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par Melvyn B. NATHANSON

Résumé. Soit $\varphi(x_1, \ldots, x_h, y) = u_1x_1 + \cdots + u_hx_h + vy$ une forme linéaire à coefficients entiers non nuls $u_1, \ldots, u_h, v$. Soient $\mathcal{A} = (A_1, \ldots, A_h)$ un h-uplet d'ensembles finis d'entiers et B un ensemble infini d'entiers. Définissons la fonction de représentation associée à la forme φ et aux ensembles $\mathcal{A}$ et B comme suit :

$$R^{(\varphi)}_{\mathcal{A},B}(n) = \mathrm{card}\left(\begin{array}{l} \Big\{(a_1, \ldots, a_h, b) \in A_1 \times \cdots \times A_h \times B : \\ \hfill \varphi(a_1, \ldots, a_h, b) = n\Big\} \end{array} \right).$$

Si cette fonction de représentation est constante, alors l'ensemble B est périodique, et la période de B est bornée en termes du diamètre de l'ensemble fini $\{\varphi(a_1, \ldots, a_h, 0) : (a_1, \ldots, a_h) \in A_1 \times \cdots \times A_h\}$. D'autres résultats sur les ensembles se complétant pour une forme linéaire sont également prouvés.

Abstract. Let $\varphi(x_1, \ldots, x_h, y) = u_1x_1 + \cdots + u_hx_h + vy$ be a linear form with nonzero integer coefficients $u_1, \ldots, u_h, v$. Let $\mathcal{A} = (A_1, \ldots, A_h)$ be an h-tuple of finite sets of integers and let B be an infinite set of integers. Define the representation function associated to the form φ and the sets $\mathcal{A}$ and B as follows :

$$R^{(\varphi)}_{\mathcal{A},B}(n) = \mathrm{card}\left(\begin{array}{l} \Big\{(a_1, \ldots, a_h, b) \in A_1 \times \cdots \times A_h \times B : \\ \hfill \varphi(a_1, \ldots, a_h, b) = n\Big\} \end{array} \right).$$

If this representation function is constant, then the set B is periodic and the period of B will be bounded in terms of the diameter of the finite set $\{\varphi(a_1, \ldots, a_h, 0) : (a_1, \ldots, a_h) \in A_1 \times \cdots \times A_h\}$. Other results for complementing sets with respect to linear forms are also proved.

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Mots clefs. Representation functions, linear forms, complementing sets, tiling by finite sets, inverse problems in additive number theory.

1. Complementing sets

Let A and B be sets of integers, and let $S(A, B)$ denote the *sumset* of A and B, that is, $S(A, B) = \{a + b : a \in A \text{ and } b \in B\}$. The pair (A, B) is called a *complementing pair* if every element of the sumset $S(A, B)$ has a unique representation as the sum of an element of A and an element of B. Equivalently, if $a_1, a_2 \in A$ and $b_1, b_2 \in B$, and if $a_1 + b_1 = a_2 + b_2$, then $a_1 = a_2$ and $b_1 = b_2$. If (A, B) is a complementing pair, then we write $A \oplus B = S(A, B)$.

A classical problem in additive number theory is the study of complementing pairs for the set of all integers, that is, pairs (A, B) such $A \oplus B = \mathbf{Z}$. There are many beautiful results and open problems about complementing sets for the integers. For example, if A is a finite set of integers and if B is an infinite set of integers such that the pair (A, B) is complementing, then B must be a periodic set, that is, a union of congruence classes modulo m for some positive integer m (Newman [7]). There are upper and lower bounds on the period m as a function of the diameter of the set A (Biro [1], Kolountzakis [3], Ruzsa [11, Appendix], Steinberger [9]), but these bounds are not sharp.

There are compactness arguments that prove that if a finite set A admits a sequence $\{B_i\}_{i=1}^{\infty}$ of finite sets that are complementary to arbitrarily long intervals of integers, then A will have an infinite complement, that is, there exists B such that $A \oplus B = \mathbf{Z}$.

In general, it is known that every pair (A, B) of complementing sets with A finite must satisfy a certain cyclotomy condition, but it is a open problem to determine if a finite set A of integers has a complement.

Complementing pairs have also been studied for sets of lattice points (Hansen [2], Nathanson [5], Niven [8]). If (A, B) is a pair of sets of lattice points such that A is finite and every lattice point has a unique representation in the form $a + b$ with $a \in A$ and $b \in B$, then it is an open problem to determine if the set B must be periodic (cf. Lagarias and Wang [4] and Szegedy [10]).

The problem of complementing pairs for the set of integers is a special case of the general problem of the representation of integers by linear forms. The object of this paper is to introduce this problem and to initiate the study of complementing sets of integers with respect to an arbitrary linear form $\varphi(x_1, \ldots, x_h, y_1, \ldots, y_\ell)$.

Notation. Let $\mathbf{Z}$ and $\mathbf{N}_0$ denote the set of integers and the set of nonnegative integers, respectively. We denote the cardinality of the set S by $|S|$ or by $\mathrm{card}(S)$. We denote the integer part of the real number x by $[x]$.

2. Representation functions for linear forms

Let $h \geq 1$ and let

$$\psi(x_1, \dots, x_h) = u_1 x_1 + \cdots + u_h x_h$$

be a linear form with nonzero integer coefficients $u_1, \dots, u_h$. Let

$$\mathcal{A} = (A_1, \dots, A_h)$$

be an h-tuple of sets of integers. The *image* of ψ with respect to $\mathcal{A}$ is the set

$$\psi(\mathcal{A}) = \{\psi(a_1, \dots, a_h) : (a_1, \dots, a_h) \in A_1 \times \cdots \times A_h\}.$$

Then $\psi(\mathcal{A}) \neq \emptyset$ if and only if $A_i \neq \emptyset$ for all $i = 1, \dots, h$. For $\psi(\mathcal{A}) \neq \emptyset$, we define the *diameter of $\mathcal{A}$ with respect to ψ* by

$$D_{\mathcal{A}}^{(\psi)} = \operatorname{diam}(\psi(\mathcal{A})) = \sup(\psi(\mathcal{A})) - \inf(\psi(\mathcal{A})).$$

We have $D_{\mathcal{A}}^{(\psi)} > 0$ if and only if $|A_i| > 1$ for some i.

For every integer n, we define the *representation function* associated to ψ by

$$R_{\mathcal{A}}^{(\psi)}(n) = \operatorname{card}\left(\{(a_1, \dots, a_h) \in A_1 \times \cdots \times A_h : \psi(a_1, \dots, a_h) = n\}\right).$$

Then $n \in \psi(\mathcal{A})$ if and only if $R_{\mathcal{A}}^{(\psi)}(n) > 0$.

Let $\ell \geq 1$ and let

$$\omega(y_1, \dots, y_\ell) = v_1 y_1 + \cdots + v_\ell y_\ell$$

be another linear form with nonzero integer coefficients $v_1, \dots, v_\ell$. Consider the linear form

$$\varphi(x_1, \dots, x_h, y_1, \dots, y_\ell) = \psi(x_1, \dots, x_h) + \omega(y_1, \dots, y_\ell).$$

Let $\mathcal{A} = (A_1, \dots, A_h)$ be an h-tuple of sets of integers and let $\mathcal{B} = (B_1, \dots, B_\ell)$ be an ℓ-tuple of sets of integers. The *image* of φ with respect to $(\mathcal{A}, \mathcal{B})$ is the set

$$\begin{aligned}\varphi(\mathcal{A}, \mathcal{B}) =& \psi(\mathcal{A}) + \omega(\mathcal{B}) \\ =& \{\psi(a_1, \dots, a_h) + \omega(b_1, \dots, b_\ell) : (a_1, \dots, a_h) \in A_1 \times \cdots \times A_h \\ & \quad \text{and } (b_1, \dots, b_\ell) \in B_1 \times \cdots \times B_\ell\}.\end{aligned}$$

We define the *representation function associated to φ, $\mathcal{A}$, and $\mathcal{B}$* by

$$\begin{aligned}R_{\mathcal{A},\mathcal{B}}^{(\varphi)}(n) =& \operatorname{card}\left(\{(a_1, \dots, a_h, b_1, \dots, b_\ell) \in A_1 \times \cdots \times A_h \times B_1 \times \cdots \times B_\ell :\right. \\ & \left. \varphi(a_1, \dots, a_h, b_1, \dots, b_\ell) = n\}\right).\end{aligned}$$

If $\ell = 1$ and $\mathcal{B} = (B)$, then we write $\varphi(\mathcal{A}, \mathcal{B}) = \varphi(\mathcal{A}, B)$ and $R_{\mathcal{A},\mathcal{B}}^{(\varphi)}(n) = R_{\mathcal{A},B}^{(\varphi)}(n)$, Let $\mathcal{A}$ be an h-tuple of sets of integers and $\mathcal{B}$ an ℓ-tuple of sets of integers. The pair $(\mathcal{A}, \mathcal{B})$ is called *complementing* with respect to φ if $R^{(\varphi)}_{\mathcal{A},\mathcal{B}}(n) = 1$ for all $n \in \mathbf{Z}$, that is, if every integer n has a unique representation in the form $n = \psi(a_1, \dots, a_h) + \omega(b_1, \dots, b_\ell)$, where $a_i \in A_i$ for $i = 1, \dots, h$ and $b_j \in B_j$ for $j = 1, \dots, \ell$. The pair $(\mathcal{A}, \mathcal{B})$ is called *t-complementing* with respect to φ if $R^{(\varphi)}_{\mathcal{A},\mathcal{B}}(n) = t$ for all $n \in \mathbf{Z}$.

Let $\mathcal{A}$ be an h-tuple of finite sets of integers. For every positive integer m, we define the *modular representation function* associated to ψ by

$$R^{(\psi)}_{\mathcal{A};m}(n) = \operatorname{card}\left(\{(a_1, \dots, a_h) \in A_1 \times \cdots \times A_h : \psi(a_1, \dots, a_h) \equiv n \pmod{m}\}\right).$$

Then $\mathcal{A}$ is called *t-complementing modulo m* with respect to ψ if $R^{(\varphi)}_{\mathcal{A};m}(\ell) = t$ for all $\ell \in \{0, 1, \dots, m-1\}$.

The pair $(\mathcal{A}, \mathcal{B})$ is called *periodic with respect to* φ if the representation function $R^{(\varphi)}_{\mathcal{A},\mathcal{B}}$ is periodic, that is, if there is a positive integer m such that $R^{(\varphi)}_{\mathcal{A},\mathcal{B}}(n+m) = R^{(\varphi)}_{\mathcal{A},\mathcal{B}}(n)$ for all integers n. The pair $(\mathcal{A}, \mathcal{B})$ is *eventually periodic with respect to* φ if the representation function $R^{(\varphi)}_{\mathcal{A},\mathcal{B}}$ is eventually periodic, that is, if there exist positive integers m and n_0 such that $R^{(\varphi)}_{\mathcal{A},\mathcal{B}}(n+m) = R^{(\varphi)}_{\mathcal{A},\mathcal{B}}(n)$ for all integers $n \geq n_0$.

We consider the case $\ell = 1$. Suppose that $\varphi(x_1, \dots, x_h, y) = \psi(x_1, \dots, x_h) + vy$ is a linear form with nonzero integer coefficients, and that $\mathcal{A}$ is an h-tuple of finite sets of integers and B is a set of integers such that the pair $(\mathcal{A}, B)$ is t-complementing with respect to φ. We shall prove that the set B is periodic, and obtain an upper bound for the period of B in terms of the diameter $D^{\psi}_{\mathcal{A}}$ of the finite set $\psi(\mathcal{A})$. We also obtain a cyclotomic condition related to t-complementing sets modulo m, and describe a compactness argument that allows us to solve an inverse problem related to representation functions associated with linear forms.

The problem of complementing sets $A \oplus B = \mathbf{Z}$ is the special case $h = 1$, $\psi(x) = x$, $\omega(y) = y$, and $\varphi(x, y) = x + y$ of the general problem of representations of integers by linear forms.

3. Linear forms and periodicity

Theorem 1. *Let $h \geq 1$ and let*

$$\psi(x_1, \dots, x_h) = u_1 x_1 + \cdots + u_h x_h$$

and

$$\varphi(x_1, \dots, x_h, y) = \psi(x_1, \dots, x_h) + vy$$

be a linear forms with nonzero integer coefficients $u_1, \ldots, u_h, v$. Let $\mathcal{A} = (A_1, \ldots, , A_h)$ be an h-tuple of nonempty finite sets of integers, and let B be an infinite set of integers. If the pair $(\mathcal{A}, B)$ is t-complementing respect to φ, then B is periodic, that is, there is a positive integer m such that B is a union of congruence classes modulo m. Moreover, $m \leq 2^d$, where $d = diam(\psi(\mathcal{A}))/|v|$.

Remark. In the case that $h = t = 1$, $\psi(x) = x$, and $v = 1$, then $\varphi(x, y) = x + y$ and the Theorem specializes to a classical result of D. J. Newman [7] for tiling by finite sets in additive number theory.

Proof. If $v < 0$, then we replace φ with $-\varphi$. Then $R^{(-\varphi)}_{\mathcal{A},B;m}(n) = R^{(\varphi)}_{\mathcal{A},B;m}(-n) = t$ for all $n \in \mathbf{Z}$, and the pair $(\mathcal{A}, B)$ is t-complementing respect to φ. Thus, we can assume without loss of generality that $v \geq 1$.

If $|A_i| = 1$ for all $i = 1, \ldots, h$, then $\text{card}(\psi(\mathcal{A})) = 1$. It follows that the linear form φ is t-complementing if and only if $t = v = 1$ and $B = \mathbf{Z}$, and so the Theorem holds with $m = 1$. Thus, we can also assume that $|A_i| > 1$ for at least one i.

Let $g_{\min} = \min(\psi(\mathcal{A}))$ and $g_{\max} = \max(\psi(\mathcal{A}))$. Since $|A_i| > 1$ for some $i \in \{1, 2, \ldots, h\}$, it follows that $g_{\min} < g_{\max}$ and

$$D^{(\psi)}_{\mathcal{A}} = \text{diam}(\psi(\mathcal{A})) = g_{\max} - g_{\min} \geq 1.$$

Let

$$G_{\min} = \{(a_1, \ldots, a_h) \in A_1 \times \cdots \times A_h : \psi(a_1, \ldots, a_h) = g_{\min}\}$$

and

$$G_{\max} = \{(a_1, \ldots, a_h) \in A_1 \times \cdots \times A_h : \psi(a_1, \ldots, a_h) = g_{\max}\}.$$

Then

$$|G_{\min}| = R^{(\psi)}_{\mathcal{A}}(g_{\min}) \geq 1$$

and

$$|G_{\max}| = R^{(\psi)}_{\mathcal{A}}(g_{\max}) \geq 1.$$

Let $\chi_B : \mathbf{R} \to \{0, 1\}$ denote the characteristic function of the set B, that is,

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B. \end{cases}$$

We have

$$\varphi(a_1, \ldots, a_h, b) = \psi(a_1, \ldots, a_h) + vb = n$$

if and only if

$$b = \frac{n - \psi(a_1, \ldots, a_h)}{v} \in B.$$

It follows that, for all $n \in \mathbf{Z}$,

$$\begin{aligned} R^{(\varphi)}_{\mathcal{A},B}(n) &= \sum_{(a_1,\ldots,a_h)\in A_1\times\cdots\times A_h} \chi_B\left(\frac{n-\psi(a_1,\ldots,a_h)}{v}\right) \\ &= \sum_{\substack{(a_1,\ldots,a_h)\in A_1\times\cdots\times A_h \\ (a_1,\ldots,a_h)\notin G_{\min}}} \chi_B\left(\frac{n-\psi(a_1,\ldots,a_h)}{v}\right) \\ &\quad + |G_{\min}|\chi_B\left(\frac{n-g_{\min}}{v}\right). \end{aligned}$$

Replacing n by $vn+g_{\min}$, we obtain the identity

$$\begin{aligned} R^{(\varphi)}_{\mathcal{A},B}(vn+g_{\min}) = &\sum_{\substack{(a_1,\ldots,a_h)\in A_1\times\cdots\times A_h \\ (a_1,\ldots,a_h)\notin G_{\min}}} \chi_B\left(\frac{vn+g_{\min}-\psi(a_1,\ldots,a_h)}{v}\right) \\ &+ |G_{\min}|\chi_B\left(n\right). \end{aligned}$$

Equivalently,

$$\begin{aligned} |G_{\min}|\chi_B(n) = &R^{(\varphi)}_{\mathcal{A},B}(vn+g_{\min}) \\ &- \sum_{\substack{(a_1,\ldots,a_h)\in A_1\times\cdots\times A_h \\ (a_1,\ldots,a_h)\notin G_{\min}}} \chi_B\left(n-\frac{\psi(a_1,\ldots,a_h)-g_{\min}}{v}\right). \end{aligned} \tag{1}$$

Since $g_{\min} < \psi(a_1,\ldots,a_h) \leq g_{\max}$ for all h-tuples $(a_1,\ldots,a_h) \notin G_{\min}$, it follows that

$$0 < \frac{1}{v} \leq \frac{\psi(a_1,\ldots,a_h)-g_{\min}}{v} \leq \frac{g_{\max}-g_{\min}}{v}.$$

Similarly,

$$\begin{aligned} R^{(\varphi)}_{\mathcal{A},B}(n) = &\sum_{\substack{(a_1,\ldots,a_h)\in A_1\times\cdots\times A_h \\ (a_1,\ldots,a_h)\notin G_{\max}}} \chi_B\left(\frac{n-\psi(a_1,\ldots,a_h)}{v}\right) + \\ &+ |G_{\max}|\chi_B\left(\frac{n-g_{\max}}{v}\right). \end{aligned}$$

Replacing n by $vn+g_{\max}$, we obtain

$$\begin{aligned} |G_{\max}|\chi_B(n) = &R^{(\varphi)}_{\mathcal{A},B}(vn+g_{\max}) \\ &- \sum_{\substack{(a_1,\ldots,a_h)\in A_1\times\cdots\times A_h \\ (a_1,\ldots,a_h)\notin G_{\max}}} \chi_B\left(n+\frac{g_{\max}-\psi(a_1,\ldots,a_h)}{v}\right). \end{aligned} \tag{2}$$

Since $g_{\min} \leq \psi(a_1,\ldots,a_h) < g_{\max}$ for $(a_1,\ldots,a_h) \notin G_{\max}$, it follows that

$$0 < \frac{1}{v} \leq \frac{g_{\max}-\psi(a_1,\ldots,a_h)}{v} \leq \frac{g_{\max}-g_{\min}}{v}.$$

Define the nonnegative integer

$$d = \left[\frac{\operatorname{diam}(\psi(\mathcal{A}))}{v}\right] = \left[\frac{g_{\max} - g_{\min}}{v}\right].$$

Formulae (1) and (2) enable us to use the representation function $R^{(\varphi)}_{\mathcal{A},B}$ to compute the characteristic function χ_B recursively for all integers n if we know the value of χ_B for any d consecutive integers.

If the pair $(\mathcal{A}, B)$ is t-complementing with respect to φ, then $R^{(\varphi)}_{\mathcal{A},B}(n) = t$ for all $n \in \mathbf{Z}$, and we can rewrite the recursion formulae (1) and (2) in the form

$$\chi_B(n) = \frac{1}{|G_{\min}|}\left(t - \sum_{\substack{(a_1,\ldots,a_h)\in A_1\times\cdots\times A_h \\ (a_1,\ldots,a_h)\notin G_{\min}}} \chi_B\left(n - \frac{\psi(a_1,\ldots,a_h) - g_{\min}}{v}\right)\right)$$

and

$$\chi_B(n) = \frac{1}{|G_{\max}|}\left(t - \sum_{\substack{(a_1,\ldots,a_h)\in A_1\times\cdots\times A_h \\ (a_1,\ldots,a_h)\notin G_{\max}}} \chi_B\left(n + \frac{g_{\max} - \psi(a_1,\ldots,a_h)}{v}\right)\right).$$

Consider the d-tuple

$$\mathcal{B}(j) = (\chi_B(j), \chi_B(j+1), \ldots, \chi_B(j+d-1)) \in \{0,1\}^d.$$

Since there only 2^d binary sequences of length d, it follows from the pigeonhole principle that there are integers j_1, j_2 such that $0 \leq j_1 < j_2 \leq 2^d$ and $\mathcal{B}(j_1) = \mathcal{B}(j_2)$. Let $m = j_2 - j_1$. Then

$$1 \leq m \leq 2^d$$

and $\chi_B(n) = \chi_B(n+m)$ for $n = j_1, \ldots, j_1 + d - 1$. The recursion formulae imply that $\chi_B(n) = \chi_B(n+m)$ for all integers n. This completes the proof. $\square$

4. Linear forms and cyclotomy

Theorem 2. *Let $h \geq 1$ and let*

$$\psi(x_1, \ldots, x_h, y) = u_1 x_1 + \cdots + u_h x_h$$

be a linear form with nonzero integer coefficients $u_1, \ldots, u_h$,. Let $\mathcal{A} = (A_1, \ldots, A_h)$ be an h-tuple of nonempty finite sets of integers. Consider the Laurent polynomials

$$F_{A_i}(z) = \sum_{a_i \in A_i} z^{a_i} \qquad \text{for } i = 1, \ldots, h.$$

For $m \geq 1$, define the polynomial

$$\Lambda_m(z) = 1 + z + z^2 + \cdots + z^{m-1}.$$

The h-tuple $\mathcal{A}$ is t-complementing modulo m with respect to ψ if and only if

$$z^L F_{A_1}(z^{u_1}) \cdots F_{A_h}(z^{u_h}) \equiv t\Lambda_m(z) \pmod{z^m - 1} \tag{3}$$

where L is any nonnegative integer such that $z^L F_{A_1}(z^{u_1}) \cdots F_{A_h}(z^{u_h})$ is a polynomial.

Proof. The function

$$F(z) = F_{A_1}(z^{u_1}) \cdots F_{A_h}(z^{u_h})$$

is a nonzero Laurent polynomial with integer coefficients. Choose a nonnegative integer L such that $z^L F(z)$ is a polynomial.

The sets $A_1, \ldots, A_h$ are finite, and so $\psi(\mathcal{A})$ is finite. We have $R_{\mathcal{A}}^{(\psi)}(n) \geq 1$ if and and only if $n \in \psi(\mathcal{A})$. For $\ell = 0, 1, \ldots, m-1$, we consider the finite set

$$\mathcal{I}_\ell = \{i \in \mathbf{Z} : R_{\mathcal{A}}^{(\psi)}(\ell + im) \geq 1\}.$$

Since $F_{A_i}(z^{u_i}) = \sum_{a_i \in A_i} z^{u_i a_i}$ for $i = 1, \ldots, h$, it follows that

$$\begin{aligned}
F(z) &= F_{A_1}(z^{u_1}) \cdots F_{A_h}(z^{u_h}) \\
&= \sum_{a_1 \in A_1} \cdots \sum_{a_h \in A_h} z^{u_1 a_1 + \cdots + u_h a_h} \\
&= \sum_{a_1 \in A_1} \cdots \sum_{a_h \in A_h} z^{\psi(a_1, \ldots, a_h)} \\
&= \sum_{n \in \psi(\mathcal{A})} R_{\mathcal{A}}^{(\psi)}(n) z^n \\
&= \sum_{\ell=0}^{m-1} \sum_{\substack{n \in \psi(\mathcal{A}) \\ n \equiv \ell \pmod{m}}} R_{\mathcal{A}}^{(\psi)}(n) z^n \\
&= \sum_{\ell=0}^{m-1} \sum_{i \in \mathcal{I}_\ell} R_{\mathcal{A}}^{(\psi)}(\ell + im) z^{\ell + im}.
\end{aligned}$$

Since

$$z^L F(z) = \sum_{\ell=0}^{m-1} \sum_{i \in \mathcal{I}_\ell} R_{\mathcal{A}}^{(\psi)}(\ell + im) z^{\ell + L + im}$$

is a polynomial, it follows that $\ell + L + im \geq 0$ for all $\ell \in \{0, 1, \ldots, m-1\}$ and $i \in \mathcal{I}_\ell$. Applying the division algorithm for integers, we can write

$$\ell + L = \alpha(\ell) + \beta(\ell) m$$

where $0 \leq \alpha(\ell) \leq m-1$ for $\ell = 0, 1, \ldots, m-1$. Moreover, if $\ell \not\equiv \ell' \pmod{m}$, then $\alpha(\ell) \neq \alpha(\ell')$ and so

$$\{\alpha(0), \alpha(1), \ldots, \alpha(m-1)\} = \{0, 1, \ldots, m-1\}.$$

Equivalently,

$$\sum_{\ell=0}^{m-1} z^{\alpha(\ell)} = \sum_{\ell=0}^{m-1} z^{\ell} = \Lambda_m(z).$$

If $i \in \mathcal{I}_\ell$, then the inequality

$$\ell + L + im = \alpha(\ell) + (\beta(\ell) + i)m \geq 0$$

implies that $\beta(\ell) + i \geq 0$. Therefore, for each $\ell \in \{0, 1, \ldots, m-1\}$ there is a polynomial $p_\ell(z)$ with nonnegative integral coefficients such that

$$\begin{aligned}
\sum_{i \in \mathcal{I}_\ell} R_{\mathcal{A}}^{(\psi)}(\ell + im) z^{\ell + L + im} &= \sum_{i \in \mathcal{I}_\ell} R_{\mathcal{A}}^{(\psi)}(\ell + im) z^{\alpha(\ell) + (\beta(\ell) + i)m} \\
&= \sum_{i \in \mathcal{I}_\ell} R_{\mathcal{A}}^{(\psi)}(\ell + im) z^{\alpha(\ell)} \left(1 + (z^m - 1)\right)^{\beta(\ell) + i} \\
&= \sum_{i \in \mathcal{I}_\ell} R_{\mathcal{A}}^{(\psi)}(\ell + im) z^{\alpha(\ell)} + (z^m - 1) p_\ell(z) \\
&= R_{\mathcal{A},m}^{(\psi)}(\ell) z^{\alpha(\ell)} + (z^m - 1) p_\ell(z).
\end{aligned}$$

It follows that

$$\begin{aligned}
z^L F(z) &= \sum_{\ell=0}^{m-1} \sum_{i \in \mathcal{I}_\ell} R_{\mathcal{A}}^{(\psi)}(\ell + im) z^{\ell + L + im} \\
&= \sum_{\ell=0}^{m-1} R_{\mathcal{A},m}^{(\psi)}(\ell) z^{\alpha(\ell)} + (z^m - 1) \sum_{\ell=0}^{m-1} p_\ell(z) \\
&= r_L(z) + (z^m - 1) q_L(z),
\end{aligned}$$

where

$$q_L(z) = \sum_{\ell=0}^{m-1} p_\ell(z)$$

and

$$r_L(z) = \sum_{\ell=0}^{m-1} R_{\mathcal{A},m}^{(\psi)}(\ell) z^{\alpha(\ell)}.$$

Since the degree of the polynomial $r_L(z)$ is at most $m-1$, the division algorithm for polynomials implies that this representation of $z^L F(z)$ is unique.

Suppose that $\mathcal{A} = (A_1, \ldots, A_h)$ is a t-complementing h-tuple modulo m. Then $R_{\mathcal{A},m}(\ell) = t$ for all ℓ, and

$$r_L(z) = \sum_{\ell=0}^{m-1} tz^{\alpha(\ell)} = t \sum_{\ell=0}^{m-1} z^{\ell} = t\Lambda_m(z).$$

Therefore,

$$z^L F(z) = t\Lambda_m(z) + (z^m - 1)q_L(z)$$

and condition (3) is satisfied.

Conversely, suppose that the generating functions $F_{A_1}(z), \ldots, F_{A_h}(z)$ satisfy condition (3) for some nonnegative integer L. By the uniqueness of the polynomial division algorithm, we have

$$\sum_{\ell=0}^{m-1} tz^{\ell} = t\Lambda_m(z) = r_L(z) = \sum_{\ell=0}^{m-1} R^{(\psi)}_{\mathcal{A},m}(\ell) z^{\alpha(\ell)}.$$

Since

$$\{\alpha(0), \alpha(1), \ldots, \alpha(m-1)\} = \{0, 1, \ldots, m-1\},$$

it follows that $R^{(\psi)}_{\mathcal{A},m}(\ell) = t$ for all $\ell \in \{0, 1, \ldots, m-1\}$, and so $\mathcal{A} = (A_1, \ldots, A_h)$ is a t-complementing h-tuple modulo m. This completes the proof. $\square$

5. An inverse problem for linear forms

Let $\varphi(x_1, \ldots, x_h, y)$ be a form in $h+1$ variables and let $f : \mathbf{Z} \to \mathbf{N}_0 \cup \{\infty\}$ be a function. If $\mathcal{A} = (A_1, \ldots, A_h)$ is an h-tuple of sets of integers, does there exist a set B such that the pair $(\mathcal{A}, B)$ satisfies $R^{(\varphi)}_{\mathcal{A},B}(n) = f(n)$ for all $n \in \mathbf{Z}$? This is the complementing set inverse problem for representation functions associated to linear forms. In this section we use a compactness argument to obtain a result in the case that $\mathcal{A} = (A_1, \ldots, A_h)$ is an h-tuple of finite sets.

Theorem 3. *Let $h \geq 1$ and let*

$$\varphi(x_1, \ldots, x_h, y) = u_1 x_1 + \cdots + u_h x_h + vy$$

be a linear form with nonzero integer coefficients $u_1, \ldots, u_h, v$. Let $\mathcal{A} = (A_1, \ldots, A_h)$ be an h-tuple of nonempty finite sets of integers. Let $f : \mathbf{Z} \to \mathbf{N}_0 \cup \{\infty\}$ be a function. Suppose that there is a strictly increasing sequence $\{k_N\}_{N=1}^{\infty}$ of positive integers with the property that, for every $N \geq 1$, there exists a set B_N of integers that satisfies

$$R^{(\varphi)}_{\mathcal{A},B_N}(n) = f(n) \qquad \text{for } |n| \leq k_N.$$

Then there exists a set B such that

$$R^{(\varphi)}_{\mathcal{A},B}(n) = f(n) \qquad \text{for all } n \in \mathbf{Z}.$$

Proof. Since $k_N \geq N$ for all $N \geq 1$, we can assume without loss of generality that $k_N = N$.

Consider the linear form

$$\psi(x_1, \ldots, x_h) = u_1 x_1 + \cdots + u_h x_h$$

Then

$$\varphi(a_1, \ldots, a_h, b) = \psi(a_1, \ldots, a_h) + vb$$

for all integers $a_1, \ldots, a_h, b$. Moreover, since the sets $A_1, \ldots, A_h$ are finite, there is a positive integer g^* such that $\psi(\mathcal{A}) \subseteq [-g^*, g^*]$. If $(a_1, \ldots, a_h) \in A_1 \times \cdots \times A_h$, if $b \in \mathbf{Z}$, and if $\varphi(a_1, \ldots, a_h, b) = n \in [-N, N]$, then

$$|b| = \left| \frac{n - \psi(a_1, \ldots, a_h)}{v} \right| \leq \frac{|n| + |\psi(a_1, \ldots, a_h)|}{|v|} \leq \frac{N + g^*}{|v|}.$$

Replacing the set B_N with $B_N \cap [-(N+g^*)/|v|, (N+g^*)/|v|]$, we can assume without loss of generality that

$$B_N \subseteq \left[-\frac{N+g^*}{|v|}, \frac{N+g^*}{|v|} \right]$$

for all $N \geq 1$.

We shall construct inductively a sequence $\{B'_N\}_{N=1}^{\infty}$ of finite sets with the following properties:

(1) $B'_N \subseteq B'_{N+1}$ for all $N \geq 1$,

(2) For every positive integer N we have

$$R^{(\varphi)}_{\mathcal{A}, B'_N}(n) = f(n) \qquad \text{for } |n| \leq N$$

(3) For every positive integer N there is a strictly increasing sequence $\{M_{j,N}\}_{j=1}^{\infty}$ of positive integers such that $N \leq M_{1,N}$ and $B'_N \subseteq B_{M_{j,N}}$ for all $j \geq 1$.

We begin by constructing the set B'_1. If $(a_1, \ldots, a_h) \in A_1 \times \cdots \times A_h$, if $b \in \mathbf{Z}$, and if $\varphi(a_1, \ldots, a_h, b) \in [-1, 1]$, then $|b| \leq (1+g^*)/|v|$. For all $N \geq 1$ we have $R^{(\varphi)}_{\mathcal{A}, B_N}(n) = f(n)$ for $|n| \leq N$, and so $R^{(\varphi)}_{\mathcal{A}, B_N}(n) = f(n)$ for $|n| \leq 1$. Let

$$B^{(1)}_N = B_N \cap \left[-\frac{1+g^*}{|v|}, \frac{1+g^*}{|v|} \right]$$

for $N \geq 1$. Then $\left\{ B^{(1)}_N \right\}_{N=1}^{\infty}$ is an infinite sequence of subsets of the finite set $[-(1+g^*)/|v|, (1+g^*)/|v|] \cap \mathbf{Z}$. By the pigeonhole principle, there is a strictly increasing sequence of positive integers $\{M_{j,1}\}_{j=1}^{\infty}$ and a set B'_1 such that

$$B'_1 = B^{(1)}_{M_{j,1}} \subseteq B_{M_{j,1}}$$

for all $j \geq 1$. The set B'_1 and the sequence $\{M_{j,1}\}_{j=1}^{\infty}$ satisfy properties (1), (2), and (3).

Suppose that we have constructed an increasing sequence of sets $B'_1 \subseteq B'_2 \subseteq \cdots \subseteq B'_N$ and sequences $\{M_{j,k}\}_{j=1}^{\infty}$ for $k = 1, \ldots, N$ that satisfy properties (1), (2), and (3). For $j \geq 1$ we define the finite set

$$B_{M_{j,N}}^{(N+1)} = B_{M_j,N} \cap \left[-\frac{N+1+g^*}{|v|}, \frac{N+1+g^*}{|v|} \right].$$

Then $\left\{ B_{M_{j,N}}^{(N+1)} \right\}_{j=1}^{\infty}$ is an infinite sequence of subsets of the finite set $[-(N+1+g^*)/|v|, (N+1+g^*)/|v|] \cap \mathbf{Z}$. By the pigeonhole principle, there is a strictly increasing sequence $\{M_{j,N+1}\}_{j=1}^{\infty}$ of positive integers and a set B'_{N+1} such that $N+1 \leq M_{1,N+1}$ and

$$B'_N \subseteq B'_{N+1} = B_{M_{j,N+1}}^{(N+1)} \subseteq B_{M_{j,N+1}}$$

for all $j \geq 1$. Properties (1), (2), and (3) are satisfied for $N+1$. This completes the induction. Moreover, the set $B = \bigcup_{N=1}^{\infty} B'_N$ satisfies $R_{\mathcal{A},B}^{(\varphi)}(n) = f(n)$ for all $n \in \mathbf{Z}$. This completes the proof. $\square$

Theorem 4. *Let $h \geq 1$ and $\varphi(x_1, \ldots, x_h, y) = u_1 x_1 + \cdots + u_h x_h + y$. Let $\mathcal{A} = (A_1, \ldots, A_h)$ be an h-tuple of nonempty finite sets of integers and let $t \geq 1$. Suppose that there is a strictly increasing sequence $\{k_N\}_{N=1}^{\infty}$ of positive integers such that, for every $N \geq 1$, there exists a set B_N of integers and a set I_N consisting of $2k_N + 1$ consecutive integers such that*

$$R_{\mathcal{A},B_N}(n) = t \qquad \text{for } n \in I_N.$$

Then there exists a set B such that

$$R_{\mathcal{A},B}(n) = t \qquad \text{for all } n \in \mathbf{Z}.$$

Proof. For every integer $N \geq 1$, there is an integer c_N such that $I_N = [c_N - k_N, c_N + k_N] \cap \mathbf{Z}$. Replace the set B_N with the set $B_N - c_N$ and apply Theorem 3. This completes the proof. $\square$

A related result appears in Nathanson [6].

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