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On the ternary Goldbach problem with primes in arithmetic progressions having a common modulus

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Résumé. Pour $\varepsilon > 0$ et n impair suffisamment grand, nous montrons que, pour presque tout $k \leq R := n^{1/5-\varepsilon}$, il existe une représentation $n = p_1 + p_2 + p_3$ avec des nombres premiers $p_i \equiv b_i$ modulo k pour presque tout triplet admissible b_1, b_2, b_3 de résidus modulo k.

Abstract. For $\varepsilon > 0$ and any sufficiently large odd n we show that for almost all $k \leq R := n^{1/5-\varepsilon}$ there exists a representation $n = p_1 + p_2 + p_3$ with primes $p_i \equiv b_i$ mod k for almost all admissible triplets b_1, b_2, b_3 of reduced residues mod k.

1. Introduction and results

Let n be a sufficiently large integer, consider an integer k and let b_1, b_2, b_3 be integers that are relatively prime to $k \geq 1$, with $0 \leq b_i < k$, $i = 1, 2, 3$.

We consider the ternary Goldbach problem of writing n as

$$n = p_1 + p_2 + p_3$$

with primes p_1, p_2 and p_3 satisfying the three congruences

$$p_i \equiv b_i \mod k, \ i = 1, 2, 3$$

for the common modulus k. One is interested in the solvability of this question for all sufficiently large n with the modulus k being as large as some power of n. This problem has been studied intensely by many authors. For an overview, see for example [2].

An obvious necessary condition for solvability is

$$n \equiv b_1 + b_2 + b_3 \mod k,$$

as otherwise no such representation of n is possible.

We call such a triplet b_1, b_2, b_3 of reduced residues mod k *admissible*, and a pair b_1, b_2 of reduced residues *admissible*, if $(n - b_1 - b_2, k) = 1$. For a given b_1 we call b_2 *admissible*, if b_1, b_2 is an admissible pair. Let us denote the number of these admissible pairs respectively triplets by $A(k)$.

We make our consideration of this strengthened ternary Goldbach problem more precise in the following way. Let

$$J_3(n) := J_{k,b_1,b_2,b_3}(n) := \sum_{\substack{m_1,m_2,m_3 \leq n \\ m_1+m_2+m_3=n \\ m_i \equiv b_i\ (k), \\ i=1,2,3}} \Lambda(m_1)\,\Lambda(m_2)\,\Lambda(m_3),$$

where Λ is von Mangoldt's function. The function $J_3(n)$ corresponds closely to the number of representations of n of the kind we are interested in.

In this paper we prove that the deviation of $J_3(n)$ from its expected main term is uniformly small for large moduli, namely in the following sense.

Theorem 1.1. *For every $A, \varepsilon > 0$, every sufficiently large n and for $D \leq n^{1/5-\varepsilon}$ we have*

$$\mathcal{E} := \sum_{D < k \leq 2D} \frac{k}{\varphi(k)} \sum_{(b_1,k)=1} \frac{1}{\varphi(k)} \sum_{b_2\ adm.} \left| J_3(n) - \frac{n^2}{k^2}\mathcal{S}(n,k) \right| \ll \frac{n^2}{(\log n)^A}.$$

Here $\mathcal{S}(n,k)$ denotes the singular series for this special Goldbach problem. In depends on k, while $J_3(n)$ depends on k, b_1, b_2; residue b_3 is simply $b_3 \equiv n - b_1 - b_2\ (k)$. Namely, see [4], for odd n we have

$$\mathcal{S}(n,k) = C(k) \prod_{p|k} \frac{p^3}{(p-1)^3+1} \prod_{\substack{p|n \\ p\nmid k}} \frac{(p-1)((p-1)^2-1)}{(p-1)^3+1} \prod_{p>2} \left(1 + \frac{1}{(p-1)^3}\right),$$

where $p > 2$ throughout, $C(k) = 2$ for odd k and $C(k) = 8$ for even k.

As a consequence of Theorem 1.1, we prove in section 2 the following result.

Theorem 1.2. *Let $A, \varepsilon > 0$ and let $n \in \mathbb{N}$ be odd and sufficiently large. Then for all $k \leq R := n^{1/5-\varepsilon}$ with at most $\ll R(\log n)^{-A}$ exceptions there exists a representation $n = p_1 + p_2 + p_3$ with primes $p_i \equiv b_i\ (k)$ for all but $\ll A(k)(\log n)^{-A}$ many admissible triplets b_1, b_2, b_3.*

So there are few exceptions for k, and also the number of exceptions of admissible triplets is small compared with the number $A(k)$ of all admissible triplets.

Let us compare this Theorem 1.2 with the result of J. Liu and T. Zhang in [4] who show the assertion for $R := n^{1/8-\varepsilon}$ and *all* admissible triplets. In another paper [3], Z. Cui improved this to $R := n^{1/6-\varepsilon}$. Further C. Bauer and Y. Wang showed in [2] the assertion for $R := n^{5/48-\varepsilon}$, but with only $\ll (\log n)^B$ many exceptions.

Here we improved the bound for R again, but at the cost of possibly a few exceptions of admissible triplets.

2. Proof of Theorem 1.2

First of all we give a lower bound for $A(k)$:

Lemma 2.1. *For odd n we have $A(k) \gg \frac{\varphi(k)^2}{(\log k)^3}$. More precisely, for every reduced residue b_1 mod k there are $\gg \frac{\varphi(k)}{(\log k)^3}$ many reduced residues b_2 mod k with $(n - b_1 - b_2, k) = 1$.*

Proof. Fix a reduced residue b_1 mod k. Now count the b_2 with $(b_2, k) = (n - b_1 - b_2, k) = 1$. So b_2 is to be chosen such that for all prime divisors $p > 2$ of k we have $b_2 \not\equiv 0 \ (p)$ and $b_2 \not\equiv n - b_1 \ (p)$. This gives $\geq p - 2$ possibilities for b_2 mod p, and $\geq p^{l-1}(p-2)$ possiblilities for b_2 mod p^l. If $p = 2$ for even k we have an odd b_1, so $n - b_1$ is even and therefore one can take $b_2 \equiv 1(2)$, so there are $2^{\nu_2(k)-1}$ possibilities for b_2 mod $2^{\nu_2(k)}$, if $2^{\nu_2(k)}||k$.

Therefore the number of b_2 is at least

$$2^{\max\{0,\nu_2(k)-1\}} \prod_{\substack{p^l||k \\ p\neq 2}} p^{l-1}(p-2) = \varphi(k) \prod_{\substack{p|k \\ p\neq 2}} \frac{p-2}{p-1}$$

with

$$\prod_{\substack{p|k \\ p\neq 2}} \frac{p-1}{p-2} = \prod_{\substack{p|k \\ p\neq 2}} \left(1 + \frac{1}{p-2}\right) \leq \prod_{p|k} \left(1 + \frac{2}{p-1}\right)$$

$$\leq \sum_{q=1}^{k} \frac{\mu(q)^2 \, 2^{\omega(q)}}{\varphi(q)} \ll \sum_{q=1}^{k} \frac{\tau(q)}{q} \log k \ll (\log k)^3.$$

$\square$

Now we deduce Theorem 1.2 as a corollary of Theorem 1.1.
Fix $A, \varepsilon > 0$ and let n be odd and sufficiently large. Consider

$$R_3(n) := \sum_{\substack{p_1,p_2,p_3 \\ p_1+p_2+p_3=n \\ p_i\equiv b_i(k), \\ i=1,2,3}} \log p_1 \log p_2 \log p_3 \quad \text{and} \quad r_3(n) := \sum_{\substack{p_1,p_2,p_3 \\ p_1+p_2+p_3=n \\ p_i\equiv b_i(k), \\ i=1,2,3}} 1.$$

Let $D < k \leq 2D$ with $D \leq R := n^{1/5-\varepsilon}$. For any admissible triplet b_1, b_2, b_3 mod k we have

$$|R_3(n) - J_3(n)| \leq (\log n)^3 W_3,$$

where W_3 denotes the number of solutions of $p^l + q^j + r^m = n$ with p, q, r prime, and where l, j or m are at least 2 such that $p^l \equiv b_1 \ (k)$, $q^j \equiv b_2 \ (k)$ and $r^m \equiv b_3 \ (k)$.

Now we prove that

$$\sum_{D<k\leq 2D} k \max_{\substack{b_1,b_2,b_3 \\ \text{admissible}}} W_3 \ll \frac{n^2}{(\log n)^{A+3}}.$$

For this, we split the number W_3 according to whether at least two of the exponents l, j, m are ≥ 2 or only one is, and for this we write $W_3 = W_1 + W_2$. There are at most $\sqrt{n}$ prime powers $\leq n$ with exponent ≥ 2, so in the first case we have $W_1 \ll n$, and the sum with W_1 replacing W_3 is $\ll D^2 W_1 \ll D^2 n \ll \frac{n^2}{(\log n)^{A+3}}$.

In the second case, if only one exponent is ≥ 2, we have $W_2 \ll \sqrt{n} \cdot \frac{n}{k} = \frac{n^{3/2}}{k}$, and so the sum with W_2 replacing W_3 is $\ll Dn^{3/2} \ll \frac{n^2}{(\log n)^{A+3}}$.

So for $D \leq n^{1/5-\varepsilon}$ it follows from Theorem 1.1 that

$$\sum_{D<k\leq 2D} \frac{k}{\varphi(k)} \sum_{(b_1,k)=1} \frac{1}{\varphi(k)} \sum_{b_2 \text{ adm.}} \left| R_3(n) - \frac{n^2}{k^2}\mathcal{S}(n,k) \right|$$

$$\leq \sum_{D<k\leq 2D} \frac{k}{\varphi(k)} \sum_{(b_1,k)=1} \frac{1}{\varphi(k)} \sum_{b_2 \text{ adm.}} \left| R_3(n) - J_3(n) \right|$$

$$+ \sum_{D<k\leq 2D} \frac{k}{\varphi(k)} \sum_{(b_1,k)=1} \frac{1}{\varphi(k)} \sum_{b_2 \text{ adm.}} \left| J_3(n) - \frac{n^2}{k^2}\mathcal{S}(n,k) \right|$$

$$\ll (\log n)^3 \sum_{D<k\leq 2D} k \max_{\substack{b_1,b_2,b_3 \\ \text{admissible}}} W_3 + \frac{n^2}{(\log n)^A}$$

$$\ll \frac{n^2}{(\log n)^A}.$$

So the formula of Theorem 1.1 holds also with $R_3(n)$ replacing $J_3(n)$.

Now for $D < k \leq 2D$ we have $A(k) := \#\{b_1, b_2 \text{ admissible mod } k\}$. Let $T(k) := \#\{b_1, b_2 \text{ admissible mod } k; \ R_3(n) = 0\}$, consider the set

$$\mathcal{K}_D := \{k; \ D < k \leq 2D, T(k) \geq A(k)(\log n)^{-A}\}$$

and let K_D be its cardinality.

Since $\mathcal{S}(n,k) \gg 1$ if it is positive, which is the case for admissible triplets and odd n (see its formula above as an Euler product), we have

$$K_D \cdot \frac{n^2}{D} \ll \sum_{\substack{D < k \leq 2D \\ k \in \mathcal{K}_D}} \frac{k}{T(k)} \sum_{\substack{b_1, b_2 \text{ adm.} \\ R_3(n) = 0}} \left| \frac{n^2}{k^2} \mathcal{S}(n,k) \right|$$

$$\ll \sum_{D < k \leq 2D} \frac{k}{A(k)} \sum_{b_1, b_2 \text{ adm.}} (\log n)^A \left| R_3(n) - \frac{n^2}{k^2} \mathcal{S}(n,k) \right|$$

$$\ll (\log n)^{A+3} \sum_{D < k \leq 2D} \frac{k}{\varphi(k)^2} \sum_{\substack{b_1, b_2 \\ \text{adm.}}} \left| R_3(n) - \frac{n^2}{k^2} \mathcal{S}(n,k) \right| \ll \frac{n^2}{(\log n)^A},$$

using Lemma 2.1 and the above. Therefore it follows that $K_D \ll D(\log n)^{-A}$, so for all $k \notin \mathcal{K}_D$ we have $R_3(n) > 0$ for all but $\ll A(k)(\log n)^{-A}$ many admissible triplets b_1, b_2, b_3, and then $r_3(n) \gg R_3(n)(\log n)^{-3}$ is positive, too. This shows Theorem 1.2, since the overall number of exceptions is

$$\ll \sum_{D = 2^i \leq R} K_D \ll (\log n) \cdot \frac{R}{(\log n)^{A+1}} = \frac{R}{(\log n)^A}.$$

$\square$

3. Proof of Theorem 1.1

Our proof is in two steps using the circle method.

Let $A, \varepsilon, \theta > 0$, $B \geq 2A + 1$ and $D \leq n^{1/4}(\log n)^{-\theta}$.

We define major arcs $\mathfrak{M} \subseteq \mathbb{R}$ by

$$\mathfrak{M} := \bigcup_{q \leq D(\log n)^B} \bigcup_{\substack{0 < a < q \\ (a,q)=1}} \left] \frac{a}{q} - \frac{D(\log n)^B}{qn}, \frac{a}{q} + \frac{D(\log n)^B}{qn} \right[$$

and minor arcs by

$$\mathfrak{m} := \left] -\frac{D(\log n)^B}{n}, 1 - \frac{D(\log n)^B}{n} \right[\setminus \mathfrak{M}.$$

For $\alpha \in \mathbb{R}$ and some residue $b \bmod k$ denote

$$S_b(\alpha) := S_{b,k}(\alpha) := \sum_{\substack{m \leq n \\ m \equiv b\,(k)}} \Lambda(m)\, e(\alpha m).$$

From the orthogonal relations for $e(\alpha m)$ it follows that

$$J_3(n) = \int_0^1 S_{b_1}(\alpha) S_{b_2}(\alpha) S_{b_3}(\alpha)\, e(-n\alpha)\, d\alpha.$$

By

$$J_3^{\mathfrak{M}}(n) := \int_{\mathfrak{M}} S_{b_1}(\alpha) S_{b_2}(\alpha) S_{b_3}(\alpha)\, e(-n\alpha)\, d\alpha$$

we denote the value of the integral for $J_3(n)$ on the major arcs $\mathfrak{M}$ and by

$$J_3^{\mathfrak{m}}(n) := J_3(n) - J_3^{\mathfrak{M}}(n)$$

its value on the minor arcs $\mathfrak{m}$.

We first consider the major arcs.

Theorem 3.1. *For $D \leq n^{1/5-\varepsilon}$ we have*

$$\mathcal{E}^{\mathfrak{M}} := \sum_{D < k \leq 2D} k \max_{\substack{b_1, b_2, b_3 \\ admissible}} \left| J_3^{\mathfrak{M}}(n) - \frac{n^2}{k^2} \mathcal{S}(n, k) \right| \ll \frac{n^2}{(\log n)^A}.$$

We can give a very short proof of Theorem 3.1 simply by adapting the result of J. Liu and T. Zhang in [4] for the major arcs used here. In fact, by following their proof we see that for $P := D(\log n)^B$ and $Q := \frac{n}{D(\log n)^B}$ and any $U \leq P$, we have to choose D such that the conditions

$$U \leq n^{1/3}(\log n)^{-E}, \qquad\qquad (UQ)^{-1} \leq U^{-3}(\log n)^{-E}$$
$$DU \leq D^{1/3-\delta} n^{1/3}(\log n)^{-E}, \qquad (UQ)^{-1} \leq D^{1-\delta}(DU)^{-3}(\log n)^{-E}$$

are satisfied for any $E > 0$ and small $\delta > 0$. The optimal choice of D is therefore given by $D \leq n^{1/5-\varepsilon}$. This proves Theorem 3.1. The improvement in this paper comes from the different intervals used as major and minor arcs such that working on the minor arcs with mean values over b_1, b_2 is still possible.

Specifically for an estimate on the minor arcs, we show the following in the next section.

Theorem 3.2. *For $D \leq n^{1/4}(\log n)^{-\theta}$ we have*

$$\mathcal{E}^{\mathfrak{m}} := \sum_{D < k \leq 2D} \frac{k}{\varphi(k)} \sum_{(b_1, k)=1} \frac{1}{\varphi(k)} \sum_{\substack{(b_2, k)=1 \\ adm.}} |J_3^{\mathfrak{m}}(n)| \ll \frac{n^2}{(\log n)^A}.$$

Theorem 1.1 is then a corollary of Theorems 3.1 and 3.2 since $\mathcal{E} \leq \mathcal{E}^{\mathfrak{M}} + \mathcal{E}^{\mathfrak{m}}$.

This Theorem is the interesting part of Theorem 1.1, where we can attain a higher power of n for the bound of D by considering the mean value over b_1, b_2 instead of the maximum. But because of this we have to allow exceptions of admissible triplets in Theorem 1.2, as we have seen in its proof.

In both Theorems 3.1 and 3.2 the resulting bound for D is the optimum with the given method, so these bounds cannot be balanced to get a larger

range than $n^{1/5}$. Also the cited method for the major arcs cannot be improved by using mean values over b_1, b_2 since the character sum estimates used are independent of b_1, b_2.

4. Proof of Theorem 3.2: the estimate on the minor arcs

Let $D \leq n^{1/4}(\log n)^{-\theta}$ and let $\sum^*$ denote a sum over reduced residues mod k. We have

$$
\mathcal{E}^{\mathfrak{m}} \ll \sum_{D < k \leq 2D} \frac{k}{\varphi(k)^2} \sum_{b_1, b_2}^* |J_3^{\mathfrak{m}}(n)|
$$

$$
\leq \sum_{D < k \leq 2D} \frac{k}{\varphi(k)^2} \sum_{b_1, b_2}^* \int_{\mathfrak{m}} |S_{b_1}(\alpha) S_{b_2}(\alpha) S_{n-b_1-b_2}(\alpha)| \, d\alpha
$$

$$
= \sum_{D < k \leq 2D} \frac{k}{\varphi(k)} \sum_{b_1}^* \int_{\mathfrak{m}} |S_{b_1}(\alpha)| \cdot \frac{1}{\varphi(k)} \sum_{b_2}^* |S_{b_2}(\alpha) S_{n-b_1-b_2}(\alpha)| \, d\alpha
$$

$$
\leq \sum_{D < k \leq 2D} \frac{k}{\varphi(k)} \int_{\mathfrak{m}} \sum_{b_1}^* |S_{b_1}(\alpha)|
$$

$$
\cdot \frac{1}{\varphi(k)} \left(\sum_{b_2 \bmod k} |S_{b_2}(\alpha)|^2 \right)^{1/2} \left(\sum_{b_2 \bmod k} |S_{n-b_1-b_2}(\alpha)|^2 \right)^{1/2} d\alpha
$$

$$
\leq \sum_{D < k \leq 2D} \frac{k}{\varphi(k)} \max_{\alpha \in \mathfrak{m}} \sum_{b_1}^* |S_{b_1}(\alpha)| \frac{1}{\varphi(k)} \sum_{b_2 \bmod k} \int_0^1 |S_{b_2}(\beta)|^2 d\beta
$$

$$
\ll n(\log n)^3 \sum_{D < k \leq 2D} \frac{1}{\varphi(k)} \max_{\alpha \in \mathfrak{m}} \sum_{b_1}^* |S_{b_1}(\alpha)|
$$

$$
\leq n(\log n)^3 \sum_{D < k \leq 2D} \max_{\alpha \in \mathfrak{m}} \left(\frac{1}{\varphi(k)} \sum_{b_1}^* |S_{b_1}(\alpha)|^2 \right)^{1/2}
$$

$$
\ll n(\log n)^3 \sum_{D < k \leq 2D} \left(\frac{n^2}{D^2 (\log n)^{2A+6}} \right)^{1/2}.
$$

In the last step we use Lemma 4.1 below, valid for $D \leq n^{1/4}(\log n)^{-\theta}$ and suitable chosen $\theta, B > 0$ depending just on $A > 0$.

Now the above is $\ll n(\log n)^3 D \frac{n}{D(\log n)^{A+3}} = \frac{n^2}{(\log n)^A}$ as was to be shown for the minor arcs. $\square$

So what is left to show is the following.

Lemma 4.1. *For all $A > 0$ and $B \geq 2A+1, \theta \geq B/2$ let $D \leq n^{1/4}(\log n)^{-\theta}$ and $\alpha \in \mathbb{R}$ with $||\alpha - \frac{u}{v}|| < \frac{1}{v^2}$ for some integers u, v with $(u, v) = 1$ and*

$D(\log n)^B \le v \le \frac{n}{D(\log n)^B}$. *Then for $D < d \le 2D$ we have*

$$\frac{1}{\varphi(d)} \sum_{c,(c,d)=1} |S_{c,d}(\alpha)|^2 \ll \frac{n^2}{D^2(\log n)^A}.$$

We remark that for $\alpha \in \mathfrak{m}$ there exist u, v with $(u, v) = 1$, $v \le \frac{n}{D(\log n)^B}$ and $||\alpha - \frac{u}{v}|| < \frac{D(\log n)^B}{vn} \le \frac{1}{v^2}$ by Dirichlet's approximation theorem, so $v \ge D(\log n)^B$ since $\alpha \in \mathfrak{m}$, and therefore the conditions of Lemma 4.1 are satisfied.

For the proof we need the following auxiliary Lemma. (See [5], Lemma 4.11 and 4.10.)

Lemma 4.2. *Let* $||\alpha - \frac{u}{v}|| \le \frac{1}{v^2}$, $(u, v) = 1$. *Then*

$$(a) \qquad \sum_{m \le X} \min(Y, ||\alpha m||^{-1}) \ll \frac{XY}{v} + Y + (X + v)(\log v),$$

$$(b) \qquad \sum_{m \le X} \min\left(\frac{Y}{m}, ||\alpha m||^{-1}\right) \ll \left(\frac{Y}{v} + X + v\right) \log(2vX)$$

Proof of Lemma 4.1. Fix n large and $D \le n^{1/4}(\log n)^{-\theta}$, and let α, u and v be as given in Lemma 4.1.

We apply Vaughan's identity on the exponential sum $S_{c,d}(\alpha)$, see for example A. Balog in [1], where a similar Lemma is given (Lemma 2 there). From that it follows that it suffices to show for any complex coefficients $|a_m|, |b_k| \le 1$ and any $M \in \mathbb{N}$ with

$$I : \qquad M \le V^2, \text{ if } b_k = 1 \text{ for all } k,$$

$$II : \qquad V \le M \le \frac{n}{V} \text{ else, where } V := D(\log n)^B,$$

we have

$$\sum_{(c,d)=1} \left| \sum_{m \sim M} \sum_{\substack{k \le n/m \\ km \equiv c(d)}} a_m b_k e(\alpha mk) \right|^2 \ll \frac{n^2}{D(\log n)^A}.$$

Here $m \sim M$ means $M < m \le M'$ for some $M' \le 2M$.

We consider first **case II**: Then the left hand side becomes (where m^* denotes the inverse of $m \bmod d$):

$$II := \sum_{(c,d)=1} \left| \sum_{\substack{m \sim M \\ (m,d)=1}} a_m \sum_{\substack{k \leq n/m \\ km \equiv c(d)}} b_k \, e(\alpha mk) \right|^2$$

$$\leq \sum_{(c,d)=1} M \sum_{\substack{m \sim M \\ (m,d)=1}} \left| \sum_{\substack{k \leq n/m \\ k \equiv cm^*(d)}} b_k \, e(\alpha mk) \right|^2$$

$$= M \sum_{\substack{m \sim M \\ (m,d)=1}} \sum_{(c,d)=1} \left| \sum_{\substack{k \leq n/m \\ k \equiv c(d)}} b_k \, e(\alpha mk) \right|^2$$

$$= M \sum_{m \sim M} \sum_{(c,d)=1} \sum_{\substack{k \leq n/m \\ k \equiv c(d)}} b_k \sum_{\substack{k' \leq n/m \\ k' \equiv k(d)}} \overline{b_{k'}} \, e(\alpha m(k - k'))$$

$$= M \sum_{m \sim M} \sum_{\substack{k \leq n/m \\ (k,d)=1}} b_k \sum_{\substack{k' \leq n/m \\ k' \equiv k(d)}} \overline{b_{k'}} \, e(\alpha m(k - k'))$$

$$= M \sum_{m \sim M} \sum_{\substack{k \leq n/m \\ (k,d)=1}} b_k \sum_{\substack{l \geq (k-n/m)/d \\ l \leq (n/m-1)/d}} \overline{b_{k-ld}} \, e(\alpha mld)$$

$$\leq M \sum_{k \leq n/M} \sum_{|l| \leq n/Md} \left| \sum_{\substack{m \sim M \\ m \leq n/k \\ m \leq n/\max\{k-ld, ld+1\}}} e(\alpha mld) \right|.$$

Now the absolute value of the exponential sum is $\ll \min(M, ||\alpha ld||^{-1})$, so

$$II \ll M \frac{n}{M} \sum_{|l| \leq n/Md} \min(M, ||\alpha ld||^{-1})$$

$$\ll n \sum_{\substack{L \leq n/M \\ d|L}} \min(M, ||\alpha L||^{-1}) + nM$$

$$\leq n \left(\sum_{\substack{L \leq n/M \\ d|L}} 1^2 \right)^{1/2} \left(\sum_{L \leq n/M} M \min(M, ||\alpha L||^{-1}) \right)^{1/2} + nM$$

$$\ll n \left(\frac{n}{Md} \right)^{1/2} M^{1/2} \left(\frac{n}{v} + M + \left(\frac{n}{M} + v \right) (\log n) \right)^{1/2} + nM,$$

because of the auxiliary Lemma 4.2 (a). So expression II is $\ll \frac{n^2}{D(\log n)^A}$ since we have $D(\log n)^B = V \le M \le n/V$ in case II, and since $D(\log n)^B \ll v \ll \frac{n}{D(\log n)^B}$ for $B \ge 2A+1$.

Now consider **case I**: Then the left hand side becomes (again m^* denotes the inverse of m mod d):

$$
I := \sum_{(c,d)=1} \Big| \sum_{\substack{m \sim M \\ (m,d)=1}} a_m \sum_{\substack{k \le n/m \\ km \equiv c(d)}} e(\alpha m k) \Big|^2
$$

$$
\le \sum_{(c,d)=1} M \sum_{\substack{m \sim M \\ (m,d)=1}} \Big| \sum_{\substack{k \le n/m \\ k \equiv cm^*(d)}} e(\alpha m k) \Big|^2
$$

$$
\le M \sum_{m \sim M} \sum_{(c,d)=1} \Big| \sum_{\substack{k \le n/m \\ k \equiv c(d)}} e(\alpha m k) \Big|^2
$$

$$
= M \sum_{m \sim M} \sum_{(c,d)=1} \sum_{\substack{k \le n/m \\ k \equiv c(d)}} e(\alpha m k) \sum_{\substack{k' \le n/m \\ k \equiv k'(d)}} e(-\alpha m k')
$$

$$
= M \sum_{m \sim M} \sum_{\substack{k \le n/m \\ (k,d)=1}} \sum_{\substack{k' \le n/m \\ k \equiv k'(d)}} e(\alpha m (k-k'))
$$

$$
\le M \sum_{m \sim M} \sum_{k \le n/m} \Big| \sum_{\substack{l \ge (k-n/m)/d \\ l \le (n/m-1)/d}} e(\alpha m d l) \Big|
$$

$$
\ll M \sum_{m \sim M} \sum_{k \le n/M} \Big(\min\Big(\frac{n}{md}, \|\alpha m d\|^{-1} \Big) + 1 \Big)
$$

$$
\ll n \sum_{m \sim M} \min\Big(\frac{n}{md}, \|\alpha m d\|^{-1} \Big) + Mn
$$

$$
\ll n \sum_{L \sim Md} \min\Big(\frac{n}{L}, \|\alpha L\|^{-1} \Big) + Mn
$$

$$
\ll n \Big(\frac{n}{v} + Md + v \Big)(\log n) + Mn,
$$

using auxiliary Lemma 4.2 (b). Now we get $I \ll \frac{n^2}{D(\log n)^A}$ since $D(\log n)^B \ll v \ll \frac{n}{D(\log n)^B}$ with $B \ge A+1$ and since $Md \ll V^2 d \ll D^3(\log n)^B \ll \frac{n}{D(\log n)^B}$ for $D \le n^{1/4}(\log n)^{-\theta}$ and $\theta \ge B/2$. So Lemma 4.1 is shown. $\quad\square$

Final remark. As was kindly pointed out to me by Z. Cui, it is possible to improve the statement on the major arcs such that Theorems 1.1, 3.1 and 3.2 hold for the improved exponent 1/4 instead of 1/5. The idea for this major arc improvement comes from Z. Cui in [3].

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