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Oscillation of Mertens' product formula

par Harold G. DIAMOND et Janos PINTZ

Résumé. La formule de Mertens affirme que

$$\prod_{p \le x} \Big(1 - \frac{1}{p}\Big) \log x \;\to\; e^{-\gamma}$$

quand $x \to \infty$. Les calculs montrent que la partie droite de la formule est supérieure à la partie gauche pour $2 \le x \le 10^8$. Par analogie avec le résultat de Littlewood sur $\pi(x) - \mathrm{li}\, x$, Rosser et Schoenfeld ont suggéré que cette inégalité et son contraire devait se produire pour des valeurs suffisamment grandes de x. Nous montrons que c'est bien le cas.

Abstract. Mertens' product formula asserts that

$$\prod_{p \le x} \Big(1 - \frac{1}{p}\Big) \log x \;\to\; e^{-\gamma}$$

as $x \to \infty$. Calculation shows that the right side of the formula exceeds the left side for $2 \le x \le 10^8$. It was suggested by Rosser and Schoenfeld that, by analogy with Littlewood's result on $\pi(x) - \mathrm{li}\, x$, this and a complementary inequality might change their sense for sufficiently large values of x. We show this to be the case.

1. Introduction.

One of the last significant prime number discoveries preceding the proof of the Prime Number Theorem was Franz Mertens' beautiful asymptotic formula [11]

$$\prod_{p \le x} \Big(1 - \frac{1}{p}\Big) \log x \;\to\; e^{-\gamma}, \quad x \to \infty. \tag{1.1}$$

Here γ denotes Euler's constant. Mertens' argument was quite complicated; simpler proofs of the formula can be found, e.g. in [8], Theorem 429; [3], Lemma 4.11; or [12], Theorem 2.7.

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Mots clefs. Mertens' product formula, oscillation, Euler's constant, Riemann hypothesis, zeta function.

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Calculations of Rosser–Schoenfeld (see [13]) have shown that

$$\frac{e^{-\gamma}}{\log x + 2/\sqrt{x}} < \prod_{p\le x}\Big(1-\frac{1}{p}\Big) < \frac{e^{-\gamma}}{\log x} \tag{1.2}$$

for $2 \le x < 10^8$. These authors raised the question of whether, by analogy with Littlewood's famous result [10] on changes of sign of $\pi(x) - \operatorname{li} x$, inequalities (1.2) also might change their sense for sufficiently large x. (This question is mentioned also in the article on Mertens' Theorem on the MathWorld website [14].) We show this to be the case.

Theorem 1.1. *The quantity*

$$\sqrt{x}\Big\{\prod_{p\le x}\Big(1-\frac{1}{p}\Big)^{-1} - \ e^{\gamma}\log x\Big\}$$

attains arbitrarily large positive and negative values as $x \to \infty$.

Mertens himself did not, to our knowledge, make any conjecture here. However, his name is associated with another famous conjecture, on the size of the summatory function of the Möbius μ function, that was subsequently disproved by Odlyzko and te Riele, using a delicate oscillation argument.

We thank Jonathan Sondow for bringing to our attention the question of Rosser–Schoenfeld and for suggestions on the write-up of this article.

2. Conversion of the problem

We begin by converting the expressions occurring in the theorem into ones that are more amenable for our analytic approach. The theorem is equivalent to showing that, for any large positive number K,

$$-\sum_{p\le x}\log\Big(1-\frac{1}{p}\Big) - \log\log x - \gamma \begin{cases} > \quad K/(\sqrt{x}\log x) \\ < \ -K/(\sqrt{x}\log x) \end{cases} \tag{2.1}$$

for sequences of x values tending to ∞. The expression on the left side of (2.1) still is not in a convenient form, so we replace it using the following two lemmas.

Lemma 2.1. *Let* $\Pi(x) = \pi(x) + \pi(x^{1/2})/2 + \pi(x^{1/3})/3 + \dots$, *where* $\pi(\cdot)$ *is the prime counting function. For* $x \ge 2$,

$$-\sum_{p\le x}\log\Big(1-\frac{1}{p}\Big) = \int_1^x \frac{d\Pi(t)}{t} + O\Big(\frac{1}{\sqrt{x}\log x}\Big).$$

Proof. By the definition of $\Pi(\cdot)$,

$$\int_1^x \frac{d\Pi(t)}{t} = \sum_{p^\alpha \le x}\frac{1}{\alpha p^\alpha}\,.$$

Thus the difference of the main terms of the lemma involves only higher prime powers:

$$\sum_{\substack{p\le x\\ p^\alpha>x}} \frac{1}{\alpha p^\alpha} \le \frac{1}{2}\sum_{p\le x}\ \sum_{\alpha>\lfloor \log x/\log p\rfloor} \frac{1}{p^\alpha} \le \sum_{p\le x} p^{-\lfloor \log x/\log p\rfloor -1}$$

$$\le \sum_{x^{1/2}<p\le x} \frac{1}{p^2} + \sum_{p\le x^{1/2}} \frac{1}{x} \ll \frac{1}{\sqrt{x}\log x} + \frac{\sqrt{x}}{x\log x},$$

which has the claimed order of magnitude. □

Next, we replace the remaining two terms in (2.1).

Lemma 2.2. *For $x>1$,*

$$\log\log x + \gamma = \int_1^x \frac{1-t^{-1}}{t\log t}dt \ - \int_x^\infty \frac{dt}{t^2\log t}\,.$$

Proof. Differentiation shows the two sides of the formula agree to within a constant. We need to show that the constant is actually γ. Since the last integral vanishes at infinity, consider the limit as $x\to\infty$ of

$$I(x) := \int_1^x \frac{1-t^{-1}}{t\log t}dt - \log\log x.$$

By integration by parts and then a change of variable,

$$\lim_{x\to\infty} I(x) = -\int_1^\infty \frac{\log\log t}{t^2}dt = -\int_0^\infty e^{-u}\log u\, du.$$

The last expression is $-\Gamma'(1)$, as we can see by differentiating the integral form of Euler's gamma function. Finally, by differentiating the product form of the gamma function, we obtain $-\Gamma'(1)=\gamma$. □

The last integral in the statement of Lemma 2.2 is $O(1/x\log x)$, which is smaller than the error term in Lemma 2.1. The two lemmas and last remark enable us to replace the left side of (2.1) by the more analytically tractable expression

$$\int_1^x \frac{d\Pi(t)}{t} - \int_1^x \frac{1-t^{-1}}{t\log t}dt + O\Big(\frac{1}{\sqrt{x}\,\log x}\Big). \tag{2.2}$$

For $x>1$, set

$$A(x) := \int_1^x \frac{d\Pi(t)}{t} - \int_1^x \frac{1-t^{-1}}{t\log t}dt.$$

We shall show that $A(x)$ attains values that greatly exceed $1/(\sqrt{x}\log x)$ in both the positive and negative directions. To do this, we introduce Mellin transforms and study their analytic properties.

By the familiar representation of $\log\zeta$ as a Mellin transform and an integration by parts, we get for $\Re s > 0$,

$$\log\zeta(s+1) = \int_1^\infty x^{-s}\frac{d\Pi(x)}{x} = s\int_1^\infty x^{-s-1}\int_1^x \frac{d\Pi(t)}{t}\,dx.$$

Also, in the same region, we have

$$\log\frac{s+1}{s} = \int_1^\infty x^{-s}\,\frac{1-x^{-1}}{x\,\log x}\,dx, \tag{2.3}$$

since the derivatives of the two sides of the formula are the same, and both sides tend to 0 as $s \to +\infty$. Then, by integration by parts,

$$\log\frac{s+1}{s} = s\int_1^\infty x^{-s-1}\int_1^x \frac{1-t^{-1}}{t\,\log t}\,dt\,dx, \quad \Re s > 0.$$

Combining the preceding formulas, we obtain the Mellin formula

$$\widehat{A}(s) := \int_1^\infty x^{-s-1}A(x)\,dx = \frac{1}{s}\log\frac{\zeta(s+1)s}{s+1}, \quad \Re s > 0.$$

3. The non-R.H. case

As with many oscillation arguments involving the zeta function, it is convenient to treat separately the cases in which the truth of the Riemann hypothesis is or is not not assumed to hold. We first assume that R.H. does not hold and apply Landau's Oscillation Theorem ([2], Theorem 11.13; [3], Theorem 6.31; or [12], Theorem 15.1).

We replace the error term in (2.2) by

$$B(x) := \frac{1-x^{-1}}{\sqrt{x}\,\log x} \asymp \frac{1}{\sqrt{x}\,\log x}, \quad x \ge 2,$$

which is better behaved near $x = 1$, and for which we can read off the Mellin transform from (2.3): for $\Re s > -1/2$,

$$\widehat{B}(s) := \int_1^\infty x^{-s-1}B(x)\,dx = \log\frac{s+3/2}{s+1/2}.$$

The strategy here is to show that, for any fixed value of K, positive or negative, $A(x) + KB(x)$ changes sign infinitely often.

If R.H. is not true, then $\log\{\zeta(s+1)s/(s+1)\}$ has a singularity at a complex point s^* with $\sigma^* = \Re s^* > -1/2$, and so the abscissa of convergence of the Mellin transform

$$\widehat{A}(s) + K\widehat{B}(s) = \frac{1}{s}\log\frac{\zeta(s+1)s}{s+1} + K\log\frac{s+3/2}{s+1/2}$$

is at least σ^*. On the other hand, $\widehat{A} + K\widehat{B}$ is continuable to all points of the real axis with $s > -1/2$, because the possible singularity at $s = 0$ is removable, as a tiny calculation shows. Thus, the real point on the line of convergence of the Mellin transform is a regular point, and so by Landau's theorem, $A(x) + KB(x)$ changes sign infinitely often for any fixed value of K. This completes the proof of Theorem 1.1 if R.H. does not hold.

We assume henceforth that the Riemann hypothesis holds. The situation here is more delicate; while Landau's theorem can show changes of sign of $A(x)$, it does not provide a measure of the oscillations, which is essential for proving Theorem 1.1. We give two arguments; the first a quickie, citing deep theorems of Littlewood and Cramér, and the second one, with more detail, that is based on a variant of the Wiener–Ikehara method and an application of almost-periodicity.

4. The R.H. case, I

By the results of §2, it suffices to show

Proposition 4.1.

$$A(x) = \Omega_{\pm}\Big(\frac{\log\log\log x}{\sqrt{x}\log x}\Big).$$

Proof. For $x \geq 2$, set

$$\mathrm{li}^* x := \int_1^x \frac{1 - t^{-1}}{\log t}\, dt = \mathrm{li}\, x - \log\log x + O(1)$$

and

$$\Delta^*(x) := \Pi(x) - \mathrm{li}^*\, x.$$

Our first goal is to express A in terms of Δ^*, which we do in formula (4.2) below. By integration by parts, we have

$$A(x) = \frac{\Pi(x) - \mathrm{li}^* x}{x} + \int_1^x \frac{\Pi(t) - \mathrm{li}^* t}{t^2}\, dt = \frac{\Delta^*(x)}{x} + \int_1^x \frac{\Delta^*(t)}{t^2}\, dt. \tag{4.1}$$

Using the logarithmic form of (1.1) (Mertens' formula) and Lemmas 2.1 and 2.2, we see that $A(x) \to 0$ as $x \to \infty$. Also, we have the crude result that

$$\frac{\Delta^*(x)}{x} \ll \frac{x/\log x}{x} \to 0, \quad x \to \infty.$$

The last two estimates together with (4.1) imply that

$$\int_1^\infty \frac{\Delta^*(t)}{t^2}\, dt = 0,$$

and hence

$$A(x) = \frac{\Delta^*(x)}{x} - \int_x^\infty \frac{\Delta^*(t)}{t^2}\, dt. \tag{4.2}$$

We have from Littlewood's famous result on sign changes of $\pi(x) - \operatorname{li} x$ [10] that

$$\frac{\Delta^*(x)}{x} = \Omega_\pm\Big(\frac{\log\log\log x}{\sqrt{x}\log x}\Big),$$

which gives the desired estimate for one term of (4.2). We shall show that the other term is suitably small by using the inequality

$$\int_x^{2x} |\psi(t) - t|\, dt \ll x^{3/2}, \tag{4.3}$$

which is an equivalent form of Cramér's bound in [4],

$$\frac{1}{x}\int_1^x \frac{|\psi(t) - t|}{\sqrt{t}}\, dt \ll 1.$$

We have, by integration by parts,

$$\begin{aligned}\Delta^*(x) &= \int_1^x \frac{1}{\log t}\,\{d\psi(t) - (1 - t^{-1})\, dt\}\\ &= \frac{\psi(x) - x}{\log x} + O(\log\log x) + \int_2^x \frac{\psi(t) - t}{t\log^2 t}\, dt.\end{aligned}$$

Break the last integral at $x^{1/4}$ and note that, trivially,

$$\int_2^{x^{1/4}} \frac{\psi(t) - t}{t\log^2 t}\, dt \ll x^{1/4}.$$

Break the remaining interval into dyadic subintervals and use (4.3) on each one to obtain

$$\begin{aligned}\int_{x^{1/4}}^x \frac{\psi(t) - t}{t\log^2 t}\, dt &\ll \Big\{\int_{x/2}^x + \int_{x/4}^{x/2} + \dots\Big\}\frac{\psi(t) - t}{t\log^2 t}\, dt\\ &\ll \frac{x^{1/2}}{\log^2 x}\Big\{1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} + \dots\Big\},\end{aligned}$$

whence

$$\Delta^*(x) = \frac{\psi(x) - x}{\log x} + O\Big(\frac{x^{1/2}}{\log^2 x}\Big).$$

By another application of the dyadic interval estimates,

$$\int_x^\infty \frac{|\psi(t) - t|}{\log t}\,\frac{dt}{t^2} \ll \frac{1}{\sqrt{x}\,\log x}\,.$$

It follows that the last term in (4.2) also satisfies this estimate. Since this quantity is smaller than the Littlewood oscillation term in (4.2), A has the claimed oscillation. ☐

5. The R.H. case, II

For the second method, we shall apply an argument along the lines of that used in [5]. The key technical device is the following variant of the Wiener–Ikehara method, which is essentially due to Ingham, [9].

Lemma 5.1. *Let F be a real valued function on $[1, \infty)$ that is continuous from the right and satisfies a one-sided bound: for some $\beta < 1$, $F(x) < \log^\beta x$ holds for all sufficiently large x or $F(x) > -\log^\beta x$ holds for all sufficiently large x. Let*

$$\widehat{F}(s) := \int_1^\infty x^{-s-1} F(x)\, dx$$

converge for $\sigma := \Re s > 0$. Let $T > 0$ and suppose that there exists a function

$$H(s) := \sum_{|\gamma_n|<T} \frac{a_n}{s - i\gamma_n}$$

(for some choice of complex numbers a_n and real γ_n) such that the family of restricted functions

$$t \mapsto \widehat{F}(\sigma + it) - H(\sigma + it) =: J_\sigma(t), \quad -T \le t \le T,$$

is Cauchy in L^1 norm as $\sigma \to 0+$, i.e.,

$$\lim_{\sigma,\sigma'\to 0+} \int_{-T}^{T} |J_\sigma(t) - J_{\sigma'}(t)|\, dt = 0.$$

Then, as $y \to \infty$,

$$\int_{u=1}^\infty F(u) K_T(y - \log u) \frac{du}{u} = \sum_{|\gamma_n|<T} a_n \Big(1 - \frac{|\gamma_n|}{T}\Big) e^{i\gamma_n y} + o_T(1).$$

Here

$$K_T(x) := \frac{1}{2\pi} \int_{-T}^{T} \Big(1 - \frac{|t|}{T}\Big) e^{ixt}\, dt, \quad x \in \mathbb{R},$$

is the Fejer kernel, and $o_T(1)$ denotes a function of y and T that, for fixed T, goes to 0 as $y \to \infty$.

The forms of this lemma given in [3] and [9] suppose that $J(s)$ has a continuation as a continuous function on the closed strip

$$\{s = \sigma + it : -T \le t \le T,\, \sigma \ge 0\},$$

while [5] assumes just the L^1 hypothesis given here. These conditions allow one, in the proof of the lemma, to shift the Fourier inversion integral for F (arbitrarily near) to the imaginary axis. By the completeness of $L^1[-T,T]$, these methods are essentially the same.

It is instructive to apply the last lemma to the function $F(x) := 1$ for $1 \le x < \infty$. Then $\widehat{F}(s) = 1/s$ and we take $H(s) := 1/s$, so that J_σ is identically 0. The lemma gives

$$\int_{u=1}^{\infty} K_T(y - \log u)\frac{du}{u} = 1 + o_T(1), \quad y \to \infty,$$

i.e., the Fejer kernel satisfies

$$\int_{-\infty}^{\infty} K_T(t)dt = 1.$$

Let us suppose that, for some real number K, $A(x)+KB(x)$ is ultimately of one sign. (Our aim is to show that this is **not** the case.) Then by Landau's Oscillation Theorem, the abscissa of convergence of $\widehat{A}(s) + K\widehat{B}(s)$ is $-1/2$, for this is the rightmost point at which the function has a singularity on the real axis. The same holds for $\widehat{B}(s)$, whence, by subtraction, the abscissa of convergence of $\widehat{A}(s)$ is at most $-1/2$; the presence of singularities with real part $-1/2$ insures that this is the abscissa of convergence of $\widehat{A}(s)$.

Since the singularities hypothesized in Lemma 5.1 are located on the imaginary axis, we work with $\widehat{A}(s - 1/2)$, which satisfies this condition. Still, we are not in a position to apply the lemma, because the singularities are logarithmic zeros, rather than poles. We can handle this problem by forming the derivative

$$\begin{aligned} -\widehat{A}'(s - 1/2) &= \int_1^\infty x^{-s-1} A(x)\, x^{1/2} \log x \, dx \\ &= \frac{1}{(s-1/2)^2} \log \frac{\zeta(s+1/2)(s-1/2)}{s+1/2} \\ &\qquad - \frac{1}{s-1/2}\Big(\frac{\zeta'(s+1/2)}{\zeta(s+1/2)} + \frac{1}{s-1/2} - \frac{1}{s+1/2}\Big). \end{aligned}$$

This expression has poles on the imaginary axis, as we would like; of course, there remain the logarithmic singularities, but as we shall see, those will count for little. Also, we remark that $s = 1/2$ is a regular point of this function, by the earlier observation about $\widehat{A}(s)$ at $s = 0$.

In Lemma 5.1 take $F(x) := A(x)\sqrt{x}\log x$. Our goal is to show that F is unbounded from above and below. Continuing with the assumption that $A(x) + KB(x)$ is ultimately of one sign, then F is bounded from one side, and one hypothesis of this lemma is satisfied. This justifies an exchange of limiting processes in the proof of the lemma.

Next, take $\widehat{F}(s) = -\widehat{A}'(s - 1/2)$, the Mellin transform of F. Given $T > 0$ (which we can assume is not the ordinate of a zeta zero), take

$$H(s) := \sum_{|\gamma_n|<T} \frac{-1}{i\gamma_n(s - i\gamma_n)},$$

where the numbers γ_n are the ordinates of the zeros of the Riemann zeta function, counted with appropriate multiplicity in case zeta has zeros of higher order. Then H matches the pole terms of

$$\frac{-1}{s-1/2}\cdot\frac{\zeta'(s+1/2)}{\zeta(s+1/2)}$$

lying on the imaginary segment $\mathcal{I} := \{s = it : -T \le t \le T\}$, and so we have

$$\begin{aligned}J_\sigma(t) :=& \widehat{F}(\sigma+it) - H(\sigma+it)\\ =& \frac{1}{(s-1/2)^2}\log\frac{\zeta(s+1/2)(s-1/2)}{s+1/2} + G(\sigma+it),\end{aligned}$$

with G analytic on an open set containing $\mathcal{I}$.

It is not hard to show that J_σ satisfies the $L^1(-T,T)$ Cauchy condition hypothesized in Lemma 5.1: the continuous portion, G, clearly satisfies the condition, and for each of the finite number of logarithmic zeros having ordinate of size at most T, we apply the following estimate.

Lemma 5.2. *Let a branch of* log *be fixed. Then*

$$\lim_{\sigma,\sigma'\to 0+}\int_{-1}^{1}|\log(\sigma+it)-\log(\sigma'+it)|\,dt = 0.$$

Proof. It suffices to show that

$$I(\sigma) := \int_0^1 |\log(\sigma+it)-\log(it)|\,dt \to 0 \quad \text{as} \quad \sigma\to 0+\,.$$

Let $\epsilon > 0$ be given. Since the integrand converges to 0 uniformly for $\epsilon \le t \le 1$, the integral over this region tends to 0 with σ. For the remainder of the range, note first that $|\arg\{\log(\sigma+it)-\log(it)\}| < \pi/2$, so that

$$I(\sigma) < \frac{\epsilon\pi}{2} + \int_0^\epsilon \log\Big|\frac{\sigma+it}{it}\Big|dt \ll \epsilon + \int_0^\epsilon \log\Big(1+\frac{\sigma^2}{t^2}\Big)dt.$$

The last integral converges to 0 with σ, since the integrand converges to 0 pointwise and the integral is dominated by

$$\int_0^\epsilon \log(1+t^{-2})\,dt < \infty.$$

Thus $I(\sigma)$ can be made arbitrarily small by choosing $\sigma > 0$ sufficiently small. □

The conditions of Lemma 5.1 are now satisfied, so we have

$$\int_1^\infty F(u)\,K_T(y-\log u)\frac{du}{u} = -\sum_{|\gamma_n|<T}\Big(1-\frac{|\gamma_n|}{T}\Big)\frac{e^{i\gamma_n y}}{i\gamma_n} + o_T(1),$$

with $F(u) := A(u)\sqrt{u}\log u$, K_T the Fejer kernel, and $o_T(1)$ a function that, for fixed $T > 0$, tends to 0 as $y \to \infty$. By the reflection principle, zeros of the zeta function occur in conjugate pairs, so we have

$$\int_1^\infty F(u)\, K_T(y - \log u)\frac{du}{u} = -2 \sum_{0<\gamma_n<T} \Big(1 - \frac{\gamma_n}{T}\Big)\frac{\sin \gamma_n y}{\gamma_n} + o_T(1). \tag{5.1}$$

We noted before that the Fejer kernel has integral 1. Also, since K_T can be written as the square of a real valued function, it is nonnegative. Thus the left side of (5.1) is an average of the function F. If this average is large positive or negative, then F itself must have this property as well. It remains to show that the right side of the formula can assume large positive and negative values for suitable choices of T and y, which we shall do in the next section.

6. Conclusion of method II

Let

$$\Sigma(y) := 2 \sum_{0<\gamma_n<T} \Big(1 - \frac{\gamma_n}{T}\Big)\frac{\sin \gamma_n y}{\gamma_n}.$$

As a trigonometrical polynomial, $\Sigma(y)$ is almost-periodic; thus any value that it assumes, it approximates arbitrarily closely for a sequence of y values tending to infinity. We shall show that $\Sigma(y)$ assumes large positive and large negative values by the beautiful device of investigating its values near the origin. This idea seems to be due to Ingham and first to have been used by Fawaz ([6], [7]). (We are indebted to Anderson–Stark [1] for these historical observations.)

We have

$$\Sigma(1/T) = \frac{2}{T} \sum_{0<\gamma_n<T} \Big(1 - \frac{\gamma_n}{T}\Big)\frac{\sin \gamma_n/T}{\gamma_n/T}.$$

The estimate $\sin x > 2x/\pi$ for $0 < x < \pi/2$ yields

$$\Sigma(1/T) > \frac{2}{T} \sum_{0<\gamma_n<T/2} \frac{1}{2}\,\frac{2}{\pi} = \frac{1}{\pi}\,\frac{N(T/2)}{T/2}\,,$$

where $N(T)$ denotes the number of zeros $\beta + i\gamma$ of Riemann's zeta function with $0 < \gamma \le T$. It is known (see [3], Theorem 8.18 or [12], Corollary 14.2) that $N(T)/T \to \infty$ as $T \to \infty$. Thus $\Sigma(1/T)$ can be made arbitrarily large by choosing T large enough. It follows from (5.1) that $F(y)$ can assume arbitrarily large negative values for a sequence of y's tending to infinity.

The preceding argument easily implies the existence of arbitrarily large positive values as well: since $\Sigma(y)$ is an odd function, we have $\Sigma(-1/T) = -\Sigma(1/T)$, and $F(y)$ can assume arbitrarily large positive values for another sequence of y's tending to infinity.

Tracing back, we see that

$$\sqrt{x}\log x \Big\{ \int_1^x \frac{d\Pi(t)}{t} - \int_1^x \frac{1-t^{-1}}{t\log t} dt \Big\}$$

assumes arbitrarily large positive and negative values and hence, by (2.1), the theorem is proved. □

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