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Mark KISIN

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Résumé. Le but de cette note est de fournir une introduction à la théorie des modèles entiers canoniques des variétés de Shimura, et de donner une esquisse de la preuve d'existence de tels modèles pour les variétés de Shimura de type Hodge, et plus généralement, de type abélien. Pour plus de détails, le lecteur est renvoyé à [Ki 3].

Abstract. The aim of these notes is to provide an introduction to the subject of integral canonical models of Shimura varieties, and then to sketch a proof of the existence of such models for Shimura varieties of Hodge and, more generally, abelian type. For full details the reader is refered to [Ki 3].

1. Shimura varieties

1.1. Shimura data. We recall the definition of a Shimura datum and the associated Shimura variety [De 1, §2.1]. Let G be a connected reductive group over $\mathbb{Q}$ and X a conjugacy class of maps of algebraic groups over $\mathbb{R}$

$$h : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}.$$

On $\mathbb{R}$-points such a map induces a map of real groups $\mathbb{C}^\times \to G(\mathbb{R})$.

We require that (G, X) satisfy the following conditions:

- Let $\mathfrak{g}$ denote the Lie algebra of $G_{\mathbb{R}}$. We require that the composite
$$\mathbb{S} \to G_{\mathbb{R}} \to G_{\mathbb{R}}^{\mathrm{ad}} \to \mathrm{GL}(\mathfrak{g})$$
defines a Hodge structure of type $(-1,1),(0,0),(1,-1)$. This means that under the action of $\mathbb{C}^\times$ on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$ we have a decomposition
$$\mathfrak{g}_{\mathbb{C}} = V^{-1,1} \oplus V^{0,0} \oplus V^{1,-1}$$
where $z \in \mathbb{C}^\times$ acts on $V^{p,q}$ via $z^{-p}\bar{z}^{-q}$.
- Conjugation by $h(i)$ induces a Cartan involution of $G_{\mathbb{R}}^{\mathrm{ad}}$ (note that $\mathrm{ad}h(-1) = 1$ on $\mathfrak{g}$ so $h(i)$ induces an involution of $G_{\mathbb{R}}^{\mathrm{ad}}$). This means that we require the real form of G defined by the involution $g \mapsto h(i)\bar{g}h(i)^{-1}$ to be compact.
- G^{ad} has no factor defined over $\mathbb{Q}$ whose real points form a compact group.

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The second condition implies that for any $h_0 \in X$ the stabilizer $K_\infty \subset G^{\mathrm{ad}}(\mathbb{R})$ (acting by conjugation) of h_0 is compact and $G^{\mathrm{ad}}(\mathbb{R})/K_\infty \xrightarrow{\sim} X$ has a complex structure.

A pair (G, X) satisfying the above conditions is called a *Shimura datum.* A morphism $(G_1, X_1) \to (G_2, X_2)$ of Shimura data is a map of groups $G_1 \to G_2$, which induces a map $X_1 \to X_2$.

Now let $K = \prod_p K_p \subset G(\mathbb{A}_f)$ be a compact open subgroup, where $\mathbb{A}_f$ denotes the finite adeles over $\mathbb{Q}$. Then a theorem of Baily-Borel asserts that

$$\mathrm{Sh}_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

has a natural structure of an algebraic variety over $\mathbb{C}$. Results of Shimura, Deligne, Milne and others imply that $\mathrm{Sh}_K(G, X)$ has a model over a number field $E = E(G, X)$ - the reflex field - which does not depend on K [Mi 2, §4,5]. We will again denote by $\mathrm{Sh}_K(G, X)$ this algebraic variety over $E(G, X)$.

Fix a prime p. We will sometimes consider the pro-variety

$$\mathrm{Sh}_{K_p}(G, X) = \varprojlim \mathrm{Sh}_K(G, X),$$

where K runs through compact open subgroups as above with a fixed factor K_p at p.

A morphism of Shimura data $(G_1, X_1) \to (G_2, X_2)$ induces a morphism of the corresponding Shimura varieties $\mathrm{Sh}_{K_1}(G_1, X_1) \to \mathrm{Sh}_{K_2}(G_2, X_2)$, provided the compact open subgroups are chosen so that K_1 maps into K_2.

1.2. Examples. (1) Let $G = \mathrm{GL}_2$, and let X be the $\mathrm{PGL}_2(\mathbb{R})$ orbit of

$$h_0 : \mathbb{C}^\times \to G(\mathbb{R}); \quad a + ib \mapsto \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right).$$

Then the map $\mathrm{ad}(g) \cdot h_0 \mapsto g \cdot i$ identifies X with the upper and lower half planes $\mathcal{H}^\pm$, compatibly with the action of $\mathrm{PGL}_2(\mathbb{R})$. Here $\mathrm{PGL}_2(\mathbb{R})$ acts on $\mathcal{H}^\pm$ in the usual way, by Möbius transformations.

(2) Fix a $\mathbb{Q}$-vector space V with a perfect alternating pairing ψ. Take $G = \mathrm{GSp}(V, \psi)$ the corresponding group of symplectic similitudes, and let $X = S^\pm$ be the Siegel double space, defined as the set of maps $h : \mathbb{S} \to G_\mathbb{R}$ such that

- The $\mathbb{C}^\times$ action on $V_\mathbb{R}$ gives rise to a Hodge structure which is of type $(-1, 0), (0, -1)$:

$$V_\mathbb{C} \xrightarrow{\sim} V^{-1,0} \oplus V^{0,-1}.$$

- $(x, y) \mapsto (x, h(i)y)$ is (positive or negative) definite on $V_\mathbb{R}$.

If $V_\mathbb{Z} \subset V$ is a $\mathbb{Z}$-lattice, and $h \in S^\pm$, then $V^{-1,0}/V_\mathbb{Z}$ is an abelian variety, and $\mathrm{Sh}_K(G, X) = \mathrm{Sh}_K(\mathrm{GSp}, S^\pm)$ has an interpretation as a moduli space for abelian varieties.

2. Integral canonical models

2.1. Hyperspecial subgroups. A compact open subgroup $K \subset G(\mathbb{Q}_p)$ is called *hyperspecial* if there exists a reductive group $\mathcal{G}$ over $\mathbb{Z}_p$ extending $G_{\mathbb{Q}_p}$ and such that $K = \mathcal{G}(\mathbb{Z}_p)$. A hyperspecial subgroup is a maximal compact open subgroup.

Such subgroups exist if G is quasi-split at p and split over an unramified extension.

2.2. The Langlands-Milne conjecture. Fix a finite prime p, and a compact open subgroup $K = \prod K_\ell \subset G(\mathbb{A}_f)$. Write $E = E(G, X)$, and let $\mathcal{O} \subset E$ be its ring of integers. For a prime $\lambda | p$ of $E(G, X)$ we will denote by $\mathcal{O}_{(\lambda)}$ the localization of $\mathcal{O}$ at λ.

The following was conjectured in a rough form by Langlands [La] and made precise by Milne [Mi 1].

Conjecture 2.2.1. *(Langlands-Milne) Suppose that $K \subset G(\mathbb{A}_f)$ is open compact and K_p is hyperspecial. Then for $\lambda | p$, $\mathrm{Sh}_K(G, X)$ has an integral canonical model over $\mathcal{O}_{(\lambda)}$.*

Let us make some remarks to try to explain the meaning of the conjecture.

First this is a statement about the tower of Shimura varieties

$$\mathrm{Sh}_{K_p}(G, X) = \varprojlim \mathrm{Sh}_{K'}(G, X)$$

where $K' = K'_p K'^p$ runs over compact open subgroups with $K'_p = K_p$. The group $G(\mathbb{A}^p_f)$ of points in the ring of finite adeles with trivial p-component acts on the pro-scheme $\mathrm{Sh}_{K_p}(G, X)$. An integral canonical model is, in particular, an extension of the tower of E-schemes $\mathrm{Sh}_{K'}(G, X)$ with its $G(\mathbb{A}^p_f)$-action to a tower of *smooth* $\mathcal{O}_{(\lambda)}$ schemes with $G(\mathbb{A}^p_f)$-action.

On its own the condition in the previous paragraph is vacuous, since it is satisfied by the tower $\mathrm{Sh}_{K_p}(G, X)$ itself ! We need another condition which expresses the integrality of the extension. If the Shimura varieties $\mathrm{Sh}_{K'}(G, X)$ happen to be proper, then we can simply insist that the extension consist of a tower of proper $\mathcal{O}_{(\lambda)}$-schemes as in [La].

In the non-proper case Milne observed that one can still formulate an extension property by using the whole tower: Namely we can require that the tower $\mathrm{Sh}_{K_p}(G, X)$ satisfy the valuative criterion with respect to any discrete valuation ring R of mixed characteristic $(0, p)$. That is, any $R[1/p]$-valued point of the tower extends to an R-valued point.

We can see why this might be a reasonable definition if we look at the example of elliptic curves, and take for R the strict henselisation of $\mathbb{Z}_{(p)}$ in a fixed algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ (to check the valuative criterion one may always replace R by a strict henselisation). The moduli theoretic description of a modular curve gives rise to a natural integral model of it which is smooth at primes where K is maximal. An $R[1/p]$-valued point of the tower gives rise to an elliptic curve $\mathcal{E}$ over $R[1/p]$ together with a basis of the ℓ-adic Tate module $T_\ell\mathcal{E}$ where $\ell \neq p$. The action of $\mathrm{Gal}(\bar{\mathbb{Q}}/R[1/p])$ on $T_\ell\mathcal{E}$ is trivial, and so in particular the action of an inertia subgroup at p is trivial. This implies that $\mathcal{E}$ has good reduction, so the point extends to an R-valued point of the tower.

We have already explained most of the features of the precise conjecture. The only difference is that Milne's version of the extension property is formulated for a more general class of schemes than discrete valuation rings (see also [Mo, §3.5]). This has the effect that one is able to show that if the integral canonical model exists then it is unique.

We remark that a proof of the conjecture for Shimura varieties of abelian type is claimed in Vasiu's papers [Va 1], [Va 2], as well as the more recent [Va 4], [Va 5]. See also Moonen's article [Mo].

2.3. Examples. (1) Take $(G, X) = (\mathrm{GSp}, S^{\pm})$, the Siegel Shimura datum defined by the symplectic space (V, ψ) as above. If we fix the $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V$, then we may consider the tower $\mathrm{Sh}_{K_p}(G, X)$ where K_p is the maximal compact open subgroup which leaves $V_{\mathbb{Z}_p} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \subset V \otimes \mathbb{Q}_p$ stable.

K_p is hyperspecial if and only if ψ induces a perfect, $\mathbb{Z}_p$-valued pairing on $V_{\mathbb{Z}_p}$, in which case it may be identified with the $\mathbb{Z}_p$-points of the group of symplectic similitudes defined by $(V_{\mathbb{Z}_p}, \psi)$, which is a reductive group over $\mathbb{Z}_p$.

Any choice of $V_{\mathbb{Z}}$ gives rise to an interpretation of $\mathrm{Sh}_K(G, X)$ as a moduli space for polarized abelian varieties, and hence to a model $\mathscr{S}_K(G, X)$ for $\mathrm{Sh}_K(G, X)$ over the ring of integers of E. The $G(\mathbb{A}_f^p)$-action extends to this model, and it will satisfy the extension property for discrete valuation rings explained above, the argument being essentially the same as for the case of elliptic curves. [1]

However the varieties $\mathrm{Sh}_K(G, X)$ will be *smooth* over $\mathcal{O}_{(\lambda)}$ if and only if the degree of the polarization in the moduli problem is prime to p. This corresponds to the condition that ψ induces a perfect pairing on $V_{\mathbb{Z}_p}$.

(2) Another example is given by Hilbert modular varieties, for which $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ for F a totally real field. If K_p is maximal compact, then it is conjugate to $\prod_{\lambda|p} \mathrm{GL}_2(\mathcal{O}_{F,\lambda})$. This is hyperspecial if and only if F is unramified at p. The corresponding integral canonical models were constructed by Deligne-Pappas [DP], and are indeed smooth if and only if F is unramified at p.

[1] On the other hand Milne's stronger form of the extension property requires more work, even in the smooth case. See [Mo, 3.6] and the references therein.

2.4. Results.

Theorem 2.4.1. *Suppose that $p > 2$ and that (G, X) is of Hodge type. That is, there exists an embedding $(G, X) \hookrightarrow (\mathrm{GSp}, S^{\pm})$. Then for any hyperspecial subgroup $K_p \subset G(\mathbb{Q}_p)$, the tower $\mathrm{Sh}_{K_p}(G, X)$ admits a canonical integral model. The corresponding extension of $\mathscr{S}_{K_p}(G, X)$ is given by taking the normalization of the closure* [2] *of $\mathrm{Sh}_{K_p}(G, X)$ in $\mathscr{S}_{K'_p}(\mathrm{GSp}, S^{\pm})$ for a suitable choice of lattice $V_{\mathbb{Z}} \subset V$ and $K'_p \subset \mathrm{GSp}(\mathbb{Q}_p)$ the stabilizer of $V_{\mathbb{Z}_p}$.*

Using the theorem one can deduce the existence of integral canonical models in many more cases (cf. [De 1, 2.7.21]). To explain this write X^{ad} for the $G^{\mathrm{ad}}(\mathbb{R})$ conjugacy class in $\mathrm{Hom}(\mathbb{S}, G^{\mathrm{ad}}_{\mathbb{R}})$ which contains the image of X. Then $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is a Shimura datum.

Corollary 2.4.2. *Suppose that K_p is hyperspecial, and that there exists a Shimura datum (G_2, X_2) of Hodge type and a central isogeny $G_2^{\mathrm{der}} \to G^{\mathrm{der}}$ inducing an isomorphism $(G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}}) \xrightarrow{\sim} (G^{\mathrm{ad}}, X^{\mathrm{ad}})$. Then $\mathrm{Sh}_{K_p}(G, X)$ has an integral canonical model.*

2.5. Remarks. (1) Deligne has given an explicit description of the $\mathbb{Q}$-simple Shimura data which satisfy the condition of the corollary. They include the cases when G is of type A, B, C and certain cases of type D [De 1, 2.3.10].

(2) We do not know whether one can always take K' in the theorem so that K'_p is hyperspecial. Thus, the theorem asserts that the normalization of the closure of $\mathrm{Sh}_K(G, X)$ in a not necessarily smooth scheme is smooth.

(3) The condition $p > 2$ does not seem essential to the method. It is needed at two technical points, which we will try to indicate below. The first of these requires this condition only if G has a factor of type B, while for the second point one can allow $p = 2$ when one can choose the embedding $(G, X) \hookrightarrow (\mathrm{GSp}, S^{\pm})$ such that all abelian varieties in the image of $\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$ have connected p-divisible groups. There seems to be some hope of removing both these restrictions and allowing $p = 2$ in general.

3. Proof of the results

3.1. The set up. We put ourselves in the situation of the theorem. Let $\mathcal{G}$ be a reductive group over $\mathbb{Z}_p$, which extends G and such that $K_p = \mathcal{G}(\mathbb{Z}_p)$.

If $V_{\mathbb{Z}_p} \subset V_{\mathbb{Q}_p}$ is a $\mathbb{Z}_p$-lattice, denote by $\mathrm{GL}(V_{\mathbb{Z}_p})$ the $\mathbb{Z}_p$-group scheme of automorphisms of $V_{\mathbb{Z}_p}$. If $V_{\mathbb{Z}_p}$ is stable by K_p, then we obviously have

$$K_p \subset \mathrm{GL}(V_{\mathbb{Z}_p})(\mathbb{Z}_p) = \mathrm{Aut}_{\mathbb{Z}_p} V_{\mathbb{Z}_p}.$$

[2] In a previous version of these notes we claimed that it was unnecessary to take the normalization. We are grateful to Ofer Gabber for explaining why this is indeed necessary.

Unfortunately this does not quite imply that $G \hookrightarrow \mathrm{GSp}(V,\psi) \subset \mathrm{GL}(V)$ extends to an embedding $\mathcal{G} \subset \mathrm{GL}(V_{\mathbb{Z}_p})$. However one can choose $V_{\mathbb{Z}_p}$ so that we do in fact have such embedding. If G has a factor of type B, then we use here that $p > 2$, (cf. [PY, Cor. 1.3], [Va 2, 3.1.2.1]).

Choose a $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V_{\mathbb{Q}}$ such that $V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p = V_{\mathbb{Z}_p}$. We will consider $K' = \prod K'_\ell \subset \mathrm{GSp}(\mathbb{A}_f)$ compact open, so that $K \subset K'$, K' leaves $V_{\mathbb{Z}} \otimes \prod_\ell \mathbb{Z}_\ell$ stable, and K'_p is the stabilizer of $V_{\mathbb{Z}_p}$ in $\mathrm{GSp}(\mathbb{Q}_p)$. The moduli theoretic interpretation of $\mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$ then gives rise to a natural extension of this scheme to a $\mathbb{Z}_{(p)}$-scheme $\mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})$.

The compact open subgroups K and K' can be so arranged that $i : \mathrm{Sh}_K(G,X) \to \mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$ is a closed embedding (cf. [De 1, 1.15]). We assume this and write $\mathscr{S}_K(G,X)$ for the normalization of the closure of $\mathrm{Sh}_K(G,X)$ in $\mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})$.

Since the tower $\varprojlim_{L'} \mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})$ with $L'_p = K'_p$ fixed, is an integral canonical model for $\mathrm{Sh}_{K'_p}(\mathrm{GSp}, S^{\pm})$, one sees that the tower $\varprojlim_L \mathrm{Sh}_K(G,X)$ with $L_p = K_p$ has an action of $G(\mathbb{A}_f^p)$ and satisfies the extension property. (At least as far as it was explained in §2). We have to show that $\mathscr{S}_K(G,X)$ is smooth over primes $\lambda | p$. We will describe it in a formal neighbourhood of a closed point $\bar{x}$ of its mod λ fibre.

3.2. The key lemma. Let $x \in \mathrm{Sh}_K(G,X)$ be a closed point with residue field $\kappa(x)$, specializing to $\bar{x}$ in the mod λ fibre of $\mathscr{S}_K(G,X)$, and $\mathcal{A}_x$ the corresponding abelian variety over $\kappa(x)$. We will again denote by λ an extension of our chosen place of E to $\kappa(x)$. The p-adic Tate module $T_p\mathcal{A}_x$ may be canonically identified with $V_{\mathbb{Z}_p}$ up to the action of K_p on $V_{\mathbb{Z}_p}$. Fix such an identification. Deligne's theorem that a Hodge cycle on $\mathcal{A}_x$ is absolutely Hodge [De 2] implies that the restriction of the representation

$$\rho_x : G_x := \mathrm{Gal}(\bar{\kappa}(x)/\kappa(x)) \to \mathrm{Aut}_{\mathbb{Z}_p} T_p\mathcal{A}_x$$

to some open subgroup $G'_x \subset G_x$ factors through $G(\mathbb{Q}_p)$ (this also follows from the earlier results of Piatetski-Shapiro and Borovoi), and hence through $G(\mathbb{Q}_p) \cap \mathrm{Aut}_{\mathbb{Z}_p} V_{\mathbb{Z}_p} = K_p$. After extending the base field E we may assume that $G'_x = G_x$. More precisely, one can show that this extension can be chosen to be independent of x, and unramified at all primes $v|p$ of E. Note that we will construct our integral canonical model as the normalization of a closure, so it suffices to prove the smoothness after such an unramified base extension.

It will be convenient to make the following convention. If R is a ring and M is a free R-module we write $M^{\otimes}$ for the direct sum of all the R-modules formed from M by taking tensor products, symmetric powers, exterior powers and duals in all possible combinations. A collection of tensors $(s_\alpha) \subset M^{\otimes}$ defines a closed subgroup of $\mathrm{GL}(M)$, namely the closed subgroup which fixes each s_α. Note that $M^\otimes \xrightarrow{\sim} M^{*\otimes}$ so that a tensor in the left hand side may be regarded in the right hand side.

Note that the closure of G in $\mathrm{GL}(V_{\mathbb{Z}_{(p)}})$ (here $V_{\mathbb{Z}_{(p)}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$) is a group whose base change to $\mathbb{Z}_p$ is $\mathcal{G}$. In particular, this group is reductive, and one can show that it is defined by a family of tensors $(s_\alpha) \subset V_{\mathbb{Z}_{(p)}}^\otimes$. Since the action of G_x on $T_p\mathcal{A}_x \xrightarrow{\sim} V_{\mathbb{Z}_p}$ factors through $K_p = \mathcal{G}(\mathbb{Z}_p)$, the s_α are all Galois invariant.

Let $\mathbb{D}_x$ be the Dieudonne module of $\mathcal{A}_x$. If $\kappa(\bar{x})$ denotes the residue field of $\bar{x}$, then $\mathbb{D}_x$ is a finite free $W(\kappa(\bar{x}))$-module. Let $G_{x,\lambda} \subset G_x$ denote a decomposition group at λ.

The p-adic comparison isomorphism is a canonical isomorphism

$$V_{\mathbb{Z}_p}^* \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}} \xrightarrow{\sim} \mathbb{D}_x \otimes B_{\mathrm{cris}}$$

which is compatible with the action of $G_{x,\lambda}$ and φ on the two sides, as well as with filtrations after tensoring by $\kappa(x)_\lambda$. Since the s_α are $G_{x,\lambda}$ invariant, they are in $\mathbb{D}_x^\otimes \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. when regarded as elements of the right hand side of this isomorphism. They are of course invariant by φ and lie in $\mathrm{Fil}^0(\mathbb{D}_x^\otimes \otimes_{W(\kappa(\bar{x}))} \kappa(x)_\lambda)$.

Lemma 3.2.1 (Key Lemma). $(s_\alpha) \subset \mathbb{D}_x^\otimes$ *(not just after inverting p) and this collection of tensors defines a reductive subgroup of* $\mathrm{GL}(\mathbb{D}_x)$.

3.3. From the key lemma to integral canonical models. It was known to experts that a statement like the key lemma should allow the construction of integral canonical models for Shimura varieties of Hodge type. The point is that using the collection of tensors in the lemma, one can define a deformation problem, which describes the complete local ring of $\mathrm{Sh}_K(G,X)$ at $\bar{x}$. On the other hand, the fact that the group defined by the (s_α) is reductive implies that this deformation problem is smooth. We sketch the argument here (cf. [Va 2, §5] and [Mo, 5.8]).

Let H be the p-divisible group of the mod p reduction $\mathcal{A}_{\bar{x}}$ of $\mathcal{A}_x$. This is the p-divisible group over $\kappa(\bar{x})$ attached to $\mathbb{D}_x$. Let $\hat{U} = \mathrm{Spf}\, R$ be the versal deformation space of H, and $\hat{H}$ a versal p-divisible group over $\hat{U}$. Then $\hat{U}$ is formally smooth, and we fix a lift of Frobenius on R.

The Lie algebra of the universal vector extension of $\hat{H}$ gives rise to a vector bundle $\mathbb{D}(\hat{H})$ over $\hat{U}$, which is equipped with a a connection, an action of Frobenius and a filtration. There is a closed, formally smooth, formal subscheme $\hat{U}_{\mathcal{G}} \subset U$ over which the tensors (s_α) extend to parallel, Frobenius invariant sections of $\mathrm{Fil}^0(\mathbb{D}(\hat{H})^\otimes)$. This is where we use the fact that $\mathcal{G}$ is reductive (cf. [Mo, §4]).

On the other hand, the composite of the inclusions $G \subset \mathrm{GSp}(V_{\mathbb{Q}}, \psi) \subset \mathrm{GL}(V_{\mathbb{Q}})$ of reductive groups over $\mathbb{Q}$ is defined by the tensors $s_\alpha \in V_{\mathbb{Z}_{(p)}}^\otimes \subset V_{\mathbb{Q}}^\otimes$. Let $\mathcal{V}$ denote the de Rham cohomology of the universal polarized abelian variety over $\mathrm{Sh}_K(G,X) \subset \mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$. Then the s_α define tensors in the de Rham cohomology of the universal abelian variety over $S^{\pm}$, and since they are invariant by $G(\mathbb{Q})$, their restrictions to X descend to $s_{\alpha,\mathrm{dR}} \in \mathrm{Fil}^0(\mathcal{V}^{\otimes})$.

Let $\hat{M}$ be the formal neighbourhood of $\bar{x} \in \mathscr{S}_K(G,X)$, and let $\hat{N} \subset \hat{M}$, be the irreducible component which contains the point x. One can show that the restrictions of $s_{\alpha,\mathrm{dR}}$ to $\hat{N}$ define parallel, Frobenius invariant sections in $\mathrm{Fil}^0(\mathbb{D}(\hat{H})^{\otimes})$, which extend $s_\alpha \in \mathbb{D}_x$. The fact that these sections are Frobenius invariant follows from a result of Blasius and Wintenberger [Mo, 5.6.3].

It follows that a map $\hat{N} \to \hat{U}$ which induces the p-divisible group of the abelian variety over $\hat{N}$ factors through $\hat{U}_{\mathcal{G}}$, so we obtain a map $\varepsilon : \hat{N} \to \hat{U}_{\mathcal{G}}$. This map is a closed embedding, because a deformation of H determines a deformation of $\mathcal{A}_{\bar{x}}$, and if the given polarization of $\mathcal{A}_{\bar{x}}$ lifts to this deformation, then it does so uniquely. On the other hand one can check that $\hat{\mathcal{N}}$ and $\hat{U}_{\mathcal{G}}$ have the same dimension. Since $\hat{U}_{\mathcal{G}}$ is formally smooth, we find that ε is an isomorphism.

4. Proof of the key lemma

4.1. Classification of crystalline representations. We recall some of the results of [Ki 1] regarding the classification of crystalline representations and p-divisible groups.

Let k be a perfect field of characteristic p, $W = W(k)$ its ring of Witt vectors and $K_0 = W(k)[1/p]$. Let K be a finite totally ramified extension of K_0, and $\mathcal{O}_K$ its ring of integers. Fix an algebraic closure $\bar{K}$ of K, and set $G_K = \mathrm{Gal}(\bar{K}/K)$.

We denote by $\mathrm{Rep}_{G_K}^{\mathrm{cris}}$ the category of crystalline G_K-representations, and by $\mathrm{Rep}_{G_K}^{\mathrm{cris}\circ}$ the category of G_K-stable lattices which span a representation in $\mathrm{Rep}_{G_K}^{\mathrm{cris}}$. For V a crystalline representation recall Fontaine's functors

$$D_{\mathrm{cris}}(V) = (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K} \quad \textit{and} \quad D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

Fix a uniformiser $\pi \in K$, and let $E(u) \in W(k)[u]$ be the Eisenstein polynomial for π. We set $\mathfrak{S} = W[\![u]\!]$ equipped with a Frobenius φ which acts as the usual Frobenius on W and sends u to u^p.

Let $\mathrm{Mod}_{/\mathfrak{S}}^{\varphi}$ denote the category of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a Frobenius semi-linear isomorphism

$$\varphi^*(\mathfrak{M})[1/E(u)] \xrightarrow{\sim} \mathfrak{M}[1/E(u)].$$

Note that this definition differs slightly from that of [Ki 1], where we insisted that the above map be induced by a map $\varphi^*(\mathfrak{M}) \to \mathfrak{M}$. This is related to the fact that in *loc. cit* we considered crystalline representations with non-negative Hodge-Tate weights, whereas here we will allow arbitrary Hodge-Tate weights.

Let $\mathcal{O}_{\mathcal{E}}$ denote the p-adic completion of $\mathfrak{S}_{(p)}$. If $\mathfrak{M}$ is in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ then $M = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ is a finite free $\mathcal{O}_{\mathcal{E}}$-module equipped with an isomorphism $\varphi^*(M) \xrightarrow{\sim} M$.

Using [Ki 1] one easily deduces the following

Theorem 4.1.1. *There exists a fully faithful tensor functor*

$$\mathfrak{M} : \mathrm{Rep}^{\mathrm{criso}}_{G_K} \to \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}.$$

If L is in $\mathrm{Rep}^{\mathrm{criso}}_{G_K}$, $V = L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and $\mathfrak{M} = \mathfrak{M}(L)$, then

- *There are canonical isomorphisms*

  $$D_{\mathrm{cris}}(V) \xrightarrow{\sim} \varphi^*(\mathfrak{M}/u\mathfrak{M})[1/p] \quad \text{and} \quad D_{\mathrm{dR}}(V) \xrightarrow{\sim} \mathfrak{M} \otimes_{\mathfrak{S}} K.$$

  Here the first isomorphism is compatible with Frobenius and in the second the map $\mathfrak{S} \to K$ is given by sending $u \mapsto \pi$.
- *There is a canonical isomorphism*

  $$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbb{Z}_p} L \xrightarrow{\sim} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{S}} \mathfrak{M},$$

  where $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ is a certain faithfully flat, and formally étale $\mathcal{O}_{\mathcal{E}}$-algebra.
- *If G is a p-divisible group over $\mathcal{O}_K$ and $L = (T_pG)^*$ is the dual of its p-adic Tate module, then $\mathfrak{M}$ is stable by φ, and there is a canonical φ-equivariant isomorphism*

  $$\mathbb{D}(G) \xrightarrow{\sim} \varphi^*(\mathfrak{M}/u\mathfrak{M}),$$

  provided either $p > 2$, or G is connected.

We remark that the condition when $p = 2$ in the final statement in the theorem is one of the two points which force us to assume that $p > 2$. When $p = 2$ and G is connected the final statement follows from [Ki 2].

4.2. Reductive groups. We now prove the key lemma. We will apply the above theory with $K = \kappa(x)_\lambda$ and $L = V^*_{\mathbb{Z}_p}$. Recall that $\mathcal{G} \subset \mathrm{GL}(V^*_{\mathbb{Z}_p})$ denotes the reductive group defined by (s_α). We may view the tensors s_α as morphisms $s_\alpha : \mathbf{1} \to L^\otimes$ in $\mathrm{Rep}^{\mathrm{criso}}_{G_K}$. Applying the functor $\mathfrak{M}$ of the theorem, we obtain morphisms $\tilde{s}_\alpha : \mathbf{1} \to \mathfrak{M}^\otimes$ in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$.

Note that the theorem immediately implies the first part of the key lemma, since specializing the $(\tilde{s}_\alpha)$ at $u = 0$ produces the tensors $s_\alpha \in \mathbb{D}_x[1/p]^\otimes$, which lie in $\mathbb{D}_x^\otimes \xrightarrow{\sim} \varphi^*(\mathfrak{M}/u\mathfrak{M})^\otimes$ by construction.

We will show that the $(\tilde{s}_\alpha)$, define a reductive subgroup of $\mathrm{GL}(\mathfrak{M})$. This will complete the proof of the theorem. Denote this closed subgroup by $\mathcal{G}_{\mathfrak{S}} \subset \mathrm{GL}(\mathfrak{M})$. It suffices to prove the statement after making the faithfully flat, formally étale base extension $\mathfrak{S} \to W(k^{\mathrm{sep}})[\![u]\!]$. Hence we will assume from now on that the residue field k is separably closed.

In fact we will prove the following stronger statement.

Proposition 4.2.1. *Let L be in $\mathrm{Rep}_{G_K}^{\mathrm{criso}}$ and $\mathfrak{M} = \mathfrak{M}(L)$. Let $(s_\alpha) \subset L^{\otimes}$ be a finite collection of G_K-invariant tensors defining a reductive $\mathbb{Z}_p$-subgroup $\mathcal{G}$ of $\mathrm{GL}(L)$, and let $(\tilde{s}_\alpha)$ be the corresponding tensors in $\mathfrak{M}^{\otimes}$.*

Let $\mathfrak{M}' = L \otimes_W \mathfrak{S}$. If k is separably closed, there is an isomorphism $\mathfrak{M} \xrightarrow{\sim} \mathfrak{M}'$ which takes each tensor $\tilde{s}_\alpha$ to s_α. In particular, the subgroup $\mathcal{G}_{\mathfrak{S}} \subset \mathrm{GL}(\mathfrak{M})$ defined by $(\tilde{s}_\alpha)$ is isomorphic to $\mathcal{G} \times_{\mathrm{Spec}\,\mathbb{Z}_p} \mathrm{Spec}\,\mathfrak{S}$.

Proof. Let $P \subset \underline{\mathrm{Hom}}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{M}')$ be the subscheme of isomorphisms between $\mathfrak{M}$ and $\mathfrak{M}'$ which take $\tilde{s}_\alpha$ to s_α. The fibres of P are either empty or a torsor under $\mathcal{G}$. We claim that P is a $\mathcal{G}$-torsor. That is, P is flat over $\mathfrak{S}$ with non-empty fibres. The claim implies the proposition since a torsor under a reductive group is étale locally trivial, while the ring $\mathfrak{S}$ is strictly henselian, so any $\mathcal{G}$ torsor over $\mathfrak{S}$ is trivial.

To prove the claim we proceed in several steps. For R a $\mathfrak{S}$-algebra, we set $P_R = P \times_{\mathrm{Spec}\,\mathfrak{S}} \mathrm{Spec}\,R$.

Step 1: $P_{\mathfrak{S}_{(p)}}$ is a $\mathcal{G}$-torsor. Since $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ is faithfully flat over $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{O}_{\mathcal{E}}$ is faithfully flat over $\mathfrak{S}_{(p)}$, it suffices to show that $P_{\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}}$ is a $\mathcal{G}$-torsor. However the isomorphism in (2) of the theorem in 4.1 shows that $P_{\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}}$ is a trivial $\mathcal{G}$-torsor.

Step 2: P_{K_0} is a $\mathcal{G}$-torsor, where we regard K_0 as a $\mathfrak{S}$-algebra via $u \mapsto 0$. This follows from (1) of the theorem in 4.1, which implies the existence of a canonical isomorphism

$$B_{\mathrm{dR}} \otimes L \xrightarrow{\sim} B_{\mathrm{dR}} \otimes \mathfrak{M}/u\mathfrak{M}.$$

Step 3: $P_{\mathfrak{S}[1/pu]}$ is a $\mathcal{G}$-torsor. Let $U \subset \mathrm{Spec}\,\mathfrak{S}[1/up]$ denote the maximal open subset over which P is flat with non-empty fibres. By Step 1, we know this subset is non-empty, since it contains the generic point. In particular, the complement of U in $\mathrm{Spec}\,\mathfrak{S}[1/up]$ contains finitely many closed points.

Let $x \in \mathrm{Spec}\,\mathfrak{S}[1/up]$ be a closed point. If $x \notin U$, we consider two cases. If $|u(x)| < |\pi|$, then since the s_α are Frobenius invariant, we have $P_{\mathfrak{S}[1/p]} \xrightarrow{\sim} \varphi^*(P_{\mathfrak{S}[1/p]})$ in a formal neighbourhood of x. Hence $P_{\mathfrak{S}}[1/p]$ cannot be a $\mathcal{G}$-torsor at $\varphi(x)$, since φ is a faithfully flat map on $\mathfrak{S}$. Repeating the argument we find $\varphi(x), \varphi^2(x), \ldots \notin U$, which gives a contradiction.

Similarly, if $|u(x)| \geq |\pi|$ consider a sequence of points $x_0, x_1, \ldots$ with $x_0 = x$, and $\varphi(x_{i+1}) = x_i$. For $i \geq 1$, we have $P_{\mathfrak{S}[1/p]} \xrightarrow{\sim} \varphi^*(P_{\mathfrak{S}[1/p]})$ in a formal neighbourhood of x_i, so we find that $x_i \notin U$ for $i \geq 1$.

Step 4: $P_{\mathfrak{S}[1/p]}$ is a $\mathcal{G}$-torsor. By Step 3, it suffices to show that the restriction of P to $K_0[\![u]\!]$ is a $\mathcal{G}$-torsor. For any $\mathfrak{N}$ in $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ there is a unique,

φ-equivariant isomorphism

$$\mathfrak{N} \otimes_{\mathfrak{S}} K_0[\![u]\!] \xrightarrow{\sim} K_0[\![u]\!] \otimes_{K_0} \mathfrak{N}/u\mathfrak{N}[1/p]$$

lifting the identity map on $\mathfrak{N}/u\mathfrak{N} \otimes_{\mathcal{O}_{K_0}} K_0$, which is functorial in $\mathfrak{N}$ (see, for example, [Ki 1, 1.2.6]). Applying this to $\mathfrak{M}$ and the morphisms $\tilde{s}_\alpha$ shows that the restriction of P to $K_0[\![u]\!]$ is isomorphic to $P_{K_0} \otimes_{K_0} K_0[\![u]\!]$, which is a $\mathcal{G}$-torsor by Step 2.

Step 5: P is a $\mathcal{G}$-torsor. Let U be the complement of the closed point in $\operatorname{Spec} \mathfrak{S}$. By Steps 1 and 4 we know that $P|_U$ is a $\mathcal{G}$-torsor. By a result of Colliot-Thélène and Sansuc [CS, Thm. 6.13] P extends to a $\mathcal{G}$-torsor over $\mathfrak{S}$ and, as we remarked above, any such torsor is trivial. Hence $P|_U$ is trivial, and there is an isomorphism $\mathfrak{M}|_U \xrightarrow{\sim} \mathfrak{M}'|_U$ taking $\tilde{s}_\alpha$ to s_α. Since any vector bundle over U has a canonical extension to $\mathfrak{S}$, obtained by taking its global sections, this isomorphism extends to $\mathfrak{S}$. This implies that P is the trivial $\mathcal{G}$-torsor, and completes the proof of the proposition and of the key lemma. □

Note that the proof of the proposition implies the following result.

Corollary 4.2.2. *Suppose that k is separably closed or finite. Then for any $i \geq 0$ there is an isomorphism $L \otimes_{\mathbb{Z}_p} W \xrightarrow{\sim} \varphi^{i*}(\mathfrak{M}/u\mathfrak{M})$ which takes $s_\alpha \in L$ to the corresponding tensor in $\varphi^{i*}(\mathfrak{M}/u\mathfrak{M}) \subset D_{\mathrm{cris}}(L \otimes \mathbb{Q}_p)$.*

Proof. Define P as in the proof of the proposition (without assuming k separably closed). Then we saw that P is a $\mathcal{G}$-torsor. If k is separably closed or finite a torsor under a reductive W-group is necessarily trivial [Sp, 4.4]. Hence P is trivial, as is $\varphi^{i*}(P)$ for $i \geq 0$. □

When L is dual to the Tate module of a p-divisible group and $\varphi^*(\mathfrak{M}/u\mathfrak{M})$ is replaced by the Dieudonne module of the p-divisible group, then the corollary is a conjecture of Milne [Mi 3]. When $p > 2$ or the p-divisible group is connected the conjecture follows from the corollary and theorem of 4.1. Vasiu [Va 3] has also claimed a proof of the conjecture.

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References

[CS] J-L. Colliot-Thélène, J-J Sansuc, *Fibrés quadratiques et composantes connexes réelles*. Math. Ann. **244** (1979), 105–134.

[De 1] P. Deligne, *Variètés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*. Automorphic forms, representations and L-functions (Corvallis 1977), Proc. Sympos. Pure Math XXXIII, 247–289, AMS, 1979.

[De 2] P. Deligne, *Hodge cycles on abelian varieties* Hodge cycles motives and Shimura varieties, Lecture notes in math. **900**, 9–100, Springer, 1982.

[DP] P. DELIGNE, G. PAPPAS, *Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant*. Compositio Math. **90** (1994), 59–79.

[Ki 1] M. KISIN, *Crystalline representations and F-crystals*. Algebraic geometry and number theory, Progr. Math **253**, 459–496, Birkhäuser, Boston, 2006.

[Ki 2] M. KISIN, *Modularity of* 2*-adic Barsotti-Tate representations*. Preprint 2007.

[Ki 3] M. KISIN, *Integral models for Shimura varieties of abelian type*. Preprint, 2008.

[La] R. LANGLANDS, *Some contemporary problems with origins in the Jugendtraum*. Mathematical developments arising from Hilbert problems (De Kalb 1974), Proc. Sympos. Pure Math. XXVIII, 401–418, AMS, 1976.

[LR] R. LANGLANDS, M. RAPOPORT, *Shimuravarietäten und Gerben*. J. Reine Angew. Math **378** (1987), 113–220.

[Mi 1] J. MILNE, *The points on a Shimura variety modulo a prime of good reduction*. The zeta functions of Picard modular surfaces, 151–253, CRM 1992.

[Mi 2] J. MILNE, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*. Automorphic forms, Shimura varieties. and L-functions I (Ann Arbor 1988), Perspectives in Math. **10**, 284–414, Academic Press, 1990.

[Mi 3] J. MILNE, *On the conjecture of Langlands and Rapoport*. Available at **arxiv.org**, 1995, 31 pages.

[Mo] B. MOONEN, *Models of Shimura varieties in mixed characteristics*. Galois representations in arithmetic algebraic geometry (Durham, 1996), London Math. Soc. Lecture Note Ser. **254**, 257–350, CUP 1998.

[Sp] T. SPRINGER, *Reductive groups*. Automorphic forms, representations and L-functions (Corvallis 1977), Proc. Sympos. Pure Math XXXIII, 3–27, AMS 1979.

[PY] G. PRASAD; J-K. YU, *On quasi-reductive group schemes. With an appendix by Brian Conrad*. J. Algebraic Geom. **15** (2006), 507–549.

[Va 1] A. VASIU, *Integral Canonical Models of Shimura Varieties of Preabelian Type*. Asian J. Math, 1999, 401–518.

[Va 2] A. VASIU, *Integral Canonical Models of Shimura Varieties of Preabelian Type* (Fully corrected version). Available at **arxiv.org**, 2003, 135 pages.

[Va 3] A. VASIU, *A motivic conjecture of Milne*. Available at **arxiv.org**, 2003, 46 pages.

[Va 4] A. VASIU, *Good reduction of Shimura varieties in arbitrary unramified mixed characteristic I*. Available at **arxiv.org**, 2007, 48 pages.

[Va 5] A. VASIU, *Good reduction of Shimura varieties in arbitrary unramified mixed characteristic II*. Available at **arxiv.org**, 2007.

Mark KISIN
Department of Mathematics
University of Chicago
5734 S. University Avenue
Chicago, IL, 60637, USA
E-mail: `kisin@math.uchicago.edu`