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Analytic and combinatoric aspects of Hurwitz polyzêtas

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Résumé. Dans ce travail, un codage symbolique des séries génératrices de Dirichlet généralisées est obtenu par les techniques combinatoires des séries formelles en variables non-commutative. Il permet d'expliciter les séries génératrices de Dirichlet généralisées 'périodiques' – donc notamment les polyzêtas colorés – comme combinaison linéaire de polyzêtas de Hurwitz. De plus, la version non commutative du théorème de convolution nous fournit une représentation intégrale des séries génératrices de Dirichlet généralisées. Celle-ci nous permet de prolonger les polyzêtas de Hurwitz comme des fonctions méromorphes à plusieurs variables.

Abstract. In this work, a symbolic encoding of generalized Dirichlet generating series is found thanks to combinatorial techniques of noncommutative rational power series. This enables to explicit periodic generalized Dirichlet generating series – particularly the coloured polyzêtas – as linear combinations of Hurwitz polyzêtas. Moreover, the noncommutative version of the convolution theorem gives easily rise to an integral representation of Hurwitz polyzêtas. This representation enables us to build the analytic continuation of Hurwitz polyzêtas as multivariate meromorphic functions.

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1. Introduction

Technologies of generating series (g.s.) and of generating functions (g.f.) on one variable are basic tools required for asymptotic, complexity and probabilistic analysis of combinatorial and discrete structures. These technologies play a central role in several application areas like in arithmetics, statistic physics, algorithmic information theory [8], analytic combinatorics [10] and analysis of algorithms, ... This makes the natural link between these structures and complex analysis and exploits intensively the association between ordinary generating functions (o.g.f.) and exponential generating functions (e.g.f.).

In this paper, we will study the association of o.g.f. and of a generalization of Dirichlet generating functions (D.g.f.) [22]. The value at 1 of a D.g.f is a Dirichlet generating serie (D.g.s); for example Riemann polyzêtas [26], defined by

$$\zeta(s_1,\ldots,s_r)=\sum_{n_1>\ldots>n_r}\frac{1}{n_1^{s_1}\ldots n_r^{s_r}}, \tag{1}$$

are D.g.s and Hurwitz polyzêtas [27], defined by

$$\zeta(s_1,\ldots,s_r;t_1,\ldots,t_r)=\sum_{n_1>\ldots>n_r}\frac{1}{(n_1-t_1)^{s_1}\ldots(n_r-t_r)^{s_r}},\tag{2}$$

are Parametrized D.g.s. Reciprocally, we show that 'periodic' Parametrized D.g.s. (and so periodic D.g.s.) can be written as a finite combination of Hurwitz polyzêtas (see Proposition 3.3). It is notably the case for the coloured polyzêtas as described by Equality (49). This equality enables us to give the expression of any Hurwitz polyzêta as a finite combination of coloured polyzêtas (Proposition 3.4).

On the other hand, the moment sum of index l of basic sign algorithm is given by [17]

$$\overline{\rho}^{(l)}=1+\frac{1}{2^l}+\frac{2}{\zeta(2l)}\sum_{d<c<2d}\frac{1}{c^ld^l}.\tag{3}$$

But

$$\sum_{d<c<2d}\frac{1}{c^ld^l}=\sum_{c<2d}\frac{1}{c^ld^l}-\sum_{c}\frac{1}{c^{2l}}-\sum_{c<d}\frac{1}{c^ld^l},$$

$$\sum_{c<2d}\frac{1}{c^ld^l}=2^{l-1}\sum_{c<e}\frac{1+(-1)^e}{c^le^l},$$

$$\sum_{0<c<e}\frac{(-1)^e}{c^le^l}=2^{-2l}\sum_{a-b,b=0}^{1}(-1)^a\sum_{n>m>0}\frac{1}{(n+a/2)^l(m+b/2)^l}.\tag{4}$$

Hence, the moment sum of index l can be expressed in terms of polyzêtas :

$$\begin{aligned}\overline{\rho}^{(l)}&=\frac{1}{2^l}-1+(2^l-2)\frac{\zeta(l,l)}{\zeta(2l)}\\&+2^{-l}\sum_{a-b,b=0}^{1}(-1)^a\frac{\zeta(l,l;-a/2,-b/2)}{\zeta(2l)}.\end{aligned}\tag{5}$$

This motivates this study of polyzêtas. First, we recall some combinatorial aspects : bi-algebra structure of Riemann polyzêtas and of Hurwitz polyzêtas [27], shuffle relation of Hurwitz polyzêtas (equality 44). This combinatorial study give easily rise to an integral representation of Hurwitz polyzêtas [21]. From this integral representation of Riemann and Hurwitz zêta function [7], we can deduce their analytic continuation and the structure of their poles. Unfortunately, in the multivariate case, several singularities appear and the 'classic' method can only treat one of them. In a preprint [20], Goncharov remarks that continuing with only one variable in turn by turn is not admissible because the result depends on the selected path. He suggests so to use a distribution to obtain the analytic continuation. This method operates in the univariate classic case with only singularities near zero [18], and still remains to be adapted in the multivariate case. Goncharov builds a tensor product of distributions, each distribution having to regularize each variable. But, in the multivariate case, there appear two singularities for any variable, and Goncharov use a product of distributions *in same variable* to expand these singularities. Moreover, to get the structures of the poles, he works with a formal development in an infinite multi-sum without considering the "problems of convergence". We define and study the distribution (the *regularized* distribution) which enables to get the continuation of Hurwitz polyzêtas. We also give the structure of their poles. Thanks to the decomposition of periodic D.g.s. into sums of Hurwitz polyzêtas (Proposition 3.3), we can derive the continuation of periodic D.g.s. and the structure of their poles. In [15], Ecalle suggests to use the equality

$$\frac{1}{(n-1)^s} = \sum_{k=0}^{+\infty} \frac{\Gamma(s+k)}{\Gamma(s)k!} n^{-s-k} \tag{6}$$

to get a relation between $\zeta(\mathbf{s})$ and its translates, in order to build the analytic continuation of the Riemann polyzêtas. So have we, in a similar way, calculated a relation between Hurwitz polyzêtas and its translates.

This paper is a continuation of [26, 27]. It is organized as follows :

- Section 2 gives the background of the classic case of single Dirichlet series (Subsection 2.1) as the guide for our developments in the next sections and the combinatorial techniques on formal power series (Subsection 2.2) as technical support for our results. We give the encoding of iterated integrals by noncommutative variables (Subsubsection 2.2.2) : to encode the polylogarithms (Equality 30) and the Riemann polyzêtas (Equality 32). In particular, we describe the convolution theorem in noncommutative version (Equality 25) in order to obtain in the next section the multiple integral representation of some special functions.
- In Section 3, we introduce the generalization of the Dirichlet generating series (D.g.s.) from Definition 3.1, of Parametrized D.g.s. (Definition 3.1), of Dirichlet generating functions (D.g.f.) in Proposition 3.1 and of Parametrized D.g.f. (Proposition 3.2), associated to sequences of complex numbers. When the sequences are periodic with same period, we give the explicit expression of generalized Parametrized D.g.f. as finite sums of Hurwitz polyzêtas (Proposition 3.3). So, we obtain the explicit expression of coloured polyzêtas as a finite combination of Hurwitz polyzêtas (Equality 49). Reciprocally, we calculate the explicit expression of Hurwitz polyzêtas as finite sum of coloured polyzêtas (Proposition 3.4). Next, we use the

convolution theorem to get the integral representation of generalized D.g.f. (Proposition 3.5) and generalized D.g.s. (Corollary 3.2) : in particular these combinatorial techniques give easily rise to an integral representation of Hurwitz polyzêtas.

- Section 4 contains the main results. In Subsection 4.1, we show the Hurwitz polyzêtas have a meromorphic continuation over the entire space. For that, we define (Definition 4.1.2) and study (Subsubsection 4.1.2) the regularization near 0, aiming at building the regularization between 0 and 1. This study enables to know the localization and the multiplicity of the poles (Theorem 4.1). In Subsection 4.2, we calculate the regularization in some case in order to get the structure of Hurwitz polyzêtas poles (Theorem 4.2). Lastly, in section 6, we note that this result give the analytic continuation of the periodic Paramatrized Dirichlet generating series, so of the colored polyzêtas functions.
- We give the relation between translates of Hurwitz polyzêtas (Proposition 5.1) in Section 5. Then we discuss about the possibility to deduce the analytic continuation from this relation.

2. Background

2.1. The univariate zêta function.

2.1.1. ***Ordinary and Dirichlet generating series.*** Any complex numbers sequence $\{f_k\}_{k\geq 1}$ can be associated to the ordinary generating series (o.g.s.) and to the Dirichlet generating series (D.g.s.)

$$F(z)=\sum_{k\geq 1} f_k z^k \quad \text{and} \quad \mathrm{Di}(F;s)=\sum_{k\geq 1}\frac{f_k}{k^s}. \tag{7}$$

We associate also $\{f_k\}_{k\geq 1}$ to the following power series generalizing the o.g.s. $F(z)$ as well as the D.g.s. $\mathrm{Di}(F;s)$ [22, 27] :

$$\mathrm{Di}_s(F|z)=\sum_{k\geq 1} f_k\frac{z^k}{k^s}, \qquad |z|<1. \tag{8}$$

Example. If $F(z)=\mathbf{1}(z)=z/(1-z)$ then $\mathrm{Di}_s(\mathbf{1}|z)$ is the classic polylogarithm $\mathrm{Li}_s(z)$ and $\mathrm{Di}(\mathbf{1};s)$ is the Riemann zêta function $\zeta(s)$ which converges in the region $\{s\in\mathbb{C}:\Re(s)>1\}$:

$$\mathrm{Di}_s(\mathbf{1}|z)=\mathrm{Li}_s(z)=\sum_{n\geq 1}\frac{z^n}{n^s} \quad \text{and} \quad \mathrm{Di}(\mathbf{1};s)=\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}.$$

In particular, if $\{f_k\}_{k\geq 1}$ is periodic of period K then $\mathrm{Di}(F;s)$ can be expressed as a linear combination of classic Hurwitz zêta functions,

$$\mathrm{Di}(F;s) = \frac{1}{K^s}\sum_{i=0}^{K-1} f_i\zeta(s;-\frac{i}{K}), \tag{9}$$

where

$$\zeta(s;t) = \sum_{n\geq 1}\frac{1}{(n-t)^s} \quad \text{and} \quad t \notin \mathbb{N}_+. \tag{10}$$

Recall also that this Hurwitz zêta function $\zeta(s;t)$, converging in the region $\{s \in \mathbb{C} : \Re(s) > 1\}$, is the special value at $z = 1$ of the classic Lerch function verifying $\Phi_s(z;0) = \mathrm{Li}_s(z)$,

$$\Phi_s(z;t) = \sum_{n\geq 1}\frac{z^n}{(n-t)^s}, \quad \text{for} \quad t \notin \mathbb{N}_+. \tag{11}$$

Example. Let $\jmath$ be a primitive cubic root of unit. Then

$$\begin{aligned}\sum_{k\geq 1}\frac{\jmath^k}{k^s} &= \sum_{n\geq 1}\frac{\jmath}{(3n-2)^s} + \sum_{n\geq 1}\frac{\jmath^2}{(3n-1)^s} + \sum_{n\geq 1}\frac{1}{(3n)^s}\\ &= \frac{1}{3^s}[\jmath\zeta(s;\frac{2}{3}) + \jmath^2\zeta(s;\frac{1}{3}) + \zeta(s)].\end{aligned}$$

2.1.2. ***Singular expansions and Mellin transform.*** To get an asymptotic expansion of the meromorphic function F, defined over an open set containing the set $\mathcal{S}$ of its poles, we use

Definition ([16])**.** The *singular expansion* of F is defined by the formal sum

$$\sum_{p\in\mathcal{S}}\sum_{k=-\infty}^{-1} c_{k,p}(z-p)^k,$$

where $c_{k,p}$ is the k-th coefficient of the Laurent series of F at the pole p, and we note

$$F(z) \asymp \sum_{p\in\mathcal{S}}\sum_{k=-\infty}^{-1} c_{k,p}(z-p)^k.$$

Example.

$$\frac{1}{z(z-1)^2} \asymp \frac{1}{z} + \frac{1}{(z-1)^2} - \frac{1}{z-1}.$$

In fact, the D.g.s. $\mathrm{Di}(F;s)$ can be obtained as the *Mellin transform* of $F(e^{-z})/\Gamma(s)$:

$$\mathrm{Di}(F;s) = \int_0^\infty \frac{F(e^{-u})}{\Gamma(s)}\frac{du}{u^{1-s}} = \int_0^1 \frac{\log^{s-1}(1/r)}{\Gamma(s)}F(r)\frac{dr}{r}$$

Example. Let $F(z) = z$ and $G(z) = z/(1-z)$. The Gamma function, $\Gamma(s)$, and the Riemann zêta function, $\zeta(s)$, can be then defined as a Mellin transform of $F(e^{-z})$ and $G(e^{-z})/\Gamma(s)$ respectively :

$$\Gamma(s) = \int_0^\infty e^{-u} \frac{du}{u^{1-s}} = \int_0^1 \log^{s-1}(1/r) dr,$$
$$\zeta(s) = \int_0^\infty \frac{G(e^{-u})}{\Gamma(s)} \frac{du}{u^{1-s}} = \int_0^1 \frac{\log^{s-1}(1/r)}{\Gamma(s)} G(r) \frac{dr}{r}.$$

There is a correspondence between the asymptotic expansions, via Mellin transform, of F at 0 and ∞ and the singularities of $\mathrm{Di}(F;s)$. Conversely, under some conditions, the poles of $\mathrm{Di}(F;s)$ induce the asymptotic expansions of F. This mapping properties are conveniently expressed in terms of singular expansions [16].

Example. Since $e^{-z} = \sum_{n\geq 0} (-z)^n/n!$ and $G(e^{-u}) = u^{-1} \sum_{k\geq 0} B_k u^k/k!$, where B_k is the k-th Bernoulli number, then $\Gamma(s)$ and $\zeta(s)$ can be expressed as follows

$$\Gamma(s) \asymp \sum_{n\geq 0} \frac{(-1)^n}{n!} \frac{1}{s+n} \quad \text{and} \quad \zeta(s) \asymp \sum_{k\geq 0} \frac{B_k}{k!\Gamma(s)} \frac{1}{s-1+k}.$$

Therefore, $\Gamma(s)$ has no zeros and the residue at the pole $s=-n$ of $\Gamma(s)$ is $(-1)^n/n!$. By cancellation, one can deduce the Riemann's result saying that $\zeta(s)$ has an analytic continuation to the complex plane as a meromorphic function, with only one simple pole of residue 1 at $s=1$.

2.2. Symbolic computations on special functions.

2.2.1. ***Formal power series.*** Let X be a finite alphabet. The free monoid generated by X is denoted by X^*. It is the set of words over X. The empty word is denoted by "ϵ". We denote by X^+ the set $X^* \setminus \{\varepsilon\}$. The shuffle of two words u and v is the polynomial recursively defined as

$$\epsilon ш u = u ш \epsilon = u \quad \text{and} \quad au ш bv = a(u ш bv) + b(au ш v), \tag{12}$$

for $a, b \in X$ and $u, v \in X^*$. The shuffle product is extended by distributivity to the shuffle product of formal power series. Let A be a commutative $\mathbb{C}$-algebra. We denote by $A\langle X\rangle$ (resp. $A\langle\langle X\rangle\rangle$) the ring of noncommutative polynomials (resp. formal power series) with coefficients in A. The $\mathbb{C}$-module $A\langle\langle X\rangle\rangle$ equipped with the shuffle product is a commutative A-algebra, denoted by $\mathrm{Sh}_A\langle X\rangle$. A formal power series S in $A\langle\langle X\rangle\rangle$ can be written as

$$S = \sum_{w\in X^*} \langle S \mid w\rangle\, w. \tag{13}$$

Let S be a *proper* formal power series (*i.e.* $\langle S|\varepsilon\rangle = 0$), the formal power series S^* and S^+ are defined as $S^* = 1 + \ldots + S^n + \ldots = 1 + SS^*$ and

$S^+ = SS^*$ respectively. The polynomials are defined as formal power series with finite support. The shuffle product is extended by distributivity to the shuffle product of formal power series as follows

$$S ⧢ T = \sum_{u,v \in X^*} \langle S|u\rangle \langle T|v\rangle \; u ⧢ v. \tag{14}$$

2.2.2. *Words and iterated integrals.* Let us associate to each letter x_i in X a 1-differential form ω_i, defined in some connected open subset $\mathcal{U}$ of $\mathbb{C}$. For all path $z_0 \rightsquigarrow z$ in $\mathcal{U}$, the *Chen iterated integral* associated to $w = x_{i_1} \cdots x_{i_k}$ along $z_0 \rightsquigarrow z$ is defined recursively as follows

$$\int_{z_0 \rightsquigarrow z} \omega_{i_1} \cdots \omega_{i_k} \int_{z_0 \rightsquigarrow z} \omega_{i_1}(z_1) \int_{z_0 \rightsquigarrow z_1} \omega_{i_2} \cdots \omega_{i_k}. \tag{15}$$

In a shortened notation, we denote this iterated integral by $\alpha_{z_0}^z(w)$ with $\alpha_{z_0}^z(\varepsilon) = 1$. More generally, if $F(z)$ is analytic, and vanishing at z_0, one puts

$$\alpha_{z_0}^z(x_{i_1} x_{i_2} \cdots x_{i_k}; F) = \int_{z_0 \rightsquigarrow z} \omega_{i_1}(t) \alpha_{z_0}^t(x_{i_2} \cdots x_{i_k}; F) \tag{16}$$

and $\alpha_{z_0}^z(\varepsilon; F) = F(z)$.

Observe that these notations are related to a choice of the differential forms ω_i associated to x_i. Recall also that $\alpha_0^z(\epsilon) \neq \alpha_0^{z_0}(\epsilon) + \alpha_{z_0}^z(\epsilon)$ (this could imply 1=1+1 !). Thus iterated integral is not ordinary integral since additivity, in particular, is not satisfied for $w = \epsilon$ and it is replaced by the rule (17) of the following properties :

- *Rule of concatenation of paths.* For any word $w \in X^*$, one has, for any $\rho \in \mathcal{U}$,

$$\alpha_{z_0}^z(w) = \sum_{u,v \in X^*, uv=w} \alpha_{z_0}^\rho(u) \alpha_\rho^z(v). \tag{17}$$

- *Rule of integration by parts.* For any words $u \in X^*$ and $v \in X^*$, one has

$$\alpha_{z_0}^z(u ⧢ v) = \alpha_{z_0}^z(u) \alpha_{z_0}^z(v). \tag{18}$$

- *Rule of inversion of path integration.* For any word $w \in X^*$, one has

$$\alpha_{z_0}^z(w) = (-1)^{|w|} \alpha_z^{z_0}(\widetilde{w}). \tag{19}$$

  Here, $\widetilde{w}$ stands for the mirror of w and $(-1)^{|w|}\widetilde{w}$ is the *antipode* of w.
- *Rule of change of variables.* For any word $w \in X^*$, one has

$$\alpha_{g(z_0)}^{g(z)}(w) = g^* \alpha_{z_0}^z(w), \tag{20}$$

where $g^*\alpha_{z_0}^z$ is the iterated integral of the path $z_0 \rightsquigarrow z$ with respect to the differential forms $g^*\omega$ (the reciprocal image g^* of ω).

2.2.3. ***Iterated integrals and noncommutative convolution theorem.*** For
any polynomial S (resp. formal power series up to convergence [30]), one defines [21]

$$\alpha_{z_0}^z(S) = \sum_{w\in X^*} \langle S|w\rangle \alpha_{z_0}^z(w). \tag{21}$$

With the previous notations, according to the rule of integration by parts, one has

$$\alpha_{z_0}^z(S \sqcup\!\sqcup T) = \alpha_{z_0}^z(S)\alpha_{z_0}^z(T). \tag{22}$$

Indeed, it is true if S and T are two words, it is also true (by distributivity) if S and T are two noncommutative formal power series. In other words,$\alpha_{z_0}^z$ is an homomorphism from $(A\langle\langle X\rangle\rangle, \sqcup\!\sqcup)$ to A.

Let φ_j be a primitive of ω_j, for $j = 0, \ldots, m$. For any *exchangeable* formal power series H, the iterated integral associated to H can be expressed as follows [21]

$$H = \sum_{n_0,\cdots,n_m\geq 0} h_{n_0,\cdots,n_m} x_0^{n_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_m^{n_m}, \tag{23}$$

$$\alpha(H) = \sum_{n_0,\cdots,n_m\geq 0} h_{n_0,\cdots,n_m} \frac{\varphi_0^{n_0}\cdots\varphi_m^{n_m}}{n_0!\cdots n_m!}. \tag{24}$$

Example. For any letters $x_j, x_k \in X$, and any $n \in \mathbb{N}$,

$$\begin{aligned}
\alpha_{z_0}^z(x_j^n x_k) &= \int_{z_0\rightsquigarrow z} \frac{[\varphi_j(z)-\varphi_j(s)]^n}{n!}\omega_k(s) \\
&= \sum_{l=0}^{n} \frac{\varphi_j^l(z)}{l!} \int_{z_0\rightsquigarrow z} \frac{[-\varphi_j(s)]^{n-l}}{(n-l)!}\omega_k(s), \\
\alpha_{z_0}^z(x_j^* x_k) &= \int_{z_0\rightsquigarrow z} e^{\varphi_j(z)-\varphi_j(s)}\omega_k(s) \\
&= e^{\varphi_j(z)} \int_{z_0\rightsquigarrow z} e^{-\varphi_j(s)}\omega_k(s).
\end{aligned}$$

More generally, if F is analytic and vanishing at z_0, the *convolution theorem* yields [21]

$$\alpha_{z_0}^z(H;F) = \int_{z_0\rightsquigarrow z} h[\varphi_0(z)-\varphi_0(s), \cdots, \varphi_m(z)-\varphi_m(s)]dF(s). \tag{25}$$

Example. For any letters $x_j, x_k \in X$, and any integers n, m,

$$\alpha_{z_0,\gamma}^z(x_j^n; F) = \sum_{l=0}^{n} \frac{\varphi_j^l(z)}{l!} \int_{z_0 \rightsquigarrow z} \frac{[-\varphi_j(s)]^{n-l}}{(n-l)!} dF(s),$$

$$\alpha_{z_0}^z(x_j^n \sqcup\!\sqcup\, x_i^m; F) = \sum_{l=0}^{n} \sum_{k=0}^{m} \frac{\varphi_j^l(z)\varphi_i^k(z)}{l!k!} \int_{z_0 \rightsquigarrow z} \frac{[-\varphi_j(s)]^{n-l}[-\varphi_i(s)]^{m-k}}{(n-l)!(m-k)!} dF(s),$$

$$\alpha_{z_0}^z(x_j^*; F) = e^{\varphi_j(z)} \int_{z_0 \rightsquigarrow z} e^{-\varphi_j(s)} dF(s),$$

$$\alpha_{z_0}^z(x_j^* \sqcup\!\sqcup\, x_i^*; F) = e^{\varphi_j(z)+\varphi_i(z)} \int_{z_0 \rightsquigarrow z} e^{-\varphi_j(s)-\varphi_i(s)} dF(s).$$

2.2.4. ***Polylogarithms, multiple harmonic sums and polyzêtas.*** The composition $\mathbf{s} = (s_1, \ldots, s_r)$, *i.e.* a sequence of positive integers, is said to have *depth* equal to r and *weight* equal to $\sum_{i=1}^r s_i$. The *empty* composition is denoted by $\mathbf{e} = ()$. The *quasi-shuffle* product of two compositions $\mathbf{r} = (r_1, \ldots, r_k) = (r_1, \mathbf{r}')$ and $\mathbf{s} = (s_1, \ldots, s_l) = (s_1, \mathbf{s}')$ is defined as

$$\mathbf{r} \uplus \mathbf{e} = \mathbf{e} \uplus \mathbf{r} = \mathbf{r} \tag{26}$$

and

$$\mathbf{r} \uplus \mathbf{s} = (r_1, \mathbf{r}' \uplus \mathbf{s}) + (s_1, \mathbf{r} \uplus \mathbf{s}') + (r_1 + s_1, \mathbf{r}' \uplus \mathbf{s}'). \tag{27}$$

To $\mathbf{s}$ we can canonically associate the word $u = x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1$ over the finite alphabet $X = \{x_0, x_1\}$. In the same way, $\mathbf{s}$ can be canonically associated to the word $v = y_{s_1} \ldots y_{s_r}$ over the infinite alphabet $Y = \{y_1, y_2, \ldots\}$. We obtain so a concatenation isomorphism from the A-algebra of compositions into the algebra $A\langle X\rangle x_1$ (resp. $A\langle Y\rangle$). We shall identify below the composition $\mathbf{s}$, the correspondent word $u \in X^*x_1$ and the correspondent word $v \in Y^*$. The word $u \in X^*$ (resp. $v \in Y^*$, resp. the composition $\mathbf{s}$) is said to be *convergent* if $s_1 > 1$.

The polylogarithm associated to the composition $\mathbf{s}$ is defined as

$$\mathrm{Li}_{\mathbf{s}}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}}, \quad \text{for} \quad |z| < 1. \tag{28}$$

Let ω_0 and ω_1 be the following differential forms

$$\omega_0(z) = \frac{dz}{z} \quad \text{and} \quad \omega_1(z) = \frac{dz}{1-z}. \tag{29}$$

One verifies that the polylogarithm $\mathrm{Li}_{\mathbf{s}}$ is the iterated integral with respect to ω_0 and ω_1 :

$$\mathrm{Li}_{\mathbf{s}}(z) = \alpha_0^z(u), \quad \text{for all} \quad |z| < 1, \tag{30}$$

where $u = x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1$ is the word corresponding to $\mathbf{s}$ in X^*x_1. >From this representation integral provides the meromorphic continuation of $\mathrm{Li}_{\mathbf{s}}$ for z over the Riemann surface of $\mathbb{C} \setminus \{0, 1\}$. Note that the polylogarithms, as iterated integrals, verify the shuffle relation :

$$\mathrm{Li}_{\mathbf{s} \sqcup\!\sqcup \mathbf{s}'}(z) = \mathrm{Li}_{\mathbf{s}}(z)\,\mathrm{Li}_{\mathbf{s}'}(z), \quad \text{for all} \quad |z| < 1. \tag{31}$$

If $\mathbf{s}$ is a convergent composition, the limit of $\mathrm{Li}_{\mathbf{s}}(z)$ when $z \to 1$ exists and is nothing but the Riemann polyzêta $\zeta(\mathbf{s})$ [41] :

$$\lim_{z \to 1} \mathrm{Li}_{\mathbf{s}}(z) = \alpha_0^1(u) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}} = \zeta(\mathbf{s}). \tag{32}$$

One thus has a encoding of the polyzêta $\zeta(\mathbf{s})$ in term of iterated integrals.

For $1 \leq r \leq N$, let $\mathbf{s} = (s_1, \ldots, s_r)$. The *finite* polyzêtas (or multiple harmonic sums) $\zeta_N(\mathbf{s})$ is defined as (see [30])

$$\zeta_N(\mathbf{s}) = \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}. \tag{33}$$

and $\zeta_N(\mathbf{s}) = 0$ for $1 \leq N < r$. For $r = 0$, we put $\zeta_0(\mathbf{s}) = 0$ and $\zeta_N(\mathbf{s}) = 1$, for any $N \geq 1$. These can be obtained as the specialization in the *quasi-monomial functions* (see [38])

$$M_{\mathbf{s}}(t) = \sum_{n_1 > \ldots > n_r > 0} t_{n_1}^{s_1} \cdots t_{n_r}^{s_r} \tag{34}$$

at $t_i = 1/i$ if $1 \leq i \leq k$ and $t_i = 0$ if $i > k$. Let us extend linearly the notation $M_{\mathbf{s}}$ when $\mathbf{s}$ is a linear combination of compositions. If $\mathbf{r}$ (resp. $\mathbf{s}$) is a composition of depth r and weight p (resp. of depth s and weight q), $M_{\mathbf{r} \uplus \mathbf{s}}$ is a quasi-monomial function of depth $r+s$ and of weight $p+q$, and one has

$$M_{\mathbf{s} \uplus \mathbf{r}} = M_{\mathbf{s}}\; M_{\mathbf{r}}. \tag{35}$$

Therefore,

$$\zeta_N(\mathbf{s} \uplus \mathbf{r}) = \zeta_N(\mathbf{s})\zeta_N(\mathbf{r}). \tag{36}$$

For $s_1 > 1$, the limit when $N \to \infty$ of $\zeta_N(\mathbf{s})$ is nothing but the polyzêta $\zeta(\mathbf{s})$ [41], and thus by an Abel's theorem,

$$\lim_{N \to \infty} \zeta_N(\mathbf{s}) = \lim_{z \to 1} \mathrm{Li}_{\mathbf{s}}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}} = \zeta(\mathbf{s}). \tag{37}$$

The asymptotic expansion of $\zeta_N(\mathbf{s})$, for $N \to \infty$, was already treated by use of Euler-Mac Laurin summation [12] or by use of a full singular expansion, at $z = 1$, of its generating series $\sum_{N \geq 0} \zeta_N(\mathbf{s}) z^N$ [11].

Let us come back to Equality (37). On the one hand, the polyzêtas $\zeta(\mathbf{s})$, as limits of the polygarithms $\mathrm{Li}_{\mathbf{s}}(z)$, verify the shuffle relation. On the other hand, the polyzêtas $\zeta(\mathbf{s})$ can be obtained also as the specialization of the quasi-monomial functions at $t_i = 1/i$, for $i \geq 1$. So, if two compositions $\mathbf{r}, \mathbf{s}$ correspond respectively to the two convergent words l_1, l_2, then we get on the convergent polyzêtas the three families of relations (see [23]) :

$$\zeta(l_1 ш l_2) = \zeta(l_1)\zeta(l_2), \tag{38}$$

$$\zeta(l_1 \,\text{ш}\!\!\!\text{-}\, l_2) = \zeta(l_1)\zeta(l_2), \tag{39}$$

$$\zeta(x_1 ш l_1 - x_1 \,\text{ш}\!\!\!\text{-}\, l_1) = 0. \tag{40}$$

The first two families are called the *double shuffle structure*. The third family involves the polynomials $x_1 ш l_1 - x_1 \,\text{ш}\!\!\!\text{-}\, l_1$ that are convergent, even when the two sums $\zeta(x_1 ш l_1)$ and $\zeta(x_1 \,\text{ш}\!\!\!\text{-}\, l_1)$ are divergent. These divergent terms can be regularized syntactically with respect to the associated shuffle products as explained in [27, 28]. Note that we only have to study that point over a generator family : the set of Lyndon word [23].

3. Generalized D.g.s. and their integral representation

By now, r stands for a positive integer. Let $T = \{t_1, \cdots, t_r\}$ and $\overline{T} = \{\bar{t}_1, \cdots, \bar{t}_r\}$ two families of parameters connected by the change of variables

$$\begin{cases} t_1 = \bar{t}_1 + \cdots + \bar{t}_r, \\ t_2 = \bar{t}_2 + \cdots + \bar{t}_r, \\ \vdots \\ t_r = \bar{t}_r. \end{cases} \iff \begin{cases} \bar{t}_1 = t_1 - t_2, \\ \bar{t}_2 = t_2 - t_3, \\ \vdots \\ \bar{t}_r = t_r. \end{cases} \tag{41}$$

For $i = 1..m$, let us consider the locally integrable function[1] F_i and the following associated differential 1-forms

$$\omega_0(z) = \frac{dz}{z}, \quad \omega_i(z) = F_i(z)\frac{dz}{z}, \quad \omega_{i,\bar{t}_i}(z) = F_i(z)\frac{dz}{z^{1+\bar{t}_i}}. \tag{42}$$

For any composition $\mathbf{s} = (s_1, \ldots, s_r)$ and for any formal variables $t_1, \ldots, t_r$, we use the short notations $\mathbf{t}$ for $(t_1, \ldots, t_r)$, $(\mathbf{s}; \mathbf{t})$ for $(s_1; t_1), \ldots, (s_r; t_r)$ and $\mathbf{F}$ for $(F_{i_1}, \ldots, F_{i_r})$.

3.1. Definitions and basic properties.

Here, we consider the case

$$F_i(z) = \sum_{n \geq 1} f_{i,n} z^n, \quad \text{for} \quad i = 1..m. \tag{43}$$

Definition. The D.g.s. associated to $\{F_i\}_{i=1..m}$ given in (43) is the sum

$$\mathrm{Di}(\mathbf{F}; \mathbf{s}) = \sum_{n_1 > \ldots > n_r > 0} \frac{f_{i_1, n_1 - n_2} \cdots f_{i_{r-1}, n_{r-1} - n_r} f_{i_r, n_r}}{n_1^{s_1} \ldots n_r^{s_r}}.$$

We get the following iterated integral interpretations :

[1] Note that, here F_1 is not necessary the function $(1-z)^{-1}$ except when $m = 1$ and it is in the case of (29).

Proposition 3.1 ([22]). *Let $i_1,\ldots,i_r = 1,..,m$ and let $w = x_0^{s_1-1}x_{i_1}\ldots$ $\ldots x_0^{s_r-1}x_{i_r}$ associated to* $\mathbf{s}$. *With the differential forms $\omega_0,\omega_1,\ldots,\omega_m$ of (42), we get*

$$\alpha_0^z(w) = \sum_{n_1>\ldots>n_r} \frac{f_{i_1,n_1-n_2}\ldots f_{i_{r-1},n_{r-1}-n_r} f_{i_r,n_r}}{n_1^{s_1}\ldots n_r^{s_r}} z^{n_1}.$$

We call Generalized *D.g.f.* $\mathrm{Di}_{\mathbf{s}}(\mathbf{F}|z)$ *associated to $\{F_i\}_{i=1..m}$ given in (43) this previous iterated integral. In particular,* $\mathrm{Di}(\mathbf{F};\mathbf{s}) = \mathrm{Di}_{\mathbf{s}}(\mathbf{F}|1) = \alpha_0^1(w)$.

Example. Let $\mathbf{z} = (z_1,\ldots,z_r)$. For the o.g.f.

$$F_i(z) = \frac{z_i z}{1 - z_i z} \quad \text{with} \quad |z_i| < 1, \text{ for } i = 1..m,$$

with the differential forms $\omega_0,\omega_1,\ldots,\omega_m$, we have [22, 27]

$$\alpha_0^z(w) = \sum_{n_1>\ldots>n_r>0} \frac{z_1^{n_1-n_2}\ldots z_{r-1}^{n_{r-1}-n_r} z_r^{n_r}}{n_1^{s_1}\ldots n_r^{s_r}} z^{n_1}.$$

and $\mathrm{Di}(F_1,\ldots,F_r;\mathbf{s})$ is nothing but the *multiple* polylogarithm [4, 19]

$$\mathrm{Li}_{\mathbf{s}}(\mathbf{z}) = \mathrm{Li}_{\mathbf{s}}(z_1,\ldots,z_r) = \sum_{n_1>\ldots>n_r>0} \frac{z_1^{n_1}(z_2/z_1)^{n_2}\ldots(z_r/z_{r-1})^{n_r}}{n_1^{s_1}\ldots n_r^{s_r}}.$$

Definition. The Parametrized D.g.s. associated to $\{F_i\}_{i=1..m}$ given in (43) is the sum

$$\mathrm{Di}(\mathbf{F_t};\mathbf{s}) = \sum_{n_1>\ldots>n_r>0} \frac{f_{i_1,n_1-n_2}\ldots f_{i_{r-1},n_{r-1}-n_r} f_{i_r,n_r}}{(n_1-t_1)^{s_1}\ldots(n_r-t_r)^{s_r}}.$$

Proposition 3.2 ([22]). *Let $i_1,\ldots,i_r = 1,..,m$ and $w = x_0^{s_1-1}x_{i_1}\ldots$ $\ldots x_0^{s_r-1}x_{i_r}$ associated to* $\mathbf{s}$. *With the differential forms $\omega_0,\omega_{1,\bar{t}_1},\ldots,\omega_{m,\bar{t}_m}$ of (42), we get*

$$\alpha_0^z(w) = \sum_{n_1>\ldots>n_r} \frac{f_{i_1,n_1-n_2}\ldots f_{i_{r-1},n_{r-1}-n_r} f_{i_r,n_r}}{(n_1-t_1)^{s_1}\ldots(n_r-t_r)^{s_r}} z^{n_1-t_1}.$$

We call Parametrized *D.g.f. associated to $\{F_i\}_{i=1..m}$ given in (43) the previous iterated integral. In particular,* $\mathrm{Di}(\mathbf{F_t};\mathbf{s}) = \alpha_0^1(w)$.

Example. Let $\mathbf{z} = (z_1,\ldots,z_r)$. For the o.g.f.

$$F_i(z) = \frac{z_i z}{1 - z_i z} \quad \text{with} \quad |z_i| < 1, \text{ for } i = 1..m,$$

with the differential forms $\omega_0,\omega_{1,\bar{t}_1},\ldots,\omega_{m,\bar{t}_m}$, we have [22, 27]

$$\alpha_0^z(w) = \sum_{n_1>\ldots>n_r} \frac{z_1^{n_1-n_2}\ldots z_{r-1}^{n_{r-1}-n_r} z_r^{n_r}}{(n_1-t_1)^{s_1}\ldots(n_r-t_r)^{s_r}} z^{n_1-t_1},$$

and $\mathrm{Di}(F_1,\ldots,F_r;\mathbf{s})$ is nothing but the *multiple* Lerch function

$$\alpha_0^1(w)=\sum_{n_1>\ldots>n_r>0}\frac{z_1^{n_1}(z_2/z_1)^{n_2}\ldots(z_r/z_{r-1})^{n_r}}{(n_1-t_1)^{s_1}\ldots(n_r-t_r)^{s_r}}.$$

Therefore, as iterated integral, the generalized and the parametrized D.g.f. and the generalized Lerch function verify the shuffle relation (38) over the alphabet $\{x_0,\ldots,x_m\}$ [22]. In particular, their special values verify this kind of relation.

3.2. Periodic D.g.s and Hurwitz polyzêtas.

Definition. The generalized Lerch function is defined as follows

$$\Phi_{\mathbf{s}}(z;\mathbf{t})=\sum_{n_1>\cdots>n_r>0}\frac{z^{n_1}}{(n_1-t_1)^{s_1}\ldots(n_r-t_r)^{s_r}}\quad\text{for}\quad |z|<1.$$

The associated Hurwitz polyzêta is defined by

$$\zeta(\mathbf{s};\mathbf{t})=\sum_{n_1>\cdots>n_r>0}\frac{1}{(n_1-t_1)^{s_1}\ldots(n_r-t_r)^{s_r}}=\Phi_{\mathbf{s}}(1;\mathbf{t}).$$

With the notation of Example 3.1, $z_1=\ldots=z_r$ and for $\mathbf{t}$ fixed, the generalized Lerch function appear as an iterated integral and so verify the shuffle product. Consequently, as z tends to 1, for any convergent compositions $\mathbf{s}$ and $\mathbf{s}'$,

$$\zeta(\mathbf{s}\shuffle\mathbf{s}';\mathbf{t})=\zeta(\mathbf{s};\mathbf{t})\zeta(\mathbf{s}';\mathbf{t}). \tag{44}$$

Note that the Hurwitz polyzêtas can be also encoded by *noncommutative rational series*

$$x_0^{s_1-1}(t_1x_0)^{*s_1}x_1\ldots x_0^{s_r-1}(t_rx_0)^{*s_r}x_1 \tag{45}$$

as given in [27] with two differential forms ω_0 and ω_1.

These polyzêtas contain divergent terms which are looked at via syntactic regularizations of divergent Hurwitz polyzêtas with respect to the associated shuffle products as in [27, 28]. Note also that if $t_1=t_2=\cdots=0$ then $\Phi_{\mathbf{s}}(z;\mathbf{0})=\mathrm{Li}_{\mathbf{s}}(z)$ and $\zeta(\mathbf{s};\mathbf{0})=\zeta(\mathbf{s})$.

Proposition 3.3. *If the* $\{f_{i,n}\}_{n\geq1}$ *for* $i=1..N$, *are* periodic of the same period K *then with the differential forms* $\omega_0,\omega_{1,\bar t_1},\ldots,\omega_{m,\bar t_m}$, *we have*

$$\mathrm{Di}(\mathbf{F_t};\mathbf{s})=\frac{1}{K^{\sum_{l=1}^r s_l}}\sum_{b_1,\ldots,b_r=0}^{K-1}f_{i_1,b_1}\ldots f_{i_r,b_r}$$
$$\zeta[(s_1;\frac{t_1-\sum_{l=1}^r b_l}{K}),\ldots,(s_r;\frac{t_r-b_r}{K})].$$

Proof. Let us sketch the proof in case of $r = 2$:

$$\begin{aligned}\mathrm{Di}(\mathbf{F_t};\mathbf{s}) &= \sum_{n_1>n_2>0} \frac{f_{i_1,n_1-n_2} f_{i_2,n_2}}{(n_1-t_1)^{s_1}(n_2-t_2)^{s_2}} \\ &= \sum_{m_1,m_2>0} \frac{f_{i_1,m_1} f_{i_2,m_2}}{(m_1+m_2-t_1)^{s_1}(m_2-t_2)^{s_2}}.\end{aligned}$$

The assumption of periodicity gives $f_{i,k+K} = f_{i,k}$ for all $k \geq 1$ and $i = 1, 2$. Therefore,

$$\begin{aligned}&\mathrm{Di}(\mathbf{F_t};\mathbf{s}) \\ &= \sum_{b_1,b_2=0}^{K-1} \sum_{m_1,m_2\geq 1} \frac{f_{i_1,m_1K+b_1} f_{i_2,m_2K+b_2}}{[(m_1+m_2)K+b_1+b_2-t_1]^{s_1}(m_2K+b_2-t_2)^{s_2}} \\ &= \sum_{b_1,b_2=0}^{K-1} f_{i_1,b_1} f_{i_2,b_2} \sum_{n_1>n_2>0} \frac{1}{(n_1K+b_1+b_2-t_1)^{s_1}(n_2K+b_2-t_2)^{s_2}} \\ &= \frac{1}{K^{s_1+s_2}} \sum_{b_1,b_2=0}^{K-1} f_{i_1,b_1} f_{i_2,b_2} \zeta\left[\left(s_1;\frac{t_1-(b_1+b_2)}{K}\right),\left(s_2;\frac{t_2-b_2}{K}\right)\right].\end{aligned}$$

The proof is easily generalized for any positive integer r. □

In other words, if $\{f_{i,n}\}_{n\geq 1}$ are periodic of the same period K then generalized D.g.s. is a linear combination of Hurwitz polyzêtas. In particular, if $f_{i_1,n} = \ldots = f_{i_r,n} = 1$, for $n \geq 1$, then one can generalize the well known result for the Riemann and Hurwitz zêta functions.

Corollary 3.1. *For $K \in \mathbb{N}_+$, one has*

$$\zeta(\mathbf{s};\mathbf{t}) = \frac{1}{K^{\sum_{l=1}^r s_l}} \sum_{b_1,\ldots,b_r=0}^{K-1} \zeta[(s_1;\frac{t_1-\sum_{l=1}^r b_l}{K}),\ldots,(s_r;\frac{t_r-b_r}{K})].$$

Now let $q = e^{2\mathrm{i}\pi/m}$ be a m-th primitive root of unity. For any integers $i_1,\ldots,i_r$, we will use the notation $q^{\mathbf{i}}$ for $q^{i_1},\ldots,q^{i_r}$. Let $\mathbb{Q}(q)$ be the cyclotomic field generated by q.

Let us introduce also the o.g.f.

$$F_i(z) = \sum_{n=1}^{\infty} q^{in} z^n, \quad \text{for} \quad i = 1..m. \tag{46}$$

The *coloured* polyzêtas are a particular case of *periodic* D.g.s. $\mathrm{Di}(\mathbf{F};\mathbf{s})$ and they are defined as follows (for $0 \leq i_1 < \ldots < i_r \leq m-1$) [3, 4, 19]

$$\zeta\begin{pmatrix}\mathbf{s}\\ q^{\mathbf{i}}\end{pmatrix} = \sum_{n_1>\ldots>n_r>0} \frac{q^{i_1n_1}\ldots q^{i_rn_r}}{n_1^{s_1}\ldots n_r^{s_r}}. \tag{47}$$

In other terms, these can be obtained as a special value at $z = 1$ of the following D.g.s. associated to $\{F_i\}_{i=1,..,m}$ given in (46)

$$\zeta\begin{pmatrix} \mathbf{s} \\ q^{i_1}, q^{i_2-i_1} \ldots, q^{i_r-i_{r-1}} \end{pmatrix} = \alpha_0^1(x_0^{s_1-1}x_{i_1}\ldots x_0^{s_r-1}x_{i_r}). \tag{48}$$

So these verify the first shuffle relation (38) over the alphabet $\{x_0, \ldots, x_m\}$. By Proposition 3.3, the periodicity of $\{q^n\}_{n\geq 1}$ enables then to express the coloured polyzêtas as linear combinations of Hurwitz polyzêtas with coefficients in $\mathbb{Q}(q)$ [27] :

$$\zeta\begin{pmatrix} \mathbf{s} \\ q^{\mathbf{i}} \end{pmatrix} = \frac{1}{m^{\sum_{l=1}^r s_l}} \sum_{a_1-a_2,\ldots,a_{r-1}-a_r,a_r=0}^{m-1} q^{\sum_{l=1}^r i_l a_l}\zeta(\mathbf{s}; -\frac{\mathbf{a}}{m}). \tag{49}$$

Here, $\mathbf{a}/m$ stands for $(a_1/m, \ldots, a_r/m)$. Consequently, these coloured polyzêtas $\zeta\binom{\mathbf{s}}{q^{\mathbf{i}}}$ (with $0 \leq i_k \leq m-1$) are linear combinations, with coefficients in $\mathbb{Q}(q)$, of the rational parameters Hurwitz polyzêtas $\zeta(\mathbf{s}; -\mathbf{a}/m)$ with $a_1 - a_2, \ldots, a_{r-1} - a_r, a_r = 1..m$. They can also be viewed as the evaluations, at the m-th primitive root of unit q of some commutative polynomials on the $\mathbb{Q}$-algebra of Hurwitz polyzêtas. The reader can find the shuffle structures of Hurwitz polyzêtas in [27] inducing then the shuffle structures of Riemann polyzêtas. Conversely, by the distribution formula, we have a partial result [30] :

Proposition 3.4. *Let $\mathbf{a} = (a_1, \ldots, a_r)$ be a composition. If the parameters a_j satisfy the conditions $1 \leq a_1 - a_2, \ldots, a_{r-1} - a_r, a_r < m$ then*

$$\zeta(\mathbf{s}; -\frac{\mathbf{a}}{m}) = m^{\sum_{l=1}^r s_l - r} \sum_{i_1,\ldots,i_r=0}^{m-1} q^{-\sum_{l=1}^r i_l a_l}\zeta\begin{pmatrix} \mathbf{s} \\ q^{\mathbf{i}} \end{pmatrix}.$$

Proof. Let us set $\mathbf{s}$ and let

$$f(\mathbf{i}) = \frac{1}{q^{i_1\cdots+i_r}}\zeta\begin{pmatrix} s_1, s_2, \ldots, s_r \\ q^{i_1}, q^{i_2-i_1} \ldots, q^{i_r-i_{r-1}} \end{pmatrix},$$
$$g(\mathbf{c}) = \zeta\left(s_1, \ldots, s_r; -a_1/m, \ldots, -a_r/m\right),$$

with $a_j = (c_j + 1) + (c_{j+1} + 1) \ldots + (c_r + 1)$, for $j = 1 \ldots r$. Let us consider the lexicographical order on the index set

$$\mathcal{I} = \{\mathbf{i} = (i_1, \ldots, i_r) \in \{0, \ldots m-1\}^r\}$$

and on the index set

$$\mathcal{C} = \{\mathbf{c} = (c_1, \ldots, c_r) \in \{0, \ldots m-1\}^r\}.$$

Then $\left(f(\mathbf{i})\right)_{\mathbf{i}\in\mathcal{I}}$ (resp. $\left(g(\mathbf{c})\right)_{\mathbf{c}\in\mathcal{C}}$) can be viewed as the entries of a column vector $\mathbf{F}$ (resp. $\mathbf{G}$) of dimension mr.

Let $M(q)$ be the matrix $\left(q^{ij}\right)_{0\le i,j\le m-1}$ of determinant

$$\det M(q) = \prod_{0\le i<j\le m-1} (q^i - q^j) \neq 0. \tag{50}$$

The inverses of $M(q)$ and of the r-th tensor product of $M(q)$ are given by

$$M(q)^{-1} = m^{-1}M(q^{-1}) \quad \text{and} \quad [M(q)^{\otimes r}]^{-1} = m^{-r}M(q^{-1})^{\otimes r}. \tag{51}$$

By Corollary 3.1, we get

$$F = m^{-(s_1+\ldots+s_r)}M(q)^{\otimes r}\mathbf{G} \quad \text{and} \quad G = m^{s_1+\ldots+s_r-r}M(q^{-1})^{\otimes r}\mathbf{F}. \tag{52}$$

Or equivalently,

$$f(\mathbf{i}) = \sum_{c_1,\ldots,c_r=0}^{m-1} \frac{q^{i_1c_1+\ldots+i_rc_r}}{m^{s_1+\ldots+s_r}} g(\mathbf{c}),$$

$$g(\mathbf{c}) = \sum_{i_1,\ldots,i_r=0}^{m-1} \frac{m^{s_1+\ldots+s_r-r}}{q^{i_1c_1+\ldots+i_rc_r}} f(\mathbf{i}).$$

Hence, by setting $b_j = c_j+1$ (thus, $b_1 = a_2-a_1, \ldots, b_{r-1} = a_{r-1}-a_r, b_r = a_r$ and $b_j = 1\ldots m$) :

$$\zeta\left(s_1,\ldots,s_r;-\frac{a_1}{m},\ldots,-\frac{a_r}{m}\right) = \sum_{i_1,\ldots,i_r=0}^{m-1} \frac{m^{s_1+\ldots+s_r-r}}{q^{i_1b_1+\ldots+i_rb_r}}\zeta\begin{pmatrix} s_1,s_2,\ldots,s_r \\ q^{i_1},q^{i_2-i_1}\ldots,q^{i_r-i_{r-1}}\end{pmatrix}.$$

By changing the indexes $j_1 = i_1$ and $j_{n+1} = i_{n+1} - i_n$, or equivalently $i_n = j_1+\ldots+j_n \pmod m$. We get $q^{i_1b_1+\ldots+i_rb_r} = q^{j_1b_1+(j_1+j_2)b_2+\ldots+(j_1+\ldots+j_r)b_r} = q^{j_1a_1+\ldots+j_ra_r}$ leading to the final result. □

Example. We get in particular :

- For $1 \le a \le m, \zeta\left(s;-\frac{a}{m}\right) = \sum_{i=0}^{m-1} \frac{m^{s-1}}{q^{ia}}\zeta\begin{pmatrix} s \\ q^i\end{pmatrix}$. Thus, some constants like Catalan number which can be expressed as numerical parametrized Hurwitz polyzêtas

$$G = \sum_{n\ge1} \frac{(-1)^n}{(2n+1)^2} = \frac{1}{16}[\zeta(2;-\frac{1}{4}) - \zeta(2;\frac{1}{4})]$$

  can so be expressed on coloured polyzêtas.
- For $1 \le a_1 - a_2,\ a_2 \le m$,

$$\zeta\left(s_1,s_2;-\frac{a_1}{m},-\frac{a_2}{m}\right) = \sum_{i_1,i_2=0}^{m-1} \frac{m^{s_1+s_2-2}}{q^{i_1a_1+i_2a_2}}\zeta\begin{pmatrix} s_1,s_2 \\ q^{i_1},q^{i_2}\end{pmatrix}.$$

Moreover, we now can express the D.g.s. of periodic sequences $\{f_{i,n}\}_{n\geq 1}$ as coloured polyzêtas, as a direct consequence of propositions 3.3 and 3.4 [30] :

$$\mathrm{Di}(\mathbf{F};\mathbf{s}) = \frac{1}{m^r}\sum_{b_1,\ldots,b_r=1}^{m}\sum_{i_1,\ldots,i_r=0}^{m-1}\frac{f_{i_1,b_1}\ldots f_{i_r,b_r}}{q^{\sum_{l=1}^{r} i_l a_l}}\zeta\binom{\mathbf{s}}{q^{\mathbf{i}}}, \tag{53}$$

where the parameters a_j are defined as the sum $b_1+\ldots+b_j$, for $j=1\ldots r$. Note that the same expression holds in the case of the first member of (53) is a coloured polyzêta. It can be viewed also as the consequence of Corollary 3.1.

3.3. Integral representation of Hurwitz polyzêtas. Let F be an o.g.f. vanishing at $z=0$. The associated D.g.f. $\mathrm{Di}_s(F|z)$ can be obtained from of $F(z)$ via the *polylogarithmic transformation* as follows [22, 27]

Lemma 3.1. *The Dirichlet function* $\mathrm{Di}_s(F|z)$ *can be represented as*

$$\int_0^1 \log^{s-1}(\frac{1}{r})\frac{F(zr)}{\Gamma(s)}\frac{dr}{r} = \int_0^\infty \frac{F(ze^{-u})}{\Gamma(s)}\frac{du}{u^{1-s}}.$$

Proof. Since

$$\mathrm{Di}_s(F|z) = \alpha_0^z(x_0^{s-1};F) = \int_0^z \frac{\log^{s-1}(z/t)}{\Gamma(s)}F(t)\frac{dt}{t}$$

then the changes of variables $t = zr$ and $r = e^{-u}$ lead to the expected expressions. □

Example. If $f_t(z) = z^{1-t}/(1-z)$ then $\mathrm{Di}_s(f_t|z)$ is linked to the Lerch function $\Phi_{\mathbf{s}}(z;t)$ by $\mathrm{Di}_s(f_t|z) = z^{-t}\Phi_{\mathbf{s}}(z;t)$. We deduce the expression of the Lerch function

$$\Phi_{\mathbf{s}}(z;t) = z^t\int_0^1 \log^{s-1}(\frac{1}{r})\frac{f_t(zr)}{\Gamma(s)}\frac{dr}{r} = z^t\int_0^\infty \frac{f_t(ze^{-u})}{\Gamma(s)}\frac{du}{u^{1-s}}$$

so the D.g.s $\mathrm{Di}(f_t;s)$ is nothing but the Hurwitz zêta function

$$\zeta(s;t) = \int_0^1 \log^{s-1}(\frac{1}{r})\frac{f_t(r)}{\Gamma(s)}\frac{dr}{r} = \int_0^\infty \frac{f_t(e^{-u})}{\Gamma(s)}\frac{du}{u^{1-s}}.$$

Recall also that

$$\frac{e^{-u(1-t)}}{1-e^{-u}} = \sum_{k\geq 0}\frac{B_k(t)}{k!}u^{k-1},$$

where $B_k(t)$ is the k-th Bernoulli polynomial. Thus, by the Mellin's transformation, one obtains, for $t<1$, the regular expansion for the Hurwitz zêta function

$$\zeta(s;t) \asymp \sum_{k\geq 0}\frac{B_k(t)}{k!\Gamma(s)}\frac{1}{s-1+k},$$

saying that $\zeta(s;t)$ has also an analytic continuation to the complex plane as a meromorphic function with only one simple pole of residue 1 at $s=1$.

We first have to consider the D.g.s. with integer arguments. But this can be studied as a complex function of $s_i = \sigma_i + \mathrm{i}\tau_i$, for $i = 1,..,r$, with $\sigma_1 > 1, \sigma_1+\sigma_2 > 1, \ldots, \sigma_1+\cdots+\sigma_r > 1$. Either, if the power series $F_i(e^{-z})$ has a suitable asymptotic expansion at 0 and at ∞ then the Dirichlet generating functions associated to the sequences $\{f_{i,n}\}_{n\geq 1, i=1,..,N}$ are meromorphic functions in $\mathbb{C}^r$ and are given by

Proposition 3.5. *The function* $\mathrm{Di}_{\mathbf{s}}(\mathbf{F}|z)$ *can be represented as*

$$\begin{aligned}\mathrm{Di}_{\mathbf{s}}(\mathbf{F}|z) &= \int_{[0,1]^r} \prod_{j=1}^r \log^{s_j-1}(\frac{1}{u_j}) \frac{F_{i_j}(z\prod_{l=1}^j u_l)}{\Gamma(s_j)} \frac{du_j}{u_j} \\ &= \int_{\mathbb{R}_+^r} \prod_{j=1}^r \frac{F_{i_j}(ze^{-\sum_{l=1}^j u_l})}{\Gamma(s_j)} \frac{du_j}{u_j^{1-s_j}}.\end{aligned}$$

where $u_{r+1} = 1$.

Proof. The proof can be obtained by induction on r and by use of Lemma 3.1. □

Therefore, by taking $z=1$, one finally gets

Corollary 3.2. *The generalized D.g.s. can be represented as*

$$\begin{aligned}\mathrm{Di}(\mathbf{F};\mathbf{s}) &= \int_{[0,1]^r} \prod_{j=1}^r \log^{s_j-1}(\frac{1}{u_j}) \frac{F_{i_j}(\prod_{l=1}^j u_l)}{\Gamma(s_j)} \frac{du_j}{u_j} \\ &= \int_{\mathbb{R}_+^r} \prod_{j=1}^r \frac{F_{i_j}(e^{-\sum_{l=1}^j u_l})}{\Gamma(s_j)} \frac{du_j}{u_j^{1-s_j}}.\end{aligned}$$

where $u_{r+1} = 1$.

For $z=1$, the polylogarithmic transformation corresponds to the *Mellin transformation* and one obtains the D.g.s. $\mathrm{Di}(F/\Gamma(s);s)$ associated to $F(\tau)/\Gamma(s)$ with $F(\tau) = \sum_{k\geq 1} f_k q^k$ and $q = e^{-\tau}$.

Example. Let $\mathbf{s}' = (s_2,\ldots,s_r)$ and $\mathbf{F}' = (F_{i_2},\ldots,F_{i_r})$. By Lemma 3.1, the Dirichlet function $\mathrm{Di}_{\mathbf{s}}(\mathbf{F}|z)$ is a polylogarithmic transform of the function $F_{i_1}(z)\,\mathrm{Di}_{\mathbf{s}'}(\mathbf{F}'|z)$:

$$\begin{aligned}\mathrm{Di}_{\mathbf{s}}(\mathbf{F}|z) &= \int_0^1 \log^{s_1-1}(\frac{1}{t}) \frac{F_{i_1}(zt)\,\mathrm{Di}_{\mathbf{s}'}(\mathbf{F}'|zt)}{\Gamma(s_1)} \frac{dt}{t} \\ &= \int_0^\infty \frac{F_{i_1}(ze^{-t})\,\mathrm{Di}_{\mathbf{s}'}(\mathbf{F}'|ze^{-t})}{\Gamma(s_1)} \frac{dt}{t^{1-s_1}}.\end{aligned}$$

and the D.g.s. $\mathrm{Di}(\mathbf{F};\mathbf{s})$ is a Mellin transform of $F_{i_1}(e^{-z})\,\mathrm{Di}_{\mathbf{s}'}(\mathbf{F}'|e^{-z})/\Gamma(s_1)$. Therefore, by Corollary 3.2, $\mathrm{Di}(\mathbf{F};\mathbf{s})$ can be viewed as a multiple Mellin transform of the multivariate function $\prod_{j=1}^r F_{i_j}(e^{-\sum_{l=1}^j u_l})/\Gamma(s_j)$.

Corollary 3.3. *Let $f(z;t) = z^{1-t}/(1-z)$. We have*

$$\begin{aligned}\zeta(\mathbf{s};\mathbf{t}) &= \int_{[0,1]^r} \prod_{j=1}^r \log^{s_j-1}(\frac{1}{u_j}) \frac{f(\prod_{l=1}^j u_l;\bar{t}_j)}{\Gamma(s_j)} \frac{du_j}{u_j} \\ &= \int_{\mathbb{R}_+^r} \prod_{i=1}^r \frac{f(e^{-\sum_{l=1}^i u_l};\bar{t}_i)}{\Gamma(s_i)} \frac{du_i}{u_i^{1-s_i}}.\end{aligned}$$

Therefore, in this way, the Hurwitz polyzêta $\zeta(\mathbf{s};\mathbf{t})$ can be viewed as a multiple Mellin transform of the multivariate function

$$\prod_{i=1}^r \frac{f(e^{-\sum_{l=1}^i u_l};\bar{t}_i)}{\Gamma(s_i)} = \prod_{i=1}^r \frac{1}{\Gamma(s_i)} \frac{e^{(1-\bar{t}_i)\sum_{l=1}^i u_l}}{1-e^{-\sum_{l=1}^i u_l}}. \tag{54}$$

Its poles are $\sum_{l=1}^i u_l$, for $i \in \{1,\ldots,r\}$. To isolate the poles, we use the next proposition.

Proposition 3.6. *Let $\mathbf{s}$ be a composition of depth $r \geq 2$ and $\mathbf{t} \in]-\infty,1[^r$. Then,*

$$\begin{aligned}\zeta(\mathbf{s};\mathbf{t}) = &\int_0^{+\infty} \frac{e^{-(1-\bar{t}_r)x_r}}{1-e^{x_r}} \frac{x_r^{\sum_{j=1}^r s_j -1}}{\Gamma(s_r)} dx_r \\ &\int_{[0,1]^{r-1}} \prod_{i=1}^{r-1} \frac{e^{-(1-\bar{t}_i)\prod_{j=i}^r x_j}}{1-e^{-\prod_{j=i}^r x_j}} \frac{x_i^{\sum_{j=1}^i s_j -1}(1-x_i)^{s_{i+1}-1}}{\Gamma(s_i)} dx_i.\end{aligned}$$

Remark 3.1. Since $1-e^{-\prod_{j=i}^r x_j} \sim \prod_{j=i}^r x_j$ near 0, the equality of Proposition 3.6 gives an holomorphic expansion of the polyzêta $\zeta(\mathbf{s};\mathbf{t})$ over the set of tuples $(\mathbf{s},\mathbf{t}) \in \mathbb{C}^r \times]-\infty,1[^r$ such that $\Re(s_1) > 1$ and $\Re(s_i) > 0$, $\Re(\sum_{j=1}^i s_j) > i$ for any $i \in \{2,\ldots,r\}$.

Proof. We use the substitution

$$u_1 = x_1 \ldots x_r,\; u_2 = (1-x_1)x_2 \ldots x_r,\; \ldots,\; u_r = (1-x_{r-1})x_r.$$

Note that $\sum_{j=1}^i u_j = \prod_{j=i}^r x_j$ for all $i \in \{1,\ldots,r\}$. The Jacobian $\mathcal{J}_r = \partial(u_1,\ldots,u_r)/\partial(x_1,\ldots,x_r)$ is equal to $\prod_{k=2}^n x_k^{k-1}$ (see Lemma 6.1 in appendix). The change of variable is so admissible for $(u_1,\ldots,u_r)$ in $[0,+\infty[^r$ i.e. when $(x_1,\ldots,x_r)$ is in $\mathcal{D}_r = [0,1]^{r-1} \times [0,+\infty[$.

$$\zeta(\mathbf{s};\mathbf{t}) \prod_{i=1}^r \Gamma(s_i)$$

$$= \int_{\mathcal{D}_r} \frac{e^{-(1-\overline{t_1})\prod_{j=1}^r x_j}}{1-e^{\prod_{j=1}^r x_j}} \prod_{j=1}^r x_j^{s_1-1}$$

$$\Big(\prod_{i=2}^r \frac{e^{-(1-\overline{t_i})\prod_{j=i}^r x_j}}{1-e^{\prod_{j=i}^r x_j}}(1-x_{i-1})^{s_i-1}\prod_{j=i}^r x_j^{s_i-1}\Big)\prod_{i=1}^r x_i^{i-1}dx_i$$

$$= \int_0^{+\infty} \frac{e^{-(1-\overline{t_r})x_r}}{1-e^{x_r}} x_r^{\sum_{j=1}^r s_j-1} dx_r$$

$$\int_{[0,1]^{r-1}} \prod_{i=1}^{r-1} \frac{e^{-(1-\overline{t_i})\prod_{j=i}^r x_j}}{1-e^{\prod_{j=i}^r x_j}} x_i^{\sum_{j=1}^i s_j-1}(1-x_i)^{s_{i+1}-1}dx_i.$$

□

4. Analytic continuation and structure of poles

In this section we assume that $\mathbf{t} \in]-\infty, 1[^r$.

4.1. Analytic continuation of Hurwitz polyzêtas.

4.1.1. *Generality over the regularization near* 0.

Notation. We denote by $f_{x_1^{k_1}\ldots x_r^{k_r}}$ the partial differentiation $\frac{\partial^{k_1+\ldots+k_r}}{\partial_{x_1}^{k_1}\cdots\partial_{x_r}^{k_r}}f$, and by $\mathbb{Z}_{<0}$ the set of negative integers.

Definition. Let f be a function C^0 over an interval I which contains 0, and C^∞ at 0. For all $s = s_r + \mathrm{i}s_i \in \mathbb{C}\setminus\mathbb{Z}_{<0}$,

$$\mathcal{R}_\rho[f](s) = \int_0^\rho x^s[f(x) - \sum_{k=0}^{n_s} f_{x^k}(0)\frac{x^k}{k!}]dx \ + \sum_{k=0}^{n_s}\frac{f_{x^k}(0)}{k!}\frac{\rho^{k+s+1}}{k+s+1}$$

is defined for any positive real $\rho \in I$ independently of any integer $n_s > -s_r - 2$. In particular, if $s_r > -1$, then

$$\mathcal{R}_\rho[f](s) = \int_0^\rho x^s f(x)dx.$$

Remark 4.1. Moreover, for any positive reals ρ_1, ρ_2 and A in I,

$$\mathcal{R}_{\rho_2}[f](s) = \int_{\rho_1}^{\rho_2} x^s f(x)dx + \mathcal{R}_{\rho_1}[f](s),$$

$$\int_{\rho_1}^A x^s f(x)dx + \mathcal{R}_{\rho_1}[f](s) = \int_{\rho_2}^A x^s f(x)dx + \mathcal{R}_{\rho_2}[f](s).$$

One can find this definition in some books about Distributions as [18].

Remark 4.2. Over any open set $\Re(s) > -n_s$, the expression of definition 4.1.1 show that $s \mapsto \mathcal{R}_\rho[f](s)$ is holomophic over $\mathbb{C}\setminus\mathbb{Z}_{<0}$ and that the poles are simples.

Proposition 4.1. *Let I be an interval which contains 0, let J be a real open set, and let f be a continuous function defined over $I \times J$. For any positive real $\rho \in I$ and for any $s \in \mathbb{C} \setminus \mathbb{Z}_{<0}$,*

(i) *$\mathcal{R}_\rho[f(x,y)](s)$ is defined for any $y \in J$ such that $f(.,y)$ is indefinitely differentiable at 0.*

(ii) *if moreover J is an interval, and if $f \in C^\infty(I \times J)$, then the function $y \mapsto \mathcal{R}_\rho[f(x,y)](s)$ is C^1 over J and*

$$\frac{\partial}{\partial y}\mathcal{R}_\rho[f(x,y)](s) = \mathcal{R}_\rho[\frac{\partial}{\partial y}f(x,y)](s).$$

Proof. (i) comes from the definition 4.1.1. Let $s \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, and an integer $n_s > -\Re(s) - 2$. The function $g(x,y) = x^s[f(x,y) - \sum_{k=0}^{n_s} f_{x^k}(0,y)x^k/k!]$ is differentiable at y over $[0,\rho] \times J$ and the partial derivative

$$\begin{aligned}\forall (x,y) \in I \times J,\ \frac{\partial g}{\partial y}(x,y) &= \frac{\partial f}{\partial y}(x,y) - \sum_{k=0}^{n_s} \frac{\partial}{\partial y}\frac{\partial^k f}{\partial x^k}(0,y))\frac{x^k}{k!} \\ &= \frac{\partial f}{\partial y}(x,y) - \sum_{k=0}^{n_s} \frac{\partial^k}{\partial x^k}\frac{\partial f}{\partial y}(0,y)\frac{x^k}{k!}\end{aligned}$$

is continuous over $[0,\rho] \times J$. So, the function $\tilde{g}$

$$\tilde{g}\ :\ y \longmapsto \int_0^\rho x^s(f(x,y) - \sum_{k=0}^{n_s} f_{x^k}(0,y)\frac{x^k}{k!})dx = \int_0^\rho g(x,y)dx$$

is C^1 over J, with derivative

$$\int_0^\rho \frac{\partial g}{\partial y}(x,y)dx.$$

Moreover, $h(y) = \sum_{k=0}^{n_s} f_{x^k}(0,y)\rho^{k+s+1}/(k!(k+s+1))$ is differentiable over $[0,\rho] \times J$ and its derivative is

$$\frac{\partial h}{\partial y}(y) = \sum_{k=0}^{n_s} \frac{1}{k!}\frac{\partial^k}{\partial x^k}\frac{\partial f}{\partial y}(0,y)\frac{\rho^{k+s+1}}{k+s+1}.$$

The (ii) follows. □

By recurrence, we deduce :

Corollary 4.1. *Let I be an interval which contains 0, let J be a real interval and let $f \in C^\infty(I \times J)$. Then, for any $s \in \mathbb{C} \setminus \mathbb{Z}_{<0}$ and any positive real $\rho \in I$, the function $y \mapsto \mathcal{R}_\rho[f(x,y)](s)$ is C^∞ over J, and, for any $k \in \mathbb{N}$,*

$$\frac{\partial^k}{\partial y^k}\mathcal{R}_\rho[f(x,y)](s) = \mathcal{R}_\rho[\frac{\partial^k}{\partial y^k}f(x,y)](s).$$

Corollary 4.2. *Let U be an open of $\mathbb{C}$, let $f \in C^\infty([0,1] \times U)$ and let $\rho \in]0,1[$. Suppose $f(x,.)$ is holomorphic over U for each $x \in [0,1]$. Then, for any $a \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, the function $s \mapsto \mathcal{R}_\rho[f(x,s)](a)$ is holomophic over U.*

Proof. Thanks to the proposition 4.1, $x_z \mapsto \mathcal{R}_\rho[f(x, x_z + \mathrm{i}y_z)](a)$ and $y_z \mapsto \mathcal{R}_\rho[f(x, x_z + \mathrm{i}y_z)](a)$ are continuously differentiable and,

$$\begin{aligned}\frac{\partial}{\partial y_z}\mathcal{R}_\rho[f(x, x_z + \mathrm{i}y_z)](a) &= \mathcal{R}_\rho[\frac{\partial}{\partial y_z} f(x, x_z + \mathrm{i}y_z)](a)\\ &= \mathcal{R}_\rho[\mathrm{i}\frac{\partial}{\partial x_z} f(x, x_z + \mathrm{i}y_z)](a)\\ &= \mathrm{i}\frac{\partial}{\partial x_z}\mathcal{R}_\rho[f(x, x_z + \mathrm{i}y_z)](a).\end{aligned}$$

□

Lemma 4.1. *Let $f \in C^\infty([0,1])$, let M be an integer and let $\rho \in]0,1[$. Then, the function*

$$\tilde{f} : s \mapsto \int_0^\rho x^s (f(x) - \sum_{k=0}^{M} f_{x^k}(0)\frac{x^k}{k!})$$

is holomorphic over the open set $\mathcal{U}_M = \{s \in \mathbb{C} / \Re(s) > -M-1\}$.

Proof. Given a compact set $\mathcal{K} \subset \mathcal{U}_M$, and the function

$$g : (x,s) \longmapsto x^s (f(x) - \sum_{k=0}^{M} f_{x^k}(0)\frac{x^k}{k!}).$$

- For any $x \in [0,\rho]$, the function $s \mapsto g(x,s)$ is holomorphic over $\mathcal{K}$.
- For any $s \in \mathcal{K}$, the function $x \mapsto g(x,s)$ is integrable over $[0,\rho]$.
- The function g is continuous over the compact $[0,\rho] \times \mathcal{K}$, so there exists $M_\mathcal{K} \in \mathbb{R}_+$ such that

  $$\forall (x,s) \in [0,\rho] \times \mathcal{K},\ |g(x,s)| \le M_\mathcal{K}.$$

  The function $x \mapsto M_\mathcal{K}$ is integrable over $[0,\rho]$.

So the function $\tilde{f} = \int_0^\rho g(x,s)dx$ is holomorphic over each compact included in $\mathcal{U}_M$ and so, over $\mathcal{U}_M$. □

Lemma 4.2. *Let $f \in \mathbb{C}^\infty([0,1]^2)$. We have the equality*

$$\mathcal{R}_{\rho_2}[\mathcal{R}_{\rho_1}[f(x,y)](a_1)](a_2) = \mathcal{R}_{\rho_1}[\mathcal{R}_{\rho_2}[f(x,y)](a_2)](a_1)$$

as meromorphic function of (a_1,a_2) over $\mathbb{C}^2$.

Proof. The functions

$$\begin{array}{rcll} f_{1,2} : (a_1,a_2) & \mapsto & \mathcal{R}_{\rho_1}[\mathcal{R}_{\rho_2}[f(x,y)](a_2)](a_1) & \text{and}\\ f_{2,1} : (a_1,a_2) & \mapsto & \mathcal{R}_{\rho_2}[\mathcal{R}_{\rho_1}[f(x,y)](a_1)](a_2) & \end{array}$$

are meromorphic in $\mathbb{C}^2$ with set of poles $(\mathbb{Z}_{<0})^2$ so discrete. But, over the open set $\Re(a_1) > 0$, $\Re(a_2) > 0$, we have

$$\begin{aligned} f_{1,2}(a_1, a_2) = \mathcal{R}_{\rho_2}[\mathcal{R}_{\rho_1}[f(x,y)](a_1)](a_2) &= \int_{[0,1]^2} x^{a_1} y^{a_2} f(x,y) dx dy \\ &= \mathcal{R}_{\rho_1}[\mathcal{R}_{\rho_2}[f(x,y)](a_2)](a_1) \\ &= f_{2,1}(a_1, a_2). \end{aligned}$$

So functions $f_{1,2}$ and $f_{2,1}$ are equal [32]. □

4.1.2. *Regularization between* 0 *and* 1.

Lemma 4.3. *Let f be a function C^0 over the interval $[0,1]$, and C^∞ at 1. For all $s = s_r + \mathrm{i}s_i \in \mathbb{C} \setminus \mathbb{Z}_{<0}$,*

$$\int_\rho^1 (1-x)^s [f(x) - \sum_{k=0}^{n_s} f_{x^k}(1) \frac{(x-1)^k}{k!}] dx + \sum_{k=0}^{n_s} (-1)^k \frac{f_{x^k}(1)}{k!} \frac{(1-\rho)^{k+s+1}}{k+s+1}$$

is defined for any $\rho \in [0,1[$ independently of any integer $n_s > -s_r - 2$ and is egal to $\mathcal{R}_{1-\rho}[f(1-x)](s)$.

Proof. The function $x \longmapsto (1-x)^s[f(x) - \sum_{k=0}^{n_s} f_{x^k}(1)(x-1)^k/k!]$ is continuous over $[1-\rho, 1[$, equivalent to $(x-1)^a$ with $a > -1$ at 1, so the expression is defined. The change of variables $x \mapsto (1-x)$ shows that its egal to $\mathcal{R}_{1-\rho}[f(1-x)](s)$, so its is independent of n_s. □

We have to study integral of type $\int_0^1 f(x) x^a (1-x)^b dx$, with a function f in $C^\infty([0,1])$. It is defined for $\Re(a) > -1$, $\Re(b) > -1$. To continue for any $(a,b) \in \mathbb{C}^2$, we decompose the integral between the singularities and use of regularized : for $\Re(a) > -1$ and $\Re(b) > -1$ (where the integral is defined), and with ρ_1 and ρ_2 in $]0,1[$,

$$\begin{aligned} \int_0^1 f(x) x^a (1-x)^b dx &= \int_0^{\rho_1} x^a (f(x)(1-x)^b) dx + \int_{\rho_1}^{\rho_2} f(x) x^a (1-x)^b dx \\ &\quad + \int_{\rho_2}^1 (1-x)^b (f(x) x^a) dx \qquad (55) \\ &= \mathcal{R}_{\rho_1}[f(x)(1-x)^b](a) + \int_{\rho_1}^{\rho_2} f(x) x^a (1-x)^b dx \\ &\quad + \mathcal{R}_{1-\rho_2}[f(1-x)(1-x)^a](b). \qquad (56) \end{aligned}$$

Definition. Let J be an interval of $\mathbb{R}$. Let $f \in C^0([0,1])$ and indefinitely differentiable at 0 and 1. The expression

$$\mathcal{R}_{\rho_1}[f(x)(1-x)^b](a) + \int_{\rho_1}^{\rho_2} f(x) x^a (1-x)^b dx + \mathcal{R}_{1-\rho_2}[f(1-x)(1-x)^a](b)$$

is defined for all $(a,b) \in \big(\mathbb{C} \setminus \mathbb{Z}_{<0}\big)^2$ independently of any reals ρ_1 and ρ_2 in $]0,1[$. We call it $\mathcal{R}_0^1[f](a,b)$.

For sake of simplicity, we will always use the above definition with $\rho_1 = \rho_2 = \rho$. Note that if $\Re(a) > -1$ and $\Re(b) > -1$, then

$$\mathcal{R}_0^1[f](a,b) = \int_0^1 f(x)x^a(1-x)^b dx. \tag{57}$$

Remark 4.3. >From Remark 4.2, we deduce that $(a,b) \mapsto \mathcal{R}_0^1[f](a,b)$ is holomorphic over $\big(\mathbb{C} \setminus \mathbb{Z}_{<0}\big)^2$ and these poles are simples.

Proposition 4.2. *Given $f \in C^\infty([0,1] \times J)$. Then, for any $(a,b) \in \big(\mathbb{C} \setminus \mathbb{Z}_{<0}\big)^2$, the function $y \mapsto \mathcal{R}_0^1[f(x,y)](a,b)$ is C^∞ over J, and, for any $k \in \mathbb{N}$, we have*

$$\frac{\partial}{\partial y^k}\mathcal{R}_0^1[f(x,y)](a,b) = \mathcal{R}_0^1[\frac{\partial}{\partial y^k} f(x,y)](a,b).$$

Proof. It comes from

$$\mathcal{R}_0^1[f(x,y)](a,b) = \mathcal{R}_\rho[f(x,y)(1-x)^b](a) + \mathcal{R}_{1-\rho}[f(1-x,y)(1-x)^a](b)$$

for $\rho \in]0,1[$, and from Corollary 4.1. □

Corollary 4.3. *Let U be an open set over $\mathbb{C}$ and let $f \in C^\infty([0,1] \times U)$. Assume $f(x,.)$ is holomorphic over U for each $x \in [0,1]$. Then, for any $(a,b) \in \big(\mathbb{C} \setminus \mathbb{Z}_{<0}\big)^2$, the function $s \mapsto \mathcal{R}_0^1[f(x,s)](a,b)$ is holomorphic over U.*

Proof. According to Proposition 4.2, $x_z \mapsto \mathcal{R}_0^1[f(x,x_z+\mathrm{i}y_z)](a,b)$ and $y_z \mapsto \mathcal{R}_0^1[f(x,x_z+\mathrm{i}y_z)](a,b)$ are continuously differentiable and,

$$\begin{aligned}\frac{\partial}{\partial y_z}\mathcal{R}_0^1[f(x,x_z+\mathrm{i}y_z)](a,b) &= \mathcal{R}_0^1[\frac{\partial}{\partial y_z} f(x,x_z+\mathrm{i}y_z)](a,b)\\ &= \mathcal{R}_0^1[\mathrm{i}\frac{\partial}{\partial x_z} f(x,x_z+\mathrm{i}y_z)](a,b)\\ &= \mathrm{i}\frac{\partial}{\partial x_z}\mathcal{R}_0^1[f(x,x_z+\mathrm{i}y_z)](a,b).\end{aligned}$$

□

Lemma 4.4. *Given $f \in \mathbb{C}^\infty([0,1]^2)$. We have the equality of meromorphic functions of (a_1,b_1,a_2,b_2) over $\mathbb{C}^4$:*

$$\mathcal{R}_0^1[\mathcal{R}_0^1[f(x,y)](a_1,b_1)](a_2,b_2) = \mathcal{R}_0^1[\mathcal{R}_0^1[f(x,y)](a_2,b_2)](a_1,b_1).$$

Proof. It comes from

$$\mathcal{R}_0^1[g(x)](a,b) = \mathcal{R}_\rho[g(x)(1-x)^b](a) + \mathcal{R}_{1-\rho}[g(1-x)(1-x)^a](b)$$

for $\rho \in]0,1[$, and from Lemma 4.2 as show in appendix (Lemma 6.2). □

4.1.3. ***Analytic continuation of Hurwitz polyzêtas.*** In order to simplify the expression of $\zeta(\mathbf{s},\mathbf{s})$ found in proposition 3.6, let $h(x,t)=xe^{-(1-t)x}/(1-e^{-x})$. The function $h(.,t)$, as product of $x/(1-e^{-x})$ and $e^{-(1-t)x}$, can be developed, for any real t, in a power series of radius of convergence 2π :

$$h(z,t)=\sum_{k=0}^{+\infty}\frac{B_k(t)}{k!}z^k \tag{58}$$

where the B_k are the Bernoulli polynomials. To simplify the expression of $h(.,\bar{t}_i)$ as a power series, we use the notation :

Notation. $\beta_k^i=B_k(\bar{t}_i)/k!$.

The function $h(.,t)$ is so in $C^{+\infty}(]-2\pi,2\pi[)$, but its definition shows that $h(.,t)\in C^{+\infty}(\mathbb{R}\setminus\{0\})$; consequently, the function $h(.,t)$ is in $C^{+\infty}(\mathbb{R})$. For $\Re(s_1)>1$, $\Re(s_i)>0$ and $\Re(\sum_{j=1}^i s_i)>i$ for all $i\in\{2,\ldots,r\}$,

$$\begin{aligned}\zeta(\mathbf{s};\mathbf{t})=\int_0^{+\infty}h(y_r,\bar{t}_r)\frac{y_r^{\sum_{j=1}^r s_j-r-1}}{\Gamma(s_r)}dy_r\\ \int_{[0,1]^{r-1}}\prod_{i=1}^{r-1}h(\prod_{k=i}^r y_i,\bar{t}_i)\frac{y_i^{\sum_{j=1}^i s_j-i-1}}{\Gamma(s_i)}(1-y_i)^{s_{i+1}-1}dy_i.\end{aligned} \tag{59}$$

By splitting the integral $\int_0^{+\infty}$ onto $\int_0^1$ and $\int_1^{+\infty}$,

$$\zeta(\mathbf{s};\mathbf{t})=\frac{1}{\prod_{i=1}^r\Gamma(s_i)}(\Phi_1(\mathbf{s};\mathbf{t})+\Phi_2(\mathbf{s};\mathbf{t})) \tag{60}$$

where, with $s_{r+1}=1$,

$$\Phi_1(\mathbf{s};\mathbf{t})=\int_{[0,1]^r}\prod_{i=1}^r h(\prod_{k=i}^r y_i,\bar{t}_i)y_i^{\sum_{j=1}^i s_j-i-1}(1-y_i)^{s_{i+1}-1}dy_i. \tag{61}$$

and,

$$\begin{aligned}\Phi_2(\mathbf{s};\mathbf{t})=\int_1^{+\infty}h(y_r,\bar{t}_r)y_r^{\sum_{j=1}^r s_j-r-1}dy_r\\ \int_{[0,1]^{r-1}}\prod_{i=1}^{r-1}h(\prod_{k=i}^r y_i,\bar{t}_i)y_i^{\sum_{j=1}^i s_j-i-1}(1-y_i)^{s_{i+1}-1}dy_i.\end{aligned} \tag{62}$$

Lemma 4.5. *The function Φ_1 defined in (61) can be extended in a meromorphic function over $\mathbb{C}^r$. Its set of poles is $s_i\in-\mathbb{N}$, $\sum_{j=1}^i s_j\in i-\mathbb{N}$, for $i\in\{1,\ldots,r-1\}$. These poles are all simple.*

For $i\in\{1,\ldots,r\}$, we put $g_i(x,y)=h(xy,\bar{t}_i)$. Note that all g_i are in $C^{\infty}(\mathbb{R}^2)$. In this way, Φ_1 can be written, for $\Re(s_1)>1$, $\Re(s_i)>0$ and

$\Re(\sum_{j=1}^{i} s_i) > i$ for all $i \in \{2, \dots, r\}$,

$$\begin{aligned}\Phi_1(\mathbf{s};\mathbf{t}) =& \mathcal{R}_0^1[h(y_r,\bar{t}_r)\mathcal{R}_0^1[h(y_r y_{r-1},\bar{t}_{r-1})\mathcal{R}_0^1[\dots\mathcal{R}_0^1[h(y_r\dots y_2,\bar{t}_2)\\ &\mathcal{R}_0^1[h(y_r\dots y_1,\bar{t}_1)](s_1-2,s_2-1)](s_1+s_2-3,s_3-1)\dots\\ &\dots](s_1+\dots+s_{r-1}-r,s_r-1)](s_1+\dots+s_r-r-1,0) \qquad (63)\\ =& \mathcal{R}_0^1[g_r(y_r,1)\mathcal{R}_0^1[g_{r-1}(y_{r-1},y_r)\mathcal{R}_0^1[\dots\mathcal{R}_0^1[g_2(y_2,y_r\dots y_3)\\ &\mathcal{R}_0^1[g_1(y_1,y_r\dots y_2)](s_1-2,s_2-1)](s_1+s_2-3,s_3-1)\dots\\ &\dots](s_1+\dots+s_{r-1}-r,s_r-1)](s_1+\dots+s_r-r-1,0) \qquad (64)\end{aligned}$$

Proof. Thanks to proposition 4.2, $y \mapsto \mathcal{R}_0^1[g_1(y_1,y)](a_1,b_1)$ exists and is in $C^\infty(\mathbb{R})$ for $(a_1,b_1) \in \left(\mathbb{C}\setminus\mathbb{Z}_{<0}\right)^2$. So

$$(y_2,y) \mapsto g_2(y_2,y)\mathcal{R}_0^1[g_1(y_1,y_2y)](a_1,b_1) \in C^\infty([0,1]\times\mathbb{R})$$

and so (proposition 4.2)

$$y \mapsto \mathcal{R}_0^1[g_2(y_2,y)\mathcal{R}_0^1[g_1(y_1,y_2y)](a_1,b_1)](a_2,b_2)$$

exists and is in $C^\infty(\mathbb{R})$. Step by step this process checks that the expression (64) is defined – and so gives an expansion of Φ_1 – for s_1-2, s_2-1, s_1+s_2-3, s_3-1, …, $s_{r-1}-1$, $s_1+\dots+s_r-r-1$ in $\mathbb{C}\setminus\mathbb{Z}_{<0}$. Moreover, thanks to Remark 4.3 and proposition 4.2, these poles are simples. □

Notation. Let $g(y_1,\dots,y_r) = \prod_{i=1}^{r} g_i(y_i, y_r\dots y_{i+1}) = \prod_{i=1}^{r} h(y_r\dots y_i,\bar{t}_i)$.

Remark 4.4. The equality (64) extended to $\mathbb{C}^r$ gives the expression of the meromorphic continuation of Φ_1, thus

$$\begin{aligned}\Phi_1(\mathbf{s};\mathbf{t}) =& \mathcal{R}_0^1[\mathcal{R}_0^1[\dots\\ &\dots\mathcal{R}_0^1[\mathcal{R}_0^1[g(y_1,\dots,y_r)](s_1-2,s_2-1)](s_1+s_2-3,s_3-1)\dots\\ &\dots](s_1+\dots+s_{r-1}-r,s_r-1)](s_1+\dots+s_r-r-1,0),\end{aligned}$$

for s_1-2, s_2-1, s_1+s_2-2, s_3-1, …, $s_{r-1}-1$, $s_1+\dots+s_r-r-1$ in $\mathbb{C}\setminus\mathbb{Z}_{<0}$. Note that, thanks to lemma 4.4, this expression is **independent** of the order of the regularized $\mathcal{R}_0^1$. In other term, any variable is privileged because the operator

$$f \longmapsto \mathcal{R}_0^1[\dots\mathcal{R}_0^1[f(y_1,\dots,u_r)](a_{\sigma(1)},b_{\sigma(1)})\dots](a_{\sigma(r)},b_{\sigma(r)})$$

is the same for all permutation σ in $\mathfrak{S}_r$.

Lemma 4.6. *The function Φ_2 defined in (62) can be continued as a meromorphic function over $\mathbb{C}^r$. Its set of poles is $s_i \in -\mathbb{N}$ for $i \in \{2,\dots,r\}$, $\sum_{j=1}^{i} s_j \in i-\mathbb{N}$ for $i \in \{1,\dots,r-1\}$. These poles are all simples.*

Note that Φ_2 verify, for $\Re(s_1) > 1$, $\Re(s_i) > 0$ for $i \in \{2, \ldots, r\}$ and $\Re(\sum_{j=1}^{i} s_i) > i$ for all $i \in \{2, \ldots, r\}$,

$$\Phi_2(\mathbf{s}, \mathbf{t}) = \mathcal{R}_0^1[\ldots \mathcal{R}_0^1[\int_1^{+\infty} g(y_1, \ldots, y_r) dy_r](s_1 - 2, s_2 - 1) \\ \ldots][s_1 + \ldots + s_{r-1} - r, s_r - 1). \tag{65}$$

Proof. The function g is C^∞ over $\mathbb{R}^r$, and, for any compact, each partial derivative $g_{y_1^{k_1} \ldots y_{r-1}^{k_{r-1}}}$ is absolutely majorized over $\mathbb{R}_+^r$ by a function $Me^{-\alpha y_r}$, with $M \in \mathbb{R}_+$ (the function $x \mapsto x/(1 - e^{-x})$ is C^∞ so the derivatives are bounded over any compact - in particular ones containing 0 - and their expression is $P(x, e^{-x})/(1 - e^{-x})^k$, where P is a polynom; so by multiplying by the derivatives of $e^{-(1-t)x}$, we get the announced form). So the function

$$(y_1, \ldots, y_{r-1}) \mapsto \int_1^{+\infty} g(y_1, \ldots, y_r) dy_r,$$

is C^∞ over $\mathbb{R}_+^{r-1}$ and the use of Proposition 4.2, applied step by step, shows that Φ_2 can be defined by equality (65) for $s_1 - 2$, $s_2 - 1$, $s_1 + s_2 - 3$, $s_3 - 1$, $\ldots$, $s_r - 1$, $s_1 + \ldots + s_{r-1} - r$ in $\mathbb{C} \setminus \mathbb{Z}_{<0}$. Moreover, Remark 4.2 and Corollary 4.3 show step by step that this extension is holomorphic over this domain, and, thanks to Remark 4.3 and Proposition 4.2, these poles are simple. $\square$

Remark 4.5. The equality (65) gives the expression of the meromorphic continuation of Φ_2 over $\mathbb{C}^r$. Note that this expression is independent of the order of the operator $\mathcal{R}_0^1$.

Theorem 4.1. *Given* $\mathbf{t} \in]-\infty, 1[^r$, *there exists an analytic continuation of the function* $\zeta(\mathbf{s}; \mathbf{t})$ *over* $\mathbb{C}^r$, *which is holomorphic over the set of* $\mathbf{s} \in \mathbb{C}^r$ *such that* $s_1 \neq 1$, $\sum_{j=1}^{i} s_j \notin i - \mathbb{N}$ *for all* $i \in \{2, \ldots, r\}$. *Moreover, the poles of this continuation are simple.*

Proof. This comes from the fact that the poles of Φ_1 and Φ_2 are simple, and so the poles $s_i \in \mathbb{N}$, $i \in \{1, \ldots, r\}$ disappear since

$$\zeta(\mathbf{s}; \mathbf{t}) = \frac{1}{\prod_{i=1}^{r} \Gamma(s_i)} (\Phi_1(\mathbf{s}; \mathbf{t}) + \Phi_2(\mathbf{s}; \mathbf{t})).$$

$\square$

4.2. Structure of poles.

Notation. We say that a series $\sum_{n=n_0}^{+\infty} f_n(z)$ verifies the Weierstrass M-test over X if there exists a positive real M such that

$$\sum_{n=n_0}^{+\infty} \sup_{x \in X} |f_n(z)| \leq M \leq +\infty.$$

4.2.1. ***Calculation of the regularization near* 0.** To know the value of the regularization for "good" power series, we need this following lemma :

Lemma 4.7. *Let $\sum_{k=0}^{+\infty} a_n z^n$ be a power series with radius of convergence $r > 0$ and let $s = s_r + \mathrm{i}s_i \in \mathbb{C}$. Let n_s be an integer such that $n_s \geq -s_r$. Then*

(i) *$\sum_{n=n_s}^{+\infty} a_n z^{n+s}$ verifies the Weierstrass M-test over any closed disc of radius strictly smaller than r.*

(ii) *If the series converge absolutely at $z = r$, $\sum_{n=n_s}^{+\infty} a_n z^{n+s}$ verify the Weierstrass M-test over the closed disc of radius r.*

Proof. For any $\rho \in]0, r[$, and for any integer $N > n_s + 1$,

$$\sum_{n=n_s}^{N} \sup_{\overline{D}(0,\rho)} |a_n z^{n+s}| \leq \sum_{n=n_s}^{N} |a_n| \rho^{n+s_r} \qquad \text{because } n + s_r \geq 0 \text{ !}$$

$$\leq \rho^{s_r} \sum_{n=n_s}^{N} |a_n| \rho^n \qquad \text{which converges}$$

We have (ii) thanks to the same calculation with $\rho = r$. □

So, now we can calculate the regularized for "good" power series :

Proposition 4.3. *Let $f(z) = \sum_{k=0}^{+\infty} a_n z^n$ be a power series with radius of convergence $r > 0$ and let $\rho \in]0, r[$. Then, for all $s \in \mathbb{C} \setminus \mathbb{Z}_{<0}$,*

$$\mathcal{R}_\rho[f](s) = \sum_{k=0}^{+\infty} a_k \frac{\rho^{k+s+1}}{k+s+1}.$$

Proof. Let n_s be an integer such that $n_s \geq -\Re(s) - 1$, then

$$\mathcal{R}_\rho[f](s) = \int_0^\rho x^s \sum_{k=n_s+1}^{+\infty} a_k z^k dx + \sum_{k=0}^{n_s} a_k \frac{\rho^{k+s+1}}{k+s+1}$$

Thanks to lemma 4.7, we know that the series $\displaystyle\sum_{k=n_s+1}^{+\infty} a_k z^{k+s}$ is uniformly convergent over $[0, \rho]$ and that

$$\mathcal{R}_\rho[f](s) = \sum_{k=n_s+1}^{+\infty} \int_0^\rho a_k z^{k+s} dx + \sum_{k=0}^{n_s} a_k \frac{\rho^{k+s+1}}{k+s+1} = \sum_{k=0}^{+\infty} a_k \frac{\rho^{k+s+1}}{k+s+1}.$$

□

Corollary 4.4. *Let ρ be a positive real, let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{+\infty} b_n z^n$ be two power series with radius of convergence strictly greater than ρ. Then, for all $s \in \mathbb{C} \setminus \mathbb{Z}_{<0}$,*

$$\mathcal{R}_\rho[f(x)g(x)](s) = \sum_{n_1, n_2 \geq 0} a_{n_1} b_{n_2} \frac{\rho^{n_1+n_2+s+1}}{n_1+n_2+s+1}.$$

Proof. $f(z)g(z)$ is the power series $\sum_{n=0}^{+\infty}\sum_{k=0}^{n} a_k b_{n-k} z^n$ and its radius of convergence is strictly greater as ρ, so thanks to Proposition 4.3,

$$\begin{aligned}\mathcal{R}_\rho[f(x)g(x)](s) &= \sum_{n=0}^{+\infty}\sum_{k=0}^{n} a_k b_{n-k}\frac{\rho^{n+s+1}}{n+s+1}\\ &= \sum_{n_1,n_2\geq 0} a_{n_1} b_{n_2}\frac{\rho^{n_1+n_2+s+1}}{n_1+n_2+s+1}.\end{aligned}$$

□

4.2.2. *Regularization between* 0 *and* 1. Near 0, we have to calculate $\mathcal{R}_\rho[f(x)(1-x)^b]$. To use result of subsubsection 4.2.1 we have to develop $(1-x)^b$, so we note :

Notation. Let $s \in \mathbb{C}$ and $n \in \mathbb{N}$. We note $(s)_n = \prod_{k=0}^{n-1}(s+k) = s\ldots(s+n-1)$ and $[s]_n = (-1)^n(-s)_n/n!$.

Let us recall that the Taylor series of $(1-z)^s$ is $\sum_{n=0}^{+\infty}[s]_n z^n$, which has 1 for radius of convergence (so they coincide for $|z|<1$).

Lemma 4.8. *Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ be a power series which verifies the Weierstrass M-test over $[0,1]$. Then, for any $(a,b) \in \left(\mathbb{C}\setminus\mathbb{Z}_{<0}\right)^2$,*

(i) $\mathcal{R}_0^1[f(x)](a,b) = \sum_{n=0}^{+\infty} a_n \mathcal{R}_0^1[x^n](a,b)$,

(ii) *If the power series $g(z) = \sum_{n=0}^{+\infty} b_n z^n$ verifies the Weierstrass M-test over $[0,1]$ then,*

$$\mathcal{R}_0^1[f(x)g(x)](a,b) = \sum_{n_1,n_2\geq 0} a_{n_1} b_{n_2}\mathcal{R}_0^1[x^{n_1+n_2}](a,b).$$

Proof. Let $(a,b) \in \left(\mathbb{C}\setminus\mathbb{Z}_{<0}\right)^2$.

(i) For any $\rho \in]0,1[$, we have $\mathcal{R}_0^1[f(z)](a,b) = \mathcal{R}_\rho[f(x)(1-x)^b](a) + \mathcal{R}_{1-\rho}[f(1-x)(1-x)^a](b)$. But, f and $(1-x)^b$ are power series of radius of convergence greater than 1, so, thanks to Corollary 4.4, we have

$$\begin{aligned}\mathcal{R}_\rho[f(x)(1-x)^b](a) &= \mathcal{R}_\rho[\sum_{n=0}^{+\infty} a_n x^n \sum_{n=0}^{+\infty}[b]_n x^n](a)\\ &= \sum_{n,k\geq 0} a_n [b]_k \frac{\rho^{n+k+a+1}}{n+k+a+1}\\ &= \sum_{n=0}^{+\infty} a_n \mathcal{R}_\rho[\sum_{k=0}^{+\infty}[b]_k x^{k+n}](a)\\ &= \sum_{n=0}^{+\infty} a_n \mathcal{R}_\rho[x^n(1-x)^b](a).\end{aligned}$$

In the same way, because $f(1-x) = \sum_{n=0}^{+\infty}[(-1)^n \sum_{k=n}^{+\infty} \binom{n}{k} a_k]x^n$ and $(1-x)^a$ are power series radius of convergence strictly greater than ρ, we have

$$\begin{aligned}
&\mathcal{R}_{1-\rho}[f(1-x)(1-x)^a](b) \\
&\quad= \sum_{n=0}^{+\infty}((-1)^n \sum_{k=n}^{+\infty}\binom{n}{k} a_k)\mathcal{R}_\rho[x^n(1-x)^b](a) \\
&\quad= \sum_{k=0}^{+\infty} a_k \sum_{n=0}^{k}(-1)^n\binom{n}{k}\mathcal{R}_\rho[x^n(1-x)^b](a) \\
&\quad= \sum_{k=0}^{+\infty} a_k \mathcal{R}_\rho[\sum_{n=0}^{k}\binom{n}{k}(-x)^n(1-x)^b](a) \\
&\quad= \sum_{k=0}^{+\infty} a_k \mathcal{R}_\rho[(1-x)^k(1-x)^a](b).
\end{aligned}$$

But $\mathcal{R}_0^1[x^n](a,b) = \mathcal{R}_\rho[x^n(1-x)^b](a) + \mathcal{R}_\rho[(1-x)^n(1-x)^a](b)$, so the (i) is proved.

(ii) $(fg)(z) = \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)z^n$ is a power series which verify the Weierstrass M-test over $[0,1]$: so we just have to apply (i).

□

To explicit the series given by lemma 4.8 we need the following lemma :

Lemma 4.9. *For any* $(a,b) \in \left(\mathbb{C}\setminus\mathbb{Z}_{<0}\right)^2$ *and any* $k \in \mathbb{N}$,

$$\mathcal{R}_0^1(x^k)(a,b) = \sum_{q=0}^{+\infty}[b]_q \frac{\rho^{q+k+a+1}}{q+k+a+1} + \sum_{q=0}^{+\infty}[a+k]_q \frac{(1-\rho)^{q+b+1}}{q+b+1}$$

for any $\rho \in]0,1[$.

Proof. Let $\rho \in]0,1[$ and let $(a,b) \in \left(\mathbb{C}\setminus\mathbb{Z}_{<0}\right)^2$. Then,

$$\begin{aligned}
\mathcal{R}_0^1[x^k](a,b) &= \mathcal{R}_\rho[x^k(1-x)^b](a) + \mathcal{R}_\rho[(1-x)^k(1-x)^a](b) \\
&= \mathcal{R}_\rho[\sum_{q=k}^{+\infty}[b]_{q-k}x^q](a) + \mathcal{R}_\rho[\sum_{q=0}^{+\infty}[a+k]_q x^q](b).
\end{aligned}$$

The power series $\sum_{q=k}^{+\infty}[b]_{q-k}x^q$ and $\sum_{q=0}^{+\infty}[a+k]_q x^q](b)$ have 1 for radius of convergence and $\rho < 1$. So, we can apply Proposition 4.3 and get the announced result. □

4.2.3. ***Structure of*** $\mathbf{\Phi_1}$**.** Moreover, we can have an explicit expression of this continuation thanks to the following lemma.

Lemma 4.10. *Let j be a positive integer, and let $a_1,\dots,a_j,b_1,\dots,b_j$ be complex numbers in $\mathbb{C}\setminus\mathbb{Z}_{<0}$. For all $x_{j+1}\in\mathbb{R}_+$,*

$$\begin{aligned}&\mathcal{R}_0^1[\sum_{k_j=1}^{+\infty}\beta_{k_j}^j x_j^{k_j}x_{j+1}^{k_j}\mathcal{R}_0^1[\dots\mathcal{R}_0^1[\sum_{k_1=1}^{+\infty}\beta_{k_1}^1 x_1^{k_1}\dots x_{j+1}^{k_1}](a_1,b_1)\dots\\&\dots](a_{j-1},b_{j-1})](a_j,b_j)\\&=\sum_{k_1,\dots,k_j\geq 0}(\prod_{i=1}^j\beta_{k_i}^i)\Big(\prod_{i=1}^j\mathcal{R}_0^1[x_i^{k_1+\dots+k_i}](a_i,b_i)\Big)x_{j+1}^{k_1+\dots+k_j}.\end{aligned}$$

Proof. The power series $\sum_{k=1}^{+\infty}(\beta_k^1x_2^k)\,x_1^k$ verifies the Weierstrass M-test over $[0,1]$ (the radius of convergence is 2π), so, thanks to Lemma 4.8 (i),

$$\begin{aligned}\mathcal{R}_0^1[\sum_{k_1=1}^{+\infty}\beta_{k_1}^1x_1^{k_1}x_2^{k_1}](a_1,b_1)&=\mathcal{R}_0^1[\sum_{k_1=1}^{+\infty}\beta_{k_1}^1x_2^{k_1}\,x_1^{k_1}](a_1,b_1)\\&=\sum_{k_1=1}^{+\infty}\beta_{k_1}^1x_2^{k_1}\mathcal{R}_0^1[x_1^{k_1}](a_1,b_1).\end{aligned}$$

This proves the lemma in the case $j=1$. Now, assume it is true for an integer $j\geq 2$. Then,

$$\begin{aligned}&\mathcal{R}_0^1[\sum_{k_j=1}^{+\infty}\beta_{k_j}^jx_j^{k_j}(x_{j+1}x_{j+2})^{k_j}\mathcal{R}_0^1[\dots\\&\quad\dots\mathcal{R}_0^1[\sum_{k_1=1}^{+\infty}\beta_{k_1}^1x_1^{k_1}\dots(x_{j+1}x_{j+2})^{k_1}](a_1,b_1)\dots\\&\quad\dots](a_{j-1},b_{j-1})](a_j,b_j)\\&=\sum_{k_1,\dots,k_j\geq 0}(\prod_{i=1}^j\beta_{k_i}^i)(\prod_{i=1}^j\mathcal{R}_0^1[x_i^{k_1+\dots+k_i}](a_i,b_i))(x_{j+1}x_{j+2})^{k_1+\dots+k_j}\\&=\sum_{k=0}^{+\infty}\Big(\sum_{\substack{k_2,\dots,k_j\geq 0\\/k_2+\dots+k_j\leq k}}(\beta_{k-k_2-\dots-k_j}^{k-k_2-\dots-k_j}\prod_{i=2}^j\beta_{k_i}^i)(\mathcal{R}_0^1[x_1^{k-k_2-\dots-k_j}](a_1,b_1)\\&\quad\times\prod_{i=2}^j\mathcal{R}_0^1[x_i^{k-k_{i+1}\dots-k_j}](a_i,b_i))\Big)x_{j+1}^k\,x_{j+2}^k.\end{aligned}$$

Consequently, thanks to Lemma 4.8 (ii),

$$\begin{aligned}&\mathcal{R}_0^1[\sum_{k_{j+1}=1}^{+\infty}\beta_{k_{j+1}}^{j+1}x_{j+1}^{k_{j+1}}x_{j+2}^{k_{j+1}}\mathcal{R}_0^1[\dots\mathcal{R}_0^1[\sum_{k_1=1}^{+\infty}\beta_{k_1}^1x_1^{k_1}\dots x_{j+1}^{k_1}](a_1,b_1)\dots\\&\dots](a_j,b_j)](a_{j+1},b_{j+1})\\&=\mathcal{R}_0^1[\end{aligned}$$

$$\sum_{k_{j+1}=1}^{+\infty}\beta_{k_{j+1}}^{j+1}x_{j+1}^{k_{j+1}}x_{j+2}^{k_{j+1}}\sum_{k=0}^{+\infty}\Big(\sum_{\substack{k_2,\ldots,k_j\geq 0\\ /k_2+\ldots+k_j\leq k}}(\beta_{k-k_2-\ldots-k_j}^{k-k_2-\ldots-k_j}\prod_{j=2}^{j}\beta_{k_i}^{i})$$

$$(\mathcal{R}[x_1^{k-k_2-\ldots-k_j}](a_1,b_1)\prod_{i=2}^{j}\mathcal{R}_0^1[x_i^{k-k_{i+1}\ldots-k_j}](a_i,b_i))x_{j+2}^{k}\Big)x_{j+1}^{k}$$

$$](a_{j+1},b_{j+1})$$

$$=\sum_{k_{j+1}=1}^{+\infty}\sum_{k=0}^{+\infty}\beta_{k_{j+1}}^{j+1}x_{j+2}^{k_{j+1}}\Big(\sum_{\substack{k_2,\ldots,k_j\geq 0\\ /k_2+\ldots+k_j\leq k}}(\beta_{k-k_2-\ldots-k_j}^{k-k_2-\ldots-k_j}\prod_{i=2}^{j}\beta_{k_i}^{i})$$

$$(\mathcal{R}[x_1^{k-k_2-\ldots-k_j}](a_1,b_1)\prod_{i=2}^{j}\mathcal{R}_0^1[x_i^{k-k_{i+1}\ldots-k_j}](a_i,b_i))x_{j+2}^{k}\Big)$$

$$\mathcal{R}_0^1[x_{j+1}^{k+k_{j+1}}](a_{j+1},b_{j+1})$$

Replacing k by $k_1=k-\sum_{i=1}^{j}k_i$ gives the result for $j+1$. □

Proposition 4.4. *The function Φ_1 defined by (61) can be continued as a meromorphic function over $\mathbb{C}^r\times]-\infty,1[^r$ with*

$$\Phi_1(\mathbf{s},\mathbf{t})=$$

$$\sum_{k_1,\ldots,k_r\geq 0}\Big(\prod_{i=1}^{r}\beta_{k_i}^{i}\Big)\prod_{i=1}^{r-1}\Big(\sum_{q_i=0}^{+\infty}[s_{i+1}-1]_{q_i}\frac{\rho_i^{\sum_{j=1}^{i}s_j-i+\sum_{j=1}^{i}k_j+q_i}}{\sum_{j=1}^{i}s_j-i+\sum_{j=1}^{i}k_j+q_i}$$

$$+\sum_{q_i=0}^{+\infty}[\sum_{j=1}^{i}s_j+\sum_{j=1}^{i}k_j-i-1]_{q_i}\frac{(1-\rho_i)^{s_{i+1}+q_i}}{s_{i+1}+q_i}\Big)$$

$$\times\frac{1}{\sum_{j=1}^{r}s_j-r+\sum_{j=1}^{r}k_j},$$

where $\rho_1,\ldots,\rho_{r-1}$ can be arbitrary chosen in $]0,1[$.

Proof. Thanks to Equality (64) and using Lemma 4.10 with $j=r$ and $x_{j+1}=1$,

$$\Phi_1(\mathbf{s},\mathbf{t})=$$

$$\sum_{k_1,\ldots,k_r\geq 0}\Big(\prod_{i=1}^{r}\beta_{k_i}^{i}\Big)\Big(\prod_{i=1}^{r}\mathcal{R}_0^1[x_i^{k_1+\ldots+k_i}](\sum_{j=1}^{i}s_j-i-1,s_{i+1}-1)\Big),$$

with the convention $s_{r+1}=1$. Lemma 4.9 ends the proof. □

4.2.4. *Structure of $\mathbf{\Phi_2}$.* Let $i\in\{1,\ldots,r-1\}$, a_1, b_1, a_{i-1}, b_{i-1}, a_{i+1}, b_{i+1}, a_{r-1}, $b_{r-1}\in\mathbb{C}$ and define the function r_i from $[0,1]$ to $\mathbb{C}$ by

$$r_i:y_i\quad\mapsto\quad\mathcal{R}_0^1[\ldots\mathcal{R}_0^1[\mathcal{R}_0^1[\ldots\mathcal{R}_0^1[\int_1^{+\infty}g(y_1,\ldots,y_r)dy_r](a_1,b_1)\ldots$$

$$\ldots](a_{i-1},b_{i-1})](a_{i+1},b_{i+1})\ldots](a_{r-1},b_{r-1}). \tag{66}$$

In Proposition 4.2, the function r_i is C^∞ over $[0,1]$ for any a_1, b_1, a_{i-1}, b_{i-1}, a_{i+1}, b_{i+1}, a_{r-1}, $b_{r-1} \in \mathbb{C}\backslash\mathbb{Z}_{<0}$. By now, let us suppose this is verified.Then, $\mathcal{R}_0^1[r_i](a_i,b_i)$ exists and, for any positive integer M, when $\Re(a_i) > -M$,

$$\begin{aligned}
&\mathcal{R}_0^1[r_i](a_i,b_i)\\
&= \mathcal{R}_\rho[(1-x)^{b_i}r_i(x)](a_i) + \mathcal{R}_{1-\rho}[(1-x)^{a_i}r_i(x)](b_i)\\
&= \sum_{k_i=0}^{M} \frac{1}{k_i!}\frac{\partial^{k_i}}{\partial y_i^{k_i}}((1-y_i)^{b_i}r_i(y_i))(0)\frac{\rho^{a_i+k_i+1}}{a_i+k_i+1}\\
&+ \int_0^\rho y_i^{a_i}\left((1-y_i)^{b_i}r_i(y_i) - \sum_{k_i=0}^{M}\frac{\partial^{k_i}}{\partial y_i^{k_i}}((1-y_i)^{b_i}r_i(y_i))(0)\frac{u_i^{k_i}}{k_i!}\right)dy_i\\
&+\mathcal{R}_{1-\rho}[(1-x)^{a_i}r_i(x)](b_i),
\end{aligned} \tag{67}$$

for any $\rho \in]0,1[$. But the function $\breve{r}_i$

$$\begin{aligned}
\breve{r}_i : a_i \ \mapsto\ & \int_0^\rho y_i^{a_i}\left((1-y_i)^{b_i}r_i(y_i) - \sum_{k_i=0}^{M}\frac{\partial^{k_i}}{\partial y_i^{k_i}}((1-y_i)^{b_i}r_i(y_i))(0)\frac{u_i^{k_i}}{k_i!}\right)dy_i\\
& +\mathcal{R}_{1-\rho}[(1-x)^{a_i}r_i(x)](b_i)
\end{aligned} \tag{68}$$

is holomorphic when $\Re(a_i) > -M$ (lemma 4.1 and corollary 4.2).

On the other hand, the first terms are given by

Lemma 4.11. *For any $i \in \{1, \ldots, r-1\}$ and any $k_i \in \mathbb{N}$,*

$$\begin{aligned}
&\frac{1}{k_i!}\frac{\partial^{k_i}}{\partial y_i^{k_i}}((1-y_i)^{b_i} r_i(y_i))(0)\\
&\quad = \sum_{k=0}^{k_i} \sum_{k'_1+\ldots+k'_i=k} \Big(\prod_{j=1}^{i} \beta^j_{k'_j}\Big)\Big(\prod_{l=1}^{i-1} \mathcal{R}_0^1[y_l^{\sum_{j=1}^{l} k'_j}](a_l, b_l)\Big)\mathcal{R}_0^1[y_{r-1}^k \ldots\\
&\quad \ldots y_{i+2}^k \mathcal{R}_0^1[y_{i+1}^k \int_1^{+\infty} y_r^k \prod_{j=i+1}^{r} h(\prod_{l=j}^{r} y_l, \bar{t}_j) dy_r](a_{i+1}, b_{i+1})] \ldots\\
&\quad \ldots](a_{r-1}, b_{r-1})\, [b_i]_{k_i-k}.
\end{aligned}$$

Proof. For any $k \in \mathbb{N}$,

$$\frac{\partial^k}{\partial y_i^k} h(\prod_{l=j}^{r} y_l, \bar{t}_j)(0) = \begin{cases} B_k(\bar{t}_j) y_j^k \ldots y_{i-1}^k y_{i+1}^k \ldots y_r^k & \text{if } j < i \\ B_k(\bar{t}_j) y_{i+1}^k \ldots y_r^k & \text{if } j = i \\ 0 & \text{if } j > i. \end{cases}$$

So, with $\mathbf{y}_i^0 = (y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_r)$,

$$\begin{aligned}
&\frac{\partial^k}{\partial y_i^k} g(y_1, \ldots, y_r)(\mathbf{y}_i^0)\\
&\quad = \sum_{k_1+\ldots+k_r=k} \frac{k!}{k_1! \ldots k_r!} \prod_{j=1}^{r} \frac{\partial^{k_j}}{\partial y_i^{k_j}} h(\prod_{l=j}^{r} y_l, \bar{t}_j)(\mathbf{y}_i^0)\\
&\quad = \sum_{k_1+\ldots+k_i=k} \frac{k!}{k_1! \ldots k_r!} \prod_{j=1}^{i} \frac{\partial^{k_j}}{\partial y_i^{k_j}} h(\prod_{l=j}^{r} y_l, \bar{t}_j)(\mathbf{y}_i^0) \prod_{j=i+1}^{r} h(\prod_{l=j}^{r} y_l, \bar{t}_j)\\
&\quad = k! \sum_{k_1+\ldots+k_i=k} \prod_{j=1}^{i} \beta^j_{k_j} \prod_{\substack{l=j\\ l\neq i}}^{r} y_l^{k_j} \prod_{j=i+1}^{r} h(\prod_{l=j}^{r} y_l, \bar{t}_j)\\
&\quad = k! \prod_{j=i+1}^{r} h(\prod_{l=j}^{r} y_l, \bar{t}_j) \sum_{k_1+\ldots+k_i=k} \prod_{j=1}^{i} \beta^j_{k_j} \prod_{\substack{l=1\\ l\neq i}}^{r} y_l^{\sum_{j=1}^{\inf(l,i)} k_j}.
\end{aligned}$$

In the proof of Lemma 4.6, we verified the conditions of differentiation under the sum sign so

$$\begin{aligned}
&\frac{1}{k!}\frac{\partial^k}{\partial y_i^k} \int_1^{+\infty} y_r^k g(y_1, \ldots, y_r) dy_r(\mathbf{y}_i^0)\\
&\quad = \sum_{k_1+\ldots+k_i=k} \prod_{j=1}^{i} \beta^j_{k_j} \prod_{\substack{l=1\\ l\neq i}}^{r-1} y_l^{\sum_{j=1}^{\inf(l,i)} k_j} \int_1^{+\infty} y_r^k \prod_{j=i+1}^{r} h(\prod_{l=j}^{r} y_l, \bar{t}_j) dy_r.
\end{aligned}$$

Thanks to proposition 4.2, we deduce,

$$\begin{aligned}\frac{1}{k!}\frac{\partial^k r_i}{\partial y_i^k}(0) &= \mathcal{R}_0^1[\ldots\mathcal{R}_0^1[\sum_{k_1+\ldots+k_i=k}\prod_{j=1}^{i}\beta_{k_j}^j\prod_{\substack{l=1\\l\neq i}}^{r-1}y_l^{\sum_{j=1}^{\inf(l,i)}k_j}\\
&\qquad\times\int_1^{+\infty}y_r^k\prod_{j=i+1}^{r}h(\prod_{l=j}^{r}y_l,\bar{t}_j)dy_r](a_1,b_1)]\ldots](a_{i-1},b_{i-1}]\\
&= \sum_{k_1+\ldots+k_i=k}\Big(\prod_{j=1}^{i}\beta_{k_j}^j\Big)\Big(\prod_{l=1}^{i-1}\mathcal{R}_0^1[y_l^{\sum_{j=1}^{l}k_j}](a_l,b_l)\Big)\mathcal{R}_0^1[y_{r-1}^k\ldots\\
&\qquad\ldots y_{i+2}^k\mathcal{R}_0^1[y_{i+1}^k\int_1^{+\infty}y_r^k\prod_{j=i+1}^{r}h(\prod_{l=j}^{r}y_l,\bar{t}_j)dy_r](a_{i+1},b_{i+1})]\ldots\\
&\qquad\ldots](a_{r-1},b_{r-1}].\end{aligned}$$

We only have to use this result in

$$\begin{aligned}&\frac{1}{k_i!}\frac{\partial^{k_i}}{\partial y_i^{k_i}}((1-y_i)^{b_i}r_i(y_i))(0)\\
&\qquad=\sum_{k=0}^{k_i}\frac{1}{k!}\frac{\partial^k r_i}{\partial y_i^k}(0)\frac{1}{(k_i-k)!}\frac{\partial^{k_i-k}(1-y_i)^{b_i}}{\partial y_i^{k_i-k}}(0).\end{aligned}$$

□

Proposition 4.5.

$$\begin{aligned}\frac{\Phi_2(\mathbf{s};\mathbf{t})}{\prod_{j=1}^r\Gamma(s_j)}\quad&\asymp\\
&\sum_{i=1}^{r-1}\frac{1}{\prod_{j=1}^r\Gamma(s_j)}\sum_{k_i\geq 0}\sum_{k_i'=0}^{k_i}[s_{i+1}-1]_{k_i-k_i'}\sum_{l_1+\ldots+l_i=k_i'}\Big(\prod_{j=1}^{i}\beta_{l_j}^j\Big)\\
&\Big(\prod_{j=1}^{i-1}\mathcal{R}_0^1[y_j^{\sum_{\iota=1}^{j}l_\iota}](s_1+\ldots+s_j-j-1,s_{j+1}-1)\Big)\mathcal{R}_0^1[y_{r-1}^{k_i'}\ldots\\
&\ldots\mathcal{R}_0^1[y_{i+2}^{k_i'}\mathcal{R}_0^1[y_{i+1}^{k_i'}\int_1^{+\infty}\prod_{j=i+1}^{r}h(\prod_{l=j}^{r}y_l,\bar{t}_j)dy_r](s_1+\ldots\\
&\ldots+s_{i+1}-i-2,s_{i+2}-1)](s_1+\ldots+s_{i+2}-i-3,s_{i+3}-1)]\ldots\\
&\ldots](s_1+\ldots+s_{r-1}-r,s_r-1]\;\frac{1}{s_1+\ldots+s_i-i+k_i}.\end{aligned}$$

Proof. Any pole of $\Phi_2(\mathbf{s})/\prod_{j=1}^r\Gamma(s_j)$, can be writen $\sum_{l=1}^i s_l=i-q$, with $i\in\{1,\ldots,r-1\}$ and $q\in\mathbb{N}$ (Lemma 4.6). Thanks to Lemma 4.4, $\Phi_2(\mathbf{s})$ is $\mathcal{R}_0^1[r_i](a_i,b_i)$ when $a_j=\sum_{l=1}^j s_l-j-1$ and $b_j=s_{j+1}-1$ for all $j\in\{1,\ldots,r\}$. Equality (67) shows that the singular part of $\mathcal{R}_0^1[r_i]$ over $\Re(a_i)>$

$-q-2$ is the sum

$$\sum_{k=0}^{q} \frac{\left(r_i(x)(1-x)^{b_i}\right)_{x^k}}{k!}(0)\frac{\rho^{a_i+k+1}}{a_i+k+1}.$$

The singular part of Laurent series at $\sum_{l=1}^{i} s_l = -i-q$ is so

$$\frac{\left(r_i(x)(1-x)^{b_i}\right)_{x^q}}{q!}(0)\frac{\rho^{a_i+q+1}}{a_i+q+1},$$

and Lemma 4.11 ends the proof. □

4.2.5. *Structure of* $\boldsymbol{\zeta}$. Using Proposition 4.4, Proposition 4.5 and Lemma 4.9, we obtain

Theorem 4.2. *The analytic expansion of $\zeta(\mathbf{s};\mathbf{t})$ has for set of poles $s_1 = 1$ and $\sum_{j=1}^{i} s_j \in i-\mathbb{N}$ for all $i \in \{2,\ldots,r\}$. These poles are simple and the residue at $\sum_{j=1}^{i} s_j = q$, $r > i > 1$ and $q \in i-\mathbb{N}$, is*

$$\begin{aligned}
&\frac{1}{\prod_{j=1}^{r}\Gamma(s_j)}\Bigg(\sum_{\substack{k_1+\ldots+k_i\le i-q\\ k_{i+1},\ldots,k_r\ge 0}}\Big(\prod_{l=1}^{r}\beta_{k_l}^{l}\Big)\\
&\prod_{\substack{l=1\\ l\neq i}}^{r-1}\Big(\sum_{q_l=0}^{+\infty}[s_{l+1}-1]_{q_l}\frac{\rho_l^{\sum_{j=1}^{l}s_j-l+\sum_{j=1}^{l}k_j+q_l}}{\sum_{j=1}^{l}s_j-l+\sum_{j=1}^{l}k_j+q_l}\\
&+\sum_{q_l=0}^{+\infty}[\sum_{j=1}^{l}s_j+\sum_{j=1}^{l}k_j-l-1]_{q_l}\frac{(1-\rho_l)^{s_{l+1}+q_l}}{s_{l+1}+q_l}\Big)\frac{[s_{i+1}-1]_{i-q-\sum_{j=1}^{i}k_j}}{\sum_{j=1}^{r}s_j-r+\sum_{j=1}^{r}k_j}\\
&+\sum_{k_i'=0}^{i-q}[s_{i+1}-1]_{i-q-k_i'}\sum_{k_1+\ldots+k_i=i-q}\Big(\prod_{j=1}^{i}\beta_{k_j}^{j}\Big)\\
&\prod_{j=1}^{i}\Big(\sum_{q_j=0}^{+\infty}[s_{j+1}-1]_{q_j}\frac{\rho_j^{\sum_{\iota=1}^{j}s_\iota-j+\sum_{\iota=1}^{j}k_\iota+q_j}}{\sum_{\iota=1}^{j}s_\iota-j+\sum_{\iota=1}^{j}k_\iota+q_j}\\
&+\sum_{q_j=0}^{+\infty}[\sum_{\iota=1}^{j}s_\iota-j-1+\sum_{\iota=1}^{j}k_\iota]_{q_j}\frac{(1-\rho_j)^{s_{j+1}+q_j}}{s_{j+1}+q_j}\Big)\mathcal{R}_0^1[y_{r-1}^{i-q}\ldots\\
&\ldots\mathcal{R}_0^1[y_{i+1}^{i-q}\int_1^{+\infty}y_r^{i-q}\prod_{j=i+1}^{r}h(\prod_{l=j}^{r}y_l,\bar{t}_j)dy_r](\sum_{\iota=1}^{i+1}s_\iota-i-2,s_{i+2}-1)]\ldots\\
&\ldots(\sum_{\iota=1}^{r-1}s_\iota-r,s_r-1)\Bigg),
\end{aligned}$$

the residue at $s_1 = 1$ *is*

$$\frac{1}{\prod_{j=2}^r \Gamma(s_j)} \Bigg(\sum_{k_2,\ldots,k_r \geq 0} \Big(\beta_0^1 \prod_{l=2}^r \beta_{k_l}^l \Big)$$
$$\prod_{l=2}^{r-1} \Big(\sum_{q_l=0}^{+\infty} [s_{l+1} - 1]_{q_l} \frac{\rho_l^{\sum_{j=2}^l s_j + 1 - l + \sum_{j=2}^l k_j + q_l}}{\sum_{j=2}^l s_j + 1 - l + \sum_{j=2}^l k_j + q_l}$$
$$+ \sum_{q_l=0}^{+\infty} [\sum_{j=2}^l s_j + 1 + \sum_{j=2}^l k_j - l - 1]_{q_l} \frac{(1-\rho_l)^{s_{l+1}+q_l}}{s_{l+1} + q_l} \Big)$$
$$\times \frac{1}{\sum_{j=2}^r s_j + 1 - r + \sum_{j=2}^r k_j}$$
$$+ \beta_0^1 \mathcal{R}_0^1 [\ldots \mathcal{R}_0^1 [\int_1^{+\infty} \prod_{j=2}^r h(\prod_{l=j}^r y_l, \bar{t}_j) dy_r](s_2 - 2, s_3 - 1)] \ldots$$
$$\ldots (\sum_{\iota=2}^{r-1} s_\iota + 1 - r, s_r - 1) \Bigg),$$

the residue at $\sum_{j=1}^r s_j = q \in r - \mathbb{N}$ *is*

$$\frac{1}{\prod_{j=1}^r \Gamma(s_j)} \Bigg(\sum_{k_1+\ldots+k_r \leq r-q} \Big(\prod_{l=1}^r \beta_{k_l}^l \Big)$$
$$\prod_{l=1}^{r-1} \Big(\sum_{q_l=0}^{+\infty} [s_{l+1} - 1]_{q_l} \frac{\rho_l^{\sum_{j=1}^l s_j - l + \sum_{j=1}^l k_j + q_l}}{\sum_{j=1}^l s_j - l + \sum_{j=1}^l k_j + q_l}$$
$$+ \sum_{q_l=0}^{+\infty} [\sum_{j=1}^l s_j + \sum_{j=1}^l k_j - l - 1]_{q_l} \frac{(1-\rho_l)^{s_{l+1}+q_l}}{s_{l+1} + q_l} \Big) \Bigg),$$

where

$$\beta_k^i = \frac{B_k(t_i - t_{i+1}}{k!} \qquad \textit{and} \qquad [s]_n = (-1)^n \frac{\prod_{k=0}^{n-1} (k - s)}{n!}$$

for all $k \in \mathbb{N}$, $i \in \{1, \ldots, r\}$ *(*$t_{r+1} = 0$*),* $s \in \mathbb{C}$, $n \in \mathbb{N}$*.*

5. A translation relation and prospects

5.1. The translation relation of Hurwitz polyzêtas. Recall that, for $s \in \mathbb{C}$, $(s)_0 = 1$ and

$$(s)_k = \frac{s(s+1)\ldots(s+k-1)}{k!} = \frac{\prod_{i=0}^{k-1}(s+i)}{k!} = \frac{\Gamma(s+k)}{k!\Gamma(s)}$$

for all $k \in \mathbb{N}^*$. To adapt Ecalle's idea to Hurwitz polyzêtas [15], we use the relation, with $t < 1$ and $n \geq 2$,

$$\frac{1}{(n-1-t)^s} = \sum_{k=0}^{+\infty} (s)_k \frac{1}{(n-t)^{k+s}} \tag{69}$$

we obtain (see Lemma 6.3 in Appendix) :

Lemma 5.1. *Let* $\mathbf{s}$ *be a convergent composition of depth* r. *If* $\mathbf{t} \in]-\infty; 1[^r$,

$$\zeta(\mathbf{s};\mathbf{t}) = \sum_{k_1,\ldots,k_r \geq 0} \left(\prod_{j=1}^{r} (s_j)_{k_j} \right) \sum_{n_1 > \ldots > n_r > 1} \frac{1}{(n_1-t_1)^{s_1+k_1} \ldots (n_r-t_r)^{s_r+k_r}}.$$

We can express the right member in Lemma 5.1 in term of polyzêtas, which gives raise to the following equality :

Proposition 5.1. *Let* $\mathbf{s}$ *be a convergent composition of depth* r *and let* $\mathbf{t} \in]-\infty; 1[^r$. *Then,*

$$\begin{aligned} 0 = & \sum_{j=1}^{r-1} \frac{(-1)^{r-j}}{(1-t_{j+1})^{s_{j+1}} \ldots (1-t_r)^{s_r}} \zeta(\mathbf{s}_j;\mathbf{t}_j) \; + \; \frac{(-1)^r}{(1-t_1)^{s_1} \ldots (1-t_r)^{s_r}} \\ & + \sum_{\substack{k_1,\ldots,k_r \geq 0 \\ (k_1;\ldots;k_r) \neq (0;\ldots;0)}} \left(\prod_{j=1}^{r} (s_j)_{k_j} \right) \Big(\zeta(\mathbf{s}+\mathbf{k};\mathbf{t}) \\ & + \sum_{j=1}^{r-1} \frac{(-1)^{r-j}}{(1-t_{j+1})^{s_{j+1}+k_{j+1}} \ldots (1-t_r)^{s_r+k_r}} \zeta(\mathbf{s}_j+\mathbf{k}_j;\mathbf{t}_j) \\ & + \frac{(-1)^r}{(1-t_1)^{s_1+k_1} \ldots (1-t_r)^{s_r+k_r}} \Big), \end{aligned}$$

where $(s)_k = \dfrac{\Gamma(s+k)}{k!\Gamma(s)}$ *for all* $k \in \mathbb{N}$.

5.2. Translation equality and analytic continuation. We want see now if the equality of Proposition 5.1 can give the analytic continuation of $\zeta(\mathbf{s};\mathbf{t})$ by induction over the depth r. So, let us have a first look at what happens with $r = 2$.

Example. When $r = 2$, Proposition 5.1 becomes :

$$\begin{aligned} 0 = & \frac{-1}{(1-t_2)^{s_2}} \zeta(s_2;t_2) + \frac{1}{(1-t_1)^{s_1}(1-t_2)^{s_2}} \\ & + \sum_{\substack{k_1,k_2 \geq 0 \\ (k_1;k_2) \neq (0;\ldots;0)}} (s_1)_{k_1} (s_2)_{k_2} \Big(\zeta(s_1+k_1, s_2+k_2;\mathbf{t}) \\ & + \frac{-1}{(1-t_2)^{s_2}} \zeta(s_2+k_2;t_2) + \frac{1}{(1-t_1)^{s_1}(1-t_2)^{s_2}} \Big) \end{aligned}$$

so,

$$\begin{aligned}
&\sum_{\substack{k_1,k_2\geq 0\\(k_1;k_2)\neq(0;\ldots;0)}} (s_1)_{k_1}(s_2)_{k_2}\zeta(s_1+k_1,s_2+k_2;\mathbf{t})\\
&=\frac{1}{(1-t_2)^{s_2}}\zeta(s_2;t_2)-\frac{1}{(1-t_1)^{s_1}(1-t_2)^{s_2}}\\
&+\sum_{\substack{k_1,k_2\geq 0\\(k_1;k_2)\neq(0;\ldots;0)}} (s_1)_{k_1}(s_2)_{k_2}\Big(\frac{1}{(1-t_2)^{s_2}}\zeta(s_2+k_2;t_2)\\
&-\frac{1}{(1-t_1)^{s_1}(1-t_2)^{s_2}}\Big).
\end{aligned}$$

Remark 5.1.

(1) The right part of the last equality can be continued as a meromorphic function over $\mathbb{C}^2$. Unfortunately, no term of the left part is *in principle* continuable as a meromorphic function over $\mathbb{C}^2$, and we can't isolate this terms with the equality only. We can not define $\zeta(\mathbf{s};\mathbf{t})$ for $\mathbf{s}$ with negative values in $\mathbb{Z}$ too because the $(s)_k$ cancel each other out (which is in agreement with our assertion that there are poles for these values).

(2) Thanks to the integral representation, we can define the function $\zeta(\mathbf{s};\mathbf{t})$ for $Re(s_1)>1$ and $Re(s_2)>0$. We can try to progress by stripe : for example, for $1\geq Re(s_1)>0$ et $Re(s_2)>0$, the terms $\zeta(s_1+k_1,s_2+k_2;\mathbf{t})$ are defined for all k_2 and all $k_1\geq 1$. Unfortunately, there are still the terms $\zeta(s_1,s_2+k_2;\mathbf{t})$, for all $k_2\in\mathbb{N}^*$ which can not be defined *in principle* over $\mathbb{C}^2$.

(3) The remarks still remain true for $r>2$.

6. Conclusion

This article gives the analytic continuation and the structure of poles of the Hurwitz poyzêta function. Thanks to Proposition 3.3, this result gives too the analytic continuation and the structure of poles of the periodic Parametrized Dirichlet generating series (so of the coloured polyzêta functions).

Howerever, some coefficients of this analytic continuation are not explicit : it would be nice to have an algorithm given explicit coefficients.

An other way is to start with a translation relation. A development of all variables *simultaneously* (as Proposition 5.1) gives an infinite sum of terms with same depth prevent from obtaining directly the analytic continuation. It seems that we need woork with a development in only *one* variable, and make a induction variable by variable and stripe by stripe.

We will study these differents points in a next article.

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Appendix

Lemma 6.1. *Consider the substitution*

$$x_1 = y_1 \dots y_r,\ x_2 = (1-y_1)y_2 \dots y_r,\ \dots,\ x_r = (1-y_{r-1})y_r. \tag{70}$$

Its Jacobian $\mathcal{J}_r = \partial(x_1,\dots,x_r)/\partial(y_1,\dots,y_r)$ *is equal to* $\prod_{k=2}^{r} y_k^{k-1}$

Proof.

$$\mathcal{J}_r = \begin{pmatrix} \prod_{k\neq 1}^{r} y_k & -\prod_{k=2}^{r} y_k & \cdots & \cdots & 0 \\ \prod_{k\neq 2}^{r} y_k & (1-y_1)\prod_{k\notin\{1,2\}}^{r} y_k & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \prod_{k\neq n-2}^{r} y_i & (1-y_1)\prod_{k\notin\{1,n-2\}}^{r} y_i & \ddots & -y_{n-1}y_n & 0 \\ \prod_{k\neq n-1}^{r} y_i & (1-y_1)\prod_{k\notin\{1,n-1\}}^{r} y_i & \cdots & (1-y_{n-2})y_n & -y_n \\ \prod_{k\neq n}^{r} y_i & (1-y_1)\prod_{k\notin\{1,n\}}^{r} y_i & \cdots & (1-y_{n-2})y_{n-1} & 1-y_{n-1} \end{pmatrix}$$

$$= \prod_{k=2}^{r} y_k \begin{pmatrix} \prod_{k\notin\{1,2\}}^{r} y_k & -\prod_{k=3}^{r} y_k & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \prod_{k\notin\{1,n-2\}}^{r} y_i & (1-y_2)\prod_{k\notin\{1,2,n-2\}}^{r} y_i & \ddots & -y_{n-1}y_n & 0 \\ \prod_{k\notin\{1,n-1\}}^{r} y_i & (1-y_2)\prod_{k\notin\{1,2,n-1\}}^{r} y_i & \cdots & (1-y_{n-2})y_n & -y_n \\ \prod_{k\notin\{1,n\}}^{r} y_i & (1-y_2)\prod_{k\notin\{1,2,n\}}^{r} y_i & \cdots & (1-y_{n-2})y_{n-1} & 1-y_{n-1} \end{pmatrix}$$

$$= \prod_{k=2}^{r} y_k^{k-1}.$$

□

Lemma 6.2. *Let $f \in \mathbb{C}^\infty([0,1]^2)$. We have the equality*

$$\mathcal{R}_0^1[\mathcal{R}_0^1[f(x,y)](a_1,b_1)](a_2,b_2) = \mathcal{R}_0^1[\mathcal{R}_0^1[f(x,y)](a_2,b_2)](a_1,b_1)$$

as meromorphic function of (a_1,b_1,a_2,b_2) over $\mathbb{C}^4$.

Proof. Using Lemma 4.2, for any $\rho_1,\rho_2 \in]0,1[$,

$$\begin{aligned} &\mathcal{R}_0^1[\mathcal{R}_0^1[f(x,y)](a_1,b_1)](a_2,b_2) \\ &= \mathcal{R}_0^1[\mathcal{R}_{\rho_1}[f(x,y)(1-x)^{b_1}](a_1) \\ &+\mathcal{R}_{1-\rho_1}[f(1-x,y)(1-x)^{a_1}](b_1)](a_2,b_2) \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{R}_{\rho_2}[(\mathcal{R}_{\rho_1}[f(x,y)(1-x)^{b_1}](a_1)\\
&+\mathcal{R}_{1-\rho_1}[f(1-x,y)(1-x)^{a_1}](b_1))(1-y)^{b_2}](a_2)\\
&+\mathcal{R}_{1-\rho_2}[(\mathcal{R}_{\rho_1}[f(x,1-y)(1-x)^{b_1}](a_1)\\
&+\mathcal{R}_{1-\rho_1}[f(1-x,1-y)(1-x)^{a_1}](b_1))(1-y)^{a_2}](b_2)\\
&= \mathcal{R}_{\rho_2}[\mathcal{R}_{\rho_1}[f(x,y)(1-x)^{b_1}(1-y)^{b_2}](a_1)](a_2)\\
&+\mathcal{R}_{\rho_2}[\mathcal{R}_{1-\rho_1}[f(1-x,y)(1-x)^{a_1}(1-y)^{b_2}](b_1)](a_2)\\
&+\mathcal{R}_{1-\rho_2}[\mathcal{R}_{\rho_1}[f(x,1-y)(1-x)^{b_1}(1-y)^{a_2}](a_1)](b_2)\\
&+\mathcal{R}_{1-\rho_2}[\mathcal{R}_{1-\rho_1}[f(1-x,1-y)(1-x)^{a_1}(1-y)^{a_2}](b_1)](b_2)\\
&= \mathcal{R}_{\rho_1}[\mathcal{R}_{\rho_2}[f(x,y)(1-x)^{b_1}(1-y)^{b_2}](a_2)](a_1)\\
&+\mathcal{R}_{1-\rho_1}[\mathcal{R}_{\rho_2}[f(1-x,y)(1-x)^{a_1}(1-y)^{b_2}](a_2)](b_1)\\
&+\mathcal{R}_{\rho_1}[\mathcal{R}_{1-\rho_2}[f(x,1-y)(1-x)^{b_1}(1-y)^{a_2}](b_2)](a_1)\\
&+\mathcal{R}_{1-\rho_1}[\mathcal{R}_{1-\rho_2}[f(1-x,1-y)(1-x)^{a_1}(1-y)^{a_2}](b_2)](b_1)\\
&= \mathcal{R}_0^1[\mathcal{R}_{\rho_2}[f(x,y)(1-y)^{b_2}](a_2)](a_1,b_1)\\
&+\mathcal{R}_0^1[\mathcal{R}_{1-\rho_2}[f(x,1-y)(1-y)^{a_2}](b_2)](a_1,b_1)\\
&= \mathcal{R}_0^1[\mathcal{R}_0^1[f(x,y)](a_2,b_2)](a_1,b_1).
\end{aligned}
$$

□

Lemma 6.3. *Let* $\mathbf{s}$ *be a convergent composition of depth* r*. If* $\mathbf{t}\in]-\infty;1[^r$,

$$\zeta(\mathbf{s};\mathbf{t}) = \sum_{k_1,\ldots,k_r\geq 0}\left(\prod_{j=1}^r (s_j)_{k_j}\right)\sum_{n_1>\ldots>n_r>1}\frac{1}{(n_1-t_1)^{s_1+k_1}\ldots(n_r-t_r)^{s_r+k_r}},$$

where $(s)_k = \dfrac{\Gamma(s+k)}{k!\Gamma(s)}$ *for all* $k\in\mathbb{N}$.

Proof. Note that, if $t<1$, for all $n\geq 2$,

$$\frac{1}{(n-1-t)^s} = \frac{1}{(n-t)^s}\frac{1}{(1-\frac{1}{n-t})^s} = \frac{1}{(n-t)^s}\sum_{k=0}^{+\infty}(s)_k\frac{1}{(n-t)^k}$$

$$= \sum_{k=0}^{+\infty}(s)_k\frac{1}{(n-t)^{k+s}} \tag{72}$$

and the serie verify the Weierstrass M-test for $n\geq 2$ (thanks to the inequalities

$$0<1/(n-t)\leq 1/(2-t)<1).$$

So, if $\mathbf{t}$ in $]-\infty;1[^r$, for any convergent composition $\mathbf{s}$,

$$\begin{aligned}\zeta(\mathbf{s};\mathbf{t}) &= \sum_{n_1>\ldots>n_r>0} \frac{1}{(n_1-t_1)^{s_1}\ldots(n_r-t_r)^{s_r}}\\ &= \sum_{n_1>\ldots>n_r>1} \frac{1}{(n_1-t_1-1)^{s_1}\ldots(n_r-t_r-1)^{s_r}}\\ &= \sum_{n_1>\ldots>n_r>1} \prod_{j=1}^{r} \sum_{k_j\geq 0} (s_j)_{k_j} \frac{1}{(n_j-t_j)^{s_j+k_j}}\\ &= \sum_{k_1,\ldots,k_r\geq 0} \sum_{n_1>\ldots>n_r>1} \prod_{j=1}^{r} (s_j)_{k_j} \frac{1}{(n_j-t_j)^{s_j+k_j}} \qquad (73)\end{aligned}$$

thanks to the Weierstrass M-test. The lemma follows. $\square$

Proposition 6.1. *Let* $\mathbf{s}$ *be a convergent composition of depth* r *and let* $\mathbf{t}\in]-\infty;1[^r$. *Then,*

$$\begin{aligned}0 = &\sum_{j=1}^{r-1} \frac{(-1)^{r-j}}{(1-t_{j+1})^{s_{j+1}}\ldots(1-t_r)^{s_r}} \zeta(\mathbf{s}_j;\mathbf{t}_j) + \frac{(-1)^r}{(1-t_1)^{s_1}\ldots(1-t_r)^{s_r}}\\ &+ \sum_{\substack{k_1,\ldots,k_r\geq 0\\ (k_1;\ldots;k_r)\neq(0;\ldots;0)}} \left(\prod_{j=1}^{r}(s_j)_{k_j}\right) \Bigg(\zeta(\mathbf{s}+\mathbf{k};\mathbf{t})\\ &+ \sum_{j=1}^{r-1} \frac{(-1)^{r-j}}{(1-t_{j+1})^{s_{j+1}+k_{j+1}}\ldots(1-t_r)^{s_r+k_r}} \zeta(\mathbf{s}_j+\mathbf{k}_j;\mathbf{t}_j)\\ &+ \frac{(-1)^r}{(1-t_1)^{s_1+k_1}\ldots(1-t_r)^{s_r+k_r}}\Bigg),\end{aligned}$$

where $(s)_k = \dfrac{\Gamma(s+k)}{k!\Gamma(s)}$ *for all* $k\in\mathbb{N}$.

Proof. For all convergent composition $\mathbf{s}$ of depth r, for all integers $k_1,\ldots,k_r$, and all $\mathbf{t}\in]-\infty;1[^r$,

$$\begin{aligned}&\sum_{n_1>\ldots>n_r>1} \frac{1}{(n_1-t_1)^{s_1+k_1}\ldots(n_r-t_r)^{s_r+k_r}}\\ &= \sum_{n_1>\ldots>n_r>0} \frac{1}{(n_1-t_1)^{s_1+k_1}\ldots(n_r-t_r)^{s_r+k_r}}\\ &\quad - \sum_{\substack{n_1>\ldots>n_{r-1}>1\\ n_r=1}} \frac{1}{(n_1-t_1)^{s_1+k_1}\ldots(n_r-t_r)^{s_r+k_r}}\\ &= \zeta(\mathbf{s}+\mathbf{k};\mathbf{t})\\ &\quad - \frac{1}{(1-t_r)^{s_r+k_r}} \sum_{n_1>\ldots>n_{r-1}>1} \frac{1}{(n_1-t_1)^{s_1+k_1}\ldots(n_{r-1}-t_{r-1})^{s_{r-1}+k_{r-1}}}.\end{aligned}$$

$$
\begin{aligned}
&= \zeta(\mathbf{s}+\mathbf{k};\mathbf{t}) + \sum_{j=1}^{r-1} \frac{(-1)^{r-j}}{(1-t_{j+1})^{s_{j+1}+k_{j+1}}\dots(1-t_r)^{s_r+k_r}}\zeta(\mathbf{s}_j+\mathbf{k}_j;\mathbf{t}_j)\\
&\quad + \frac{(-1)^r}{(1-t_1)^{s_1+k_1}\dots(1-t_r)^{s_r+k_r}}.
\end{aligned}
\tag{74}
$$

Injecting equality (74) in equality of lemma 6.3, we obtain, for $\mathbf{t} \in]-\infty;1[^r$ and for convergent composition $\mathbf{s}$,

$$
\begin{aligned}
\zeta(\mathbf{s};\mathbf{t}) = \sum_{k_1,\dots,k_r \geq 0} \left(\prod_{j=1}^{r} (s_j)_{k_j}\right) \Big(& \zeta(\mathbf{s}+\mathbf{k};\mathbf{t})\\
&+ \sum_{j=1}^{r-1} \frac{(-1)^{r-j}}{(1-t_{j+1})^{s_{j+1}+k_{j+1}}\dots(1-t_r)^{s_r+k_r}}\zeta(\mathbf{s}_j+\mathbf{k}_j;\mathbf{t}_j)\\
&+ \frac{(-1)^r}{(1-t_1)^{s_1+k_1}\dots(1-t_r)^{s_r+k_r}}\Big).
\end{aligned}
\tag{75}
$$

Note that, if $k_1 = \dots = k_r = 0$, $\prod_{j=1}^{r}(s_j)_{k_j} = 1$ and $\zeta(\mathbf{s}+\mathbf{k};\mathbf{t}) = \zeta(\mathbf{s};\mathbf{t})$. The lemma follows. □

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