a Galois cohomology group, and in particular is a richer object than the Galois cohomology group $H^1(K, A[P]^\circ)$ of §2.1.

The relation between Δ and the period-index problem is as follows. Let $\eta \in H^1(K, A)[P]$, and let ξ be any lift of η to $H^1(K, A[P])$. We again apply the principle of descent to characterize $H^1(K, A[P])$ as parameterizing a set of twisted forms of A endowed with extra structure. This time our fundamental object is the morphism into projective space determined by the ample, basepoint free line bundle $\mathcal{L}$: $\varphi : A \to \mathbb{P}^N$. It follows from the definition of the theta group scheme that $A[P]$ is precisely the group of translations τ_x of A which extend via φ to linear automorphisms $\gamma(\tau_x)$ of $\mathbb{P}^N$, i.e., which render the following diagram commutative:

$$\begin{array}{ccc} A & \xrightarrow{\tau_x} & A \\ \varphi\downarrow & & \downarrow\varphi \\ \mathbb{P}^N & \xrightarrow{\gamma(\tau_x)} & \mathbb{P}^N \end{array}$$

The twisted forms of φ are morphisms $X \to V$, where X is a torsor under A and V is a Severi-Brauer variety. There is therefore a corresponding class $[V]$ in the Brauer group of K. In [WCII, §5] it is shown that $\Delta(X \to V) = [V]$. The key step is that $\Delta : H^1(K, A[P]) \to H^2(K, \mathbb{G}_m)$ factors through $H^1_{\mathrm{fl}}(K, \mathrm{PGL}_N) = H^1_{\mathrm{et}}(K, \mathrm{PGL}_N)$ (smoothness again). The map $H^1(K, \mathrm{PGL}_N) \to H^2(K, \mathbb{G}_m)$ can be computed explicitly in terms of cocycles as in §5 of *loc. cit.*

This has the following consequence: suppose that $\eta \in H^1(K, A)$ is a class for which there exists at least one Kummer lift of η to $\xi \in H^1(K, A[P])$ such that $\Delta(\xi) = 0$. Then ξ corresponds to a morphism $f : X \to \mathbb{P}^N$ of degree equal to the degree of $\varphi_{\mathcal{L}}$, namely $(g!)P^g$. Intersecting the image of f with a suitable linear subvariety of $\mathbb{P}^N$ and pulling back via f, we get a K-rational zero cycle on X of degree $g!P^g$. We conclude that $I(\eta) \leq (g!)\ P^g$.

Theorem 14. *The period-index obstruction map is a quadratic map on abelian groups: the associated map*

$$B : H^1(K, A[P]) \times H^1(K, A[P]) \to \operatorname{Br}(K),\ (x, y) \mapsto \Delta(x+y) - \Delta(x) - \Delta(y)$$

is bilinear.

Proof. Again, the key is that $\Delta : H^1(K, A[P]) \to H^2(K, \mathbb{G}_m)$ factors through $H^1(K, \mathrm{PGL}_N)$, so Δ satisfies the same formal properties as the connecting homomorphism Δ' for the central short exact sequence of smooth group schemes

$$1 \to \mathbb{G}_m \to GL_N \to \mathrm{PGL}_N \to 1.$$

Finally, it follows from a theorem of Zarhin that Δ' is a quadratic map [Za]. $\square$

Corollary 15. *Define P^* to be P if P is odd and $2P$ if P is even. Then*

$$\Delta(H^1(K, A[P])) \subset \operatorname{Br}(K)[P^*].$$

Proof. This holds for any quadratic map between abelian groups: [WCII, §6.6]. $\square$

This has as a consequence the following relation between Brauer groups and WC-groups, a characteristic-unrestricted version of [WCII, Thm. 6].

Theorem 16. *Suppose that a field K is Br(d) for some $d \in \mathbb{N}$. Then K is almost weakly pp WC(d+1).*

Proof. Let L/K be a finite extension, and let $(A, \lambda)_{/L}$ be a principally polarized abelian variety of dimension g. By Corollary 13, up to replacing L by an extension field of degree at most 2^{2g}, we may assume that the polarization is strong.[4] Let $\eta \in H^1(L, A)$ be any class, and choose any lift of η to $\xi \in H^1(L, A[P])$. By Corollary 15, $\Delta(\xi) \in \operatorname{Br}(L)[2P]$, so by our Br(d) assumption, there exists a splitting field M/L for $\Delta(\xi)$ of degree at most $2^d P^d$. Since $\Delta(\xi|_M) = (\Delta\xi)|_M = 0$, it follows that there exists an M-rational zero-cycle on the corresponding torsor X of degree $(g!)P^g$, so altogether η has a splitting field of degree at most

$$2^{2g} 2^d P^d \cdot (g! P^g) = (2^{2g+d} g!) P^{\dim A + (d+1) - 1}.$$

$\square$

[4]We will need to use this observation several times in the sequel. We will not further belabor the point but just include a correction factor of $2^{2 \dim A}$ whenever we make use of the period-index obstruction map.

In the sequel, we will use the following special case: if $\mathrm{Br}(K) = 0$, then for any principally polarized abelian variety $A_{/K}$ and $\eta \in H^1(K, A)$, $I(\eta) \leq 2^{2g}(g!)\ P(\eta)^{\dim A}$.

3. Proof of the Main Theorem

3.1. Work of Gabber-Liu-Lorenzini, Liu-Lorenzini-Raynaud.

Theorem 17. *(*[LLR, Prop. 8.1]*) Let K be a discretely valued field with valuation ring R and perfect residue field k. Let $A_{/K}$ be an abelian variety with good reduction. Then any torsor V under A admits a proper regular model $X_{/R}$ endowed with an action $A \times_R X \to X$ extending the structure of a torsor under $A_{/K}$ on the generic fiber and such that the map $A \times_R X \to X \times_R X$ given by $(a, x) \mapsto (ax, x)$ is surjective. Then the reduced subscheme $X^{\mathrm{red}}_{/k}$ of the special fiber is a torsor under an abelian variety isogenous to $A_{/k}$.*

We will call a model $X_{/R}$ as in the statement of Theorem 17 an **LLR model**.

Theorem 18. *(Index Specialization Theorem* [GLL]*) Let K be a Henselian discretely valued field with valuation ring R and residue field k. Let X be regular scheme equipped with a proper flat morphism $X \to \operatorname{Spec} R$. Denote by $X_{/K}$ (resp. $X_{/k}$) the generic (resp. special) fiber of $X \to \operatorname{Spec} R$. Write X_k as $\sum_{i=1}^n r_i \Gamma_i$, where each Γ_i is irreducible and of multiplicity r_i in $X_{/k}$. Then*

$$I(X_{/K}) = \gcd_i r_i I(\Gamma_i^{\mathrm{reg}}).$$

3.2. Beginning of the proof.

Suppose K is complete, discretely valued field with perfect residue field k. If $\mathrm{char}(k) = 0$, we are assuming the existence of $i \in \mathbb{N}$ and a function $c : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that: for all finite extensions l/k, all nontrivial abelian varieties $A_{/l}$ and all torsors $\eta \in H^1(l, A)$, we have

$$I(\eta) \leq c(\dim A) P(\eta)^{\dim A + i - 1}.$$

If $\mathrm{char}(k) > 0$, we assume that there exists $i \in \mathbb{N}$ and a function $c : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that for all abelian varieties $A_{/k}$ and all torsors $\eta \in H^1(k, A)$,

$$I(\eta) \leq c(\dim A) P(\eta)^i.$$

In either case we wish to show that there exists a function $C : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that for all finite extensions L/K, all principally polarizable abelian varieties $A_{/L}$ and all torsors $\eta \in H^1(L, A)$,

$$I(\eta) \leq C(\dim A) P(\eta)^{\dim A + i}.$$

Both the hypothesis and the conclusion are stable under finite base extensions, so it is no loss of generality to assume $L = K$. Thus, let $A_{/K}$ be a principally polarizable abelian variety and $\eta \in H^1(K, A)$. Let $(X\, mu)$ be the corresponding torsor under A. By (common) abuse of notation we will omit the action μ in what follows.

3.3. Good reduction.

Suppose $A_{/K}$ has good reduction.

In the case when k is WC(0) we can prove somewhat sharper results, so we begin there and then discuss modifications necessary to establish the general case.

Let P_{unr} (resp. I_{unr}) denote the period (resp. the index) of the torsor X extended to the field K^{unr}. As for every field extension, we have $P_{\mathrm{unr}} \mid P$ and $I_{\mathrm{unr}} \mid I$. By Lang's theorem $\mathrm{Br}(K^{\mathrm{unr}}) = 0$, so by Theorem 16, $I_{\mathrm{unr}} \leq 2^{2g}(g!)\ P_{\mathrm{unr}}^g$.

Therefore the following result implies the Main Theorem in this case.

Proposition 19. *Suppose that A has good reduction and k is WC(0). Then*

$$P = P_{\mathrm{unr}},\ I = I_{\mathrm{unr}}.$$

Proof. Step 1: We claim that V does not split in K^{unr}. (In other words, since V is arbitrary, we claim that the relative WC-group $H^1(K^{\mathrm{unr}}/K, A)$ is trivial in the case of good reduction.) Indeed, suppose to the contrary that X admits a K^{unr}-rational point. Let $A_{/R}$ be the Néron model of A and $X_{/R}$ an LLR model. It follows that the special fiber of X is smooth. Now we use our WC(0): $X(k) \neq \varnothing$. By Hensel's Lemma, $V(K) \neq \varnothing$.

Step 2: Now consider the class $\eta' = P_{\mathrm{unr}}\eta$ in $H^1(K, A)$. We have

$$\eta'|_{K^{\mathrm{unr}}} = P_{\mathrm{unr}}\eta|_{K^{\mathrm{unr}}} = P_{\mathrm{unr}}(\eta_{K^{\mathrm{unr}}}) = 0,$$

so by Step 1 $\eta' = 0$. It follows that $P \mid P_{\mathrm{unr}}$, so $P = P_{\mathrm{unr}}$.

Step 3: We appeal to the LLR-model $X_{/R}$ of Theorem 17. The condition WC(0) implies that $X_{/k}^{\mathrm{red}}$ has a k-rational point. By the Index Specialization Theorem, the index of X is equal to the multiplicity of the special fiber. This quantity does not change upon unramified base extension, so $I = I_{\mathrm{unr}}$. This completes the proof of Proposition 19, and with it the good reduction case of Theorem 1.5a). □

Virtually the same argument works if k *almost* WC(0): i.e., if there exists a constant $c(g)$ such that for all principally polarized $A_{/k}$ and $\eta \in H^1(k, A)$, $I(\eta) \leq c(g)$. Let X be the torsor corresponding to η and $X_{/R}$ its LLR-model. By assumption, after making a field extension of degree at most $c(g)$, the reduced special fiber has a rational point. Again by index specialization, it follows that we can further trivialize the class by making a totally ramified extension of degree $I_{\text{unr}} \leq 2^{2g} g! P_{\text{unr}}^g \leq 2^{2g} g! P^g$. Thus overall V can be split by an extension of degree at most $2^{2g}(g!)\ c(g)\ P^g$.

Next we recall the following closely related result, which is essentially due to Lang and Tate (c.f. [LaTa, Cor. 1]).

Corollary 20. *Let K be a CDVF with perfect WC(0)-residue field k. Let $A_{/K}$ an abelian variety and $\eta \in H^1(K, A)$. If $\operatorname{char}(k)$ does not divide $P(\eta)$, then a finite field extension L/K splits X if and only if $e(L/K) \mid P(\eta)$.*

Proof. Let L/K be a finite extension splitting V. As for any finite extension of a local field, we can decompose it into a tower $L/M/K$, where M/K is unramified and L/M is totally ramified. By our previous results, we know that the unramified extension M/K is index-nonreducing: $I(V_{/L}) = I = P$. So certainly P divides $[L : M] = e(L/K)$. Conversely, suppose L/K is a finite extension with $P \mid e(L/K)$; we wish to show that L splits V. Decomposing L/K into $L/M/K$ as above, it is enough to show that L splits $V_{/M}$, i.e., we reduce to the case in which L/K is totally tamely ramified (ttr). By the structure of ttr extensions, there exists a unique degree P subextension, so we may further assume that $[L : K] = P$. But we know that there is at least one degree P ttr splitting field, and although there are in general several ttr extensions of degree P, the compositum of any two of them with K^{unr} coincide. It follows that each such L/K is a splitting field. □

Remark 3.3.1: In the case in which $k = \overline{k}$, we can say even more: by the criterion of Néron-Ogg-Shafarevich, we have $A[P] = A[P](K)$, and $A(K)$ is P-divisible, so the Kummer sequence trivializes to give isomorphisms

$$H^1(K, A)[P] = \operatorname{Hom}(\mathfrak{g}_K, A[P]) \cong \operatorname{Hom}(\mathfrak{g}_k, \mathbb{Z}/P\mathbb{Z})^{2g}$$

which are functorial with respect to restriction to finite extensions. Since tame ramification groups are cyclic, we get $\operatorname{Hom}(\mathfrak{g}_K, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/P\mathbb{Z}$, i.e., every element of $H^1(K, A)[P]$ is split by $k((t^{\frac{1}{P}}))$, the unique degree P extension of K.

Next, suppose that k is almost WC(i). Arguing as above, the Index Specialization Theorem gives us

$$I(\eta) = I_{\text{unr}}(\eta) I(X_{/k}^{\text{red}}). \tag{3.1}$$

Combining the estimate $I_{\text{unr}}(\eta) \leq 2^{2g}(g!)\ P(\eta)^{\text{unr}}$ of Theorem 16 and

$$I(X_{/k}^{\text{red}}) \leq c(g) P(X_{/k}^{\text{red}}))^i \leq c(g) P(\eta)^i,$$

we conclude

$$I(\eta) \leq 2^{2g}(g!)\ c(g)\ P(\eta)^{g+i},$$

establishing the Main Theorem in this case.

Finally, suppose that $\mathrm{char}(k) = 0$. Then by Corollary 20,

$$I_{\mathrm{unr}}(\eta) = P_{\mathrm{unr}}(\eta) \mid P(\eta).$$

Substituting this into (3.1) and using our hypothesis that $I(X^{\mathrm{red}}_{/k}) \leq c(g)P(\eta)^{g+i-1}$, we conclude that

$$I(\eta) \leq c(g)P(\eta)^{g+i}.$$

3.4. Purely toric reduction.

Proposition 21. *(Gerritzen) Let K be any field. Let $\tilde{A}$ be a $\mathfrak{g}_K$-module and $\Gamma \subseteq \tilde{A}$ a $\mathfrak{g}_K$-submodule which is torsionfree as a $\mathbb{Z}$-module and such that $\Gamma^{\mathfrak{g}_k} = \Gamma$; put $A := \tilde{A}/\Gamma$. Suppose also that $H^1(L, \tilde{A}) = 0$ for all finite extensions L/K. Let $\eta \in H^1(K, A)$ be a class of period P. Then:*
a) η has a unique *minimal splitting field $L = L(\eta)$.*
b) The extension L/K is abelian of exponent P.
c) $I(\eta) \mid P^g$, where $g = \dim_{\mathbb{Q}}(\Gamma \otimes \mathbb{Q})$ is the rank of Γ.

Proof. See [Ge] or [WCII, Prop. 16]. □

Now let $A_{/K}$ be a g-dimensional abelian variety with split toric reduction: the identity component of the Néron special fiber is $\mathbb{G}_m^g$. Then A admits an analytic uniformization: it is isomorphic, as a rigid K-analytic group, to $\Gamma \backslash \mathbb{G}_m^g$, where $\Gamma \cong \mathbb{Z}^g$ is a discrete subgroup.

Corollary 22. *Let K be any complete, discretely valued field, and let $A_{/K}$ be a g-dimensional analytically uniformized abelian variety. For any $\eta \in H^1(K, A)$, $I(\eta) \mid P(\eta)^g$.*

Proof. We apply Proposition 21 with $\tilde{A} = \mathbb{G}_m^g(K^{\mathrm{sep}})$. By Hilbert 90, $H^1(L, \tilde{A}) = 0$ for all finite L/K by Hilbert 90. The result follows immediately. □

Now assume A has purely toric reduction, not necessarily split. Then there exists a function $F_1(g)$ such that the torus splits over an extension of degree dividing $F_1(g)$. Thus, at the cost of possibly multiplying the ratio I/P by $F_1(g)$, we can reduce to the previous case, getting $I \mid F_1(g) \cdot P^g$.

3.5. General case.

We now recall a result due (in essence) to Bosch-Xarles [BoXa]. In its precise statement we follow [ClXa, Thm. 7], wherein the reader will also find an explanation of why it follows immediately from the work of [BoXa].

Theorem 23. *(Uniformization Theorem) Let $A_{/K}$ be a g-dimensional abelian variety over a complete field. Then there exists a semiabelian variety $S_{/K}$ of dimension g, whose abelian part has potentially good reduction,* *a $\mathfrak{g}_K$-module M whose underlying abelian group is torsion free of rank equal to the toric rank of S and an exact sequence of $\mathfrak{g}_K$-modules*

$$0 \to M \to S(K^{\mathrm{sep}}) \to A(K^{\mathrm{sep}}) \to 0. \tag{3.2}$$

Moreover $\mathrm{rank}_{\mathbb{Z}} M^{\mathfrak{g}_K} = s$*, the split toric rank of the Néron special fiber of A. For every finite extension L/K, the* identity components *of the Néron special fibers of S/L and $A_{/L}$ are isomorphic.*

Moreover, by making a base extension of degree depending only on g, we can achieve split semistable reduction:

Lemma 24. *For any positive integer g, there exists an integer $F(g)$ such that for any g-dimensional abelian variety A over a CDVF K, there exists an extension L/K of degree at most $F_1(g)$ such that $A_{/L}$ has split semistable reduction.*

Proof. To get semistable reduction, we can trivialize the Galois action on $A[3]$ (if the residue characteristic is not 3) or on $A[4]$ (if the residue characteristic is not 2), and to split the toric part of the reduction we need to trivialize the Galois action on a rank $d \leq g$ finite free $\mathbb{Z}$-module, which can be done over an extension of degree at most $\#GL_d(\mathbb{Z}/3\mathbb{Z}) \leq \#GL_g(\mathbb{Z}/3\mathbb{Z})$. Thus we could take

$$F_1(g) = (\max \#GSp_{2g}(\mathbb{Z}/3\mathbb{Z}), \#GSp_{2g}(\mathbb{Z}/4\mathbb{Z})) \cdot \#GL_g(\mathbb{Z}/3\mathbb{Z}).$$

□

After making the reduction split semistable, the exact sequence (3.2) above becomes

$$0 \to \mathbb{Z}^{\mu} \to S(K^{\mathrm{sep}}) \to A(K^{\mathrm{sep}}) \to 0, \tag{3.3}$$

where $S_{/K}$ is a semiabelian variety of the form

$$0 \to \mathbb{G}_m^{\mu} \to S \to B \to 0, \tag{3.4}$$

and B is abelian with good reduction. Taking $\mathfrak{g}_K$-cohomology of (3.3) gives

$$0 \to H^1(K,S) \to H^1(K,A) \xrightarrow{\delta} H^2(K,\mathbb{Z})^{\mu} \to \ldots.$$

Moreover, taking $\mathfrak{g}_K$-cohomology of (3.4) and applying Hilbert 90, we get an injection

$$H^1(K,S) \hookrightarrow H^1(K,B).$$

Finally, we have

$$H^2(K,\mathbb{Z})^{\mu} = \mathrm{Hom}(K,\mathbb{Q}/\mathbb{Z})^{\mu}.$$

So, starting with any class $\eta \in H^1(K,A)[P]$, put $\xi = \delta(\eta) \in H^2(K,\mathbb{Z})$. There is an abelian extension L/K of exponent dividing P and degree dividing P^{μ} which splits ξ, so that $\eta|_L \in H^1(L,S) \hookrightarrow H^1(L,B)$. Since B

is an abelian variety of dimension $g - \mu$ with good reduction, the work of Section §3.2 applies to show that $\eta|_L$ can be split over an extension M/L of degree at most $C(g-\mu)P^{g-\mu+i}$, so overall η can be split by an extension M of degree at most $C(g-\mu)F_1(g)P^{g+i} \leq C'(g)F_1(g)P^{g+i}$, where $C'(g) = \max_{1 \leq j \leq g} C(j)$. This completes the proof of the Main Theorem.

3.6. $K = \mathbb{C}_g$.

We now give the proof of Theorems 6 and 7.

First, let $K = \mathbb{C}((t))$ and let $A_{/K}$ be an abelian variety of dimension g. In Corollary 20 we established that if A has good reduction, any class $\eta \in H^1(K, A)$ has period equals index. At the other extreme, if A has split multiplicative reduction, then $H^1(K, A)[P]$ injects into $X = \operatorname{Hom}(\mathfrak{g}_K, (\mathbb{Z}/P\mathbb{Z})^g)$. However, because $\mathfrak{g}_K \cong \hat{\mathbb{Z}}$ is procyclic, any element of $\operatorname{Hom}(\mathfrak{g}_K, (\mathbb{Z}/P\mathbb{Z})^g)$ splits over a degree P field extension, thus period equals index in this case as well. Finally, after making an unramified field extension of bounded degree $f(g)$ to attain split semistable reduction of A, the dévissage argument of §3.4 shows that we can split any class $\eta \in H^1(K, A)$ by first splitting its purely toric part and then splitting its good reduction part, getting overall a splitting field of degree at most $f(g)P^2$. This completes the proof of theorem 6.

Now let $K = \mathbb{C}_2 = \mathbb{C}((t_1))((t_2))$. This time $\mathfrak{g}_K \cong \hat{\mathbb{Z}}^2$, so that if A has split multiplicative reduction the sharp upper bound on the index is P^2. Once again, after making a base extension of degree $f(g)$ to ensure split semistable reduction, the critical case is getting an upper bound on the index in the case of good reduction. But now something extremely fortunate occurs: both the residue field k and the maximal unramified extension K^{unr} are isomorphic to $\mathbb{C}_1 = \mathbb{C}((t))$! So, by a now familiar argument with the Index Specialization Theorem, we can split a class of period P by an extension of degree at most $f(2)P^2 \cdot P = f(2)P^3$. The general case follows by an obvious inductive argument.

4. Appendix: Some Related Field Arithmetic

4.1. PAC and I1 fields.

A field k is **PAC** (pseudo-algebraically closed) if every geometrically integral k-variety has a k-rational point. A field k which admits a geometrically integral variety $V_{/k}$ of positive dimension with only finitely many k-rational points cannot be PAC: apply the definition to the complement of the set of rational points! In particular a finite field is not PAC.

Remark 4.1.1: Since every geometrically integral variety of positive dimension over an infinite field contains a geometrically integral curve [FA, Cor. 10.5.3], it suffices to verify the PAC condition on geometrically integral curves.

Example 4.1.2: All algebraically and separably closed fields are PAC. An infinite algebraic extension of a finite field is PAC. A nonprincipal ultraproduct of finite fields is PAC. Certain large algebraic extensions of $\mathbb{Q}$ are PAC. Any algebraic extension of a PAC field is PAC.

A field k has property **I1** if every geometrically irreducible variety over k has a k-rational zero-cycle of degree one. Again, it suffices to check this for curves.

Example 4.1.3: Any PAC field is I1. A theorem of F.K. Schmidt [Sc] implies that every geometrically integral curve over a finite field $\mathbb{F}$ has a zero-cycle of degree 1. Applying the above remark about curves lying on higher dimensional varieties to the maximal pro-p and maximal pro-q extensions of $\mathbb{F}$ for primes $p \neq q$, one sees that $\mathbb{F}$ is I1.

Example 4.1.4: Geyer and Jarden define a **weakly PAC** field to be a field k such that for every geometrically integral variety V such that $V_{/\overline{k}}$ is birational either to projective space ("Type 0") or to an abelian variety ("Type 1") already has a K-rational point. Evidently weakly PAC implies WC(0), so the following result gives many examples of WC(0) fields:

Proposition 25. *(Geyer-Jarden* [GeJa, Lemma 2.5]*) For any countable field k_0, there exists a countable extension field k with the following properties:*
(i) k is weakly PAC.
(ii) For every smooth curve $C_{/k_0}$ of genus at least 2, $C(k) = C(k_0)$.

4.2. Two implications.

It is well known to the experts that the property I1 implies the property Br(0). In this section we will factor this implication through the property WC(0).

Proposition 26. *Let k be a I1 field, and let $G_{/k}$ be a connected commutative algebraic group. Then $H^1(k, G) = 0$. In particular, k is WC(0).*

Proof. The elements of $H^1(k, G)$ correspond to torsors X under the group G, and as in the case of WC-groups, the period of X – i.e., its order in the torsion group $H^1(k, G)$ – divides its index, which is equal to the least positive degree of a zero-cycle on X. The result follows immediately. □

Remark 4.2.1: It seems to be unknown whether the converse holds, nor even whether all of the WC(0)-fields constructed by Geyer-Jarden have the I1 property.

Theorem 27. *A perfect WC(0) field k has the property Br(0).*

Proof. Step 0: It is enough to show that for any finite extension k'/k and any prime number ℓ, $\mathrm{Br}(k')[\ell] = 0$. However, by our hypotheses are stable under finite base change, so we may as well assume $k' = k$. Further, let $p \geq 0$ be the characteristic of k. In case $p > 0$, the assumed perfection of k implies that $\mathrm{Br}(k)[p] = 0$ [CG, §II.3.1], and hence $\mathrm{Br}(k')[p] = 0$. Thus we may assume that ℓ is different from the characteristic of k.

Step 1: We recall the following celebrated theorem of Merkurjev-Suslin: Let k be a field of characteristic $p \geq 0$ and n is a positive integer indivisible by p. If k contains the nth roots of unity, then $\mathrm{Br}(k)[n]$ is generated by the order n norm-residue symbols $\langle a, b\rangle_n$ as a, b run though $k^\times/k^{\times n}$ [MeSu].

Step 2: We recall a consequence of the theory of the period-index obstruction map Δ: let ℓ be a prime number, M a field of characteristic not equal to ℓ, and $E_{/M}$ an elliptic curve with full ℓ-torsion rational over M: $\#E(M)[\ell] = \ell^2$. (By the Galois-equivariance of the Weil pairing, this implies that M contains the ℓth roots of unity.) Then $H^1(M, E[\ell]) \cong (M^\times/M^{\times \ell})^2$, and for $\ell > 2$, the map $\Delta : H^1(M, E[\ell]) \rightarrow \mathrm{Br}(M)$ is of the form $(a, b) \mapsto \langle C_1 a, C_2 b\rangle_\ell - \langle C_1,\ C_2\rangle_\ell$ for suitable $C_1, C_2 \in K^\times$. (More precise results are now known, but this weak version has the advantage of treating $\ell = 2$ and $\ell > 2$ uniformly, so is useful for our present purpose.) It follows that the image of Δ contains every norm residue symbol, and therefore, by Merkurjev-Suslin, the subgroup generated by the image is all of $\mathrm{Br}(M)[\ell]$. So, if for our particular elliptic curve $E_{/M}$ we have $H^1(M, E)[\ell] = 0$, it follows that $\mathrm{Br}(M)[\ell] = 0$.

Step 3: If $\ell = 2$, this strategy succeeds in a straightforward manner: over every field k of characteristic different from 2, there exists an elliptic curve with full 2-torsion, namely $y^2 = x(x-1)(x-2)$. So if every hyperelliptic quartic curve over k has a rational point, every conic over k has a rational point.

Step 4: If $\ell > 2$, we need not have an elliptic curve $E_{/k}$ with full ℓ-torsion, since in particular we need not have the ℓth roots of unity rational over k. So we employ a trick. Let $E_{/k}$ be any elliptic curve with complex multiplication by any imaginary quadratic field. Let $m = k(E[\ell])$. Then $[m : k]$ divides either $2(\ell^2 - 1)$ or $2(\ell - 1)^2$, so in particular is prime to ℓ. It follows that the restriction map $\mathrm{Br}(k)[\ell] \rightarrow \mathrm{Br}(m)[\ell]$ is injective, so it suffices to show that $\mathrm{Br}(m)[\ell] = 0$. By hypothesis we have $H^1(m, E)[\ell] = 0$, so that applying Step 2 we conclude that $\mathrm{Br}(m)[\ell] = 0$ and hence $\mathrm{Br}(k)[\ell] = 0$. □

Example 4.2.2: Br(0) does *not* imply WC(0). This goes back to work of Ogg and Shafarevich: neither $\mathbb{C}(t)$ nor $\mathbb{C}((t))$ is a WC(0) field. In fact, if $A_{/\mathbb{C}((t))}$ is a g-dimensional abelian variety with good reduction, then consideration of the Kummer sequence

$$0 \rightarrow A(k)/nA(k) \rightarrow H^1(k, A[n]) \rightarrow H^1(k, A)[n] \rightarrow 0$$

swiftly yields $H^1(k, A) \cong (\mathbb{Q}/\mathbb{Z})^{2g}$.

Remark 4.2.3: By a restriction of scalars argument, the proof easily gives that for any ℓ prime to the characteristic of k, if $\mathrm{Br}(k)[\ell] \neq 0$, then there exists a principally polarized abelian variety $A_{/k}$ of dimension at most $2(\ell^2 - 1)$ such that $H^1(k, A)[\ell] \neq 0$.

Remark 4.2.4: It follows easily that weakly WC(0) implies $\mathrm{Br}(k)[\ell] = 0$ for all sufficiently large primes ℓ. Indeed, weakly WC(0) implies that there exists a fixed prime ℓ_0 such that for all $\ell > \ell_0$ and all principally polarized abelian varieties $A_{/k}$ we have $H^1(k, A)[\ell] = 0$.

Remark 4.2.5: It seems likely that the result is true without the assumption on the perfection of k. The characteristic p analogue of the Merkurjev-Suslin theorem is the (much earlier and easier) theorem of Teichmüller: for all $r \geq 1$, $\mathrm{Br}(k)[p^r]$ is generated by cyclic algebras [Te], [GiSz]. To make use of this we need to know the explicit form of the period-index obstruction map in the flat case, so we defer the issue to [WCV].

Example 4.2.6: Let $\mathbb{F}$ be a finite field. By elementary arguments involving the zeta function, F.K. Schmidt showed that the index of any nice curve over a finite field is 1. It follows from Remark 4.1.1 that finite fields are I1. Using Proposition 26 and Theorem 27 we deduce two more famous theorems: Lang's theorem that a torsor under a connected, commutative algebraic group has a rational point, and Wedderburn's theorem that the Brauer group of a finite field is trivial.

Next we recall the following theorem:

Theorem 28. *(Steinberg,* [St]*) Let k be a perfect $Br(0)$ field and $G_{/k}$ a smooth, connected algebraic group. Then $H^1(k, G) = 0$.*

Corollary 29. *If k is a perfect field such that $H^1(k, A) = 0$ for all abelian varieties A/k, then $H^1(k, G) = 0$ for all connected algebraic groups $G_{/k}$.*

Proof. Recall that every connected algebraic group G over a perfect field k admits a **Chevalley decomposition** [BLR, Ch. IX]: there is a (unique) normal linear subgroup L of G such that $G/L = A$ is an abelian variety. The result now follows "by dévissage", using the exact sequence of [CG, § I.5.5, Prop. 38]. □

Finally, the proof of Theorem 27 leads naturally to the following question.

Question 1. *Let k be a field. Characterize the set of all elements in the Brauer group which arise as the obstruction associated to a k-rational divisor class on some genus one curve $C_{/k}$. Is it, for instance, the class of all cyclic algebras?*

Remark 4.2.7: It is also possible to ask the question on the level of central simple algebras. The Severi-Brauer variety associated to a degree $n \geq 2$ divisor class on a genus one curve C is a twisted form of $\mathbb{P}^{n-1}$, and the corresponding central simple algebra has degree $n - 1$. The problem can also be stated geometrically: find all Severi-Brauer varieties $V_{/K}$ such that there exists a genus one curve $C_{/K}$ and a morphism $C \to V$.

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