

JOURNAL

de Théorie des Nombres

de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 22, n° 2 (2010), p. 483-512.

http://jtnb.cedram.org/item?id=JTNB_2010__22_2_483_0

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The local Jacquet-Langlands correspondence via Fourier analysis

par JARED WEINSTEIN

RÉSUMÉ. Soit F un corps local non archimédien et localement compact, et soit B/F un corps de quaternions. La correspondance de Jacquet-Langlands fournit une bijection entre les représentations lisses et irréductibles de B^\times de dimension > 1 et les représentations cuspidales et irréductibles de $\mathrm{GL}_2(F)$. Nous présentons une nouvelle construction de cette bijection pour laquelle la préservation des facteurs epsilon est automatique. Nous construisons une famille de paires (\mathcal{L}, ρ) , où $\mathcal{L} \subset M_2(F) \times B$ est un ordre et ρ est une représentation d'une certaine sous-groupe de $\mathrm{GL}_2(F) \times B^\times$ qui contient \mathcal{L}^\times . Soit $\pi \otimes \pi'$ une représentation irréductible de $\mathrm{GL}_2(F) \times B^\times$; nous prouvons que $\pi \otimes \pi'$ contient une telle ρ si et seulement si π est cuspidale et correspond à $\check{\pi}'$ sous la correspondance de Jacquet-Langlands. On y voit tous les π et les π' . L'égalité des facteurs epsilon est réduite à un calcul Fourier-analytique sur un anneau quotient de \mathcal{L} .

ABSTRACT. Let F be a locally compact non-Archimedean field, and let B/F be a division algebra of dimension 4. The Jacquet-Langlands correspondence provides a bijection between smooth irreducible representations π' of B^\times of dimension > 1 and irreducible cuspidal representations of $\mathrm{GL}_2(F)$. We present a new construction of this bijection in which the preservation of epsilon factors is automatic. This is done by constructing a family of pairs (\mathcal{L}, ρ) , where $\mathcal{L} \subset M_2(F) \times B$ is an order and ρ is a finite-dimensional representation of a certain subgroup of $\mathrm{GL}_2(F) \times B^\times$ containing \mathcal{L}^\times . Let $\pi \otimes \pi'$ be an irreducible representation of $\mathrm{GL}_2(F) \times B^\times$; we show that $\pi \otimes \pi'$ contains such a ρ if and only if π is cuspidal and corresponds to $\check{\pi}'$ under Jacquet-Langlands, and also that every π and π' arises this way. The agreement of epsilon factors is reduced to a Fourier-analytic calculation on a finite ring quotient of \mathcal{L} .

1. Introduction

Let F be a non-Archimedean local field, *i.e.* a finite extension either of \mathbf{Q}_p or of the field of Laurent series over the finite field \mathbf{F}_p . Let B/F be a central

division algebra of dimension n^2 . The Jacquet-Langlands correspondence assigns to each irreducible admissible representation π' of B^\times a square-integrable representation π of $\mathrm{GL}_n(F)$. The passage $\pi' \mapsto \pi$ is characterized by a character relation. It also manifests as a relationship between epsilon factors, see for instance [DKV84]. When $n = 2$, the collection of epsilon factors of the twists of π by characters determines π up to isomorphism, so that the Jacquet-Langlands correspondence is characterized completely by its preservation of epsilon factors. In this case the reciprocity between $\mathrm{GL}_n(F)$ and B^\times was proved by Jacquet and Langlands [JL70] in both the local and global settings. In the case of a division algebra B in characteristic 0 it was established for all n by [Rog83]. The case of a general inner form of $\mathrm{GL}_n(F)$ was carried out by Deligne, Kazhdan and Vignéras in [DKV84] in characteristic 0 and Badulescu [Bad02] in characteristic p . Each of these cases was accomplished by embedding the local problem into a global one and then applying trace formula methods.

There has also been a great deal of effort to construct the Jacquet-Langlands correspondence in an explicit manner using purely local techniques. The simplest case is when π' and π are both associated to a so-called “admissible pair” (E, θ) , where E/F is a field extension of degree n and θ is a character of E^\times . (All supercuspidal π will arise this way if $p \nmid n$.) In this case the corresponding π was constructed explicitly by Howe [How77]; Gérardin [Gér79] constructed the representation π' and proved that the epsilon factors of π and π' agree. Henniart [Hen93b] showed that if n is a prime distinct from p the representations π and π' so constructed have the correct character identity. Using the technology of types laid down by Bushnell and Kutzko in [BK93], Henniart and Bushnell construct the explicit correspondence in the case of $n = p$ in [BH00]. The case of n a power of p with p odd and π totally ramified is carried out in [BH05].

In this paper we present a novel approach to the passage $\pi' \mapsto \pi$ in the case $n = 2$ in such a way that the preservation of epsilon factors is manifest in the construction. Our approach is entirely Fourier-analytic, and there is no special treatment needed for the case $p = 2$. In that sense it is similar to Gérardin-Li [GL85]. Unlike that paper, however, our method is linked to the theory of strata developed for GL_n in [BK93]. That theory is summarized in Section 2.2. Roughly speaking, a stratum for GL_2 is a certain sort of character of a compact open subgroup of $\mathrm{GL}_2(F)$. Then irreducible representations of $\mathrm{GL}_2(F)$ can be conveniently classified according to which strata they contain. There is a notion of simple stratum: these are parametrized by certain regular elliptic elements $\beta \in \mathrm{GL}_2(F)$. It can be shown that an admissible representation of $\mathrm{GL}_2(F)$ contains a simple stratum if and only if it is supercuspidal. A similar notion of stratum exists for B^\times , and strata for B^\times are easily seen to be more or less the same

objects as simple strata for $GL_2(F)$. It is therefore natural to try to define the correspondence $\pi' \mapsto \pi$ relative to each stratum.

Let S be a simple stratum associated to the regular elliptic element $\beta \in GL_2(F)$, and let S' be the stratum in B^\times corresponding to S . We choose an embedding of the field $E = F(\beta)$ into B . Let $\Delta: E \rightarrow M_2(F) \times B$ be the diagonal map. We construct what we have called a “linking order” \mathcal{L}_S inside $M_2(F) \times B$; this is a $\Delta(\mathcal{O}_E)$ -order defined by certain congruence conditions. We then define an irreducible (and thus finite-dimensional) representation ρ_S of the unit group \mathcal{L}_S^\times which is trivial on $\Delta(\mathcal{O}_F^\times)$. In the case where E/F is unramified, the construction of ρ_S comes from the Weil representation of SL_2 over a finite field. However, we also give a geometric construction using ℓ -adic cohomology which is well-suited to our purposes. Then loosely speaking, the induction of ρ_S to $GL_2(F) \times B^\times$ will realize the Jacquet-Langlands correspondence for those representations π which contain S .

To make this precise, we must pay careful attention to the role of the center $Z = F^\times \times F^\times$ of $GL_2(F) \times B^\times$. Choose a character ω of $F^\times = F^\times \times 1$ which extends $\rho_S|_{(F^\times \times 1) \cap \mathcal{L}_S^\times}$. We will give a recipe for an extension of ρ_S to the group $\mathcal{K}_S = \Delta(E^\times)Z\mathcal{L}_S^\times \subset GL_2(F) \times B^\times$ whose restriction to Z is $(g, h) \mapsto \omega(gh^{-1})$. Call this representation $\rho_{S,\omega}$.

Let $\Pi_{S,\omega}$ be the compactly supported induction of $\rho_{S,\omega}$ up to $GL_2(F) \times B^\times$. Then $\Pi_{S,\omega}$ is the direct sum of irreducible representations $\pi \otimes \pi'$ of $GL_2(F) \times B^\times$; here π must have central character ω and π' must have central character ω^{-1} . We show that Π_S realizes the Jacquet-Langlands correspondence relative to the stratum S and the character ω in the following sense. First, we show that a representation π of $GL_2(F)$ (resp., B^\times) of central character ω (resp., ω^{-1}) appears in Π_S if and only if π (resp., the contragredient $\check{\pi}$) contains S (resp., S'). Then, we show that an irreducible admissible representation $\pi \otimes \check{\pi}'$ of $GL_2(F) \times B^\times$ appears inside of $\Pi_{S,\omega}$ if and only if the epsilon factors of twists of π and π' agree up to a minus sign:

$$(1.0.1) \quad \varepsilon(\pi\chi, s, \psi) = -\varepsilon(\pi'\chi, s, \psi).$$

Here χ runs through sufficiently many characters of F^\times to determine π and π' uniquely. Therefore if π is a given supercuspidal irreducible representation of $GL_2(F)$ which contains the stratum S , then $\text{Hom}_{GL_2(F)}(\pi, \Pi_S)$ is a sum of copies of a single supercuspidal representation π' of B^\times . Then the contragredient representation of π' is the one corresponding to π under the Jacquet-Langlands correspondence. It must be stressed that our approach does not yield a proof of the Jacquet-Langlands correspondence *de novo*. One must be able to deduce the correct character identity from Eq. 1.0.1. For this, we refer the reader to [BH06], §56, where a proof of the correspondences is sketched in a series of exercises. Our approach may nonetheless

be of interest because it avoids the computation of any particular epsilon factors.

The linking orders \mathcal{L}_S are constructed in Section 4. We also define corresponding additive characters ψ_S of the ring $M_2(F) \times B$ for which the \mathcal{O}_F -module

$$\mathcal{L}_S^* = \left\{ x \in M_2(F) \times B \mid \psi_S(x\mathcal{L}_S) = 1 \right\}$$

happens to be a two-sided ideal in \mathcal{L}_S . The required representation ρ_S of \mathcal{L}_S^\times is inflated from a representation of the unit group of the finite k -algebra $\mathcal{R}_S = \mathcal{L}_S/\mathcal{L}_S^*$. The additive character ψ_S descends to a nondegenerate additive character of this ring, so that we have a theory of Fourier transforms $f \mapsto \mathcal{F}_S f$ for functions f on \mathcal{R}_S . The characteristic property of ρ_S is that its matrix coefficients f , considered as functions on \mathcal{R}_S supported on \mathcal{R}_S^\times , satisfy the functional equation

$$(1.0.2) \quad \mathcal{F}_S f(y) = \pm f(y^{-1})$$

for $y \in \mathcal{R}_S^\times$; see Prop. 5.2.1 and Theorem 5.0.3. (The sign in this equation depends only on S .) The functional equation in Eq. 1.0.2 on the level of finite rings is used in Section 6 to deduce the functional equation in Eq. 1.0.1 concerning constituents of the induced representation of ρ_S up to $\mathrm{GL}_2(F) \times B^\times$.

The reader may be wondering if this sort of strategy may be extended to the general case of GL_n , where one still lacks a complete local proof of the existence of the correspondences. It will not be difficult to extend the definitions of \mathcal{L}_S , ρ_S , and $\Pi_{S,\omega}$ to this context. In doing so one would produce a recipe for some sort of correspondence $\pi' \mapsto \pi$ for π supercuspidal which satisfies Eq. 1.0.1 for a certain collection of characters χ . For $n = 3$, we do not know if this collection of characters is enough to characterize the Jacquet-Langlands correspondence. And for $n > 4$, the establishment of Eq. 1.0.1 for *all* characters is not enough to characterize the correspondence. Indeed, the epsilon factors of pairs of representations are required to characterize the isomorphism class of a given representation π , see [Hen93a]. One would have to work harder to obtain access to the characters of the representations π and π' so constructed in order to prove the right identity.

The present effort fits into a larger program concerning the geometry of Lubin-Tate curves. Suppose F has uniformizer π_F and residue field k . Let \mathcal{F}_0 be a formal \mathcal{O}_F -module of height 2 over the algebraic closure of the residue field k of F . For each $m \geq 0$, consider the functor that assigns to each complete local Noetherian $\hat{\mathcal{O}}_{F^{\mathrm{nr}}}$ -algebra A having residue field \bar{k} the set of one-dimensional formal \mathcal{O}_F -modules \mathcal{F} over A equipped with an isomorphism $\mathcal{F}_0 \rightarrow \mathcal{F}_{\bar{k}}$ and a Drinfeld π_F^m -level structure. This functor is represented by a formal curve X_m over $\hat{\mathcal{O}}_{F^{\mathrm{nr}}}$. The inverse system of curves

$(X_m)_{m \geq 1}$ admits an action by a subgroup \mathcal{G} of the triple product group $\mathrm{GL}_2(F) \times B^\times \times W_F$ of “index \mathbf{Z} ”. It is known by the theorems of Deligne and Carayol, see [Car86], that the ℓ -adic étale cohomology of this curve realizes (up to some benign modifications) both the Jacquet-Langlands correspondence $\pi' \mapsto \pi$ and the local Langlands correspondence $\sigma \mapsto \pi(\sigma)$ for the discrete series of $\mathrm{GL}_2(F)$.

It would be very interesting to compute a system of semistable models of the curves X_m over a ramified extension of $\hat{\mathcal{O}}_{F^{\mathrm{nr}}}$; then the special fiber of the system ought to realize the supercuspidal parts of the correspondences in its cohomology. This has already been done in the case of $m = 1$ by Bouw-Wewers [BW04]; the generalization of this case for GL_n was carried out by Yoshida [Yos09]. But for $m \geq 2$ the structure of this special fiber is still unknown. Ignore the Weil group for the moment and consider the action of $(\mathrm{GL}_2(F) \times B^\times) \cap \mathcal{G}$ on the semi-stable reduction of the system $(X_m)_{m \geq 1}$. We conjecture that for a simple stratum S arising from an elliptic element $\beta \in \mathrm{GL}_2(F)$, the special fiber contains a smooth component X_S whose stabilizer is exactly $\Delta(E^\times)\mathcal{L}_S^\times$, such that for primes $\ell \neq p$, the ℓ -adic versions of the representations ρ_S appear in the action of this group on $H^1(X_S, \overline{\mathbf{Q}}_\ell)$. In light of the preceding paragraphs this would be consistent with the theorems of Deligne-Carayol. In future work we intend to give a description of the structure of the special fiber of the stable reduction of X_m which includes the action of the Weil group W_F .

A different approach to the Jacquet-Langlands correspondence for GL_2 has been advanced in A. Snowden’s thesis [Sno09].

This work was supported by a grant from the National Science Foundation.

2. Preparations: The representation theory of $\mathrm{GL}_2(F)$ and B^\times

2.1. Basic Notations. In this paper, F will be a finite extension of \mathbf{Q}_p , or else a finite extension of $\mathbf{F}_p((T))$. For a finite extension E of F (possibly F itself), we use the notation \mathcal{O}_E , \mathfrak{p}_E , and k_E for the ring of integers, maximal ideal, and quotient field of E . Let $q_E = \#k_E$, and let $q = q_F$. We fix a uniformizer π_F for F . Let $|\cdot|_F$ be the absolute value on F^* for which $|\pi_F|_F = q^{-1}$.

We also fix a character ψ_F of F of level 1; this means that ψ_F vanishes on \mathfrak{p}_F but not on \mathcal{O}_F .

Let B/F be a division algebra of dimension 4; this is unique up to isomorphism. Let \mathcal{O}_B be its unique maximal order. We use $N_{B/F}$ and $\mathrm{Tr}_{B/F}$ to denote the reduced norm and trace, respectively, from B to F ; sometimes we will omit the “ B/F ” from this notation. If G is the group $\mathrm{GL}_2(F)$ or B^\times , and $g \in G$, we will use the notation $\|g\|$ to mean $|\det g|_F$ or $|\mathrm{N} g|_F$ as appropriate.

Let A be the algebra $M_2(F)$ or B . For any additive character ψ of F , let ψ_A be the character of A defined by $\psi_A(x) = \psi(\text{Tr}_{A/F} x)$. Let μ_{ψ_A} (or just μ_ψ) be the measure on A which is self-dual with respect to ψ .

Let μ_ψ^\times be the corresponding Haar measure on A^\times : $\mu_\psi^\times(g) = \|g\|_{A^\times}^{-2} \mu_\psi(g)$.

2.2. Chain Orders and Strata. In this subsection, A is the algebra $M_2(F)$ or B . We will closely follow the notation of [BH06] concerning chain orders and strata for GL_2 , where the situation is somewhat simpler than the general case of GL_n .

First consider the case $A = M_2(F)$. A *lattice chain* is an F -stable family of lattices $\Lambda = \{L_i\}$ with each $L_i \subset F \oplus F$ an \mathcal{O}_F -lattice and $L_{i+1} \subset L_i$, all integers i . Let $e(\Lambda)$ be the unique integer for which $\pi_F L_i = L_{i+e(\Lambda)}$. Let \mathfrak{A}_Λ be the stabilizer in A of Λ ; that is, $\mathfrak{A}_\Lambda = \{a \in A \mid aL_i \subset L_i, \text{ all } i\}$. A *chain order* in A is an \mathcal{O}_F -order $\mathfrak{A} \subset A$ equal to \mathfrak{A}_Λ for some lattice chain Λ . We set $e_\mathfrak{A} = e_\Lambda$.

For example, suppose E/F is a quadratic field extension of ramification index e . Identify E with $F \oplus F$ as F -vector spaces. Then $\Lambda = \{\mathfrak{p}_E^i\}$ is a lattice chain with $e_\Lambda = e$. Up to conjugation by an element of A^\times , every lattice chain arises in this manner. We have the following description of \mathfrak{A} , again only up to A^\times -conjugation:

$$\mathfrak{A} = \begin{cases} M_2(\mathcal{O}_F), & e = 1, \\ \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathcal{O}_F \end{pmatrix}, & e = 2. \end{cases}$$

Note also that $\mathfrak{A}^\times \subset A^\times$ is normalized by $E^\times \subset \text{GL}_2(F)$, and that $\mathcal{O}_E \subset \mathfrak{A}$.

For a chain order $\mathfrak{A} \subset M_2(F)$, let $\mathcal{K}_\mathfrak{A}$ be its normalizer in $\text{GL}_2(F)$. This equals $F^*M_2(\mathcal{O}_F)$ if $e_\mathfrak{A} = 1$. If $e_\mathfrak{A} = 2$ then $\mathcal{K}_\mathfrak{A}$ is the semidirect product of \mathfrak{A}^\times with the cyclic group generated by a prime element of \mathfrak{A} .

Let $\mathfrak{P}_\mathfrak{A}$ be the Jacobson radical of \mathfrak{A} : this equals $\pi_F M_2(\mathcal{O}_F)$ for $\mathfrak{A} = M_2(\mathcal{O}_F)$ and $\begin{pmatrix} \mathfrak{p}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix}$ in the case that $\mathfrak{A} = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathcal{O}_F \end{pmatrix}$. We have a filtration of \mathfrak{A}^\times by the subgroups $U_\mathfrak{A}^n = 1 + \mathfrak{P}_\mathfrak{A}^n$. This filtration is normalized by $\mathcal{K}_\mathfrak{A}$.

All of the above constructions have obvious (and simpler) analogues in the quaternion algebra B : If $\mathfrak{A} = \mathcal{O}_B$ is the maximal order in B , then the normalizer of \mathfrak{A}^\times in B^\times is all of B^\times . The Jacobson radical $\mathfrak{P}_\mathfrak{A}$ is the unique maximal two-sided ideal of \mathfrak{A} , generated by a prime element π_B ; we let $U_\mathfrak{A}^n = 1 + \mathfrak{P}_\mathfrak{A}^n$ and $e_\mathfrak{A} = 2$.

Definition 2.2.1. Let A be the matrix algebra $M_2(F)$ or the quaternion algebra B . A *stratum* in A is a triple $(\mathfrak{A}, n, \alpha)$, where \mathfrak{A} is a chain order if $A = M_2(F)$ (resp. \mathcal{O}_B if $A = B$), n is an integer, and $\alpha \in \mathfrak{P}_\mathfrak{A}^{-n}$. Two strata $(\mathfrak{A}, n, \alpha)$ and $(\mathfrak{A}, n, \alpha')$ are equivalent if $\alpha \equiv \alpha' \pmod{\mathfrak{P}_\mathfrak{A}^{1-n}}$. The

stratum $(\mathfrak{A}, n, \alpha)$ is *ramified simple* if $E = F(\alpha)$ is a ramified quadratic extension of F , n is odd, and $\alpha \in E$ has valuation exactly $-n$. The stratum is *unramified simple* if E is an unramified quadratic extension of F , $\alpha \in E$ has valuation exactly $-n$, and the minimal polynomial of $\pi_F^n \alpha$ is irreducible mod \mathfrak{p}_F . Finally, the stratum is *simple* if it is ramified simple or unramified simple.

There is a correspondence $\mathfrak{S}' \mapsto \mathfrak{S}$ between simple strata in B and simple strata in $M_2(F)$. Given the simple stratum $\mathfrak{S}' = (\mathfrak{A}', n', \alpha')$, let $E = F(\alpha')$. Choose an embedding $E \hookrightarrow M_2(F)$, and let α be the image of α' . Finally, let $\mathfrak{A} \subset M_2(F)$ be a chain order associated to E . Then $\mathfrak{S} = (\mathfrak{A}, n, \alpha)$. The correspondence $\mathfrak{S}' \rightarrow \mathfrak{S}$ is a bijection between conjugacy classes of simple strata in B and in $M_2(F)$, respectively. The relationship between n' and n is as follows: $n' = n$ if E/F is ramified and $n' = 2n$ if E/F is unramified.

Let π be an irreducible admissible representation of $\mathrm{GL}_2(F)$. The level $\ell(\pi)$ is defined to be the least value of n/e , where (n, e) runs over pairs of integers for which there exists a chain order \mathfrak{A} of ramification index e such that π contains the trivial character of $U_{\mathfrak{A}}^{n+1}$. If π is a representation of B^\times , we define $\ell(\pi)$ to be $n/2$, where n is the least integer for which π contains the trivial character of $U_{\mathcal{O}_B}^{n+1}$.

We shall call π *minimal* if its level cannot be lowered by twisting by one-dimensional characters of F^\times .

When $n \geq 1$, a stratum $S = (\mathfrak{A}, n, \alpha)$ of $M_2(F)$ or B determines a nontrivial character ψ_α of $U_{\mathfrak{A}}^n/U_{\mathfrak{A}}^{n+1}$ by $\psi_\alpha(1+x) = \psi_F(\mathrm{Tr}_{A/F}(\alpha x))$. This character only depends on the equivalence class of S .

If S is a stratum, we say that π *contains the stratum* S if $\pi|_{U_{\mathfrak{A}}^n}$ contains the character ψ_α . From [BH06], 14.5 Theorem, we have the following classification of supercuspidal representations of $\mathrm{GL}_2(F)$:

Theorem 2.2.2. *A minimal irreducible representation π of $\mathrm{GL}_2(F)$ is supercuspidal if and only if exactly one of the following conditions holds:*

- (1) π has level 0, and π contains a representation of $\mathrm{GL}_2(\mathcal{O}_F)$ inflated from an irreducible cuspidal representations of $\mathrm{GL}_2(k_F)$.
- (2) π has level $\ell > 0$, and π contains a simple stratum.

The classification of representations of B^\times is analogous:

Theorem 2.2.3. *A minimal irreducible representation π of B^\times of dimension greater than one satisfies exactly one of the following properties:*

- (1) π has level 0, and π contains a representation of \mathcal{O}_B^\times inflated from a character χ of k_B^\times not factoring through the norm map $k_B^\times \rightarrow k^\times$.
- (2) π has level $\ell > 0$, and π contains a simple stratum.

By k_B we mean the finite field $\mathcal{O}_B/\mathfrak{P}_B$: this is a quadratic extension of k .

The supercuspidal representations of $\mathrm{GL}_2(F)$ and B^\times are all induced from irreducible representations of open compact-mod-center subgroups in a manner which can be made explicit. Suppose $S = (\mathfrak{A}, n, \alpha)$ is a simple stratum in $M_2(F)$ or B . Let $E \subset \mathrm{GL}_2(F)$ be the subfield $F(\alpha)$. The definition of ψ_α given above is well-defined on the subgroup $U_{\mathfrak{A}}^{\lfloor n/2 \rfloor + 1}$. Let $J_S \subset \mathrm{GL}_2(F)$ denote the group $E^\times U_{\mathfrak{A}}^{\lfloor (n+1)/2 \rfloor}$ and let $C(\psi_\alpha, \mathfrak{A})$ denote the set of isomorphism classes of irreducible representations $\Lambda \in \hat{J}_S$ for which $\Lambda|_{U_{\mathfrak{A}}^{\lfloor n/2 \rfloor + 1}}$ is a multiple of ψ_α .

Definition 2.2.4. A *cuspidal inducing datum* in A^\times is a pair (\mathfrak{A}, Ξ) , where \mathfrak{A} is a chain order in A and Ξ is a representation of $\mathcal{K}_{\mathfrak{A}}$ of one of the following types:

- (1) $A = M_2(F)$, $\mathfrak{A} \cong M_2(\mathcal{O}_F)$, and the restriction of Ξ to $\mathrm{GL}_2(\mathcal{O}_F)$ is inflated from a cuspidal representation of $\mathrm{GL}_2(k)$.
- (2) $A = B$, and the restriction of Ξ to \mathcal{O}_B^\times contains a character of inflated from a character of k_B^\times not factoring through the norm map $k_B^\times \rightarrow k^\times$.
- (3) There is a simple stratum $(\mathfrak{A}, n, \alpha)$ and a representation $\Lambda \in C(\psi_\alpha, \mathfrak{A})$ for which $\Xi = \mathrm{Ind}_{J_S}^{\mathcal{K}_{\mathfrak{A}}} \Lambda$.
- (4) The representation Ξ is the twist of a representation of one of the above types by a character of F^\times .

In the first two cases we will say that (\mathfrak{A}, Ξ) has level zero.

The following construction of supercuspidal representations is found in Section 15.5 of [BH06] in the case of $A = M_2(F)$:

Theorem 2.2.5. *If (\mathfrak{A}, Ξ) is a cuspidal inducing datum then $\pi_\Xi = \mathrm{Ind}_{\mathcal{K}_{\mathfrak{A}}}^{A^\times} \Xi$ is an irreducible supercuspidal representation of A^\times . Conversely, every supercuspidal representation of A^\times arises in this manner. The cuspidal inducing datum (\mathfrak{A}, Ξ) has level zero if and only if π_Ξ has level zero. Furthermore, (\mathfrak{A}, Ξ) arises from the simple stratum S if and only if π_Ξ contains S .*

2.3. Zeta functions and local constants. In this section we follow Godement and Jacquet [GJ72], §3. Let A be the algebra B or $M_2(F)$, and let $G = A^\times$. Let $\psi \in \hat{F}$ be an additive character of F . Let π be a supercuspidal (not necessarily irreducible) representation of G , realized on the space W . Let $\tilde{\pi}$ be the contragredient representation, with underlying space \check{W} . When $w \in W$, $\check{w} \in \check{W}$, we let $\gamma_{\check{w}, w}: G \rightarrow \mathbf{C}$ denote the function

$$g \mapsto \langle \check{w}, \pi(g)w \rangle.$$

Let $\mathcal{C}(\pi)$ denote the \mathbf{C} -span of the functions $\gamma_{\check{w}, w}$ for $w \in W$, $\check{w} \in \check{W}$. These functions are compactly supported modulo the center Z of G .

Let $C_c^\infty(A)$ be the space of locally constant compactly supported complex-valued functions on A . For $\Phi \in C_c^\infty(A)$ and $f \in \mathcal{C}(\pi)$, define the zeta function

$$\zeta(\Phi, f, s) = \int_G \Phi(g) f(g) \|g\|^s d\mu_\psi^\times(g).$$

When π is irreducible (and still cuspidal), there is a rational function $\varepsilon(\pi, s, \psi) \in \mathbf{C}(q^{-s})$ satisfying

$$\zeta(\hat{\Phi}, \check{f}, \frac{3}{2} - s) = \varepsilon(\pi, s, \psi) \zeta\left(\Phi, f, \frac{1}{2} + s\right),$$

where $\hat{\Phi}$ is the Fourier transform of Φ with respect to ψ . (Since π is cuspidal, its L -function vanishes.) See [GJ72], Thm. 3.3.

The local constant further satisfies

$$(2.3.1) \quad \varepsilon(\pi, s, \psi) \varepsilon(\check{\pi}, 1 - s, \psi) = \omega_\pi(-1)$$

where ω_π is the central character of π ([GJ72], p. 33).

2.4. Converse Theory. By the converse theorem, a supercuspidal representation of $\mathrm{GL}_2(F)$ of B^\times is determined by the epsilon factors of all of its twists by one-dimensional characters. We need an effective version of this theorem, which states that a supercuspidal representation is determined up to isomorphism by the data of its level together with the epsilon factors of twists of π by a collection of characters of F^\times of bounded level.

Next, we observe that epsilon factors have the “stability” property. If χ is a character of F^* , let the level $\ell(\chi)$ be the least integer n such that χ vanishes on $1 + \mathfrak{p}_F^{n+1}$. Then if π is an irreducible representation of $\mathrm{GL}_2(F)$ or B^\times , and χ is a character of F^\times with $\ell(\chi) > \ell(\pi)$, then $\varepsilon(\pi\chi, s, \psi)$ only depends on χ and the central character of π (and of course ψ). This is Prop. 3.8 of [JL70] in the case of $\mathrm{GL}_2(F)$ and Prop. 2.2.5 of [GL85] in the case of B^\times .

As χ varies through all characters of F^\times , the quantities $\varepsilon(\chi\pi, s, \psi)$ determine π up to isomorphism. We may therefore conclude the following explicit converse theorem:

Theorem 2.4.1. *Let π_1 and π_2 be two minimal supercuspidal representations of $\mathrm{GL}_2(F)$ or B^\times having the same central character and equal level ℓ . Then $\pi_1 \cong \pi_2$ if and only if*

$$(2.4.1) \quad \varepsilon(\pi_1\chi, s, \psi) = \varepsilon(\pi_2\chi, s, \psi)$$

for all characters $\chi \in \hat{F}^\times$ for which $\ell(\chi) \leq \ell$.

Definition 2.4.2. For minimal supercuspidal representations π' and π of B^\times and $\mathrm{GL}_2(F)$ having the same central character, we say that π' and π *correspond* if the following conditions hold:

- (1) π and π' have the same level ℓ .

(2) The equation

$$\varepsilon(\pi\chi, s, \psi) = -\varepsilon(\pi'\chi, s, \psi)$$

holds for all characters χ with $\ell(\chi) \leq \ell$.

In view of Theorem 2.4.1, at most one π can correspond to a given π' , and vice versa.

3. Zeta functions for $GL_2(F) \times B^\times$.

In this section we adopt the abbreviations $A_1 = M_2(F)$, $A_2 = B$, $G_1 = GL_2(F)$, $G_2 = B^\times$.

Let $\mathbf{A} = A_1 \times A_2$. Let $\mathbf{G} = \mathbf{A}^\times = GL_2(F) \times B^\times$. We will define zeta functions for representations of \mathbf{G} and use them to give a criterion for when such a representation “realizes the Jacquet-Langlands correspondence.” We will adopt the convention that if $g \in \mathbf{G}$, then g_1 and g_2 are its projections in $GL_2(F)$ and B^\times respectively. Let Π be an admissible cuspidal representation of \mathbf{G} . For $\Phi \in C_c^\infty(\mathbf{A})$ and $f \in \mathcal{C}(\Pi)$, define the zeta function

$$\zeta(\Phi, f, s) = \int_{\mathbf{G}} \Phi(g)f(g) \|g_1\|^s \|g_2\|^{2-s} d\mu^\times(g),$$

where μ^\times is a Haar measure on \mathbf{G} .

Let ψ be an additive character of F , and let $\mu_\psi^\times = \mu_{\mathbf{A},\psi}^\times = \mu_{A_1,\psi}^\times \times \mu_{A_2,\psi}^\times$; this is a Haar measure on \mathbf{G} . Let $\psi_{\mathbf{A}}$ be the additive character $(x_1, x_2) \mapsto \psi_{A_1}(x_1)\psi_{A_2}(-y_1)$. The Fourier transform of a decomposable test function $\Phi = \Phi_1 \otimes \Phi_2 \in C_c^\infty(\mathbf{A})$ is $\hat{\Phi}(x_1, x_2) = \hat{\Phi}_1(x_1)\hat{\Phi}_2(-x_2)$. Consequently if $f = f_1 \otimes f_2 \in \mathcal{C}(\pi_1 \otimes \pi_2)$ is a decomposable matrix coefficient for a tensor product representation $\pi_1 \otimes \pi_2$ of \mathbf{G} , then

$$(3.0.1) \quad \zeta(\hat{\Phi}, f, s) = \omega_{\pi_2}(-1)\zeta(\hat{\Phi}_1, f_1, s)\zeta(\hat{\Phi}_2, f_2, 2 - s),$$

where ω_{π_2} is the central character of π_2 .

Proposition 3.0.1. *Let Π be an admissible cuspidal semisimple (not necessarily irreducible) representation of $GL_2(F) \times B^\times$. The following are equivalent:*

- (1) *For every irreducible representation $\pi_1 \otimes \pi_2$ of $GL_2(F) \times B^\times$ appearing in Π , we have*

$$\varepsilon(\pi_1, s, \psi) = -\varepsilon(\check{\pi}_2, s, \psi).$$

- (2) *The functional equation*

$$(3.0.2) \quad \zeta(\Phi, f, s) = -\zeta(\hat{\Phi}, \check{f}, 2 - s)$$

holds for all $\Phi \in C_c^\infty(\mathbf{A})$, $f \in \mathcal{C}(\Pi)$. (Here the integral is taken with respect to the measure $\mu_{\mathbf{A},\psi}^\times$, and the Fourier transform is taken with respect to the character $\psi_{\mathbf{A}}$.)

Proof. It will simplify our notation if we set $s_1 = s, s_2 = 2 - s$. Let $\pi_1 \otimes \pi_2$ be any irreducible representation of $G_1 \times G_2$ appearing in Π . For $i = 1, 2$, let $\Phi_i \in C_c^\infty(G_i)$ and $f_i \in \mathcal{C}(\pi_i)$ be such that $\zeta(\Phi_i, f_i, s_i) \neq 0$. Let $\Phi = \Phi_1 \otimes \Phi_2$ and $f = f_1 \otimes f_2$. The respective functional equations for π_1 and π_2 are

$$\zeta(\hat{\Phi}_i, \check{f}_i, 2 - s_i) = \varepsilon\left(\pi_i, s_i - \frac{1}{2}, \psi\right) \zeta(\Phi_i, f_i, s_i), \quad i = 1, 2.$$

Multiplying these together and applying Eq. 3.0.1 yields

$$\omega_{\pi_2}(-1)\zeta(\hat{\Phi}, \check{f}, 2 - s) = \varepsilon\left(\pi_1, s - \frac{1}{2}, \psi\right) \varepsilon\left(\pi_2, \frac{3}{2} - s, \psi\right) \zeta(\Phi, f, s).$$

Therefore Eq. 3.0.2 holds if and only if

$$\varepsilon\left(\pi_1, s - \frac{1}{2}, \psi\right) \varepsilon\left(\pi_2, \frac{3}{2} - s, \psi\right) = -\omega_{\pi_2}(-1).$$

Combining this with the standard relation

$$\varepsilon\left(\pi_2, \frac{3}{2} - s, \psi\right) \varepsilon\left(\check{\pi}_2, s - \frac{1}{2}, \psi\right) = \omega_{\pi_2}(-1)$$

yields

$$\varepsilon\left(\pi_1, s - \frac{1}{2}, \psi\right) = -\varepsilon\left(\check{\pi}_2, s - \frac{1}{2}, \psi\right).$$

We see now that (2) \implies (1): Apply Eq. 3.0.2 to an arbitrary matrix coefficient $f = f_1 \otimes f_2$ belonging to $\pi_1 \otimes \pi_2 \subset \Pi$. For the converse, one need only note that every $\Phi \in C_c^\infty(\mathbf{A})$ and $f \in \mathcal{C}(\Pi)$ is a finite sum of pure tensors, and $\zeta(\Phi, f, s)$ is linear in Φ and f . \square

Combining Prop. 3.0.1 with the Converse Theorem 2.4.1 gives a necessary and sufficient condition for a representation Π of $\mathrm{GL}_2(F) \times B^\times$ to realize the Jacquet-Langlands correspondence. When $f \in C_c^\infty(\mathbf{G})$ and $\chi \in \hat{F}^\times$, we let χf be the function $g \mapsto \chi(\det(g_1)N(g_2)^{-1})f(g)$.

Corollary 3.0.2. *Let Π be an admissible cuspidal semisimple representation of $\mathrm{GL}_2(F) \times B^\times$ on which the diagonally-embedded group $\Delta(F^\times)$ acts trivially. Assume either that every irreducible representation of $\mathrm{GL}_2(F)$ (resp., B^\times) appearing in Π is minimal of the same level ℓ . Then the following are equivalent:*

- (1) Π is the direct sum of irreducible representations of \mathbf{G} of the form $\pi_1 \otimes \check{\pi}_2$, where π_1 and π_2 correspond.
- (2) The functional equation

$$(3.0.3) \quad \zeta(\Phi, \chi f, s) = -\zeta(\hat{\Phi}, \chi^{-1} \check{f}, 2 - s)$$

holds for all $\Phi \in C_c^\infty(\mathbf{A})$, $f \in \mathcal{C}(\Pi)$, and for all characters $\chi \in \hat{F}^\times$ for which $\ell(\chi) \leq \ell$.

Proof. That (1) \implies (2) is clear from Prop. 3.0.1. Therefore assume (2). Suppose $\pi_1 \otimes \check{\pi}_2$ appears in Π . Since Π vanishes on $\Delta(F^\times)$, the central characters of π_1 and π_2 agree. By Prop. 3.0.1 we find that $\varepsilon(\pi_1 \chi, s, \psi) =$

$-\varepsilon(\pi_2\chi, s, \psi)$ for all characters χ of level no greater than ℓ , so π_1 and π_2 correspond. □

4. Linking orders and congruence subgroups of $\mathrm{GL}_2(F) \times B^\times$

Our goal now is to produce, for each simple stratum S in $M_2(F)$, a certain semisimple representation Π_S of $\mathrm{GL}_2(F) \times B^\times$ having the following properties:

- (1) Π_S vanishes on the diagonal subgroup $\Delta(F^\times) \subset \mathrm{GL}_2(F) \times B^\times$.
- (2) The restriction of Π_S to the first factor $\mathrm{GL}_2(F)$ is a sum of exactly those irreducible representations which contain S . Similarly, the restriction of Π_S to the second factor B^\times is a sum of exactly those irreducible representations of B^\times which contain the corresponding stratum S' in B .
- (3) Matrix coefficients for Π_S satisfy the functional equation in Eq. 3.0.3 for sufficiently many χ .

We will present a similar construction for representations of level zero. In light of Cor. 3.0.2, such a family $\{\Pi_S\}$ is sufficient to establish the Jacquet-Langlands correspondence.

The strategy for producing Π_S is as follows: We will first define an order $\mathcal{L}_S \subset M_2(F) \times B$. The required representation Π_S will be induced from a certain representation of \mathcal{L}_S^\times . In this section we construct the orders \mathcal{L}_S and gather some geometric properties in preparation for proving the properties listed above.

4.1. Geometric preparations: $M_2(F)$ and B . Let E/F be a separable quadratic extension field of ramification degree e . Let \mathcal{O}_E be its ring of integers, \mathfrak{p}_E its maximal ideal, k_E its quotient field and σ the nontrivial element of $\mathrm{Gal}(E/F)$.

Let A be the ring $M_2(F)$ or B . Define an order $\mathfrak{A} \subset A$ as follows: if $A = M_2(F)$, let \mathfrak{A} be the chain order equal to the endomorphism ring of the lattice chain $\{\mathfrak{p}_E^i\}$, as in Section 2.2. If $A = B$, let $\mathfrak{A} = \mathcal{O}_B$. Either way, we may identify \mathcal{O}_E with an \mathcal{O}_F -subalgebra of \mathfrak{A} in such a way that $\mathfrak{A} \cap E = \mathcal{O}_E$.

There is a nondegenerate pairing $A \times A \rightarrow F$ given by $(x, y) \mapsto \mathrm{Tr}_{A/F}(xy)$. Let C be the complement of E in A with respect to this pairing, so that $A = E \oplus C$. Let $s_A: A \rightarrow E$ be the projection onto the first factor. Note that both the space C and the map s_A are stable under multiplication by E on either side. C is a (left and right!) E -vector space of dimension 1. It satisfies the property that $\alpha v = v\alpha^\sigma$ for all $v \in C$, $\alpha \in E$. Let $\mathfrak{C} = \mathfrak{A} \cap C$.

Lemma 4.1.1. *We have*

$$\mathfrak{C} = \begin{cases} \mathfrak{p}_E, & E/F \text{ unramified and } A = B \\ \mathcal{O}_E, & \text{all other cases.} \end{cases}$$

Proof. Since elements of E commute with $\mathfrak{C}\mathfrak{C}$, we must have $\mathfrak{C}\mathfrak{C} \subset E$; since $\mathfrak{C} \subset \mathfrak{A}$ this implies $\mathfrak{C}\mathfrak{C} \subset E \cap \mathfrak{A} = \mathcal{O}_E$. Thus $\mathfrak{C}\mathfrak{C}$ is an \mathcal{O}_E -submodule of \mathcal{O}_E ; i.e. it is an ideal of \mathcal{O}_E .

If $A = M_2(F)$ then \mathfrak{A} is the endomorphism ring of the lattice chain $\{\mathfrak{p}_E^i\}$. Consider the element $\sigma \in \text{Gal}(E/F)$: this certainly preserves each \mathfrak{p}_E^i and therefore belongs to \mathfrak{A} . For any $\alpha \in E$, we have that $(\alpha\sigma)^2 = N_{E/F}(\alpha)$ belongs to the center $F \subset M_2(F)$, but $\alpha\sigma$ does not itself belong to F , implying that $\text{Tr}_{A/F}(\alpha\sigma) = 0$ and therefore that $\sigma \in C$. So $\sigma \in C \cap \mathfrak{A} = \mathfrak{C}$. Consequently $\mathfrak{C}\mathfrak{C}$ contains $\sigma^2 = 1$, whence it is the unit ideal.

Now suppose $A = B$. Let $v_B: B^\times \rightarrow \mathbf{Z}$ denote the valuation on B . If E/F is ramified, then a uniformizer π_E of E has $v_B(\pi_E) = 1$, so that if $x \in \mathfrak{C}$ has valuation n , then $\pi_E^{-n}x \in \mathfrak{C}$ is a unit. This implies that $\mathfrak{C}\mathfrak{C}$ is the unit ideal.

On the other hand if E/F is unramified, then every element of E has even valuation in B . Considering that $A = E \oplus C$, this means that \mathfrak{C} contains an element π_B of valuation 1, so that $\mathfrak{C} = \mathcal{O}_E\pi_B$. Then $\mathfrak{C}\mathfrak{C} = \mathcal{O}_E\pi_B^2 = \mathfrak{p}_E$ as required. \square

Now suppose that $S = (\mathfrak{A}, n, \alpha)$ is a simple stratum in A with $E = F(\alpha)$. By replacing α with a sufficiently nearby element of \mathfrak{A} , it may be assumed that E/F is a separable field extension. This may be done without changing the character ψ_α of $U_{\mathfrak{A}}^{\lfloor n/2 \rfloor + 1}$. Choose an additive character ν of E vanishing on \mathfrak{p}_E^{n+1} but not on \mathfrak{p}_E^n . Assume that $\nu = \nu^\sigma$ if $e = 1$. Then define a character ν_S of A by $\nu_S(x) = \nu(s_A(x))$.

Whenever W is an \mathcal{O}_E -stable lattice of A , we may define the annihilator of W with respect to ν_S :

$$W^* = \{x \in A \mid \nu_S(xW) = 1\};$$

then W^* is also an \mathcal{O}_E -module. Note that $(\mathfrak{p}_E^k W)^* = \mathfrak{p}_E^{-k} W^*$.

Lemma 4.1.2. *The \mathcal{O}_E -module \mathfrak{C}^* equals $E \oplus \mathfrak{p}_E^n \mathfrak{C}$ if E/F is unramified and $A = B$. It equals $E \oplus \mathfrak{p}_E^{n+1} \mathfrak{C}$ in all other cases.*

Proof. Certainly we have $E \subset \mathfrak{C}^*$; all that remains is to find $\mathfrak{C}^* \cap \mathfrak{C}$. This last is an \mathcal{O}_E -submodule of the free rank-one \mathcal{O}_E -module \mathfrak{C} , so that it equals $I\mathfrak{C}$ for an ideal $I \subset \mathcal{O}_E$. For an element $x \in \mathcal{O}_E$ to belong to I the condition is $\nu_S(s_A(x\mathfrak{C}\mathfrak{C})) = \nu(I\mathfrak{C}\mathfrak{C}) = 1$. The lemma now follows from Lemma 4.1.1 and the definition of ν . \square

For an integer $m \geq 1$, we define an \mathcal{O}_E -submodule $V_A^m \subset \mathfrak{C}$ as follows:

$$V_A^m = \begin{cases} \mathfrak{p}_E^{\lfloor m/2 \rfloor} \mathfrak{C}, & A = B \text{ and } E/F \text{ unramified} \\ \mathfrak{p}_E^{\lfloor (m+1)/2 \rfloor} \mathfrak{C}, & \text{all other cases.} \end{cases}$$

The next proposition shows that $V_A^n \subset \mathfrak{C}$ is nearly a “square root” of the ideal \mathfrak{p}_E^n :

Proposition 4.1.3. *The module V_A^n has the following properties:*

- (1) $V_A^n V_A^n \subset \mathfrak{p}_E^n$. More precisely, if E/F is unramified then the value of $V_A^n V_A^n$ is given by the following table:

	<i>n even</i>	<i>n odd</i>
$A = M_2(F)$	\mathfrak{p}_E^n	\mathfrak{p}_E^{n+1}
$A = B$	\mathfrak{p}_E^{n+1}	\mathfrak{p}_E^n

- (2) If E/F is ramified, then $V_A^n = V_A^{n+1}$.
 (3) If E/F is unramified, then the dimension of V_A^n/V_A^{n+1} as a k_E -vector space is given by the following table:

	<i>n even</i>	<i>n odd</i>
$A = M_2(F)$	1	0
$A = B$	0	1

- (4) With respect to the character ν_S , we have $(V_A^n)^* = E \oplus V_A^{n+1}$.

Proof. Claim (1) follows from Lemma 4.1.1. For claim (2): Since E/F is ramified, n must be odd by definition of simple stratum; then $\lfloor (n+1)/2 \rfloor = \lfloor ((n+1)+1)/2 \rfloor$. For claim (3), assume E/F is unramified. When $A = M_2(F)$ we have $V_A^n = \mathfrak{p}_E^{\lfloor (n+1)/2 \rfloor} \mathfrak{C}$, so that there is an isomorphism of k_E -vector spaces $V_A^n/V_A^{n+1} \approx \mathfrak{p}_E^{\lfloor (n+1)/2 \rfloor} / \mathfrak{p}_E^{\lfloor (n+2)/2 \rfloor}$, and this has dimension 1 or 0 as n is even or odd, respectively. When $A = B$ we have $V_A^n = \mathfrak{p}_E^{\lfloor n/2 \rfloor} \mathfrak{C}$, so that there is an isomorphism of k_E -vector spaces $V_A^n/V_A^{n+1} = \mathfrak{p}_E^{\lfloor n/2 \rfloor} / \mathfrak{p}_E^{\lfloor (n+1)/2 \rfloor}$, and this has dimension 0 or 1 as n is even or odd, respectively.

Claim (4) follows directly from Lemma 4.1.2. □

4.2. Congruence subgroups and cuspidal representations. Keeping the notations from the previous subsection, we let

$$H_S = 1 + \mathfrak{p}_E^n + V_A^n$$

$$H_S^1 = 1 + \mathfrak{p}_E^n + V_A^{n+1}.$$

These are subgroups of \mathfrak{A}^\times because V_A^n is an \mathcal{O}_E -module and because $V_A^n V_A^n \subset \mathfrak{p}_E^n$ by Prop. 4.1.3. Note the inclusions $U_{\mathfrak{A}}^n \subset H_S^1 \subset H_S \subset J_S$ and $H_S^1 \subset U_{\mathfrak{A}}^{\lfloor n/2 \rfloor + 1}$.

Proposition 4.2.1. *For a representation $\Lambda \in C(\psi_\alpha, \mathfrak{A})$, we have that $\Lambda|_{H_S}$ is irreducible. Further, $\Lambda|_{H_S}$ is the unique irreducible representation of H_S whose restriction to H_S^1 is a sum of copies of $\psi_\alpha|_{H_S^1}$.*

Proof. If E/F is ramified, the claims in the proposition are trivial, because $H_S = H_S^1$ and Λ is a one-dimensional character. If E/F is unramified, then the same is true in the case that $A = M_2(F)$ and n is odd, and as well in the case that $A = B$ and n is even.

Therefore assume that E/F is unramified, and that $A = M_2(F)$ and n is even, or else that $A = B$ and n is odd. Then V_A^n/V_A^{n+1} is a k_E -module of dimension 1. Let ψ_α^1 denote the restriction of ψ_α to H_S^1 . We have an exact sequence

$$1 \rightarrow H_S^1/\ker \psi_\alpha^1 \rightarrow H_S/\ker \psi_\alpha^1 \rightarrow V_A^n/V_A^{n+1} \rightarrow 1$$

in which $H_S^1/\ker \psi_\alpha^1$ is the center. Thus $H_S/\ker \psi_\alpha^1$ is a discrete Heisenberg group. By the discrete Stone-von Neumann Theorem, there is a unique irreducible representation $\tilde{\psi}_\alpha$ of H_S lying over ψ_α^1 .

If $\Lambda \in C(\psi_\alpha, \mathfrak{A})$, then $\Lambda|_{H_S}$ is a q -dimensional representation of H_S whose restriction to H_S^1 is a multiple of ψ_α^1 . By the uniqueness property of $\tilde{\psi}_\alpha$, we must have $\Lambda|_{H_S} = \tilde{\psi}_\alpha$. The proposition follows. \square

4.3. Linking Orders. It is time to investigate the geometry of the product algebra $M_2(F) \times B$. It will be helpful to use the abbreviations $A_1 = M_2(F)$, $A_2 = B$, $\mathbf{A} = M_2(F) \times B$. Suppose $S = S_1 = (\mathfrak{A}_1, n_1, \alpha_1)$ is a simple stratum in $M_2(F)$. Choose an embedding $E = F(\alpha_1) \hookrightarrow B$ and let $\alpha_2 \in B^\times$ be the image of α_1 so that $S_2 = (\mathfrak{A}_2, n_2, \alpha_2)$ is the simple stratum in B which corresponds to S . Here $\mathfrak{A}_2 = \mathcal{O}_B$. For convenience of notation we set $n = n_1$. Let $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2$ and let $\Delta: E \rightarrow \mathbf{A}$ be the diagonal map $\Delta(a) = (a, a)$. We denote by s_1 and s_2 the projections $A_1 \rightarrow E$, $A_2 \rightarrow E$, respectively. Let C_i be the complement of E in A_i .

Let ν be an additive character of E as in Section 4.1. We define a character ν_S of \mathbf{A} by

$$\nu_S(x_1, x_2) = \nu(s_1(x_1) - s_2(x_2)).$$

Lemma 4.3.1. *With respect to ν_S , the annihilator of the diagonally embedded subring $\Delta(\mathcal{O}_E) \subset \mathfrak{A}$ is*

$$(\Delta(\mathcal{O}_E))^* = \Delta(E) + \mathfrak{p}_E^{n+1} \times \mathfrak{p}_E^{n+1} + C_1 \times C_2.$$

Proof. Suppose $(x_1, x_2) \in (\Delta(\mathcal{O}_E))^*$; then for all $\beta \in \mathcal{O}_E$, $v(\beta(s_1(x_1) - s_2(x_2))) = 1$. This means exactly that $s(x_1) \equiv s(x_2) \pmod{\mathfrak{p}_E^{n+1}}$, so that the pair $(s(x_1), s(x_2))$, being equal to $(s(x_1), s(x_1)) + (0, s(x_2) - s(x_1))$, lies in $\Delta(E) + \mathfrak{p}_E^{n+1} \times \mathfrak{p}_E^{n+1}$ as required. \square

Let $\mathbf{V}^n = V_A^n \times V_B^n \subset \mathfrak{A}$. The following properties of \mathbf{V}^n follow directly from Prop. 4.1.3:

Proposition 4.3.2. *The module \mathbf{V}^n has the following properties:*

- (1) $\mathbf{V}^n \mathbf{V}^n \subset \mathfrak{p}_E^n \times \mathfrak{p}_E^n$. Furthermore, if E/F is unramified then $\mathbf{V}^n \mathbf{V}^n$ equals $\mathfrak{p}_E^n \times \mathfrak{p}_E^{n+1}$ or $\mathfrak{p}_E^{n+1} \times \mathfrak{p}_E^n$ as n is even or odd, respectively.

- (2) If E/F is unramified, then $\mathbf{V}^n/\mathbf{V}^{n+1}$ is a left and right k_E -vector space of dimension 1, with the property that $\alpha v = v\alpha^q$ for $\alpha \in k_E$, $v \in \mathbf{V}^n/\mathbf{V}^{n+1}$.
- (3) If E/F is ramified, then $\mathbf{V}^n = \mathbf{V}^{n+1}$.
- (4) With respect to ψ_S , the annihilator of \mathbf{V}^n is $(E \times E) \oplus \mathbf{V}^{n+1}$.

Definition 4.3.3. The linking order \mathcal{L}_S is defined by

$$\mathcal{L}_S = \Delta(\mathcal{O}_E) + \mathfrak{p}_E^n \times \mathfrak{p}_E^n + \mathbf{V}^n.$$

Then \mathcal{L}_S is a (left and right) \mathcal{O}_E -submodule of \mathfrak{A} . It is easy to check that \mathcal{L}_S is indeed an order; this is a consequence of item (1) of the previous paragraph. We will also have use for a smaller subspace $\mathcal{L}_S^\circ \subset \mathcal{L}_S$, defined by

$$\mathcal{L}_S^\circ = \Delta(\mathfrak{p}_E) + \mathfrak{p}_E^{n+1} \times \mathfrak{p}_E^{n+1} + \mathbf{V}^{n+1}.$$

Proposition 4.3.4. *The linking order \mathcal{L}_S has the following properties:*

- (1) The group \mathcal{L}_S^\times is normalized by $\Delta(E^\times)$.
- (2) With respect to ν_S , the annihilator of \mathcal{L}_S is \mathcal{L}_S° .
- (3) \mathcal{L}_S° is a double-sided ideal of \mathcal{L}_S .
- (4) If E/F is ramified, then $\mathcal{L}_S/\mathcal{L}_S^\circ$ is a commutative ring of order q^2 , isomorphic to $k[X]/(X^2)$.
- (5) If E/F is unramified, then $\mathcal{L}_S/\mathcal{L}_S^\circ$ is a noncommutative ring of order q^6 whose isomorphism class depends only on q (and not n).
- (6) $\mathcal{L}_S^\times \cap \text{GL}_2(F) = H_{S_1}$, and $\mathcal{L}_S^\times \cap B^\times = H_{S_2}$.

Proof. Claim (1) is easy to check. For claim (2), we calculate the annihilator of \mathcal{L}_S as follows:

$$\begin{aligned} \mathcal{L}_S^* &= [\Delta(\mathcal{O}_E) + \mathfrak{p}_E^n \times \mathfrak{p}_E^n + \mathbf{V}^n]^* \\ &= \Delta(\mathcal{O}_E)^* \cap (\mathfrak{p}_E^n \times \mathfrak{p}_E^n)^* \cap (\mathbf{V}^n)^* \end{aligned}$$

The three terms to be intersected are

$$\begin{aligned} \Delta(\mathcal{O}_E)^* &= \Delta(E) + \mathfrak{p}_E^{n+1} \times \mathfrak{p}_E^{n+1} + C_1 \times C_2, \text{ by Lemma 4.3.1} \\ (\mathfrak{p}_E^n \times \mathfrak{p}_E^n)^* &= \mathfrak{p}_E \times \mathfrak{p}_E + C_1 \times C_2 \\ (\mathbf{V}^n)^* &= (E \times E) \oplus \mathbf{V}^{n+1}, \text{ by Lemma 4.3.2} \end{aligned}$$

We claim the intersection is \mathcal{L}_S° . Indeed, for a pair (x_1, x_2) to lie in \mathcal{L}_S^* , the first two equations imply $s_1(x_1), s_2(x_2) \in \mathfrak{p}_E$ and $s_1(x_1) \equiv s_2(x_2) \pmod{\mathfrak{p}_E^{n+1}}$, and the third implies $(x_1 - s_1(x_1), x_2 - s_2(x_2)) \in \mathbf{V}^{n+1}$.

Claim (3) follows from the inclusion $\mathbf{V}^n \mathbf{V}^{n+1} \subset \mathfrak{p}_E^{n+1} \times \mathfrak{p}_E^{n+1}$, which is easily checked.

For claims (4) and (5), let $\mathcal{R}_S = \mathcal{L}_S/\mathcal{L}_S^\circ$. Fix a uniformizer π_E of E .

In the case that E/F is ramified, we have $\mathbf{V}^n = \mathbf{V}^{n+1}$, so there is an isomorphism

$$\mathcal{R}_S \cong \frac{\Delta(\mathcal{O}_E) + \mathfrak{p}_E^n \times \mathfrak{p}_E^n}{\Delta(\mathfrak{p}_E) + \mathfrak{p}_E^{n+1} \times \mathfrak{p}_E^{n+1}}.$$

The “numerator” of the right-hand side is the ring of pairs $(x, x + \pi_E^n y) \in \mathcal{O}_E \times \mathcal{O}_E$ with $x, y \in \mathcal{O}_E$. Define a map

$$\begin{aligned} \mathcal{R}_S &\rightarrow k \times k \\ (x, x + \pi_E^n y) &\mapsto (\bar{x}, \bar{y}), \end{aligned}$$

where if $z \in \mathcal{O}_E$ we have put $\bar{z} = z \pmod{\mathfrak{p}_E}$. It is easily checked that this map is an isomorphism of (additive) groups; the multiplication law induced on $k \times k$ is $(x_1, y_1)(x_2, y_2) = (x_1x_2, x_1y_2 + x_2y_1)$, which is to say that $\mathcal{R}_S \cong k[X]/(X^2)$.

Now suppose E/F is unramified. In this case $V = \mathbf{V}^n/\mathbf{V}^{n+1}$ is a vector space over k_E of dimension 1. We have $\mathbf{V}^n\mathbf{V}^n \subset \mathfrak{p}_E^n \times \mathfrak{p}_E^n$. On the other hand the image of $\mathfrak{p}_E^n \times \mathfrak{p}_E^n$ in \mathcal{R}_S may be identified with k_E via $(x_1, x_2) \mapsto \pi_E^{-n}(x_1 - x_2)$. For $v, w \in V$, let $v \cdot w$ be the image of $vw \in \mathfrak{p}_E^n \times \mathfrak{p}_E^n$ under this latter map. Then $(v, w) \mapsto v \cdot w$ is a pairing $V \times V \rightarrow k_E$ which is k_E -linear in the first variable and satisfies $w \cdot v = (v \cdot w)^q$. This pairing is nondegenerate by part (1) of Lemma 4.3.2: One of the factors of $\mathbf{V}^n\mathbf{V}^n$ is always \mathfrak{p}_E^n . Choose an isomorphism $\phi: V \rightarrow k_E$ of k_E vector spaces in such a way that $v \cdot w = \phi(v)\phi(w)^q$.

We are now ready to describe the ring \mathcal{R}_S : let R be the k -algebra of matrices

$$[\alpha, \beta, \gamma] = \begin{pmatrix} \alpha & \beta & \gamma \\ & \alpha^q & \beta^q \\ & & \alpha \end{pmatrix},$$

where $\alpha, \beta, \gamma \in k_E$. Any element of \mathcal{L}_S is of the form $(x, x + \pi_E^n y) + v$, where $x, y \in \mathcal{O}_E$ and $v \in \mathbf{V}^n$. Define a map

$$\begin{aligned} \mathcal{L}_S &\rightarrow R \\ (x, x + \pi_E^n y) + v &\mapsto [\bar{x}, \bar{y}, \phi(v)]; \end{aligned}$$

it is easy to see that this map descends to a ring isomorphism $\mathcal{R}_S \rightarrow R$. Therefore \mathcal{R}_S is a noncommutative ring of order q^6 whose isomorphism class is independent of n .

For claim (6), we begin with the fact that any element b of \mathcal{L}_S^\times is of the form $(x + \pi^n y, x) + v$, with $x \in \mathcal{O}_E^\times$, $y \in \mathcal{O}_E$, and $v \in \mathbf{V}^n = \mathbf{V}_1^n \times \mathbf{V}_2^n$. If such an element has B -component 1 we must have $x = 1$ and $v = (v_1, 0)$, which is to say that $b = (1 + \pi^n y, 1) + (v_1, 0) \in (1 + \mathfrak{p}_E^n + V^n) \times \{1\}$ is an element of H_S . The argument for B^\times is similar. \square

In the sequel, we will construct a representation ρ_S of the unit group \mathcal{L}_S^\times inflated from a representation of the finite group $(\mathcal{L}_S/\mathcal{L}_S^{\mathcal{O}})^\times$. Then when ρ_S

is extended to $\Delta(E^\times)(F^\times \times F^\times)\mathcal{L}^\times$ and induced up to $\mathrm{GL}_2(F) \times B^\times$, the result will realize the Jacquet-Langlands correspondence for representations of $\mathrm{GL}_2(F)$ containing the stratum S . For completeness' sake, we also want to construct the correspondence for supercuspidal representations of level 0. To this end we define the linking order of level 0 by

$$\mathcal{L}_0 = M_2(\mathcal{O}_F) \times \mathcal{O}_B$$

and its double-sided ideal by

$$\mathcal{L}_0^\circ = \mathfrak{p}_F M_2(\mathcal{O}_F) \times \mathfrak{P}_B.$$

Let E be the unique unramified quadratic extension of F and choose embeddings $E \hookrightarrow M_2(F)$, $E \hookrightarrow B$ so that $M_2(\mathcal{O}_F) \cap E = \mathcal{O}_B \cap E = \mathcal{O}_E$. Let $s_1: M_2(\mathcal{O}_F) \rightarrow E$ and $s_2: B \rightarrow E$ be the projections as in the previous section, let ν be an additive character of E vanishing on \mathfrak{p}_E but not on \mathcal{O}_E , and let $\nu_0: \mathbf{A} \rightarrow \mathbf{C}^\times$ be the character $\nu_0(x_1, y_1) = \nu(s_1(x_1) - s_2(y_1))$. Then Prop. 4.3.4 has the following analogue in level zero:

Proposition 4.3.5. *The linking order \mathcal{L}_0 has the following properties:*

- (1) \mathcal{L}_0^\times is normalized by $\Delta(E^\times)$.
- (2) With respect to ν_0 , the annihilator of \mathcal{L}_0 is \mathcal{L}_0° .
- (3) $\mathcal{L}_0/\mathcal{L}_0^\circ \cong M_2(k_F) \times k_E$.
- (4) $\mathcal{L}_0^\times \cap \mathrm{GL}_2(F) = \mathrm{GL}_2(\mathcal{O}_F)$, and $\mathcal{L}_0^\times \cap B^\times = \mathcal{O}_B^\times$.

5. Representations of \mathcal{L}_S^\times and the Fourier transform.

Keep the notations from the previous section: Let $S = (\mathfrak{A}_1, n_1, \alpha_1)$ be a simple stratum in $\mathrm{GL}_2(F)$, let $S' = (\mathfrak{A}_2, n_2, \alpha_2)$ be its corresponding simple stratum in B^\times , let $n = n_1$, let \mathcal{L}_S be the associated linking order, let \mathcal{R}_S be its quotient ring by the ideal \mathcal{L}_S° , and let ν_S be the associated additive character on $\mathbf{A} = M_2(F) \times B$. Let $\mathbf{G} = \mathrm{GL}_2(F) \times B^\times$. For $g = (g_1, g_2) \in \mathbf{G}$, write

$$\|g\| = |\det g_1|_F |\mathbb{N} g_2|_F.$$

We let μ_S be the unique Haar measure on the additive group \mathbf{A} which is self-dual with respect to ν_S , and let \mathcal{F}_S be the Fourier transform with respect to ψ_S :

$$\mathcal{F}_S f(y) = \int_{\mathbf{A}} f(x) \nu_S(xy) d\mu_S(x).$$

There are translation operators $L, R: \mathbf{G} \rightarrow \mathrm{Aut} C_c^\infty(\mathbf{G})$, defined by $L_g f(y) = f(g^{-1}y)$ and $R_h f(y) = f(yh)$; we have the rules

$$(5.0.1) \quad L_g \mathcal{F}_S = \|g\|^2 \mathcal{F}_S R_g, \quad R_h \mathcal{F}_S = \|h\|^{-2} \mathcal{F}_S L_h.$$

Let \mathcal{R}_S be the k_E -algebra $\mathcal{L}_S/\mathcal{L}_S^\circ$ as in the proof of Prop. 4.3.4.

Proposition 5.0.1. *The measure of \mathcal{L}_S° with respect to μ_S is $\#\mathcal{R}_S^{-1/2}$.*

Proof. Let $\chi_{\mathcal{L}_S}$ be the characteristic function of \mathcal{L}_S . Then

$$\mathcal{F}_S \chi_{\mathcal{L}_S}(y) = \int_{\mathcal{L}_S} \nu_S(xy) d\mu_S(x)$$

is supported on $\mathcal{L}_S^\perp = \mathcal{L}_S^\circ$ and equals $\mu_S(\mathcal{L}_S)$ there; *i.e.* $\mathcal{F}_S \chi_{\mathcal{L}_S} = \mu_S(\mathcal{L}_S) \chi_{\mathcal{L}_S^\circ}$. Similarly $\mathcal{F}_S^2 \chi_{\mathcal{L}_S} = \mu_S(\mathcal{L}_S) \mu_S(\mathcal{L}_S^\circ) \chi_{\mathcal{L}_S}$. On the other hand, since μ_S is self-dual, we must have $\mathcal{F}_S^2 \chi_{\mathcal{L}_S} = \chi_{\mathcal{L}_S}$, implying $\mu_S(\mathcal{L}_S) \mu_S(\mathcal{L}_S^\circ) = 1$. Since $\mu_S(\mathcal{L}_S) = \#\mathcal{R}_S \mu_S(\mathcal{L}_S^\circ)$, the result follows. \square

Let $\mathcal{C}(\mathcal{R}_S)$ be the space of complex-valued functions on \mathcal{R}_S . Note that the character ν_S vanishes on \mathcal{L}_S° and therefore induces a well-defined additive character of \mathcal{R}_S . We identify $\mathcal{C}(\mathcal{R}_S)$ with a subspace of $C_c^\infty(\mathbf{A})$.

Prop. 5.0.1 together with the key property that \mathcal{L}_S and \mathcal{L}_S° are dual lattices imply the following:

Proposition 5.0.2. *The Fourier transform $f \mapsto \mathcal{F}_S f$ preserves the space $\mathcal{C}(\mathcal{R}_S)$. For $f \in \mathcal{C}(\mathcal{R}_S)$, we have*

$$(5.0.2) \quad \mathcal{F}_S f(y) = \#\mathcal{R}_S^{-1/2} \sum_{x \in \mathcal{R}_S} f(x) \nu_S(xy).$$

Recall that the data of S and S' determine characters ψ_{α_1} and ψ_{α_2} of the subgroups $U_{\mathfrak{A}_1}^{n_1}$ and $U_{\mathfrak{A}_2}^{n_2}$ of \mathfrak{A}_1^\times and \mathfrak{A}_2^\times , respectively. The product group $U_{\mathfrak{A}_1}^{n_1} \times U_{\mathfrak{A}_2}^{n_2} = 1 + \mathfrak{p}_E^n \mathfrak{A}_1 \times \mathfrak{p}_E^n \mathfrak{A}_2$ is a subgroup of \mathcal{L}_S^\times , and the product character $\psi_S = \psi_{\alpha_1} \times \psi_{\alpha_2}^{-1}$ vanishes on $(U_{\mathfrak{A}_1}^{n_1} \times U_{\mathfrak{A}_2}^{n_2}) \cap (1 + \mathcal{L}_S^\circ) = U_{\mathfrak{A}_1}^{n_1+1} \times U_{\mathfrak{A}_2}^{n_2+1}$. Therefore if we let \mathbf{U}_S be the image of $U_{\mathfrak{A}_1}^{n_1} \times U_{\mathfrak{A}_2}^{n_2}$ in \mathcal{R}_S , then ψ_S induces a well-defined nontrivial character of \mathbf{U}_S .

We are now ready to construct the special representation ρ_S . Its relevant properties are as follows:

Theorem 5.0.3. *There exists an irreducible representation ρ_S of \mathcal{R}_S^\times satisfying the conditions:*

- (1) ρ_S vanishes on $k^\times \subset \mathcal{R}_S^\times$.
- (2) $\rho_S|_{\mathbf{U}_S}$ is a sum of copies of ψ_S .
- (3) If $f \in \mathcal{C}(\rho_S)$ is a matrix coefficient, then $\mathcal{F}_S f$ is supported on \mathcal{R}_S^\times and satisfies $\mathcal{F}_S f(y) = \pm f(y^{-1})$, all $y \in \mathcal{R}_S^\times$. The sign is 1 if E/F is ramified and -1 otherwise.

Remark 5.0.4. These three properties correspond to the three desired properties of the representation Π_S listed at the beginning of Section 4.

Proof. First, consider the case where $E = F(\alpha)$ is a ramified extension of F . Then by Prop. 4.3.4 we have an isomorphism $\mathcal{R}_S \cong k[X]/(X^2)$ with respect to which ν_S is a nontrivial additive character which vanishes on $k \subset \mathcal{R}_S$. The subgroup $\mathbf{U}_S \subset \mathcal{R}_S^\times$ corresponds to $\{1 + aX \mid a \in k\}$. There

is obviously a unique character ρ_S of \mathcal{R}_S^\times lifting ψ_S and vanishing on k^\times . It takes the form

$$\rho_S(a + bX) = \Psi(a^{-1}b),$$

where $\Psi: k \rightarrow \mathbf{C}^\times$ is a nontrivial character determined by ψ_S . That ρ_S satisfies claim (3) is a simple calculation in the commutative ring \mathcal{R}_S .

The case of $e = 1$ is far more subtle. The required representation ρ_S is related to the construction of the Weil representation of a symplectic group over a finite field. We present a self-contained version of the construction in the following section. □

5.1. Fourier transforms on the Heisenberg group. In this section, k is the finite field with q elements and k_2/k is a quadratic field extension. As in the proof of Prop. 4.3.4, let R be the k -algebra of matrices of the form

$$[\alpha, \beta, \gamma] = \begin{pmatrix} \alpha & \beta & \gamma \\ & \alpha^q & \beta^q \\ & & \alpha \end{pmatrix},$$

where $\alpha, \beta, \gamma \in k_2$. Let $U \subset R^\times$ be the subgroup of matrices of the form $[1, 0, \gamma]$, and let $U^1 \subset U$ be the subgroup consisting of those $[1, 0, \gamma]$ for which $\text{Tr}_{k_2/k} \gamma = 0$. Note that the center of R^\times is $k^\times U$.

Let ℓ be a prime not dividing q , and let $\nu_k: k \rightarrow \overline{\mathbf{Q}}_\ell^\times$ be a nontrivial additive character. Define an additive character ν_R of R by $\nu_R([\alpha, \beta, \gamma]) = \nu_k(\text{Tr}_{k_2/k} \gamma)$. Let \mathcal{F} be the Fourier transform with respect to ν_R .

Theorem 5.1.1. *For each character ψ of U which is nontrivial on U^1 , there exists a representation ρ_ψ of R^\times satisfying the properties:*

- (1) ρ_ψ is trivial on k^\times .
- (2) $\rho_\psi|_U$ is a multiple of ψ .
- (3) For a matrix coefficient $f \in \mathcal{C}(\rho_\psi)$, the Fourier transform $\mathcal{F}f$ is supported on R^\times and satisfies $\mathcal{F}f(y) = -f(y^{-1})$ for $y \in R^\times$.

The proof will occupy the rest of the section. To construct ρ_ψ , we will build a nonsingular projective curve X/\overline{k} admitting an action of R^\times , and find ρ_ψ in the ℓ -adic cohomology of X .

First, we recognize a relationship between R^\times and the unitary group GU_3 . Let Φ be the matrix

$$\Phi = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix},$$

and let $\text{GU}_3(k)$ be the subgroup of matrices $M \in \text{GL}_3(k_2)$ satisfying $M^* \Phi M = \lambda(M) \Phi$ for a scalar $\lambda(M)$. (Here M^* is the conjugate transpose of M .) Then a large part of the Borel subgroup of $\text{GU}_3(k)$ is contained in

R^\times . Indeed, if $M \in R^\times$, we can measure the defect of M from lying in $\mathrm{GU}_3(k)$ by a homomorphism $\delta: R^\times \rightarrow k$ defined by

$$(5.1.1) \quad \Phi^{-1}M^*\Phi M = \lambda(M) \begin{pmatrix} 1 & \delta(M) \\ & 1 \\ & & 1 \end{pmatrix}.$$

Explicitly, $\delta([\alpha, \beta, \gamma]) = \alpha\gamma^q + \alpha^q\gamma - \beta^{q+1}$. Let $R^1 = \ker \delta$; then $R^1 \subset \mathrm{GU}_3(\mathbf{F}_q)$.

The algebraic group GU_3 acts on the projective plane \mathbf{P}_k^2 in the usual manner; the group $\mathrm{GU}_3(k)$ preserves the equation $y^{q+1} = x^qz + xz^q$ in projective coordinates. This equation defines a nonsingular projective curve X^1 of genus $q(q-1)/2$ with an action of $\mathrm{GU}_3(k)$. Let $X = R^\times \times_{R^1} X^1$; this is a smooth projective curve with an action of R^\times . Let ℓ be a prime distinct from the characteristic of k , and let $\rho: R^\times \rightarrow H^1(X, \overline{\mathbf{Q}}_\ell)$ be the representation of R^\times on the first cohomology of X . The degree of ρ is $q^2(q-1)$. Note that ρ is trivial on $k^\times \subset R^\times$.

Since U lies in the center of R^1 , we have a decomposition $\rho = \bigoplus_\psi \rho_\psi$ of ρ into its irreducible ψ -isotypic components, where ψ runs over characters of U which are nontrivial on U^1 ; each has dimension q . We claim that ρ_ψ is irreducible. By the discrete Stone-von Neumann theorem there is a unique irreducible representation ς of the p -Sylow subgroup $H \subset R^\times$ which lies over ψ , and furthermore $\deg \varsigma = q$. Since the restriction of ρ_ψ to H lies over ψ and has degree q , it must agree with ς . Therefore ρ_ψ is irreducible.

Let $T \subset R^\times$ be the subgroup of diagonal matrices, so that $T \cong k_2^*$. The Lefschetz fixed-point theorem can easily be used to compute the restriction of ρ_ψ to T :

Proposition 5.1.2. *The restriction of ρ_ψ to T is exactly the direct sum of those characters χ of T which are nontrivial on T/k^\times .*

For a matrix coefficient $f \in \mathcal{C}(\rho_\psi)$, we consider the Fourier transform $\mathcal{F}f$. We claim that the Fourier transform $\mathcal{F}f$ is supported on R^\times . Indeed, if $y \in R$ is not invertible then $uy = y$ for all $u \in U$. It follows from this that $\mathcal{F}f(uy) = \mathcal{F}f(uy) = \psi(u)^{-1}\mathcal{F}f(y)$ for all $u \in U$; since ψ is nontrivial we see that $\mathcal{F}f(y) = 0$.

Next we claim that for $y \in R^\times$ we have

$$(5.1.2) \quad \mathcal{F}f(y) = -f(y^{-1}).$$

Formally, we have $\mathcal{F}f(y) = f(y^{-1})\mathcal{F}(1)$, so in fact it suffices to show that

$$(5.1.3) \quad \mathcal{F}f(1) = -f(1).$$

It is enough to prove Eq. 5.1.3 in the case that f equals the character of ρ_χ . This is because the character of ρ_ψ generates $\mathcal{C}(\rho_\psi)$ as an $(R^\times \times R^\times)$ -module, and because the property in Eq. 5.1.2 is invariant when we replace f by

any of its $(R^\times \times R^\times)$ -translates. Therefore let $f = \text{Tr } \rho_\psi$ be the character of ρ_ψ .

We have

$$\mathcal{F}f(1) = \frac{1}{q^3} \sum_{x \in R^\times} \text{Tr } \rho_\psi(x) \nu_R(x).$$

We observe that the term $\text{Tr } \rho_\psi(x) \nu_R(x)$ only depends on the conjugacy class of x in R^\times . We first dispense with those terms in the above sum for which x has eigenvalues in k^\times . The sum over these terms vanishes, because for such an x we have $\text{Tr } \rho_\psi(xu) \nu_R(xu) = \psi(u) \text{Tr } \rho_\psi(x) \nu_R(x)$ for all $u \in U^1$. All that remains are the elements $x = [\alpha, \beta, \gamma]$ with $\alpha \in k_2^\times \setminus k^\times$, and each of these are conjugate to a unique element of the form tu , with $t \in T \setminus k^\times$ and $u \in U$. Each such conjugacy class has cardinality q^2 , and the value of $\text{Tr } \rho_\psi(tu)$ on such a class is $-\psi(u)$. Therefore

$$\mathcal{F}f(1) = -\frac{1}{q} \sum_{t \in T \setminus k^\times} \sum_{u \in U} \psi(u) \nu_r(tu).$$

This reduces to $-q = -f(1)$ by a simple calculation, thus completing the proof of Theorem 5.1.1.

Remark 5.1.3. The curve X is isomorphic (over \bar{k}) to the Fermat curve $x^{q+1} + y^{q+1} + z^{q+1} = 0$. It appears in the construction of the so-called unipotent representation of $\text{GU}_3(k)$; see [Lus78].

There is also a connection to the theory of the discrete Weil representation. We have $R^\times = T \rtimes H$, where H is the p -Sylow subgroup of R^\times . Furthermore, $U \cap H = U^1$ is the center of H . Write ψ^1 for the (nontrivial) restriction of ψ to U^1 . The group $H/\ker \psi^1$ is a discrete Heisenberg group. By the Stone-von Neumann theorem, there is a unique irreducible representation V_ψ of H lying over ψ .

The group T embeds as a nonsplit torus in $\text{SL}_2(k)$, and the conjugation action of T on $H/\ker \psi^1$ extends to an action of $\text{SL}_2(k)$ in a manner which fixes each element of U^1 . The uniqueness property of V_ψ means that if $\alpha \in \text{SL}_2(k)$ and ${}^\alpha V_\psi$ is the conjugate representation $g \mapsto V_\psi(\alpha(g))$, then there is an isomorphism $W(\alpha): {}^\alpha V_\psi \cong V_\psi$ which is well-defined up to a scalar. The operators $W(\alpha)$ give an *a priori* projective representation of $\text{SL}_2(k)$ on the underlying space of V_ψ which in fact lifts to a proper representation W , the Weil representation. See for instance [Gér77]. The operators $W(\alpha)$ together with the representation V_ψ give a q -dimensional representation of $\text{SL}_2(k) \rtimes H$; restricting this to $T \rtimes H/\ker \psi^1 = R^\times/\ker \psi^1$ gives the representation ρ_ψ we have constructed in Theorem 5.1.1.

When W is restricted to a nonsplit torus of $\text{SL}_2(k)$, each nontrivial character appears at most once, see Theorem 3 of [GH08]; this implies the property of ρ_ψ given in Prop. 5.1.2. The equation of part (3) of Thm. 5.1.1 may

be established once one has a formula for the character of the “Heisenberg-Weil representation” of $SL_2(k) \rtimes H$; for this, see Theorem 2.2.1 of [GH07]. We have chosen to provide a cohomological proof, however, because of its relative simplicity and because we believe the curve X appears as a connected component of the stable reduction of the Lubin-Tate curve, *cf.* the introduction.

The case of $e = 1$ in Theorem 5.0.3 follows from Theorem 5.1.1 once we observe the following:

- (1) There exists an isomorphism $\mathcal{R}_S \rightarrow R$.
- (2) Under this isomorphism, ν_S is identified with an additive character of the form ν_R described above.
- (3) The subgroup $\mathbf{U}_S \in \mathcal{R}_S^\times$ is identified with $U \subset R^\times$.
- (4) Choose an isomorphism $\iota: \mathbf{C} \rightarrow \overline{\mathbf{Q}}_\ell$, then the complex character ψ_S of \mathbf{U}_S is identified with an ℓ -adic character ψ of U .
- (5) The condition that $S = (M_2(\mathcal{O}_F), n, \alpha)$ be a simple stratum implies that the reduction of $\pi_F^n \alpha$ has irreducible characteristic polynomial, which in turn implies that ψ is nontrivial on U^1 .
- (6) The ℓ -adic representation ρ_ψ constructed in Theorem 5.1.1 with respect to the data of ν_R and ψ may be transported via ι^{-1} to a complex representation of \mathcal{R}_S^\times which satisfies the requirements of Theorem 5.0.3.

5.2. The case of level 0. The linking order of level 0 is $\mathcal{L}_0 = M_2(\mathcal{O}_F) \times \mathcal{O}_B$, and its quotient ring \mathcal{R}_0 is $M_2(k) \times k_E$. The additive character ν_0 is of the form

$$\nu_0(x, y) = \nu(\mathrm{Tr}_{M_2(k)/k} x - \mathrm{Tr}_{k_E/k} y),$$

where ν is a nontrivial additive character of k , and \mathcal{F}_0 is the Fourier transform with respect to this character. Let θ be a character of k_E^\times . Assume that θ is *regular*, meaning that it does not factor through the norm map $k_E^\times \rightarrow k^\times$. It is well-known that there is an irreducible cuspidal representation η_θ of $GL_2(k_F)$ corresponding to θ . The character of this representation takes the value $-(\theta(\alpha) + \theta(\alpha^q))$ on an element $g \in GL_2(k_F)$ with distinct eigenvalues $\alpha, \alpha^q \in k_E$ not lying in k_F .

Let ρ_θ be the character $\eta_\theta \otimes \theta^{-1}$ of $\mathcal{R}_0^\times = GL_2(k_F) \times k_E^\times$. The following proposition concerns the Fourier transforms of matrix coefficients of ρ_θ .

Proposition 5.2.1. *For $f \in \mathcal{C}(\rho_\theta)$ we have that $\mathcal{F}_0 f$ is supported on \mathcal{R}_0^\times and satisfies $\mathcal{F}_0 f(y) = -f(y^{-1})$ for $y \in \mathcal{R}_0^\times$.*

Proof. We reduce this to two calculations relative to the rings $M_2(k)$ and k_E , respectively. Let $R_1 = M_2(k)$, $R_2 = k_E$, and for $i = 1, 2$ let ν_i be the additive character of R_i defined by $\nu_i(x) = \nu_0(\mathrm{Tr}_{R_i/k} x)$, so that $\nu_S(x, y) = \nu_1(x)\nu_2(-y)$.

Write $\tau_{\theta,\nu}$ for the Gauss sum $\sum_{\alpha \in k_E^\times} \theta(\alpha)\nu(\text{Tr}_{k_E/k_F} \alpha)$. We claim that for all $f \in \mathcal{C}(\eta_\theta)$ we have that $\mathcal{F}_1 f$ is supported on $R_1^\times = \text{GL}_2(k)$ and satisfies

$$\mathcal{F}_1 f(y) = -\tau_{\theta,\nu} f(y^{-1}).$$

This is a straightforward calculation. It is a special case of a calculation of epsilon factors of irreducible representations of GL_n which appears in [Kon63]; these can always be expressed as a product of Gauss sums. See also [Mac73], Chap. IV.

The corresponding analysis for $R_2 = k_E$ is simpler: define a Fourier transform \mathcal{F}_2 on $\mathcal{C}(R_2)$ by $\mathcal{F}f(y) = q^{-1} \sum_{x \in k_E^\times} f(x)\nu_2(-xy)$. Then the Fourier transform of the character θ^{-1} is supported on k_E^\times and equals $q^{-1}\tau_{\theta^{-1},\nu^{-1}}\theta$.

We may now complete the proof of the proposition. For a decomposable element $f = f_1 \otimes f_2$ of $\mathcal{C}(R_1 \times R_2)$, we have $\mathcal{F}_0 f = \mathcal{F}_1 f_1 \otimes \mathcal{F}_2 f_2$. If this same f is a matrix coefficient for $\rho_\theta = \eta_\theta \otimes \theta^{-1}$ then we must have $\mathcal{F}_0 f = -q^{-2}\tau_{\theta,\nu}\tau_{\theta^{-1},\nu^{-1}}f(y^{-1})$. We now use the classical identity of Gauss sums $\tau_{\theta,\nu}\tau_{\theta^{-1},\nu^{-1}} = \#k_E = q^2$, and the proof is complete. \square

6. Construction of the Jacquet-Langlands Correspondence

The construction of the family of rings \mathcal{L}_S together with the representations ρ_S of \mathcal{L}_S^\times will now be used to construct certain representations Π_S of $\text{GL}_2(F) \times B^\times$. We will then use Cor. 3.0.2 to show that the family Π_S realizes the Jacquet-Langlands Correspondence. This will involve showing that the matrix coefficients of Π_S satisfy the functional equation in Eq. 3.0.2 for sufficiently many χ . The heart of that calculation has already been completed in Theorem 5.0.3.

Recall that $\mathbf{G} = \text{GL}_2(F) \times B^\times$; this group has center $Z(\mathbf{G}) = F^\times \times F^\times$. Let $S = (\mathfrak{A}, n, \alpha)$ be a simple stratum in $M_2(F)$, and let $S' = (\mathfrak{A}', n', \alpha')$ be its corresponding simple stratum in B . From these data we have constructed a linking order \mathcal{L}_S and an irreducible representation ρ_S of \mathcal{L}_S^\times . Let $\ell = n/e$, so that every supercuspidal representation of GL_2 containing S has level ℓ , and likewise for B^\times . The intersection of $Z(\mathbf{G})$ with \mathcal{L}_S^\times is

$$Z(\mathbf{G}) \cap \mathcal{L}_S^\times = \left\{ (z_1, z_2) \in \mathcal{O}_F^\times \times \mathcal{O}_F^\times \mid v_F(z_1 - z_2) \geq \ell \right\}.$$

Here v_F is the valuation on F . By Theorem 5.0.3, ρ_S vanishes on the diagonally embedded subgroup $\Delta(F^\times) \cap \mathcal{L}_S^\times$. Choose a character ω of $Z(\mathbf{G})$ which vanishes on $\Delta(F^\times)$ and agrees with the central character of ρ_S on $Z(\mathbf{G}) \cap \mathcal{L}_S^\times$. We identify ω with a character of F^\times via its restriction to $F^\times \times \{1\}$.

We now extend ρ_S to a representation on a larger group which contains $Z(\mathbf{G})$ and which intertwines ρ_S . Define a group \mathcal{K}_S by

$$\mathcal{K}_S = Z(\mathbf{G})\Delta(E^\times)\mathcal{L}_S^\times.$$

(Recall that $\Delta(E^\times)$ normalizes \mathcal{L}_S^\times , so this is indeed a group.) There is a unique extension of ρ_S to a representation $\rho_{S,\omega}$ of \mathcal{K}_S which satisfies the conditions:

- (1) $\rho_{S,\omega}|_{Z(\mathbf{G})} = \omega$,
- (2) For $\beta \in E^\times$, $\rho_{S,\omega}(\Delta(\beta)) = (-1)^{v_E(\beta)}$ if E/F is ramified,
- (3) For $\beta \in E^\times$, $\rho_{S,\omega}(\Delta(\beta)) = 1$ if E/F is unramified.

The group \mathcal{K}_S is open and compact modulo its center. We may now define the representation $\Pi_{S,\omega}$ of \mathbf{G} as the induction of $\rho_{S,\omega}$ with compact support:

$$\Pi_{S,\omega} = \text{Ind}_{\mathcal{K}_S}^{\mathbf{G}} \rho_{S,\omega}.$$

We wish to confirm that $\Pi_{S,\omega}$ satisfies the desired properties (1)-(3) listed at the beginning of Section 4. It is already apparent that (1) $\Pi_{S,\omega}$ vanishes on $\Delta(F^\times)$. For property (2) we have the following:

Theorem 6.0.1. *$\Pi_{S,\omega}$ is the direct sum of representations of \mathbf{G} of the form $\pi \otimes \tilde{\pi}'$, where π (resp., π') is a minimal supercuspidal irreducible representation of $\text{GL}_2(F)$ (resp., B^\times) having central character ω and containing the stratum S (resp., S'). Every representation of either group having the above properties is contained in $\Pi_{S,\omega}$.*

Proof. Note that $\mathcal{K}_S \subset J_S \times J_{S'}$ is a subgroup of finite index. Let

$$M = \text{Ind}_{\mathcal{K}_S}^{J_S \times J_{S'}} \rho_{S,\omega}.$$

Then M is a direct sum of irreducible representations of $J_S \times J_{S'}$ of the form $\Lambda \otimes \tilde{\Lambda}'$. By Theorem 5.0.3, such a $\Lambda \otimes \tilde{\Lambda}'$ lies over the character $\psi_S = \psi_\alpha \otimes \psi_{\alpha'}^{-1}$ of $U_S \times U_{S'}$. Therefore we have $\Lambda \in C(\psi_\alpha, \mathfrak{A})$ and $\Lambda' \in C(\psi_{\alpha'}, \mathfrak{A}')$. By Theorem 2.2.5, $\pi = \text{Ind}_{J_S}^{\text{GL}_2(F)} \Lambda$ is an irreducible supercuspidal representation of $\text{GL}_2(F)$ containing S . Since $\rho_{S,\omega}$ has central character ω , the same is true of π . The reasoning is similar for $\pi' = \text{Ind}_{J_{S'}}^{B^\times} \Lambda'$.

Now assume π is an irreducible supercuspidal representation of $\text{GL}_2(F)$ containing S with central character ω . We claim that π is contained in $\Pi_{S,\omega}|_{\text{GL}_2(F)}$. Since $\Pi_{S,\omega}$ is induced from the representation $\rho_{S,\omega}$ of \mathcal{K}_S , the restriction of $\Pi_{S,\omega}$ to $\text{GL}_2(F)$ contains $\text{Ind}_{\mathcal{K}_S \cap \text{GL}_2(F)}^{\text{GL}_2(F)} \rho_{S,\omega}$. Therefore to show that π is contained in $\Pi_{S,\omega}|_{\text{GL}_2(F)}$ it suffices to prove that $\pi|_{\mathcal{K}_S \cap \text{GL}_2(F)}$ meets $\rho_{S,\omega}|_{\mathcal{K}_S \cap \text{GL}_2(F)}$. By Prop. 4.3.4 we have

$$\mathcal{K}_S \cap \text{GL}_2(F) = F^\times H_S.$$

The central characters of π and $\rho_{S,\omega}$ agree on F^\times by hypothesis. Therefore it suffices to show that $\pi|_{H_S}$ meets $\rho_S|_{H_S}$. By Theorem 2.2.5, π contains a representation $\Lambda \in C(\mathfrak{A}, \psi_\alpha)$. This means that the restriction of π to H_S contains $\Lambda|_{H_S}$, which must agree with $\rho_S|_{H_S}$ by Theorem 4.2.1. The case of a representation of B^\times is similar. □

The third required property of $\Pi_{S,\omega}$, concerning the zeta functions attached to matrix coefficients of this representation, shall follow from Prop. 5.0.3. We will start by translating Prop. 5.0.3 into a statement concerning the Fourier transforms of matrix coefficients of $\Pi_{S,\omega}$.

For a function f on \mathbf{G} , and a real number s , let f_s be the function

$$f_s(g) = f(g) \|g_1\|^{s-2} \|g_2\|^{-s}.$$

If $f \in \mathcal{C}(\Pi_{S,\omega})$, we wish to consider Fourier transforms of the functions f_s . The functions f_s are supported on \mathcal{K}_S , which is not compact, so their Fourier transforms do not *a priori* converge. Nonetheless we may formally define the Fourier transform \widehat{f}_s by integrating $f_s(x)\psi_{\mathbf{A}}(xy)$ over each of the (compact) cosets of \mathcal{L}_S^\times in \mathbf{G} . Since f is a linear combination of $\mathbf{G} \times \mathbf{G}$ -translates of vectors in $\mathcal{C}(\rho_S)$, which are in turn supported on \mathcal{L}_S^\times , we see that the integral vanishes on all but finitely many of the cosets. We now evaluate \widehat{f}_s .

Proposition 6.0.2. *For a matrix coefficient $f \in \mathcal{C}(\Pi_{S,\omega})$, we have*

$$(6.0.1) \quad \widehat{f}_s = -\check{f}_{2-s}.$$

Proof. We will first prove the corresponding statement relative to the Fourier transform \mathcal{F}_S :

$$(6.0.2) \quad \mathcal{F}_S f_s = \pm \check{f}_{2-s},$$

where the sign is 1 if E/F is ramified and -1 otherwise. It will suffice to prove Eq. 6.0.2 for matrix coefficients $f \in \mathcal{C}(\rho_S)$ supported on the group \mathcal{L}_S^\times . Indeed, glancing at the rules in Eq. 5.0.1 shows that the validity of Eq. 6.0.2 is unchanged upon replacing f by $L_g R_h f$ for elements $g, h \in \mathbf{G}$, and these translates span $\mathcal{C}(\Pi_{S,\omega})$ as f runs through $\mathcal{C}(\rho_S)$. But for $f \in \mathcal{C}(\rho_S)$, Eq. 6.0.2 follows from Theorem 5.0.3, because $f_s = f$.

To derive Eq. 6.0.1 from Eq. 6.0.2 we must compare the Fourier transforms \widehat{f} and $\mathcal{F}_S f$. The first transform is taken relative to the additive character $\psi_{\mathbf{A}}$, while the second is taken relative to the character ν_S . The characters are related by $\nu_S(x) = \psi_{\mathbf{A}}(\Delta(\beta)^{-1}x)$ for an element $\beta \in E^\times$ of valuation n ; formally we have $\widehat{f} = \|\Delta(\beta)\|^{-1} R_\beta \mathcal{F}_S f$. Applying this to the function f_s , we see that

$$\begin{aligned} \widehat{f}_s &= \|\Delta(\beta)\|^{-1} R_\beta \mathcal{F}_S f_s \\ &= \pm \|\Delta(\beta)\|^{-1} R_\beta (\check{f})_{2-s} \\ &= \pm (R_\beta \check{f})_{2-s}, \end{aligned}$$

where the sign is positive if and only if E/F is ramified. If E/F is ramified, then $\beta \in E^\times$ has odd valuation, and $R_\beta \check{f} = -\check{f}$ because $\rho_{S,\omega}$ takes the value -1 on such elements. If E/F is unramified, then $\rho_{S,\omega}(\Delta(\beta)) = 1$, and therefore $R_\beta \check{f} = \check{f}$. The proposition follows. □

We are ready to prove the appropriate functional equation for the zeta functions attached to $\Pi_{S,\omega}$. Recall that for an admissible representation Π of \mathbf{G} , and for $\Phi \in \mathbf{C}_c^\infty(\mathbf{A})$, $f \in \mathcal{C}(\Pi_{S,\omega})$, we defined the zeta function

$$\begin{aligned} \zeta(\Phi, f, s) &= \int_{\mathbf{G}} \Phi(g) f(g) \|g_1\|^s \|g_2\|^{2-s} d\mu^\times(g) \\ &= \int_{\mathbf{G}} \Phi(g) f_s(g) d\mu(g) \end{aligned}$$

where μ is a Haar measure on \mathbf{A} .

Theorem 6.0.3. *For all $\Phi \in \mathbf{C}_c^\infty(\mathbf{A})$ all $f \in \mathcal{C}(\Pi_\omega)$, and all characters χ of F^\times of conductor not exceeding ℓ , we have*

$$\zeta(\Phi, \chi f, s) = -\zeta(\hat{\Phi}, \chi^{-1} \check{f}, 2 - s).$$

Proof. It suffices to prove the claim for $\chi = 1$. Indeed, if $f \in \mathcal{C}(\Pi_{S,\omega})$, then χf lies in $\mathcal{C}(\Pi_{S',\chi^2\omega})$ for a different simple stratum $S' = (\mathcal{A}_1, n_1, \alpha'_1)$. (Explicitly: let $\beta \in \mathfrak{p}_E^n$ be such that $(\chi \circ N_{E/F})(1+x) = \psi_F(\text{Tr}_{E/F} \beta x)$ for all $x \in \mathfrak{p}_E^n$; then $\alpha'_1 = \alpha_1 + \beta$.)

Assume therefore that $\chi = 1$. We will take the measure $d\mu$ to equal $d\mu_\psi$, the measure dual to the character $\psi_{\mathbf{A}}$. Since $\hat{\Phi}(x) = \Phi(-x)$ we have that $\zeta(\hat{\Phi}, f, s) = \zeta(\Phi, f, s)$ by a change of variable $g \mapsto -g$ in the integral. Now we apply Prop. 6.0.1:

$$\begin{aligned} \zeta(\Phi, f, s) &= \zeta(\hat{\Phi}, f, s) \\ &= \int_{\mathbf{G}} \hat{\Phi}(g) f_s(g) d\mu_\psi(g) \\ &= \int_{\mathbf{G}} \hat{\Phi}(g) \hat{f}_s(g) d\mu_\psi(g) \\ &= - \int_{\mathbf{G}} \hat{\Phi}(g) \check{f}_{2-s}(g) d\mu_\psi(g) \\ &= -\zeta(\hat{\Phi}, \check{f}, 2 - s). \end{aligned}$$

□

6.1. The construction in level zero. The preceding constructions carry over easily to the case of level zero. Let E be an unramified quadratic extension of F . Letting θ denote a regular character of k_E^\times , we constructed in Section 5.2 a representation ρ_χ of the unit group of the linking order \mathcal{L}_0 . Choose a central character ω of $F^\times \times F^\times$ which agrees with the central character of ρ_θ on $(F^\times \times F^\times) \cap \mathcal{L}_0^\times = \mathcal{O}_F^\times \times \mathcal{O}_F^\times$. Extend ρ_θ to a representation $\rho_{\theta,\omega}$ of $\mathcal{K}_0 = (F^\times \times F^\times) \mathcal{L}_0$ agreeing with ω on the center. Finally, let $\Pi_{\theta,\omega}$ be the induced representation of $\rho_{\theta,\omega}$ from \mathcal{K}_0 up to $\text{GL}_2(F) \times B^\times$.

Then Thm. 6.0.1 has the following analogue:

Theorem 6.1.1. *Let π be a minimal irreducible admissible representation of $\mathrm{GL}_2(F)$ (resp., B^\times) with central character ω (resp., ω^{-1}). The following are equivalent:*

- (1) π has level zero, and the restriction of π to $\mathrm{GL}_2(\mathcal{O}_F)$ (resp., \mathcal{O}_B^\times) contains a representation inflated from the representation η_θ of $\mathrm{GL}_2(k)$ (resp., the character θ of k_E^\times .)
- (2) π is contained in $\Pi_{\theta,\omega}|_{\mathrm{GL}_2(F)}$ (resp., $\check{\pi}$ is contained in $\Pi_{\theta,\omega}|_{B^\times}$).

Similarly, Prop. 6.0.3 has this analogue:

Theorem 6.1.2. *For $\Phi \in C_c^\infty(\mathbf{A})$, $f \in \mathcal{C}(\Pi_{\omega,\theta})$, we have*

$$\zeta(\Phi, \chi f, s) = -\zeta(\hat{\Phi}, \chi^{-1} \check{f}, 2 - s)$$

for all characters χ of F^\times which are trivial on $1 + \mathfrak{p}_F$.

The proofs of Thm. 6.1.1 and Prop. 6.1.2 run exactly the same as those of Thm. 6.0.1 and Prop. 6.0.3.

6.2. Conclusion of the construction. Our construction of the Jacquet-Langlands correspondence is nearly complete.

Theorem 6.2.1. *For every irreducible representation π' of B^\times of dimension greater than one, there is a supercuspidal representation π of $\mathrm{GL}_2(F)$ for which π and π' correspond. Every supercuspidal representation of $\mathrm{GL}_2(F)$ arises this way.*

Proof. By Theorem 2.2.2 we may twist π' to assume either that $\check{\pi}'$ contains a simple stratum S' , or else that it is level zero. In the first case, let $S = (\mathfrak{A}, n, \alpha)$ be the corresponding stratum in $M_2(F)$. Applying Theorem 6.0.1, $\check{\pi}'$ is contained in $\Pi_{S,\omega}|_{B^\times}$, where ω is the central character of π' . Suppose π is a representation of $\mathrm{GL}_2(F)$ appearing in $\mathrm{Hom}_{B^\times}(\check{\pi}, \Pi_{S,\omega})$. Then $\pi \otimes \check{\pi}'$ appears in $\Pi_{S,\omega}$.

Applying Theorem 6.0.1 again, we find that π contains S . Combining Cor. 3.0.2 with Prop. 6.0.3 shows that π' and π correspond.

The logic is the same if π' has level zero: In this case $\check{\pi}'$ contains a character of \mathcal{O}_B^\times inflated from a character θ of a quadratic extension of k , so that $\check{\pi}'$ is contained in $\Pi_{\theta,\omega}|_{B^\times}$. Proceeding as above, we find a representation π of $\mathrm{GL}_2(F)$ corresponding to π' .

If π is a given supercuspidal representation of $\mathrm{GL}_2(F)$, the argument above may be reversed to find a representation π' of B^\times which corresponds to it. This concludes the proof. \square

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