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On the parity of generalized partition functions, III

par Fethi BEN SAÏD, Jean-Louis NICOLAS et Ahlem ZEKRAOUI

Résumé. Dans cet article, nous complétons les résultats de J.-L. Nicolas [15], en déterminant tous les éléments de l'ensemble $\mathcal{A} = \mathcal{A}(1+z+z^3+z^4+z^5)$ pour lequel la fonction de partition $p(\mathcal{A}, n)$ (c-à-d le nombre de partitions de n en parts dans $\mathcal{A}$) est paire pour tout $n \geq 6$. Nous donnons aussi un équivalent asymptotique à la fonction de décompte de cet ensemble.

Abstract. Improving on some results of J.-L. Nicolas [15], the elements of the set $\mathcal{A} = \mathcal{A}(1+z+z^3+z^4+z^5)$, for which the partition function $p(\mathcal{A}, n)$ (i.e. the number of partitions of n with parts in $\mathcal{A}$) is even for all $n \geq 6$ are determined. An asymptotic estimate to the counting function of this set is also given.

1. Introduction.

Let $\mathbb{N}$ (resp. $\mathbb{N}_0$) be the set of positive (resp. non-negative) integers. If $\mathcal{A} = \{a_1, a_2, ...\}$ is a subset of $\mathbb{N}$ and $n \in \mathbb{N}$ then $p(\mathcal{A}, n)$ is the number of partitions of n with parts in $\mathcal{A}$, i.e., the number of solutions of the diophantine equation

$$a_1x_1 + a_2x_2 + \ldots = n, \tag{1.1}$$

in non-negative integers $x_1, x_2, \ldots$. As usual we set $p(\mathcal{A}, 0) = 1$.
The counting function of the set $\mathcal{A}$ will be denoted by $A(x)$, i.e.,

$$A(x) = \mid \{n \leq x,\ n \in \mathcal{A}\} \mid . \tag{1.2}$$

Let $\mathbb{F}_2$ be the field with 2 elements, $P = 1+\epsilon_1 z^1+\ldots+\epsilon_N z^N \in \mathbb{F}_2[z],\ N \geq 1$. Although it is not difficult to prove (cf. [14], [5]) that there is a unique subset

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Mots clefs. Partitions, periodic sequences, order of a polynomial, orbits, 2-adic numbers, counting function, Selberg-Delange formula.

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$\mathcal{A} = \mathcal{A}(P)$ of $\mathbb{N}$ such that the generating function $F(z)$ satisfies

$$F(z) = F_{\mathcal{A}}(z) = \prod_{a\in\mathcal{A}} \frac{1}{1-z^a} = \sum_{n\geq 0} p(\mathcal{A},n)z^n \equiv P(z) \pmod 2, \tag{1.3}$$

the determination of the elements of such sets for general $P's$ seems to be hard.

Let the decomposition of P into irreducible factors over $\mathbb{F}_2$ be

$$P = P_1^{\alpha_1} P_2^{\alpha_2} ... P_l^{\alpha_l}. \tag{1.4}$$

We denote by $\beta_i = \mathrm{ord}(P_i)$, $1 \leq i \leq l$, the order of P_i, that is the smallest positive integer β_i such that $P_i(z)$ divides $1 + z^{\beta_i}$ in $\mathbb{F}_2[z]$. It is known that β_i is odd (cf. [13]). We set

$$\beta = \mathrm{lcm}(\beta_1, \beta_2, ..., \beta_l). \tag{1.5}$$

Let $\mathcal{A} = \mathcal{A}(P)$ satisfy (1.3) and $\sigma(\mathcal{A}, n)$ be the sum of the divisors of n belonging to $\mathcal{A}$, i.e.,

$$\sigma(\mathcal{A}, n) = \sum_{d|n,\, d\in\mathcal{A}} d = \sum_{d|n} d\chi(\mathcal{A}, d), \tag{1.6}$$

where $\chi(\mathcal{A}, .)$ is the characteristic function of the set $\mathcal{A}$, i.e, $\chi(\mathcal{A}, d) = 1$ if $d \in \mathcal{A}$ and $\chi(\mathcal{A}, d) = 0$ if $d \notin \mathcal{A}$. It was proved in [6] (see also [4], [12]) that for all $k \geq 0$, the sequence $(\sigma(\mathcal{A}, 2^k n) \bmod 2^{k+1})_{n\geq 1}$ is periodic with period β defined by (1.5), in other words,

$$n_1 \equiv n_2 \pmod \beta \Rightarrow \forall k \geq 0,\ \sigma(\mathcal{A}, 2^k n_1) \equiv \sigma(\mathcal{A}, 2^k n_2) \pmod{2^{k+1}}. \tag{1.7}$$

Moreover, the proof of (1.7) in [6] allows to calculate $\sigma(\mathcal{A}, 2^k n) \bmod 2^{k+1}$ and to deduce the value of $\chi(\mathcal{A}, n)$ where n is any positive integer. Indeed, let

$$S_{\mathcal{A}}(m, k) = \chi(\mathcal{A}, m) + 2\chi(\mathcal{A}, 2m) + \ldots + 2^k\chi(\mathcal{A}, 2^k m). \tag{1.8}$$

If n writes $n = 2^k m$ with $k \geq 0$ and m odd, (1.6) implies

$$\sigma(\mathcal{A}, n) = \sigma(\mathcal{A}, 2^k m) = \sum_{d\,|\,m} dS_{\mathcal{A}}(d, k), \tag{1.9}$$

which, by Möbius inversion formula, gives

$$mS_{\mathcal{A}}(m, k) = \sum_{d\,|\,m} \mu(d)\sigma(\mathcal{A}, \frac{n}{d}) = \sum_{d\,|\,\overline{m}} \mu(d)\sigma(\mathcal{A}, \frac{n}{d}), \tag{1.10}$$

where $\overline{m} = \prod_{p\,|\,m} p$ denotes the radical of m with $\overline{1} = 1$.

In the above sums, $\frac{n}{d}$ is always a multiple of 2^k, so that, from the values of $\sigma(\mathcal{A}, \frac{n}{d})$, by (1.10), one can determine the value of $S_{\mathcal{A}}(m, k) \bmod 2^{k+1}$ and by (1.8), the value of $\chi(\mathcal{A}, 2^i m)$ for all i, $i \leq k$.

Let β be an odd integer ≥ 3 and $(\mathbb{Z}/\beta\mathbb{Z})^*$ be the group of invertible elements modulo β. We denote by $< 2 >$ the subgroup of $(\mathbb{Z}/\beta\mathbb{Z})^*$ generated by 2 and consider its action $\star$ on the set $\mathbb{Z}/\beta\mathbb{Z}$ given by $a \star x = ax$ for all $a \in < 2 >$ and $x \in \mathbb{Z}/\beta\mathbb{Z}$. The quotient set will be denoted by $(\mathbb{Z}/\beta\mathbb{Z})/_{<2>}$ and the orbit of some n in $\mathbb{Z}/\beta\mathbb{Z}$ by $O(n)$. For $P \in \mathbb{F}_2[z]$ with $P(0) = 1$ and $\mathrm{ord}(P) = \beta$, let $\mathcal{A} = \mathcal{A}(P)$ be the set obtained from (1.3). Property (1.7) shows (after [3]) that if n_1 and n_2 are in the same orbit then

$$\sigma(\mathcal{A}, 2^k n_1) \equiv \sigma(\mathcal{A}, 2^k n_2) \pmod{2^{k+1}},\ \forall k \geq 0. \tag{1.11}$$

Consequently, for fixed k, the number of distinct values that $(\sigma(\mathcal{A}, 2^k n) \mod 2^{k+1})_{n\geq 1}$ can take is at most equal to the number of orbits of $\mathbb{Z}/\beta\mathbb{Z}$.

Let φ be the Euler function and s be the order of 2 modulo β, i.e., the smallest positive integer s such that $2^s \equiv 1 \pmod{\beta}$. If $\beta = p$ is a prime number then $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic and the number of orbits of $\mathbb{Z}/p\mathbb{Z}$ is equal to $1 + r$ with $r = \frac{\varphi(p)}{s} = \frac{p-1}{s}$. In this case, we have

$$(\mathbb{Z}/p\mathbb{Z})/_{<2>} = \{O(g),\ O(g^2), ...,\ O(g^r) = O(1),\ O(p)\}, \tag{1.12}$$

where g is some generator of $(\mathbb{Z}/p\mathbb{Z})^*$. For $r = 2$, the sets $\mathcal{A} = \mathcal{A}(P)$ were completely determined by N. Baccar, F. Ben Saïd and J.-L. Nicolas ([2], [8]). Moreover, N. Baccar proved in [1] that for all $r \geq 2$, the elements of $\mathcal{A}$ of the form $2^k m$, $k \geq 0$ and m odd, are determined by the 2-adic development of some root of a polynomial with integer coefficients. Unfortunately, his results are not explicit and do not lead to any evaluation of the counting function of the set $\mathcal{A}$. When $r = 6$, J.-L. Nicolas determined (cf. [15]) the odd elements of $\mathcal{A} = \mathcal{A}(1 + z + z^3 + z^4 + z^5)$. His results (which will be stated in Section 2, Theorem 2.1) allowed to deduce a lower bound for the counting function of $\mathcal{A}$. In this paper, we will consider the case $p = 31$ which satisfies $r = 6$. In $\mathbb{F}_2[z]$, we have

$$\frac{1 - z^{31}}{1 - z} = P^{(1)} P^{(2)} ... P^{(6)}, \tag{1.13}$$

with

$$P^{(1)} = 1+z+z^3+z^4+z^5,\ P^{(2)} = 1+z+z^2+z^4+z^5, P^{(3)} = 1+z^2+z^3+z^4+z^5,$$

$$P^{(4)} = 1 + z + z^2 + z^3 + z^5,\ P^{(5)} = 1 + z^2 + z^5,\ P^{(6)} = 1 + z^3 + z^5.$$

In fact, there are other primes p with $r = 6$. For instance, $p = 223$ and $p = 433$.

In Section 2, for $\mathcal{A} = \mathcal{A}(P^{(1)})$, we evaluate the sum $S_{\mathcal{A}}(m, k)$ which will lead to results of Section 3 determining the elements of the set $\mathcal{A}$. Section 4 will be devoted to the determination of an asymptotic estimate to the counting function $A(x)$ of $\mathcal{A}$. Although, in this paper, the computations are only carried out for $P = P^{(1)}$, the results could probably be extended to any $P^{(i)}$, $1 \le i \le 6$, and more generally, to any polynomial P of order p and such that $r = 6$.

Notation. We write $a \mod b$ for the remainder of the euclidean division of a by b. The ceiling of the real number x is denoted by

$$\lceil x \rceil = \inf\{n \in \mathbb{Z},\ x \le n\}.$$

2. The sum $S_{\mathcal{A}}(m,k)$, $\mathcal{A} = \mathcal{A}(1 + z + z^3 + z^4 + z^5)$.

From now on, we take $\mathcal{A} = \mathcal{A}(P)$ with

$$P = P^{(1)} = 1 + z + z^3 + z^4 + z^5. \tag{2.1}$$

The order of P is $\beta = 31$. The smallest primitive root modulo 31 is 3 that we shall use as a generator of $(\mathbb{Z}/31\mathbb{Z})^*$. The order of 2 modulo 31 is $s = 5$ so that

$$(\mathbb{Z}/31\mathbb{Z})/_{<2>} = \{O(3),\ O(3^2), ...,\ O(3^6) = O(1),\ O(31)\}, \tag{2.2}$$

with

$$O(3^j) = \{2^k 3^j,\ 0 \le k \le 4\},\ 1 \le j \le 6 \tag{2.3}$$

and

$$O(31) = \{31\}. \tag{2.4}$$

For $k \ge 0$ and $0 \le j \le 5$, we define the integers $u_{k,j}$ by

$$u_{k,j} = \sigma(\mathcal{A}, 2^k 3^j) \mod 2^{k+1}. \tag{2.5}$$

The Graeffe transformation. Let $\mathbb{K}$ be a field and $\mathbb{K}[[z]]$ be the ring of formal power series with coefficients in $\mathbb{K}$. For an element

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

of this ring, the product

$$f(z)f(-z) = b_0 + b_1 z^2 + b_2 z^4 + \ldots + b_n z^{2n} + \ldots$$

is an even power series. We shall call $\mathcal{G}(f)$ the series

$$\mathcal{G}(f)(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots. \tag{2.6}$$

It follows immediately from the above definition that for $f,\ g \in \mathbb{K}[[z]]$,

$$\mathcal{G}(fg) = \mathcal{G}(f)\mathcal{G}(g). \tag{2.7}$$

Moreover if q is an odd integer and $f(z) = 1 - z^q$, we have $\mathcal{G}(f) = f$. We shall use the following notation for the iterates of f by $\mathcal{G}$:

$$f_{(0)} = f,\ \ f_{(1)} = \mathcal{G}(f),\ \ \ldots,\ \ f_{(k)} = \mathcal{G}(f_{(k-1)}) = \mathcal{G}^{(k)}(f). \tag{2.8}$$

More details about the Graeffe transformation are given in [6]. By making the logarithmic derivative of formula (1.3), we get (cf. [14]):

$$\sum_{n=1}^{\infty}\sigma(\mathcal{A},n)z^n = z\frac{F'(z)}{F(z)} \equiv z\frac{P'(z)}{P(z)} \pmod{2}, \tag{2.9}$$

which, by Propositions 2 and 3 of [6], leads to

(2.10)

$$\sum_{n=1}^{\infty}\sigma(\mathcal{A},2^k n)z^n \equiv z\frac{P'_{(k)}(z)}{P_{(k)}(z)} = \frac{z}{1-z^{31}}\left(P'_{(k)}(z)W_{(k)}(z)\right) \pmod{2^{k+1}},$$

with $P'_{(k)}(z) = \frac{\mathrm{d}}{\mathrm{d}z}(P_{(k)}(z))$ and

$$W(z) = (1-z)P^{(2)}(z)...P^{(6)}(z). \tag{2.11}$$

Formula (2.10) proves (1.11) with $\beta = 31$, and the computation of the k-th iterates $P_{(k)}$ and $W_{(k)}$ by the Graeffe transformation yields the value of $\sigma(\mathcal{A},2^k n) \mod 2^{k+1}$. For instance, for $k = 11$, we obtain:

$u_{k,0} = 1183,\ u_{k,1} = 1598,\ u_{k,2} = 1554,\ u_{k,3} = 845,\ u_{k,4} = 264,\ u_{k,5} = 701.$

A divisor of $2^k 3^j$ is either a divisor of $2^{k-1}3^j$ or a multiple of 2^k. Therefore, from (2.5) and (1.6), $u_{k,j} \equiv u_{k-1,j} \pmod{2^k}$ holds and the sequence $(u_{k,j})_{k\geq 0}$ defines a 2-adic integer U_j satisfying for all k's:

$$U_j \equiv u_{k,j} \pmod{2^{k+1}},\ 0 \leq j \leq 5. \tag{2.12}$$

It has been proved in [1] that the U_j's are the roots of the polynomial

$$R(y) = y^6 - y^5 + 3y^4 - 11y^3 + 44y^2 - 36y + 32.$$

Note that $R(y)^5$ is the resultant in z of $\phi_{31}(z) = 1 + z + ... + z^{30}$ and $y + z + z^2 + z^4 + z^8 + z^{16}$.

Let us set

$$\theta = U_0 = 1 + 2 + 2^2 + 2^3 + 2^4 + 2^7 + 2^{10} + ...$$

It turns out that the Galois group of $R(y)$ is cyclic of order 6 and therefore the other roots $U_1,...,U_5$ of $R(y)$ are polynomials in θ. With Maple, by factorizing $R(y)$ on $\mathbb{Q}[\theta]$ and using the values of $u_{11,j}$, we get

$$U_0 = \theta \equiv 1183 \pmod{2^{11}}$$

$$U_1 = \frac{1}{32}(3\theta^5 + 5\theta^3 - 36\theta^2 + 84\theta) \equiv 1598 \pmod{2^{11}}$$

$$U_2 = \frac{1}{32}(-3\theta^5 - 5\theta^3 + 20\theta^2 - 100\theta) \equiv 1554 \pmod{2^{11}}$$

$$U_3 = \frac{1}{32}(-\theta^5 - 7\theta^3 + 12\theta^2 - 44\theta + 32) \equiv 845 \pmod{2^{11}}$$

$$U_4 = \frac{1}{32}(-\theta^5 + 4\theta^4 + \theta^3 + 24\theta^2 - 68\theta + 96) \equiv 264 \pmod{2^{11}}$$

$$U_5 = \frac{1}{16}(\theta^5 - 2\theta^4 + 3\theta^3 - 10\theta^2 + 48\theta - 48) \equiv 701 \pmod{2^{11}}. \tag{2.13}$$

For convenience, if $j \in \mathbb{Z}$, we shall set

$$U_j = U_{j \bmod 6}. \tag{2.14}$$

We define the completely additive function $\ell : \mathbb{Z} \setminus 31\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ by

$$\ell(n) = j \quad if\ n \in O(3^j), \tag{2.15}$$

so that $\ell(n_1 n_2) \equiv \ell(n_1) + \ell(n_2) \pmod{6}$. We split the odd primes different from 31 into six classes according to the value of ℓ. More precisely, for $0 \le j \le 5$,

$$p \in \mathcal{P}_j \iff \ell(p) = j \iff p \equiv 2^k 3^j \pmod{31},\ k = 0, 1, 2, 3, 4. \tag{2.16}$$

We take $L : \mathbb{N} \setminus 31\mathbb{N} \longrightarrow \mathbb{N}_0$ to be the completely additive function defined on primes by

$$L(p) = \ell(p). \tag{2.17}$$

We define, for $0 \le j \le 5$, the additive function $\omega_j : \mathbb{N} \longrightarrow \mathbb{N}_0$ by

$$\omega_j(n) = \sum_{p|n,\ p \in \mathcal{P}_j} 1 = \sum_{p|n,\ \ell(p)=j} 1, \tag{2.18}$$

and $\omega(n) = \omega_0(n) + ... + \omega_5(n) = \sum_{p|n} 1$. We remind that additive functions vanish on 1.

From (2.5), (2.3), (1.11) and (2.12), it follows that if $n = 2^k m \in O(3^j)$ (so that $j = \ell(n) = \ell(m)$),

$$\sigma(\mathcal{A}, n) = \sigma(\mathcal{A}, 2^k m) \equiv U_{\ell(m)} \pmod{2^{k+1}}. \tag{2.19}$$

We may consider the 2-adic number

$$S(m) = S_{\mathcal{A}}(m) = \chi(\mathcal{A}, m) + 2\chi(\mathcal{A}, 2m) + ... + 2^k \chi(\mathcal{A}, 2^k m) + ... \tag{2.20}$$

satisfying from (1.8),

$$S(m) \equiv S_{\mathcal{A}}(m, k) \pmod{2^{k+1}}. \tag{2.21}$$

Then (1.10) implies for $\gcd(m, 31) = 1$,

$$mS(m) = \sum_{d \mid m} \mu(d) U_{\ell(\frac{m}{d})}. \tag{2.22}$$

If 31 divides m, it was proved in [3, (3.6)] that, for all k's,

$$\sigma(\mathcal{A}, 2^k m) \equiv -5 \pmod{2^{k+1}}. \tag{2.23}$$

Remark 2.1. *No element of $\mathcal{A}$ has a prime factor in $\mathcal{P}_0$. This general result has been proved in* [3]*, but we recall the proof on our example: let us assume that $n = 2^k m \in \mathcal{A}$, where m is an odd integer divisible by some prime p in $\mathcal{P}_0$, in other words $\omega_0(m) \geq 1$. (1.10) gives*

$$\begin{aligned} mS_{\mathcal{A}}(m,k) &= \sum_{d\,|\,m} \mu(d)\sigma\left(\mathcal{A}, \frac{n}{d}\right) = \sum_{d\,|\,\overline{m}} \mu(d)\sigma\left(\mathcal{A}, 2^k\frac{m}{d}\right) \\ &= \sum_{d\,|\,\frac{\overline{m}}{p}} \mu(d)\sigma\left(\mathcal{A}, 2^k\frac{m}{d}\right) + \sum_{d\,|\,\frac{\overline{m}}{p}} \mu(pd)\sigma\left(\mathcal{A}, 2^k\frac{m}{pd}\right) \\ &= \sum_{d\,|\,\frac{\overline{m}}{p}} \mu(d)\left(\sigma\left(\mathcal{A}, 2^k\frac{m}{d}\right) - \sigma\left(\mathcal{A}, 2^k\frac{m}{pd}\right)\right). \end{aligned}$$

In the above sum, both $\frac{m}{d}$ and $\frac{m}{pd}$ are in the same orbit, so that from (1.11), $\sigma(\mathcal{A}, 2^k\frac{m}{d}) \equiv \sigma(\mathcal{A}, 2^k\frac{m}{pd}) \pmod{2^{k+1}}$ and therefore $mS_{\mathcal{A}}(m,k) \equiv 0 \pmod{2^{k+1}}$. Since m is odd and (cf. (1.8)) $0 \leq S_{\mathcal{A}}(m,k) < 2^{k+1}$ then $S_{\mathcal{A}}(m,k) = 0$, so that by (1.8), $2^h m \notin \mathcal{A}$, for all $0 \leq h \leq k$.

In [15], J.-L. Nicolas has described the odd elements of $\mathcal{A}$. In fact, he obtained the following:

Theorem 2.1. *(*[15]*)*

(a) *The odd elements of $\mathcal{A}$ which are primes or powers of primes are of the form p^λ, $\lambda \geq 1$, satisfying one of the following four conditions:*

$$\begin{aligned} &p \in \mathcal{P}_1 \quad \text{and} \quad \lambda \equiv 1,3,4,5 \pmod 6 \\ &p \in \mathcal{P}_2 \quad \text{and} \quad \lambda \equiv 0,1 \pmod 3 \\ &p \in \mathcal{P}_4 \quad \text{and} \quad \lambda \equiv 0,1 \pmod 3 \\ &p \in \mathcal{P}_5 \quad \text{and} \quad \lambda \equiv 0,2,3,4 \pmod 6. \end{aligned}$$

(b) *No odd element of $\mathcal{A}$ is a multiple of 31^2. If m is odd, $m \neq 1$, and not a multiple of 31, then*

$$m \in \mathcal{A} \quad \text{if and only if} \quad 31m \in \mathcal{A}.$$

(c) *An odd element $n \in \mathcal{A}$ satisfies $\omega_0(n) = 0$ and $\omega_3(n) = 0$ or 1; in other words, n is free of prime factor in $\mathcal{P}_0$ and has at most one prime factor in $\mathcal{P}_3$.*

(d) *The odd elements of $\mathcal{A}$ different from 1, not divisible by 31, which are not primes or powers of primes are exactly the odd $n's$, $n \neq 1$, such that (where $\overline{n} = \prod_{p|n} p$):*
   (1) *$\omega_0(n) = 0$ and $\omega_3(n) = 0$ or 1.*
   (2) *If $\omega_3(n) = 1$ then $\ell(n) + \ell(\overline{n}) \equiv 0$ or $1 \pmod 3$.*

(3) *If* $\omega_3(n) = 0$ *and* $\omega_1(n) + \ell(n) - \ell(\overline{n})$ *is even then*

$$2\ell(n) - \ell(\overline{n}) \equiv 2 \text{ or } 3 \text{ or } 4 \text{ or } 5 \pmod{6}.$$

(4) *If* $\omega_3(n) = 0$ *and* $\omega_1(n) + \ell(n) - \ell(\overline{n})$ *is odd then*

$$2\ell(n) - \ell(\overline{n}) \equiv 0 \text{ or } 4 \pmod{6}.$$

Remark 2.2. *Point (b) of Theorem 2.1 can be improved in the following way: No element of* $\mathcal{A}$ *is a multiple of* 31^2. *Indeed, from (1.10), we have for* m *odd,* $k \geq 0$ *and* $\tau \geq 2$,

$$\begin{aligned} 31^\tau m S_{\mathcal{A}}(31^\tau m, k) &= \sum_{d \mid 31^\tau m} \mu(d)\sigma\left(\mathcal{A}, 2^k 31^\tau \frac{m}{d}\right) \\ &= \sum_{d \mid 31\overline{m}} \mu(d)\sigma\left(\mathcal{A}, 2^k 31^\tau \frac{m}{d}\right) \\ &= \sum_{d \mid \overline{m}} \mu(d)\left\{\sigma\left(\mathcal{A}, 2^k 31^\tau \frac{m}{d}\right) - \sigma\left(\mathcal{A}, 2^k 31^{\tau-1} \frac{m}{d}\right)\right\}. \end{aligned}$$

Since $31^\tau \frac{m}{d}$ *and* $31^{\tau-1}\frac{m}{d}$ *are in the same orbit* $O(31)$ *then (1.11) and (2.23) give* $\sigma(\mathcal{A}, 2^k 31^\tau \frac{m}{d}) \equiv \sigma(\mathcal{A}, 2^k 31^{\tau-1}\frac{m}{d}) \equiv -5 \pmod{2^{k+1}}$, *so that we get* $S_{\mathcal{A}}(31^\tau m, k) \equiv 0 \pmod{2^{k+1}}$. *Hence, from (1.8),* $S_{\mathcal{A}}(31^\tau m, k) = 0$ *and for all* $0 \leq h \leq k$ *and all* $\tau \geq 2$, $2^h 31^\tau m$ *does not belong to* $\mathcal{A}$.

In view of stating Theorem 2.2 which will extend Theorem 2.1 , we shall need some notation. The radical $\overline{m}$ of an odd integer $m \neq 1$, not divisible by 31 and free of prime factors belonging to $\mathcal{P}_0$ will be written

(2.24)
$$\overline{m} = p_1 \dots p_{\omega_1} p_{\omega_1+1} \dots p_{\omega_1+\omega_2} p_{\omega_1+\omega_2+1} \dots \dots p_{\omega_1+\omega_2+\omega_3+\omega_4+1} \dots p_\omega,$$

where $\ell(p_i) = j$ for $\omega_1 + ... + \omega_{j-1} + 1 \leq i \leq \omega_1 + ... + \omega_j$, $\omega_j = \omega_j(m) = \omega_j(\overline{m})$ and $\omega = \omega(m) = \omega(\overline{m}) \geq 1$. We define the additive functions from $\mathbb{Z} \setminus 31\mathbb{Z}$ into $\mathbb{Z}/12\mathbb{Z}$:

$$\alpha = \alpha(m) = 2\omega_5 - 2\omega_1 + \omega_4 - \omega_2 \bmod 12, \tag{2.25}$$

$$a = a(m) = \omega_5 - \omega_1 + \omega_2 - \omega_4 \bmod 12. \tag{2.26}$$

Let $(v_i)_{i\in\mathbb{Z}}$ be the periodic sequence of period 12 defined by

$$v_i = \begin{cases} \frac{2}{\sqrt{3}} \cos(i\frac{\pi}{6}) & \text{if } i \text{ is odd} \\ 2\cos(i\frac{\pi}{6}) & \text{if } i \text{ is even.} \end{cases} \tag{2.27}$$

The values of $(v_i)_{i\in\mathbb{Z}}$ are given by:

$i =$	0	1	2	3	4	5	6	7	8	9	10	11
$v_i =$	2	1	1	0	-1	-1	-2	-1	-1	0	1	1

Note that

$$v_{i+6} = -v_i, \tag{2.28}$$

$$v_i + v_{i+2} = \begin{cases} v_{i+1} & \text{if } i \text{ is odd} \\ 3v_{i+1} & \text{if } i \text{ is even,} \end{cases} \tag{2.29}$$

$$v_{2i} \equiv -2^i \pmod{3} \tag{2.30}$$

and

$$v_i \equiv v_{i+3} \equiv v_{2i} \pmod{2}. \tag{2.31}$$

From the U_j's (cf. (2.12) and (2.13)), we introduce the following 2-adic integers:

$$E_i = \sum_{j=0}^{5} v_{i+2j} U_j, \;\; i \in \mathbb{Z}, \tag{2.32}$$

$$F_i = \sum_{j=0}^{5} v_{i+4j} U_j, \;\; i \in \mathbb{Z}, \tag{2.33}$$

$$G = \sum_{j=0}^{5} (-1)^j U_j. \tag{2.34}$$

From (2.28), we have

$$E_{i+6} = -E_i, \;\; E_{i+12} = E_i, \;\; F_{i+6} = -F_i, \;\; F_{i+12} = F_i. \tag{2.35}$$

From (2.29), it follows that, if i is odd,

$$E_i + E_{i+2} = E_{i+1}, \;\; F_i + F_{i+2} = F_{i+1}, \tag{2.36}$$

while, if i is even,

$$E_i + E_{i+2} = 3E_{i+1}, \;\; F_i + F_{i+2} = 3F_{i+1}, \tag{2.37}$$

The values of these numbers are given in the following array:

Z		Z mod 2^{11}
$E_0 =$	$\frac{1}{32}(11\theta^5 - 8\theta^4 + 29\theta^3 - 124\theta^2 + 500\theta - 256)$	1157
$E_1 =$	$\frac{1}{16}(3\theta^5 - 2\theta^4 + 9\theta^3 - 26\theta^2 + 136\theta - 64)$	1533
$E_2 =$	$3E_1 - E_0$	1394
$E_3 =$	$2E_1 - E_0$	1909
$E_4 =$	$3E_1 - 2E_0$	237
$E_5 =$	$E_1 - E_0$	376
$F_0 =$	$\frac{1}{32}(-3\theta^5 - 21\theta^3 + 36\theta^2 - 36\theta + 64)$	1987
$F_1 =$	$\frac{1}{32}(-3\theta^5 - 4\theta^4 - 13\theta^3 + 24\theta^2 - 28\theta - 64)$	166
$F_2 =$	$3F_1 - F_0$	559
$F_3 =$	$2F_1 - F_0$	393
$F_4 =$	$3F_1 - 2F_0$	620
$F_5 =$	$F_1 - F_0$	227
$G =$	$\frac{1}{4}(-\theta^5 + \theta^4 - \theta^3 + 11\theta^2 - 34\theta + 20)$	1905

TABLE 1

Lemma 2.1. *The polynomials* $(U_j)_{0\le j\le 5}$ *(cf.* (2.13)*) form a basis of* $\mathbb{Q}[\theta]$*. The polynomials* $E_0,\ E_1,\ F_0,\ F_1,\ G,\ U_0$ *form another basis of* $\mathbb{Q}[\theta]$*. For all i's,* E_i *and* F_i *are linear combinations of respectively* E_0 *and* E_1 *and* F_0 *and* F_1*.*

Proof. With Maple, in the basis $1, \theta, \dots, \theta^5$, we compute determinant $(U_0, ..., U_5) = \frac{1}{1024}$. From (2.32), (2.33) and (2.34), the determinant of $(E_0, E_1,\ F_0, F_1, G, U_0)$ in the basis $U_0, U_1, \dots, U_5$ is equal to 12. The last point follows from (2.36) and (2.37). □

We have

Theorem 2.2. *Let* $m \neq 1$ *be an odd integer not divisible by* 31 *with* $\overline{m}$ *of the form (2.24). Under the above notation and the convention*

$$0^\omega = \begin{cases} 1 & \text{if } \omega = 0 \\ 0 & \text{if } \omega > 0, \end{cases} \tag{2.38}$$

we have:

(1) *The 2-adic integer* $S(m)$ *defined by (2.20) satisfies*

$$mS(m) = 2^{\omega_3 - 1} 3^{\lceil \frac{\omega_2+\omega_4}{2} - 1\rceil} E_{\alpha - 2\ell(m)} + \frac{0^{\omega_3}}{2} 3^{\lceil \frac{\omega}{2} - 1\rceil} F_{a-4\ell(m)}$$

$$+ \frac{0^{\omega_2+\omega_4}}{3} 2^{\omega - 1}(-1)^{\ell(m)} G. \tag{2.39}$$

(2) *The 2-adic integer* $S(31m)$ *satisfies*

$$S(31m) = -31^{-1} S(m), \tag{2.40}$$

where 31^{-1} *is the inverse of* 31 *in* $\mathbb{Z}_2$*. In particular, for all* $k \in \{0,1,2,3,4\}$*, we have*

$$2^k m \in \mathcal{A} \iff 31 \cdot 2^k m \in \mathcal{A},$$

since the inverse of 31 *modulo* 2^{k+1} *is* -1 *for* $k \leq 4$*.*

Proof of Theorem 2.2 (1). From (2.22), we have

$$mS(m) = \sum_{d \mid \overline{m}} \mu(d) U_{\ell(\frac{m}{d})} = \sum_{d \mid \overline{m}} \mu(d) U_{\ell(m)-\ell(d)}. \tag{2.41}$$

Further, (2.41) becomes

$$mS(m) = \sum_{j=0}^{5} T(m,j) U_{\ell(m)-j} = \sum_{j=0}^{5} T(m, \ell(m)-j) U_j, \tag{2.42}$$

with

$$T(m,j) = T(\overline{m}, j) = \sum_{d \mid \overline{m},\ \ell(d) \equiv j \pmod{6}} \mu(d). \tag{2.43}$$

Therefore (2.39) will follow from (2.42) and from the following lemma. ☐

Lemma 2.2. *The integer* $T(m,j)$ *defined in (2.43) with the convention (2.38) and the definitions (2.18) and (2.24)-(2.27), for* $m \neq 1$*, is equal to*

$$T(m,j) = 2^{\omega_3 - 1} 3^{\lceil \frac{\omega_2+\omega_4}{2} - 1 \rceil} v_{\alpha - 2j} + \frac{0^{\omega_3}}{2} 3^{\lceil \frac{\omega}{2} - 1 \rceil} v_{a-4j}$$

$$+ 0^{\omega_2+\omega_4} \frac{(-1)^j}{3} 2^{\omega-1}. \tag{2.44}$$

Proof of Lemma 2.2. Let us introduce the polynomial

$$f(X) = (1-X)^{\omega_1}(1-X^2)^{\omega_2} \ldots (1-X^5)^{\omega_5} = \sum_{\nu \geq 0} f_\nu X^\nu. \tag{2.45}$$

If the five signs were plus instead of minus, $f(X)$ would be the generating function of the partitions in at most ω_1 parts equal to 1, ..., at most ω_5 parts equal to 5. More generally, the polynomial

$$\widetilde{f}(X) = \prod_{i=1}^{\omega} (1 + a_i X^{b_i}) = \sum_{\nu \geq 0} \widetilde{f_\nu} X^\nu$$

is the generating function of

$$\widetilde{f_\nu} = \sum_{\epsilon_1,\ldots,\epsilon_\omega \in \{0,1\},\ \sum_{i=1}^{\omega} \epsilon_i b_i = \nu} \ \prod_{i=1}^{\omega} a_i^{\epsilon_i}.$$

To the vector $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_\omega) \in \mathbb{F}_2^\omega$, we associate

$$d = \prod_{i=1}^{\omega} p_i^{\epsilon_i},\ \mu(d) = \prod_{i=1}^{\omega} (-1)^{\epsilon_i},\ L(d) = \sum_{i=1}^{\omega} \epsilon_i \ell(p_i)$$

where L is the arithmetic function defined by (2.17) and we get

$$f_\nu = \sum_{d|\overline{m},\ L(d)=\nu} \mu(d). \tag{2.46}$$

Consequently, by setting $\xi = \exp(\frac{i\pi}{3})$, (2.43), (2.45) and (2.46) give

$$\begin{aligned} T(m,j) &= \sum_{\nu,\ \nu\equiv j \pmod 6} \ \sum_{d|\overline{m},\ L(d)=\nu} \mu(d) \\ &= \sum_{\nu\equiv j \pmod 6} f_\nu \\ &= \frac{1}{6}\sum_{i=0}^{5} \xi^{-ij} f(\xi^i) \\ &= \frac{1}{6}\sum_{i=1}^{5} \xi^{-ij} f(\xi^i) \\ &= \frac{1}{6}\sum_{i=1}^{5} \xi^{-ij}(1-\xi^i)^{\omega_1}(1-\xi^{2i})^{\omega_2}(1-\xi^{3i})^{\omega_3}(1-\xi^{4i})^{\omega_4}(1-\xi^{5i})^{\omega_5}. \qquad (2.47) \end{aligned}$$

By observing that

$$1-\xi = \xi^5,\ 1-\xi^2 = \varrho = \sqrt{3}(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}),\ 1-\xi^3 = 2,\ 1-\xi^4 = \overline{\varrho},\ 1-\xi^6 = 0,$$

the sum of the terms in $i = 1$ and $i = 5$ in (2.47), which are conjugate, is equal to

$$\frac{2}{6}\mathcal{R}(\xi^{-j}\xi^{5\omega_1}\varrho^{\omega_2}2^{\omega_3}\overline{\varrho}^{\omega_4}\xi^{\omega_5}) = \frac{2^{\omega_3}}{3}\sqrt{3}^{\omega_2+\omega_4}\cos\frac{\pi}{6}(2\omega_5 - 2\omega_1 + \omega_4 - \omega_2 - 2j). \tag{2.48}$$

Now, the contribution of the terms in $i = 2$ and $i = 4$ is

$$\begin{aligned} \frac{2}{6}\mathcal{R}(\xi^{-2j}\varrho^{\omega_1}\overline{\varrho}^{\omega_2}0^{\omega_3}\varrho^{\omega_4}\overline{\varrho}^{\omega_5}) &= 0^{\omega_3}\frac{\sqrt{3}^{\omega_1+\omega_2+\omega_4+\omega_5}}{3} \\ &\quad \times \cos\frac{\pi}{6}(\omega_2+\omega_5-\omega_1-\omega_4-4j) \\ &= 0^{\omega_3}\frac{\sqrt{3}^{\omega}}{3}\cos\frac{\pi}{6}(\omega_2+\omega_5-\omega_1-\omega_4-4j). \qquad (2.49) \end{aligned}$$

Finally, the term corresponding to $i = 3$ in (2.47) is equal to

$$\frac{1}{6}(-1)^j 2^{\omega_1}0^{\omega_2}2^{\omega_3}0^{\omega_4}2^{\omega_5} = 0^{\omega_2+\omega_4}\frac{(-1)^j}{6}2^{\omega_1+\omega_3+\omega_5} = 0^{\omega_2+\omega_4}\frac{(-1)^j}{6}2^{\omega}. \tag{2.50}$$

Consequently, by using our notation (2.24)-(2.26), (2.47) becomes

$$T(m,j) = \frac{2^{\omega_3}}{3}\sqrt{3}^{\omega_2+\omega_4}\cos\frac{\pi}{6}(\alpha - 2j) + 0^{\omega_3}\frac{\sqrt{3}^{\omega}}{3}\cos\frac{\pi}{6}(a-4j)$$

$$+\,0^{\omega_2+\omega_4}\frac{(-1)^j}{6}2^{\omega}. \tag{2.51}$$

Observing that $\alpha - 2j$ has the same parity than $\omega_2+\omega_4$ and similarly for $a-4j$ and ω (when $\omega_0=\omega_3=0$), via (2.27), we get (2.44). □

Proof of Theorem 2.2 (2). For all $k \geq 0$, from (1.10), we have

$$\begin{aligned} 31mS_{\mathcal{A}}(31m,k) &= \sum_{d\,|\,31m}\mu(d)\sigma(\mathcal{A},31\cdot 2^k\frac{m}{d}) = \sum_{d\,|\,31\overline{m}}\mu(d)\sigma(\mathcal{A},31\cdot 2^k\frac{m}{d}) \\ &= \sum_{d\,|\,\overline{m}}\mu(d)\sigma(\mathcal{A},31\cdot 2^k\frac{m}{d}) - \sum_{d\,|\,\overline{m}}\mu(d)\sigma(\mathcal{A},2^k\frac{m}{d}) \\ &= \sum_{d\,|\,\overline{m}}\mu(d)\sigma(\mathcal{A},31\cdot 2^k\frac{m}{d}) - mS_{\mathcal{A}}(m,k). \end{aligned} \tag{2.52}$$

Since for all d dividing $\overline{m}$, $31\cdot 2^k\frac{m}{d}\in O(31)$ then, from (2.23), $\sigma(\mathcal{A},31\cdot 2^k\frac{m}{d})\equiv\sigma(\mathcal{A},31\cdot 2^k)\equiv -5 \pmod{2^{k+1}}$, so that (2.52) gives

$$31mS_{\mathcal{A}}(31m,k)+mS_{\mathcal{A}}(m,k)\equiv -5\sum_{d\,|\,\overline{m}}\mu(d) \pmod{2^{k+1}}. \tag{2.53}$$

Since $\overline{m}\neq 1$, $31mS_{\mathcal{A}}(31m,k)+mS_{\mathcal{A}}(m,k)\equiv 0 \pmod{2^{k+1}}$. Recalling that m is odd, by using (2.20), (2.21) and their similar for $S(31m)$, we obtain the desired result. □

3. Elements of the set $\mathcal{A}=\mathcal{A}(1+z+z^3+z^4+z^5)$.

In this section, we will determine the elements of the set $\mathcal{A}$ of the form $n=2^k31^\tau m$, where $\overline{m}\neq 1$ satisfies (2.24) and $\tau\in\{0,1\}$, since from Remark 2.2, $2^k31^\tau m\notin\mathcal{A}$ for all $\tau\geq 2$. The elements of the set $\mathcal{A}(1+z+z^3+z^4+z^5)$ of the form $31^\tau 2^k$, $\tau=0$ or 1, were shown in [1] to be solutions of 2-adic equations. More precisely, the following was proved in that paper.

1) The elements of the set $\mathcal{A}(1+z+z^3+z^4+z^5)$ of the form 2^k, $k\geq 0$, are given by the 2-adic solution

$$\sum_{k\geq 0}\chi(\mathcal{A},2^k)\,2^k = S(1) = U_0 = 1+2+2^2+2^3+2^4+2^7+2^{10}+2^{11}+...$$

of the equation

$$y^6-y^5+3y^4-11y^3+44y^2-36y+32=0.$$

Note that $S(1)=U_0$ follows from (2.22).

2) The elements of the set $\mathcal{A}(1+z+z^3+z^4+z^5)$ of the form $31\cdot 2^k$, $k\geq 0$, are given by the solution

$$\sum_{k\geq 0}\chi(\mathcal{A},31\cdot 2^k)\,2^k = S(31) = y = 2^2+2^5+2^{11}+...$$

of the equation

$$31^5y^6+31^5y^5+13\cdot 31^4y^4+91\cdot 31^3y^3+364\cdot 31^2y^2+796\cdot 31y+752=0,$$

since, from (2.53) with $m=1$, we have $31S(31)=-5-U_0$, so that

$$S(31)=\frac{5+U_0}{1-32}=(1+4+U_0)(1+2^5+2^{10}+...)=2^2+2^5+2^{11}+...$$

Theorem 3.1. *Let $m\neq 1$ be an odd integer not divisible by any prime $p\in\mathcal{P}_0$ (cf. (2.16)) neither by 31^2. Then the sum $S(m)$ defined by (2.20) does not vanish. So we may introduce the 2-adic valuation of $S(m)$:*

$$\gamma=\gamma(m)=v_2(S(m)). \tag{3.1}$$

Then, if 31 does not divide m, we have

$$\gamma(31m)=\gamma(m). \tag{3.2}$$

Let us assume now that m is coprime with 31. We shall use the quantities $\omega_i=\omega_i(m)$ defined by (2.18), $\ell(m)$, $\alpha=\alpha(m)$, $a=a(m)$ defined by (2.15), (2.25) and (2.26),

(3.3)
$$\alpha'=\alpha'(m)=\alpha-2\ell(m)\bmod 12=2\omega_5-2\omega_1+\omega_4-\omega_2-2\ell(m)\bmod 12,$$

$$a'=a'(m)=a-4\ell(m)\bmod 12=\omega_5-\omega_1+\omega_2-\omega_4-4\ell(m)\bmod 12, \tag{3.4}$$

$$\begin{aligned} t=t(m)&=\left\lceil\frac{\omega_1+\omega_5+\omega_2+\omega_4}{2}-1\right\rceil-\left\lceil\frac{\omega_2+\omega_4}{2}-1\right\rceil\\ &=\begin{cases}\lceil\frac{\omega_1+\omega_5}{2}\rceil & \text{if}\quad \omega_1+\omega_5\equiv\ \omega_2+\omega_4\equiv 1 \pmod 2\\ \lceil\frac{\omega_1+\omega_5}{2}-1\rceil & \text{if}\quad \text{not}.\end{cases}\end{aligned} \tag{3.5}$$

We have:

(i) *if $\omega_3\neq 0$ and $\omega_2+\omega_4\neq 0$, the value of $\gamma=\gamma(m)$ is given by*

$$\gamma=\begin{cases}\omega_3-1 & \text{if}\quad \alpha'\equiv 0,1,3,4 \pmod 6\\ \omega_3 & \text{if}\quad \alpha'\equiv 2 \pmod 6\\ \omega_3+2 & \text{if}\quad \alpha'\equiv 5 \pmod 6).\end{cases}$$

(ii) *If $\omega_2+\omega_4=0$ and $\omega_3\geq 1$, we set $\alpha''=\alpha'+6\ell(m)\bmod 12$ and $\delta(i)=v_2(E_i+2^{v_2(E_i)}G)$ and we have*

$$\begin{array}{lll} \text{if}\quad \omega_1+\omega_5<v_2(E_{\alpha''}), & \text{then} & \gamma=\omega_3-1+\omega_1+\omega_5,\\ \text{if}\quad \omega_1+\omega_5=v_2(E_{\alpha''}), & \text{then} & \gamma=\omega_3-1+\delta(\alpha''),\\ \text{if}\quad \omega_1+\omega_5>v_2(E_{\alpha''}), & \text{then} & \gamma=\omega_3-1+v_2(E_{\alpha''}).\end{array}$$

(iii) *If $\omega_3=0$ and $\omega_2+\omega_4\neq 0$, we have*

$$\gamma=-1+v_2(E_{\alpha'}+3^tF_{a'}).$$

(iv) *If* $\omega_3 = \omega_2 = \omega_4 = 0$ *and* $\omega_1 + \omega_5 \neq 0$*, we have*

$$\gamma = -1 + v_2(E_{\alpha'} + 3^t F_{a'} + 2^{\omega_1+\omega_5}(-1)^{\ell(m)} G).$$

Proof. We shall prove that $S(m) \neq 0$ in each of the four cases above. Assuming $S(m) \neq 0$, it follows from Theorem 2.2, (2) that $S(31m) \neq 0$ and that $\gamma(31m) = \gamma(m)$, which sets (3.2).

Proof of Theorem 3.1 (i). In this case, formula (2.39) reduces to

$$mS(m) = 2^{\omega_3-1} 3^{\lceil \frac{\omega_2+\omega_4}{2} - 1 \rceil} E_{\alpha'}.$$

Since $E_{\alpha'} \neq 0$, $S(m)$ does not vanish; we have

$$\gamma = v_2(S(m)) = \omega_3 - 1 + v_2(E_{\alpha'})$$

and the result follows from the values of $E_{\alpha'}$ modulo 2^{11} given in Table 1.

Proof of Theorem 3.1 (ii). If $\omega_2 + \omega_4 = 0$ and $\omega_3 \neq 0$, formula (2.39) becomes (since, cf. (2.35), $E_{i+6} = -E_i$ holds)

$$\begin{aligned} mS(m) &= \frac{2^{\omega_3-1}}{3} \left(E_{\alpha'} + 2^{\omega_1+\omega_5}(-1)^{\ell(m)} G \right) \\ &= (-1)^{\ell(m)} \frac{2^{\omega_3-1}}{3} \left(E_{\alpha''} + 2^{\omega_1+\omega_5} G \right). \end{aligned}$$

As displaid in Table 1, E_i is a linear combination of E_0 and E_1 so that, from Lemma 2.1, $S(m)$ does not vanish and $\gamma = \omega_3 - 1 + v_2\left(E_{\alpha''} + 2^{\omega_1+\omega_5} G\right)$, whence the result. The values of $v_2(E_i)$ and $\delta(i)$ calculated from Table 1 are given below.

i	0	1	2	3	4	5	6	7	8	9	10	11
$v_2(E_i)$	0	0	1	0	0	3	0	0	1	0	0	3
$\delta(i)$	1	1	2	1	1	8	2	2	4	2	2	4

Proof of Theorem 3.1 (iii). If $\omega_3 = 0$ and $\omega_2 + \omega_4 \neq 0$ it follows, from (2.39) and the definition of t above, that

$$mS(m) = \frac{1}{2}\, 3^{\lceil \frac{\omega_2+\omega_4}{2} - 1 \rceil} (E_{\alpha'} + 3^t F_{a'}).$$

But E_i and F_i are non-zero linear combinations of, respectively, E_0 and E_1 and F_0 and F_1; by Lemma 2.1, $E_{\alpha'} + 3^t F_{a'}$ does not vanish and $\gamma = -1 + v_2(E_{\alpha'} + 3^t F_{a'})$.

Proof of Theorem 3.1 (iv). If $\omega_3 = \omega_2 = \omega_4 = 0$ and $m \neq 1$, formula (2.39) gives

$$mS(m) = \frac{1}{6} \left(E_{\alpha'} + 3^t F_{a'} + 2^{\omega_1+\omega_5}(-1)^{\ell(m)} G \right).$$

From Lemma 2.1, we obtain $E_{\alpha'} + 3^t F_{a'} + 2^{\omega_1+\omega_5}(-1)^{\ell(m)} G \neq 0$, which implies $S(m) \neq 0$ and $\gamma = -1 + v_2\left(E_{\alpha'} + 3^t F_{a'} + 2^{\omega_1+\omega_5}(-1)^{\ell(m)} G\right)$. □

Theorem 3.2. *Let m be an odd integer satisfying $m \neq 1$, $\gcd(m,31)=1$, and with $\overline{m}$ of the form (2.24). Let $\gamma=\gamma(m)$ as defined in Theorem 3.1 and $Z(m)$ be the odd part of the right hand-side of (2.39), so that*

$$mS(m) = 2^{\gamma(m)}Z(m). \tag{3.6}$$

(i) *If $k<\gamma$, then $2^k m \notin \mathcal{A}$ and $2^k 31m \notin \mathcal{A}$.*
(ii) *If $k=\gamma$, then $2^k m \in \mathcal{A}$ and $2^k 31m \in \mathcal{A}$.*
(iii) *If $k=\gamma+r$, $r\geq 1$, then we set $\mathcal{S}_r=\{2^r+1,2^r+3,...,2^{r+1}-1\}$ and we have*

$$2^{\gamma+r}m \in \mathcal{A} \iff \exists\, l \in \mathcal{S}_r,\ m \equiv l^{-1}Z(m) \pmod{2^{r+1}},$$

$$2^{\gamma+r}31m \in \mathcal{A} \iff \exists\, l \in \mathcal{S}_r,\ m \equiv -(31l)^{-1}Z(m) \pmod{2^{r+1}}.$$

Proof of Theorem 3.2, (i). We remind that m is odd and (cf. 2.21) $S(m)\equiv S_{\mathcal{A}}(m,k)(\bmod\ 2^{k+1})$. It is obvious from (3.6) that if $\gamma>k$ then $S_{\mathcal{A}}(m,k)\equiv 0(\bmod\ 2^{k+1})$. So that from (1.8), $S_{\mathcal{A}}(m,k)=0$ and $2^h m\notin\mathcal{A}$, for all h, $0\leq h\leq k$. To prove that $2^k 31m \notin \mathcal{A}$, it suffices to use this last result and (2.40) modulo 2^{k+1}.

Proof of Theorem 3.2, (ii). If $\gamma=k$ then the same arguments as above show that

$$mS_{\mathcal{A}}(m,k) \equiv 2^k Z(m)(\bmod\ 2^{k+1}).$$

So that, by using Theorem 3.2, (i) and (1.8), we obtain

$$2^k m\chi(\mathcal{A},2^k m) \equiv 2^k Z(m)(\bmod\ 2^{k+1}).$$

Since both m and $Z(m)$ are odd, we get $\chi(\mathcal{A},2^k m)\equiv 1(\bmod\ 2)$, which shows that $2^k m\in\mathcal{A}$. Once again, to prove that $2^k 31m\in\mathcal{A}$, it suffices to use this last result and (2.40) modulo 2^{k+1}.

Proof of Theorem 3.2, (iii). Let us set $k=\gamma+r$, $r\geq 1$. (3.6) and (2.21) give

$$mS_{\mathcal{A}}(m,k) \equiv 2^{\gamma}Z(m)(\bmod\ 2^{\gamma+r+1}). \tag{3.7}$$

So that, by using Theorem 3.2, (i) and (ii), we get

$$m(2^{\gamma}+2^{\gamma+1}\chi(\mathcal{A},2^{\gamma+1}m)+\ldots+2^{\gamma+r}\chi(\mathcal{A},2^{\gamma+r}m)) \equiv 2^{\gamma}Z(m)(\bmod\ 2^{\gamma+r+1}),$$

which reduces to

$$m(1+2\chi(\mathcal{A},2^{\gamma+1}m)+\ldots+2^{r}\chi(\mathcal{A},2^{\gamma+r}m)) \equiv Z(m)(\bmod\ 2^{r+1}).$$

By observing that $2^{\gamma+r}m\in\mathcal{A}$ if and only if $l=1+2\chi(\mathcal{A},2^{\gamma+1}m)+\ldots+2^r\chi(\mathcal{A},2^{\gamma+r}m)$ is an odd integer in $\mathcal{S}_r$, we obtain

$$2^{\gamma+r}m \in \mathcal{A} \iff m \equiv l^{-1}Z(m) \pmod{2^{r+1}},\ l\in\mathcal{S}_r.$$

To prove the similar result for $2^{\gamma+r}31m$, one uses the same method and (2.40) modulo 2^{k+1}. □

4. The counting function.

In Theorem 4.1 below, we will determine an asymptotic estimate to the counting function $A(x)$ (cf. (1.2)) of the set $\mathcal{A} = \mathcal{A}(1 + z + z^3 + z^4 + z^5)$. The following lemmas will be needed.

Lemma 4.1. *Let K be any positive integer and $x \geq 1$ be any real number. We have*

$$|\{n \leq x : \ \gcd(n, K) = 1\}| \leq 7\frac{\varphi(K)}{K}x,$$

where φ is the Euler function.

Proof. This is a classical result from sieve theory: see Theorems $3-5$ of [11]. □

Lemma 4.2. *(Mertens's formula) Let θ and η be two positive coprime integers. There exists an absolute constant C_1 such that, for all $x > 1$,*

$$\pi(x; \theta, \eta) = \prod_{p \leq x, \ p \equiv \theta (\bmod \ \eta)} (1 - \frac{1}{p}) \leq \frac{C_1}{(\log x)^{\frac{1}{\varphi(\eta)}}}.$$

Proof. For θ and η fixed, Mertens's formula follows from the Prime Number Theorem in arithmetic progressions. It is proved in [9] that the constant C_1 is absolute. □

Lemma 4.3. *For $i \in \{2, 3, 4\}$, let*

$$K_i = K_i(x) = \prod_{p \leq x, \ \ell(p) \in \{0, i\}} p = \prod_{p \leq x, \ p \in \mathcal{P}_0 \cup \mathcal{P} i} p,$$

where ℓ, $\mathcal{P}_0$ and $\mathcal{P}_i$ are defined by (2.15)-(2.16). Then for x large enough,

$$|\{n : 1 \leq n \leq x, \ \gcd(n, K_i) = 1\}| = \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right).$$

Proof. By Lemma 4.1 and (2.16), we have

$$\begin{aligned} |\{n : n \leq x, \ \gcd(n, K_i) = 1\}| &\leq 7x\frac{\varphi(K_i)}{K_i} \\ &= 7x \prod_{0 \leq j \leq 4, \ \tau \in \{0, i\}} \ \prod_{\substack{p \leq x, \\ p \equiv 2^j 3^\tau (\bmod \ 31)}} (1 - \frac{1}{p}). \end{aligned}$$

So that by Lemma 4.2, for all $i \in \{2, 3, 4\}$ and x large enough,

$$|\{n : n \leq x, \ \gcd(n, K_i) = 1\}| \leq \frac{7C_1^{10}x}{(\log x)^{\frac{10}{\varphi(31)}}} = \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right).$$

□

Lemma 4.4. *Let $r, u \in \mathbb{N}_0$, ℓ and α' be the functions defined by (2.15) and (3.3), ω_j be the additive function given by (2.18). We take ξ to be a Dirichlet character modulo 2^{r+1} with ξ_0 as principal character and we let ϱ be the completely multiplicative function defined on primes p by*

$$\varrho(p) = \begin{cases} 0 & \text{if } \ell(p) = 0 \text{ or } p = 31 \\ 1 & \text{otherwise.} \end{cases} \tag{4.1}$$

If y and z are respectively some 2^u-th and 12-th roots of unity in $\mathbb{C}$, and if x is a real number > 1, we set

$$S_{y,z,\xi}(x) = \sum_{2^{\omega_3(n)} n \le x} \varrho(n)\xi(n) y^{\omega_2(n)+\omega_4(n)} z^{\alpha'(n)}. \tag{4.2}$$

Then, when x tends to infinity, we have

- *If $\xi \neq \xi_0$,*

$$S_{y,z,\xi}(x) = \mathcal{O}\left(x \frac{\log\log x}{(\log x)^2}\right). \tag{4.3}$$

- *If $\xi = \xi_0$,*

$$S_{y,z,\xi_0}(x) = \frac{x}{(\log x)^{1-f_{y,z}(1)}} \left(\frac{H_{y,z,\xi_0}(1) C_{y,z}}{\Gamma\left(f_{y,z}(1)\right)} + \mathcal{O}\left(\frac{\log\log x}{\log x}\right)\right), \tag{4.4}$$

where Γ is the Euler gamma function,

$$f_{y,z}(s) = \frac{5}{\varphi(31)} \sum_{1 \le j \le 5} g_{j,y,z}(s), \tag{4.5}$$

$$\begin{aligned} &g_{1,y,z}(s) = z^8, && g_{2,y,z}(s) = yz^7, && g_{3,y,z}(s) = \frac{z^6}{2^s}, \\ &g_{4,y,z}(s) = yz^5, && g_{5,y,z}(s) = z^4, \end{aligned} \tag{4.6}$$

$$H_{y,z,\xi}(s) = \prod_{1 \le j \le 5} \prod_{p,\ \ell(p)=j} \left(1 + \frac{g_{j,y,z}(s)\xi(p)}{p^s - z^{-2j}\xi(p)}\right) \left(1 - \frac{\xi(p)}{p^s}\right)^{g_{j,y,z}(s)}, \tag{4.7}$$

$$C_{y,z} = \prod_{1 \le j \le 5} \left\{ \prod_{p,\ \ell(p)=j} (1 - \frac{1}{p})^{-g_{j,y,z}(1)} \prod_{p} (1 - \frac{1}{p})^{\frac{g_{j,y,z}(1)}{30}} \right\}. \tag{4.8}$$

Proof. The evaluation of such sums is based, as we know, on the Selberg-Delange method. In [7], one finds an application towards direct results on such problems. In our case, to apply Theorem 1 of that paper, one should start with expanding, for complex number s with $\mathcal{R}s > 1$, the Dirichlet series

$$F_{y,z,\xi}(s) = \sum_{n \ge 1} \frac{\varrho(n)\xi(n) y^{\omega_2(n)+\omega_4(n)} z^{\alpha'(n)}}{(2^{\omega_3(n)} n)^s}$$

in an Euler product given by

$$F_{y,z,\xi}(s) = \prod_{1\le j\le 5}\ \prod_{p,\ \ell(p)=j}\left(1+\sum_{m=1}^{\infty}\frac{\xi(p^m)y^{\omega_2(p^m)+\omega_4(p^m)}z^{\alpha'(p^m)}}{\left(2^{\omega_3(p^m)}p^m\right)^s}\right)$$
$$= \prod_{1\le j\le 5}\ \prod_{p,\ \ell(p)=j}\left(1+\frac{g_{j,y,z}(s)\xi(p)}{p^s - z^{-2j}\xi(p)}\right),$$

which can be written

$$F_{y,z,\xi}(s) = H_{y,z,\xi}(s)\prod_{1\le j\le 5}\ \prod_{p,\ \ell(p)=j}\left(1-\frac{\xi(p)}{p^s}\right)^{-g_{j,y,z}(s)},$$

where $g_{j,y,z}(s)$ and $H_{y,z,\xi}(s)$ are defined by (4.6) and (4.7). To complete the proof of Lemma 4.4, one has to show that $H_{y,z,\xi}(s)$ is holomorphic for $\mathcal{R}s > \frac{1}{2}$ and, for y and z fixed, that $H_{y,z,\xi}(s)$ is bounded for $\mathcal{R}s \ge \sigma_0 > \frac{1}{2}$, which can be done by adapting the method given in [7] (Preuve du Théorème 2, p. 235). □

Lemma 4.5. *We keep the above notation and we let $\mathcal{G}$ be the set of integers of the form $n = 2^{\omega_3(m)}m$ with the following conditions:*

- *m odd and* $\gcd(m, 31) = 1$,
- *$m = m_1m_2m_3m_4m_5$, where all prime factors p of m_i satisfy $\ell(p) = i$.*

If $G(x)$ is the counting function of the set $\mathcal{G}$ then, when x tends to infinity,

$$G(x) = \frac{Cx}{(\log x)^{1/4}}\left(1+\mathcal{O}\left(\frac{\log\log x}{\log x}\right)\right), \tag{4.9}$$

where

$$C = \frac{H_{1,1,\xi_0}(1)C_{1,1}}{\Gamma\left(f_{1,1}(1)\right)} = 0.61568378..., \tag{4.10}$$

$H_{1,1,\xi_0}(1)$, $C_{1,1}$ and $f_{1,1}(1)$ are defined by (4.7),(4.8) and (4.5).

Proof. We apply Lemma 4.4 with $y = z = 1$, $\xi = \xi_0$ and remark that $G(x) = S_{1,1,\xi_0}(x)$. By observing that $(1+\frac{1}{p-1})(1-\frac{1}{p}) = 1$, we have

$$H_{1,1,\xi_0}(1) = \prod_{p\in\mathcal{P}_3}\left(1+\frac{1}{2(p-1)}\right)\left(1-\frac{1}{p}\right)^{\frac{1}{2}} = \prod_{p\in\mathcal{P}_3}\left(1-\frac{1}{2p}\right)\left(1-\frac{1}{p}\right)^{-\frac{1}{2}}$$
$$\approx 1.000479390466,$$

$$C_{1,1} = \lim_{x\to\infty}\prod_{\substack{p\in\mathcal{P}_1\cup\mathcal{P}_2\cup\mathcal{P}_4\cup\mathcal{P}_5,\\ p\le x}}\left(1-\frac{1}{p}\right)^{-1}\prod_{\substack{p\in\mathcal{P}_3,\\ p\le x}}\left(1-\frac{1}{p}\right)^{-\frac{1}{2}}\prod_{p\le x}\left(1-\frac{1}{p}\right)^{\frac{3}{4}}$$
$$\approx 0.75410767606.$$

The numerical value of the above Eulerian products has been computed by the classical method already used and described in [7]. Since $\Gamma\left(f_{1,1}(1)\right) = \Gamma(\frac{3}{4}) = 1.225416702465...$, we get (4.10).

□

Lemma 4.6. *We keep the notation introduced in Lemmas 4.4 and 4.5. If* $(y,z) \in \{(1,1),(-1,-1)\}$, *we have*

$$S_{y,z,\xi_0}(x) = \frac{C\,x}{(\log x)^{1/4}}\left(1 + \mathcal{O}\left(\frac{\log\log x}{\log x}\right)\right), \tag{4.11}$$

while, if $(y,z,\xi) \notin \{(1,1,\xi_0),(-1,-1,\xi_0)\}$, *we have*

$$S_{y,z,\xi}(x) = \mathcal{O}_r\left(\frac{x}{(\log x)^{1/4+2^{-2u-3}}}\right). \tag{4.12}$$

Proof. For $y = z = 1$, Formula (4.11) follows from Lemma 4.5. For $y = z = -1$ (which does not occur for $u = 0$), it follows from (4.4) and by observing that the values of $g_{j,y,z}(s), f_{y,z}(s), H_{y,z,\xi}(s), C_{y,z}$ do not change when replacing y by $-y$ and z by $-z$.

Let us define

$$M_{y,z} = \Re(f_{y,z}(1)) = \frac{1}{6}\Re(z^6(z^2 + z^{-2} + \frac{1}{2} + y(z + z^{-1}))).$$

When $\xi \neq \xi_0$, (4.3) implies (4.12) while, if $\xi = \xi_0$, it follows from (4.4) and from the inequality to be proved

$$M_{y,z} \leq \frac{3}{4} - \frac{1}{2^{2u+3}}, \qquad (y,z) \notin \{(1,1),(-1,-1)\}. \tag{4.13}$$

To show (4.13), let us first recall that z is a twelfth root of unity.

If $z \neq \pm 1$, $6f_{y,z}(1)$ is equal to one of the numbers $-3/2 \pm y\sqrt{3}$, $-1/2 \pm y$, $3/2$ so that

$$M_{y,z} \leq |f_{y,z}(1)| \leq \frac{1}{6}\left(\frac{3}{2} + \sqrt{3}\right) < 0.55 \leq \frac{3}{4} - \frac{1}{2^{2u+3}}$$

for all $u \geq 0$, which proves (4.13).

If $z = 1$ and $y \neq 1$ (which implies $u \geq 1$), we have

$$\Re y \leq \cos\frac{2\pi}{2^u} = 1 - 2\sin^2\frac{\pi}{2^u} \leq 1 - 2\left(\frac{2}{\pi}\frac{\pi}{2^u}\right)^2 = 1 - \frac{8}{2^{2u}},$$

and

$$M_{y,1} = \frac{5}{12} + \frac{1}{3}\Re y \leq \frac{3}{4} - \frac{8}{3\cdot 2^{2u}} < \frac{3}{4} - \frac{1}{2^{2u+3}}.$$

If $z = -1$ and $y \neq -1$, (4.13) follows from the preceding case by observing that $f_{y,z}(1) = f_{-y,-z}(1)$, which completes the proof of (4.13). □

Lemma 4.7. *Let $\mathcal{G}$ be the set defined in Lemma 4.5, ω_j and α' be the functions given by (2.18) and (3.3). For $0 \le j \le 11$, r, u, λ, $t \in \mathbb{N}_0$ such that t is odd, we let $\mathcal{G}_{j,r,u,\lambda,t}$ be the set of integers $n = 2^{\omega_3(m)}m$ in $\mathcal{G}$ with the following conditions:*

- $\alpha'(m) \equiv j \pmod{12}$,
- $\omega_2(m) + \omega_4(m) \equiv \lambda \pmod{2^u}$,
- $m \equiv t \pmod{2^{r+1}}$.

If ρ is the function given by (4.1), the counting function $G_{j,r,u,\lambda,t}(x)$ of the set $\mathcal{G}_{j,r,u,\lambda,t}$ is equal to

$$G_{j,r,u,\lambda,t}(x) = \sum_{\substack{2^{\omega_3(m)}m \le x,\ m \equiv t \pmod{2^{r+1}} \\ \alpha'(m)\equiv j \pmod{12},\ \omega_2(m)+\omega_4(m)\equiv\lambda \pmod{2^u}}} \rho(m).$$

If $u \ge 1$ and $\lambda \not\equiv j \pmod 2$, $\mathcal{G}_{j,r,u,\lambda,t}$ is empty while, if $\lambda \equiv j \pmod 2$, when x tends to infinity, we have

$$G_{j,r,u,\lambda,t}(x) = \frac{C}{6 \cdot 2^{r+u}} \frac{x}{(\log x)^{\frac{1}{4}}} \left(1 + \mathcal{O}\left(\frac{1}{(\log x)^{2^{-2u-3}}}\right)\right),$$

where C is the constant given by (4.10).
If $u = 0$, then

$$G_{j,r,0,0,t}(x) = \frac{C}{12 \cdot 2^{r}} \frac{x}{(\log x)^{\frac{1}{4}}} \left(1 + \mathcal{O}\left(\frac{1}{(\log x)^{1/8}}\right)\right),$$

Proof. If $u \ge 1$, it follows from (3.3) that $\alpha'(m) \equiv \omega_2(m)+\omega_4(m) \pmod 2$; therefore, if $j \not\equiv \lambda \pmod 2$, then $\mathcal{G}_{j,r,u,\lambda,t}$ is empty. Let us set

$$\zeta = e^{\frac{2i\pi}{2^u}},\ \mu = e^{\frac{2i\pi}{12}}.$$

By using the relations of orthogonality:

$$\sum_{j_2=0}^{11} \mu^{j_2\alpha'(m)}\mu^{-jj_2} = \begin{cases} 12 & \text{if } \alpha' \equiv j \pmod{12} \\ 0 & \text{if not,} \end{cases}$$

$$\sum_{j_1=0}^{2^u-1} \zeta^{-\lambda j_1}\zeta^{j_1(\omega_2(m)+\omega_4(m))} = \begin{cases} 2^u & \text{if } \omega_2(m)+\omega_4(m) \equiv \lambda \pmod{2^u} \\ 0 & \text{if not,} \end{cases}$$

$$\sum_{\xi \bmod 2^{r+1}} \overline{\xi}(t)\xi(m) = \begin{cases} \varphi(2^{r+1}) = 2^r & \text{if } m \equiv t \pmod{2^{r+1}} \\ 0 & \text{if not,} \end{cases}$$

we get

$$G_{j,r,u,\lambda,t}(x) = \frac{1}{12 \cdot 2^{r+u}} \sum_{\xi \bmod 2^{r+1}} \sum_{j_1=0}^{2^u-1} \sum_{j_2=0}^{11} \overline{\xi}(t)\zeta^{-\lambda j_1}\mu^{-jj_2} S_{\zeta^{j_1},\mu^{j_2},\xi}(x).$$

In the above triple sums, the main contribution comes from $S_{1,1,\xi_0}(x)$ and $S_{-1,-1,\xi_0}(x)$, and the result follows from (4.11) and (4.12).

If $u = 0$, we have

$$G_{j,r,0,0,t}(x) = \frac{1}{12 \cdot 2^r} \sum_{\xi \bmod 2^{r+1}} \sum_{j_2=0}^{11} \overline{\xi}(t) \mu^{-jj_2} S_{1,\mu^{j_2},\xi}(x)$$

and, again, the result follows from Lemma 4.6. □

Theorem 4.1. *Let $\mathcal{A} = \mathcal{A}(1 + z + z^3 + z^4 + z^5)$ be the set given by (1.3) and $A(x)$ be its counting function. When $x \to \infty$, we have*

$$A(x) \sim \kappa \frac{x}{(\log x)^{\frac{1}{4}}},$$

where $\kappa = \frac{74}{31} C = 1.469696766...$ and C is the constant of Lemma 4.5 defined by (4.10).

Proof. Let us define the sets $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ containing the elements $n = 2^k m$ (m odd) of $\mathcal{A}$ with the restrictions:

$$\begin{aligned} \mathcal{A}_1 :&\quad \omega_3(m) \neq 0 \text{ and } \omega_2(m) + \omega_4(m) \neq 0 \\ \mathcal{A}_2 :&\quad \omega_3(m) \neq 0 \text{ and } \omega_2(m) = \omega_4(m) = 0 \\ \mathcal{A}_3 :&\quad \omega_3(m) = 0 \text{ and } \omega_2(m) + \omega_4(m) \neq 0 \\ \mathcal{A}_4 :&\quad \omega_2(m) = \omega_3(m) = \omega_4(m) = 0. \end{aligned}$$

We have

$$A(x) = A_1(x) + A_2(x) + A_3(x) + A_4(x). \tag{4.14}$$

Further, for $i = 2,\ 3,\ 4$, it follows from Lemma 4.3 that $A_i(x) = \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right)$ and therefore

$$A(x) = A_1(x) + \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right). \tag{4.15}$$

Now, we split $\mathcal{A}_1$ in two parts $\mathcal{B}$ and $\widehat{\mathcal{B}}$ by putting in $\mathcal{B}$ the elements $n \in \mathcal{A}_1$ which are coprime with 31 and in $\widehat{\mathcal{B}}$ the elements $n \in \mathcal{A}_1$ which are multiples of 31. Let us recall that, from Remark 2.2, no element of $\mathcal{A}$ is a multiple of 31^2. Therefore,

$$A_1(x) = \mathcal{B}(x) + \widehat{\mathcal{B}}(x) \tag{4.16}$$

with

$$\mathcal{B}(x) = \sum_{n=2^k m \in \mathcal{A}_1,\ n \le x} \rho(m), \quad \widehat{\mathcal{B}}(x) = \sum_{n=2^k 31 m \in \mathcal{A}_1,\ n \le x} \rho(m). \tag{4.17}$$

Let us consider $\mathcal{B}(x)$; the case of $\widehat{\mathcal{B}}$ will be similar. We define

$$\nu_i = v_2(E_i) - 1 = \begin{cases} -1 & \text{if} \quad i \equiv 0,\ 1,\ 3,\ 4 \pmod 6 \\ 0 & \text{if} \quad i \equiv 2 \pmod 6 \\ 2 & \text{if} \quad i \equiv 5 \pmod 6 \end{cases} \tag{4.18}$$

so that, if $\widehat{E_i}$ is the odd part of E_i (cf. (2.32) and Table 1), we have

$$\widehat{E_i} = 2^{-1-\nu_i} E_i. \tag{4.19}$$

In view of Theorem 3.1 (i), if $i = \alpha'(m) \bmod 12$ then

$$\gamma(m) - \omega_3(m) = \nu_i. \tag{4.20}$$

Further, an element $n = 2^k m$ (m odd) belonging to $\mathcal{A}_1$ is said of index $r \geq 0$ if $k = \gamma(m) + r$. For $r \geq 0$ and $0 \leq i \leq 11$,

(4.21)

$$T_r^{(i)}(x) = \sum_{\substack{n=2^{\gamma(m)+r}m \in \mathcal{A}_1,\ n \leq x \\ \alpha'(m) \equiv i \pmod{12}}} \rho(m) = \sum_{\substack{n=2^{\gamma(m)+r}m \in \mathcal{A}_1,\ 2^{\omega_3(m)}m \leq 2^{-r-\nu_i}x \\ \alpha'(m) \equiv i \pmod{12}}} \rho(m)$$

will count the number of elements of $\mathcal{A}_1$ up to x of index r and satisfying $\alpha'(m) \equiv i \pmod{12}$, so that

$$\mathcal{B}(x) = \sum_{r \geq 0} \sum_{i=0}^{11} T_r^{(i)}(x). \tag{4.22}$$

Since $\gamma(m) \geq 0$, from the first equality in (4.21), each n counted in $T_r^{(i)}(x)$ is a multiple of 2^r, hence the trivial upper bound

$$\sum_{i=0}^{11} T_r^{(i)}(x) \leq \frac{x}{2^r} \cdot \tag{4.23}$$

Since $\nu_i \geq -1$, the second equality in (4.21) implies

$$\sum_{i=0}^{11} T_r^{(i)}(x) \leq G(2^{1-r}x) \tag{4.24}$$

with G defined in Lemma 4.5. Moreover, from Lemma 4.5, there exists an absolute constant K such that, for $x \geq 3$,

$$G(x) \leq K \frac{x}{(\log x)^{\frac{1}{4}}} \cdot \tag{4.25}$$

Now, let R be a large but fixed integer; R' is defined in terms of x by $2^{R'-1} \leq \sqrt{x} < 2^{R'}$ and $R" = \frac{\log x}{\log 2}$. Since $T_r^{(i)}(x)$ is a non-negative integer, (4.23) implies that $T_r^{(i)}(x) = 0$ for $r > R"$. If x is large enough, $R < R' < R''$ holds. Setting

$$\mathcal{B}_R(x) = \sum_{r=0}^{R} \sum_{i=0}^{11} T_r^{(i)}(x), \tag{4.26}$$

from (4.22), we have

$$\mathcal{B}(x) - \mathcal{B}_R(x) = S' + S",$$

with

$$S' = \sum_{r=R+1}^{R'} \sum_{i=0}^{11} T_r^{(i)}(x), \qquad S" = \sum_{r=R'+1}^{R"} \sum_{i=0}^{11} T_r^{(i)}(x).$$

The definition of R' and (4.23) yield

$$S" \le \sum_{r=R'+1}^{R"} \frac{x}{2^r} \le \sum_{r=R'+1}^{\infty} \frac{x}{2^r} = \frac{x}{2^{R'}} \le \sqrt{x},$$

while (4.24), (4.25) and the definition of R' give

$$S' \le \sum_{r=R+1}^{R'} G\left(\frac{x}{2^{r-1}}\right) \le \sum_{r=R+1}^{R'} \frac{2Kx}{2^r \left(\log \frac{x}{2^{R'-1}}\right)^{\frac{1}{4}}}$$
$$\le \frac{2^{\frac{5}{4}} Kx}{(\log x)^{\frac{1}{4}}} \sum_{r=R+1}^{R'} \frac{1}{2^r} \le \frac{3Kx}{2^R (\log x)^{\frac{1}{4}}},$$

so that, for x large enough, we have

$$0 \le \mathcal{B}(x) - \mathcal{B}_R(x) \le \sqrt{x} + \frac{3Kx}{2^R (\log x)^{\frac{1}{4}}}. \tag{4.27}$$

We now have to evaluate $T_r^{(i)}(x)$; we shall distinguish two cases, $r = 0$ and $r \ge 1$.

Calculation of $T_0^{(i)}(x)$.

From (4.21), we have

$$T_0^{(i)}(x) = \sum_{\substack{n=2^{\gamma(m)}m \in \mathcal{A}_1,\ n \le x \\ \alpha'(m) \equiv i \pmod{12}}} \rho(m) = \sum_{\substack{n=2^{\gamma(m)}m \in \mathcal{A},\ n \le x,\ \omega_3 \ne 0,\ \omega_2+\omega_4 \ne 0 \\ \alpha'(m) \equiv i \pmod{12}}} \rho(m).$$

From Theorem 3.2, we know that $2^{\gamma(m)}m \in \mathcal{A}$. Hence,

$$T_0^{(i)}(x) = \sum_{\substack{2^{\gamma(m)}m \le x,\ \omega_3 \ne 0,\ \omega_2+\omega_4 \ne 0 \\ \alpha'(m) \equiv i \pmod{12}}} \rho(m),$$

which, by use of (4.20), gives

$$T_0^{(i)}(x) = \sum_{\substack{2^{\omega_3(m)}m \le 2^{-\nu_i}x,\ \omega_3 \ne 0,\ \omega_2+\omega_4 \ne 0 \\ \alpha'(m) \equiv i \pmod{12}}} \rho(m).$$

But, at the cost of an error term $\mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right)$, Lemma 4.3 allows us to remove the conditions $\omega_3 \ne 0$, $\omega_2 + \omega_4 \ne 0$, and to get from the second part of Lemma 4.7,

$$T_0^{(i)}(x) = G_{i,0,0,0,1}\left(\frac{x}{2^{\nu_i}}\right) + \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right)$$

$$= \frac{C}{12}\frac{x}{2^{\nu_i}(\log x)^{\frac{1}{4}}}\left(1+\mathcal{O}\left(\frac{1}{(\log x)^{1/12}}\right)\right). \tag{4.28}$$

Calculation of $T_r^{(i)}(x)$ for $r \geq 1$.

Under the conditions $\omega_3 \neq 0$ and $\omega_2+\omega_4 \neq 0$, from (3.6), (2.39), (3.3), (4.19) and (4.20), we get

$$Z(m) = 3^{\lceil \frac{\omega_2+\omega_4}{2} - 1\rceil}\widehat{E}_{\alpha'(m)}.$$

From (4.21), it follows that

$$T_r^{(i)}(x) = \sum_{\substack{n=2^{\gamma(m)+r}m\in\mathcal{A},\ n\leq x,\ \omega_3\neq 0,\ \omega_2+\omega_4\neq 0 \\ \alpha'(m)\equiv i \pmod{12}}} \rho(m).$$

Now, by Theorem 3.2, we know that $2^{\gamma(m)+r}m$ belongs to $\mathcal{A}$ if there is some $l \in \mathcal{S}_r = \{2^r+1, ..., 2^{r+1}-1\}$ such that $m \equiv l^{-1}Z(m) \bmod 2^{r+1}$. Note that the order of 3 modulo 2^{r+1} is 2^{r-1} if $r \geq 2$ and 2^r if $r=1$. We choose

$$u = r+1$$

so that $\omega_2+\omega_4 \equiv \lambda \pmod{2^{r+1}}$ implies $3^{\lceil \frac{\lambda}{2}-1\rceil} \equiv 3^{\lceil \frac{\omega_2+\omega_4}{2}-1\rceil} \pmod{2^{r+1}}$. Therefore, we have

$$T_r^{(i)}(x) = \sum_{l\in\mathcal{S}_r}\sum_{\lambda=0}^{2^{r+1}-1} \sum_{\substack{2^{\omega_3(m)}m\leq 2^{-\nu_i-r}x,\ \omega_3\neq 0,\ \omega_2+\omega_4\neq 0 \\ \alpha'(m)\equiv i \pmod{12},\ \omega_2+\omega_4\equiv\lambda \pmod{2^{r+1}} \\ m\equiv l^{-1}3^{\lceil\frac{\lambda}{2}-1\rceil}\widehat{E}_i \pmod{2^{r+1}}}} \rho(m).$$

As in the case $r=0$, we can remove the conditions $\omega_3 \neq 0$ and $\omega_2+\omega_4 \neq 0$ in the last sum by adding a $\mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right)$ error term, and we get by Lemma 4.7 for r fixed

$$T_r^{(i)}(x) = \sum_{l\in\mathcal{S}_r}\sum_{\substack{\lambda=0 \\ \lambda\equiv i \pmod{2}}}^{2^{r+1}-1} G_{i,r,r+1,\lambda,l^{-1}3^{\lceil\frac{\lambda}{2}-1\rceil}\widehat{E}_i}\left(\frac{x}{2^{\nu_i+r}}\right) + \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right)$$

$$= \frac{C}{24}\frac{x}{2^{\nu_i+r}(\log x)^{\frac{1}{4}}}\left(1+\mathcal{O}\left(\frac{1}{(\log x)^{2^{-2r-5}}}\right)\right). \tag{4.29}$$

From (4.26), (4.28), (4.29) and (4.18), we have

$$\begin{aligned}\mathcal{B}_R(x) &= \frac{Cx}{12(\log x)^{\frac14}}\left(\left(\sum_{i=0}^{11}\frac{1}{2^{\nu_i}}\right)\left(1+\frac12\sum_{r=1}^{R}\frac{1}{2^r}\right)+\mathcal{O}\left(\frac{1}{(\log x)^{2^{-2R-5}}}\right)\right)\\ &= \frac{37}{24}\frac{Cx}{(\log x)^{\frac14}}\left(\frac32-\frac{1}{2^R}\right)\left(1+\mathcal{O}\left(\frac{1}{(\log x)^{2^{-2R-5}}}\right)\right).\end{aligned}$$

By making R going to infinity, the above equality together with (4.27) show that

$$\mathcal{B}(x) \sim \frac{37}{16}\frac{Cx}{(\log x)^{\frac14}},\ x\to\infty. \tag{4.30}$$

In a similar way, we can show that $\widehat{\mathcal{B}}(x)$ defined in (4.17) satisfies

$$\widehat{\mathcal{B}}(x) \sim \frac{1}{31}\mathcal{B}(x) \sim \frac{37}{16\cdot 31}\frac{x}{(\log x)^{\frac14}}$$

which, with (4.16) and (4.15), completes the proof of Theorem 4.1 with

$$\kappa = \frac{37}{16}\left(1+\frac{1}{31}\right)C = \frac{74}{31}C = 1.469696766....$$

□

Numerical computation of $A(x)$.

There are three ways to compute $A(x)$. The first one uses the definition of $\mathcal{A}$ and simultaneously calculates the number of partitions $p(\mathcal{A},n)$ for $n \le x$; it is rather slow. The second one is based on the relation (1.10) and the congruences (2.19) and (2.23) satisfied by $\sigma(\mathcal{A},n)$. The third one calculates $\omega_j(n),\ 0\le j\le 5$, in view of applying Theorem 2.2 . The two last methods can be encoded in a sieving process

The following table displays the values of $A(x)$, $A_1(x)$, ..., $A_4(x)$ as defined in (4.14) and also

$$c(x) = \frac{A(x)(\log x)^{\frac14}}{x},\ c_1(x) = \frac{A_1(x)(\log x)^{\frac14}}{x}.$$

It seems that $c(x)$ and $c_1(x)$ converge very slowly to $\kappa = 1.469696766\ldots$, which is impossible to guess from the table.

x	$A(x)$	$c(x)$	$A_1(x)$	$c_1(x)$	$A_2(x)$	$A_3(x)$	$A_4(x)$
10^3	480	0.7782	20	0.032	44	233	183
10^4	4543	0.7914	361	0.063	532	2294	1356
10^5	43023	0.7925	5087	0.094	5361	21810	10765
10^6	411764	0.7939	60565	0.117	52344	208633	90222
10^7	3981774	0.7978	680728	0.136	506199	2007168	787679
10^8	38719773	0.8022	7403138	0.153	4887357	19390529	7038749

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