JOURNAL
de Théorie des Nombres
de BORDEAUX
anciennement Séminaire de Théorie des Nombres de Bordeaux

Cécile DARTYGE, András SÁRKÖZY et Mihály SZALAY

On the number of prime factors of summands of partitions

Tome 18, n^{o} 1 (2006), p. 73-87.
<http://jtnb.cedram.org/item?id=JTNB_2006__18_1_73_0>

Journal de Théorie des Nombres
de Bordeaux **18** (2006), 73-87

On the number of prime factors of summands of partitions

par Cécile DARTYGE, András SÁRKÖZY et Mihály SZALAY

Résumé. Nous présentons plusieurs résultats sur le nombre de facteurs premiers des parts d'une partition d'un entier. Nous étudions la parité, les ordres extrémaux et nous démontrons un théorème analogue au théorème de Hardy-Ramanujan. Ces résultats montrent que pour presque toutes les partitions d'un entier, la suite des parts vérifie des propriétés arithmétiques similaires à la suite des entiers naturels.

Abstract. We present various results on the number of prime factors of the parts of a partition of an integer. We study the parity of this number, the extremal orders and we prove a Hardy-Ramanujan type theorem. These results show that for almost all partitions of an integer the sequence of the parts satisfies similar arithmetic properties as the sequence of natural numbers.

1. Introduction

The aim of this paper is to study the number of prime factors of parts of partitions of n. For $n \in \mathbb{N}$ denote the set of all partitions of n by $\mathcal{P}(n)$ and the cardinality of $\mathcal{P}(n)$ by $p(n)$.

For a partition $\gamma = (\gamma_1, \dots, \gamma_s)$ of $n = \gamma_1 + \cdots + \gamma_s$ with $\gamma_1 \geqslant \ldots \geqslant \gamma_s \geqslant 1$, $s(\gamma) := s$ denotes the number of the parts of γ. We will also use the standard notations $\omega(n)$ for the number of distinct prime factors of n and $\Omega(n)$ for the number of prime factors of n counted with multiplicity.

The first result is a Hardy-Ramanujan type theorem.

Theorem 1.1. *Let Φ be a non-decreasing function with $\lim_{N\to\infty} \Phi(N) = +\infty$. For $n \to \infty$, for all but $o(p(n))$ partitions of n the number of parts γ_j with*

$$|\omega(\gamma_j) - \log\log n| > \Phi(n)\sqrt{\log\log n} \tag{1.1}$$

is $o(\sqrt{n}\log n)$.

Manuscrit reçu le 31 mars 2004.
Research partially supported by the Hungarian National Foundation for Scientific Research, Grant T043623 and by French-Hungarian EGIDE-OMKFHÁ exchange program Balaton F-2/03.

The main tool of the proof of this theorem is a mean value theorem from [DSS] related to arithmetic progressions. We will also study the maximum of the Ω and ω functions over the parts of a partition.

Proposition 1.2. *Let $\varepsilon > 0$. Then for $n \to \infty$, for almost all partitions $\gamma \in \mathcal{P}(n)$ we have*

$$(1.2) \qquad (1-\varepsilon)\frac{\log n}{2\log 2} \leqslant \max_{1\leqslant j\leqslant s(\gamma)} \Omega(\gamma_j) \leqslant (1+\varepsilon)\frac{\log n}{2\log 2}$$

and

$$(1.3) \qquad (1-\varepsilon)\frac{\log n}{2\log\log n} \leqslant \max_{1\leqslant j\leqslant s(\gamma)} \omega(\gamma_j) \leqslant (1+\varepsilon)\frac{\log n}{2\log\log n}.$$

Finally, we will show that in almost all partitions γ of n, about half of the parts have even and half of them odd number of prime factors, *i. e.*, the Liouville function $\lambda(N) = (-1)^{\Omega(N)}$ assumes the values $+1$ and -1 with about the same frequency over the parts $\gamma_1, \ldots, \gamma_s$. Let $s_0 = s_0(\gamma)$ and $s_1 = s_1(\gamma)$ denote the number of parts with even, resp. odd number of prime factors in the partition γ.

Theorem 1.3. *If $f(n) \to \infty$ and $f(n) = o(\log n)$, then for almost all partitions γ of n we have*

$$(1.4) \qquad \frac{s_0(\gamma)}{s_1(\gamma)} = 1 + O\Big(\frac{f(n)}{\log n}\Big).$$

By (2.1) and Lemma 5.1 below, it is easy to see that (1.4) is sharp : for arbitrarily large constant c, there exist $(\exp(-2\pi c/\sqrt{6}) + o(1))p(n)$ partitions of n containing 1 as summand $2[c\sqrt{n}]$-times at least. Omitting $2[c\sqrt{n}]$ of the 1's and increasing the number of 2's with $[c\sqrt{n}]$ we obtain

$$\frac{s_0(\gamma')}{s_1(\gamma')} = \frac{s_0(\gamma) - 2[c\sqrt{n}]}{s_1(\gamma) + [c\sqrt{n}]} < \frac{s_0(\gamma)}{s_1(\gamma)} - \frac{c}{\log n}.$$

In the next paragraph we will recall some results about $s(\gamma)$, and the three other paragraphs are devoted to the proofs of Theorem 1.1, Proposition 1.2 and Theorem 1.3 respectively.

2. On the number of parts of partitions : results of Erdős, Lehner, Szalay and Turán

In 1941, Erdős and Lehner [EL] proved that if f tends to $+\infty$ arbitrarily slowly then the inequality

$$(2.1) \qquad \Big|s(\gamma) - \frac{\sqrt{6n}}{2\pi}\log n\Big| \leqslant \sqrt{n}f(n)$$

holds for almost all partitions of n. Szalay and Turán showed a strong form of this theorem.

Theorem 2.1. (Szalay and Turán, [ST2]**, Theorem IV, p. 388)** *If $f(n)$ tends arbitrarily slowly to $+\infty$ and satisfies $f(n) = o(\log n)$ then the number of partitions with*

$$\left|s(\gamma) - \frac{\sqrt{6n}}{2\pi}\log n\right| > c_1\sqrt{n}f(n) \tag{2.2}$$

is at most $c_2 p(n)\exp(-f(n))$.

In the proof of Theorem 1.1 we will also need the following result of Szalay and Turán:

Lemma 2.2. (Szalay and Turán, [ST3]**, Lemma 4, p.137)** *The inequality $s(\gamma) \leqslant \frac{5\sqrt{6}}{2\pi}\sqrt{n}\log n$ holds with the exception of at most $O(p(n)n^{-2})$ partitions γ.*

3. Proof of Theorem 1.1

3.1. A mean square value argument.

We define

$$\omega^*(\gamma_j) := \sum_{\substack{p|\gamma_j \\ p<n^{1/10}}} 1.$$

We have

$$0 \leqslant \omega(\gamma_j) - \omega^*(\gamma_j) \leqslant 10.$$

To prove Theorem 1.1 it is sufficient to show that

$$\begin{aligned} D = D(n) &:= \sum_{\gamma\in\mathcal{P}(n)} \sum_{j=1}^{s(\gamma)} (\omega^*(\gamma_j) - \log\log n)^2 \\ &\ll p(n)\sqrt{n}\log n \log\log n. \end{aligned} \tag{3.1}$$

Indeed, if the Theorem 1.1 is not true then there exist $Ap(n)$ (with $A > 0$) partitions γ of n such that for at least $B\sqrt{n}\log n$ ($B > 0$) parts we have

$$|\omega(\gamma_j) - \log\log n| > \Phi(n)\sqrt{\log\log n}.$$

This implies that for $n > n_0(\Phi)$ the inequality

$$\begin{aligned} |\omega^*(\gamma_j) - \log\log n| &\geqslant |\omega(\gamma_j) - \log\log n| - 10 \\ &> \frac{\Phi(n)}{2}\sqrt{\log\log n} \end{aligned}$$

holds for at least $B\sqrt{n}\log n$ parts of these partitions γ. This gives for D :

$$D > \frac{AB}{4}p(n)\sqrt{n}(\log n)\Phi^2(n)\log\log n.$$

This contradicts (3.1).

It remains to prove (3.1). The square in (3.1) is

$$
\begin{aligned}
(\omega^*(\gamma_j)-\log\log n)^2 &= \Big(\sum_{\substack{p\mid\gamma_j\\ p<n^{1/10}}}1\Big)\Big(\sum_{\substack{q\mid\gamma_j\\ q<n^{1/10}}}1\Big)-2\log\log n\Big(\sum_{\substack{p\mid\gamma_j\\ p<n^{1/10}}}1\Big)\\
&\quad+(\log\log n)^2\\
&=\sum_{\substack{pq\mid\gamma_j\\ p<n^{1/10},q<n^{1/10}\\ p\neq q}}1-(2\log\log n-1)\sum_{\substack{p\mid\gamma_j\\ p<n^{1/10}}}1\\
&\quad+(\log\log n)^2.
\end{aligned}
\tag{3.2}
$$

For a partition γ, some integers d, r with $d\geqslant 1$, $1\leqslant r\leqslant d$, and a real number $\Gamma\geqslant 1$ we define :

$$
S(n,\gamma,\Gamma;d,r):=\sum_{\substack{\gamma_j\equiv r \pmod d\\ \gamma_j\geqslant\Gamma}}1.
$$

By (3.2) and after changing the order of summations, we have

$$
\begin{aligned}
D(n) &= \sum_{\substack{p,q<n^{1/10}\\ p\neq q}}\sum_{\gamma\in\mathcal{P}(n)}S(n,\gamma,pq;pq,pq)\\
&\quad-(2\log\log n-1)\sum_{p<n^{1/10}}\sum_{\gamma\in\mathcal{P}(n)}S(n,\gamma,p;p,p)\\
&\quad+(\log\log n)^2\sum_{\gamma\in\mathcal{P}(n)}s(\gamma)\\
&=D_1-D_2+D_3
\end{aligned}
\tag{3.3}
$$

by definition.

3.2. The term $\mathbf{D_3}$.

Let $L:=\sqrt{6n}\log n/(2\pi)$ the normal order of $s(\gamma)$. We cut the sum D_3 in three subsums. The first one contains the partitions γ such that $s(\gamma)$ is near L; this will be the main contribution of D_3. The second subsum is over the partitions γ such that $s(\gamma)-L$ is not *small* but with $s(\gamma)$ still $\ll\sqrt{n}\log n$. We will give an upper bound for it with Theorem 2.1 applied with $f(n)=(\log\log n)^2$. The third subsum contains the remaining partitions *i.e.,* the partitions with many parts. We will give an upper bound for it with Lemma 2.2.

$$D_3 \leqslant (\log\log n)^2 \sum_{\substack{\gamma\in\mathcal{P}(n)\\ |s(\gamma)-L|\leqslant c_1(\log\log n)^2\sqrt{n}}} s(\gamma)$$
$$+ (\log\log n)^2 \sum_{\substack{\gamma\in\mathcal{P}(n)\\ |s(\gamma)-L|>c_1(\log\log n)^2\sqrt{n}\\ s(\gamma)\leqslant \frac{5\sqrt{6n}}{2\pi}\log n}} \frac{5\sqrt{6n}}{2\pi}\log n$$
$$+ (\log\log n)^2 \sum_{\substack{\gamma\in\mathcal{P}(n)\\ s(\gamma)>\frac{5\sqrt{6n}}{2\pi}\log n}} n$$
$$\leqslant (\log\log n)^2 p(n)\Big(\frac{\sqrt{6n}}{2\pi}\log n + c_1\sqrt{n}(\log\log n)^2\Big)$$
$$+ c_2 p(n)\exp(-(\log\log n)^2)(\log\log n)^2\frac{5\sqrt{6n}}{2\pi}\log n$$
$$+ O\Big(\frac{p(n)}{n}(\log\log n)^2\Big).$$

This finally gives

$$D_3 \leqslant (\log\log n)^2 p(n)\frac{\sqrt{6n}\log n}{2\pi} + O(p(n)\sqrt{n}(\log\log n)^4). \tag{3.4}$$

3.3. Parts in arithmetic progressions : the terms $\mathbf{D_1}$ and $\mathbf{D_2}$.

The estimates of D_1 and D_2 rely on a result concerning the mean value of $S(n,\gamma,\Gamma;d,r)$. We define

$$M(n) := \frac{1}{p(n)}\sum_{\gamma\in\mathcal{P}(n)} S(n,\gamma,\Gamma;d,r).$$

Theorem 3.1. ([DSS]**, Lemma 1)** *For $n\geqslant n_0$, $d\leqslant n$ and $\Gamma\leqslant \frac{\sqrt{6n}}{\pi}\log n$ we have*

$$M(n) = (E(n)+F(n))\Big(1+O\Big(\frac{(\log n)^2}{\sqrt{n}}\Big)\Big) + O(R(n)) + O(n^{-3})$$

where the functions E, F and R are defined by

$$E(n) := -\frac{\sqrt{6n}}{\pi d}\log\big(1-\exp(-\frac{\pi(r+(\Gamma'+1)d)}{\sqrt{6n}})\big),$$
$$F(n) := \big(\exp(\pi(r+\Gamma' d)/\sqrt{6n})-1\big)^{-1},$$
$$R(n) := \Big(\exp\big(\frac{\pi(r+(\Gamma'+1)d)}{\sqrt{6n}}\big)-1\Big)^{-1},$$

where Γ' is the smallest integer $\geqslant (\Gamma-r)/d$.

We apply this result to estimate D_2. We take $\Gamma = r = d = p$. This gives $\Gamma' = 0$. We remark that for $p < n^{1/10}$ we have the elementary facts

$$1 - \exp\Big(-\frac{2\pi p}{\sqrt{6n}}\Big) = \frac{2\pi p}{\sqrt{6n}}(1 + O(n^{-2/5})),$$
$$-\log\Big(1 - \exp(-\frac{2\pi p}{\sqrt{6n}})\Big) = \log\Big(\frac{\sqrt{6n}}{2\pi p}\Big) + O(n^{-2/5}) = \frac{\log n}{2} - \log p + O(1)$$

and

$$\big(\exp(\frac{\pi p}{\sqrt{6n}}) - 1\big)^{-1} = \frac{\sqrt{6n}}{\pi p} + O(1).$$

This gives for D_2

$$D_2 = p(n)(2\log\log n - 1)\sum_{p<n^{1/10}}\{\frac{\sqrt{6n}\log n}{2\pi p} - \frac{\log p\sqrt{6n}}{\pi p} + O(\frac{\sqrt{n}}{p})\}$$
$$= p(n)\big(2(\log\log n)^2\frac{\sqrt{6n}}{2\pi}\log n + O(\sqrt{n}\log n\log\log n)\big).$$

The upper bound for D_1 is similar :

$$D_1 = p(n)\sum_{\substack{p,q<n^{1/10}\\ p\neq q}}\frac{\sqrt{6n}}{\pi pq}(\frac{\log n}{2} - \log(pq) + O(1))$$
$$\leqslant p(n)\big(\frac{\sqrt{6n}\log n}{2\pi}(\log\log n)^2 + O(\sqrt{n}\log n\log\log n)\big).$$

Finally we have

$$D_1 - D_2 + D_3 \ll p(n)\sqrt{n}\log n\log\log n.$$

This ends the proof of Theorem 1.1.

4. Proof of Proposition 1.2

It follows from Theorem 2.1 that for almost all partitions the greatest part is less than $L(1 + o(1))$ (where L was defined at the beginning of paragraph 3.2). Furthermore, it is well known (e. g., see [T], p. 85) that for $k \to \infty$ we have $\Omega(k) \leqslant (1 + o(1))\log k/\log 2$ and $\omega(k) \leqslant (1 + o(1))\log k/\log\log k$. The upper bounds in (1.2) and (1.3) follow from these facts. The lower bounds in these two inequalities are direct consequences of the following lemma:

Lemma 4.1. **([DSS], Corollary 1 of Theorem 1)** *Let* $0 < \varepsilon < 1/2$, $1 \leqslant d \leqslant n^{1/2-\varepsilon}$. *For almost all partitions of* n *we have (for* $n \to \infty$*)*

$$\sum_{\substack{1\leqslant j\leqslant s(\gamma)\\ \gamma_j\equiv 0 \ (\mathrm{mod}\ d)}} 1 = (1+o(1))\frac{\sqrt{6n}}{\pi d}\log\Big(\frac{\sqrt{n}}{d}\Big). \tag{4.1}$$

It follows from this lemma that for all $d \leqslant n^{1/2-\varepsilon}$, for almost all partitions γ there exists at least one part $\gamma_j \equiv 0 \pmod d$. We apply this with $d = 2^k$, $k = \big[(1/2-\varepsilon)\log n/\log 2\big]$ to show the lower bound in (1.2). To prove the lower bound in (1.3) we take d as the largest integer less than $n^{1/2-\varepsilon}$ of the form $d = p_1 \cdots p_k$ where p_j is the j-th prime number. We have $k > (1/2-2\varepsilon)\frac{\log n}{\log\log n}$. This ends the proof of Proposition 1.2.

5. Proof of Theorem 1.3

5.1. A probabilistic approach.

Since by Theorem 2.1, the opposite of inequality (2.2) holds for almost all partitions γ, and since

$$\begin{aligned} s_0(\gamma) &= \sum_{\substack{1\leqslant i\leqslant s(\gamma)\\ \lambda(\gamma_i)=1}} 1 \\ &= \sum_{1\leqslant i\leqslant s(\gamma)} \frac{\lambda(\gamma_i)+1}{2} \\ &= \frac{1}{2}\sum_{1\leqslant i\leqslant s(\gamma)} \lambda(\gamma_i) + \frac{s(\gamma)}{2}, \end{aligned}$$

thus clearly it suffices to show that

$$\sum_{1\leqslant i\leqslant s(\gamma)} \lambda(\gamma_i) = O(f(n)\sqrt{n}) \tag{5.1}$$

for almost all $\gamma \in \mathcal{P}(n)$. We will prove this by a probabilistic argument. Let ξ_n denote the random variable which assigns the value

$$S(n,\gamma) = \sum_{1\leqslant i\leqslant s(\gamma)} \lambda(\gamma_i)$$

to the partition $\gamma \in \mathcal{P}(n)$ selected with probability $1/p(n)$. Now we will estimate the mean value $M(\xi_n)$, the second moment $M(\xi_n^2)$ and the standard deviation $D(\xi_n)$. The computation will be similar to the one in [DSS] (only the handling of the new "weights" $\lambda(\gamma_i)$ needs special attention), thus we will leave some details to the reader, and we will refer to [DSS] frequently.

We will need

Lemma 5.1. *For $n \geqslant 2$ and $1 \leqslant t \leqslant n^{5/8}$ we have*

$$\frac{p(n-t)}{p(n)} = (1+O(n^{-3/4}))\Big(1+\frac{t}{n}-\frac{\pi t^2}{4n\sqrt{6n}}-\frac{3\pi t^3}{8n^2\sqrt{6n}}+\frac{\pi^2 t^4}{192n^3}\Big)\exp\big(-\frac{\pi t}{\sqrt{6n}}\big).$$

Proof of Lemma 5.1 : this is Lemma 3 in [DSS].

5.2. Estimation of the mean value.

We will prove an analogue of Lemma 4 in [DSS]. In what follows we will write

$$z := \Big[6\frac{\sqrt{6n}}{\pi}\log n\Big]+1. \tag{5.2}$$

Lemma 5.2. *For $n \to \infty$ we have*

$$M(\xi_n) = \sum_{a\leqslant z}\lambda(a)\sum_{1\leqslant t\leqslant z/a}\big(1+\frac{at}{n}-\frac{\pi(at)^2}{4n\sqrt{6n}}\big)\exp\big(-\frac{\pi at}{\sqrt{6n}}\big)$$
$$+O(n^{-1/4}(\log n)^2).$$

Proof. Denote the multiplicity of the part a in $\gamma \in \mathcal{P}(n)$ by $m(a,\gamma)$. Then we have

$$\begin{aligned} M(\xi_n) &= \frac{1}{p(n)}\sum_{\gamma\in\mathcal{P}(n)} S(n,\gamma) \\ &= \frac{1}{p(n)}\sum_{\gamma\in\mathcal{P}(n)}\sum_{i=1}^{s(\gamma)}\lambda(\gamma_i) \\ &= \frac{1}{p(n)}\sum_{\gamma\in\mathcal{P}(n)}\sum_{a\leqslant n} m(a,\gamma)\lambda(a). \end{aligned} \tag{5.3}$$

Next we observe that

$$\sum_{\gamma\in\mathcal{P}(n)}\sum_{a\leqslant n} m(a,\gamma)\lambda(a) = \sum_{a\leqslant n}\lambda(a)\sum_{1\leqslant t\leqslant n/a} p(n-at). \tag{5.4}$$

Inserting this in (5.3) we get

$$\begin{aligned} M(\xi_n) &= \sum_{a\leqslant n}\lambda(a)\sum_{1\leqslant t\leqslant n/a}\frac{p(n-at)}{p(n)} \\ &= \sum_{a\leqslant z}\lambda(a)\sum_{1\leqslant t\leqslant z/a}\frac{p(n-at)}{p(n)} + \sum_{a=1}^{n}\lambda(a)\sum_{z/a<t\leqslant n/a}\frac{O(p(n-z))}{p(n)}. \end{aligned}$$

Whence, by Lemma 5.1, we obtain

$$M(\xi_n) = \sum_{a\leqslant z} \lambda(a) \sum_{1\leqslant t\leqslant z/a} (1+O(n^{-3/4})) \\ \times \Big(1+\frac{at}{n}-\frac{\pi(at)^2}{4n\sqrt{6n}}-\frac{3\pi(at)^3}{8n^2\sqrt{6n}}+\frac{\pi^2(at)^4}{192n^3}\Big)\exp\Big(-\frac{\pi at}{\sqrt{6n}}\Big) \\ + \sum_{a=1}^{n}\sum_{t\leqslant n} O(\exp(-6\log n)).$$

By standard computations it follows

$$\begin{aligned} M(\xi_n) &= \sum_{a\leqslant z} \lambda(a) \sum_{1\leqslant t\leqslant z/a} (1+O(n^{-3/4})) \\ &\quad \times \Big(1+\frac{at}{n}-\frac{\pi(at)^2}{4n\sqrt{6n}}+O\Big(\frac{(\log n)^4}{n}\Big)\Big)\exp\Big(-\frac{\pi at}{\sqrt{6n}}\Big) \\ &\quad + O(n^{-4}) \\ &= \sum_{a\leqslant z} \lambda(a) \sum_{1\leqslant t\leqslant z/a} (1+O(n^{-3/4}))\Big(1+\frac{at}{n}-\frac{\pi(at)^2}{4n\sqrt{6n}}\Big)\exp\Big(-\frac{\pi at}{\sqrt{6n}}\Big) \\ &\quad + O(n^{-4}) \\ &= \sum_{a\leqslant z} \lambda(a) \sum_{1\leqslant t\leqslant z/a} \Big(1+\frac{at}{n}-\frac{\pi(at)^2}{4n\sqrt{6n}}\Big)\exp\Big(-\frac{\pi at}{\sqrt{6n}}\Big) \\ &\quad + O\Big(\sum_{a\leqslant z}\sum_{t\leqslant z/a} n^{-3/4}\Big) + O(n^{-4}). \end{aligned}$$

Here the error term is

$$\begin{aligned} O\Big(n^{-3/4}\sum_{a\leqslant z}\frac{z}{a}\Big)+O(n^{-4}) &= O(n^{-3/4}z\log z)+O(n^{-4}) \\ &= O(n^{-1/4}(\log n)^2) \end{aligned}$$

which completes the proof of the lemma.

Lemma 5.3. *For $n\to\infty$ we have*

$$M(\xi_n) = O(\sqrt{n}). \tag{5.5}$$

Proof. By Lemma 5.2 we have

$$M(\xi_n) = \sum_{a\leqslant z} \lambda(a) \sum_{1\leqslant t\leqslant z/a} \Big(1+\frac{at}{n}-\frac{\pi(at)^2}{4n\sqrt{6n}}\Big)\exp\Big(-\frac{\pi at}{\sqrt{6n}}\Big) \\ + O(n^{-1/4}(\log n)^2)$$

$$(5.6)\qquad \begin{aligned}
&= \sum_{a\leqslant z}\lambda(a)\sum_{1\leqslant t\leqslant z/a}\Big(\exp\Big(-\frac{\pi a t}{\sqrt{6n}}\Big)+O\Big(\frac{(\log n)^2}{\sqrt{n}}\Big)\Big)\\
&\quad + O(n^{-1/4}(\log n)^2)\\
&= \sum_{a\leqslant z}\lambda(a)\sum_{1\leqslant t\leqslant z/a}\exp\Big(-\frac{\pi a t}{\sqrt{6n}}\Big)+O\Big(\frac{(\log n)^2}{\sqrt{n}}\sum_{a\leqslant z}\sum_{1\leqslant t\leqslant z/a}1\Big)\\
&\quad + O(n^{-1/4}(\log n)^2)\\
&= \sum_{a\leqslant z}\lambda(a)\frac{1-\exp\big(-\frac{\pi([z/a]+1)a}{\sqrt{6n}}\big)}{1-\exp\big(-\frac{\pi a}{\sqrt{6n}}\big)}-\sum_{a\leqslant z}\lambda(a)\\
&\quad + O\Big(\frac{(\log n)^2}{\sqrt{n}}z\log z\Big)+O(n^{-1/4}(\log n)^2)\\
&= \sum_{a\leqslant z}\frac{\lambda(a)}{1-\exp\big(-\frac{\pi a}{\sqrt{6n}}\big)}-\sum_{a\leqslant z}\lambda(a)\frac{\exp\big(-\frac{\pi([z/a]+1)a}{\sqrt{6n}}\big)}{1-\exp\big(-\frac{\pi a}{\sqrt{6n}}\big)}\\
&\quad -\sum_{a\leqslant z}\lambda(a)+O((\log n)^4)\\
&= U-V-\sum_{a\leqslant z}\lambda(a)+O((\log n)^4).
\end{aligned}$$

Write

$$(5.7)\qquad L(a):=\sum_{i=1}^{a}\lambda(i)$$

so that by the prime number theorem we have for $a\geqslant 2$,

$$(5.8)\qquad L(a)=O\Big(\frac{a}{(\log a)^2}\Big)$$

(and indeed, much more is true but this is enough for our purpose). Then we obtain by partial summation :

$$\begin{aligned}
U &= \sum_{a=1}^{z}(L(a)-L(a-1))\frac{1}{1-\exp\big(-\frac{\pi a}{\sqrt{6n}}\big)}\\
&= \sum_{a=1}^{z}L(a)\Big(\frac{1}{1-\exp\big(-\frac{\pi a}{\sqrt{6n}}\big)}-\frac{1}{1-\exp\big(-\frac{\pi(a+1)}{\sqrt{6n}}\big)}\Big)\\
&\quad+\frac{L(z)}{1-\exp\big(-\frac{\pi(z+1)}{\sqrt{6n}}\big)}.
\end{aligned}$$

By (5.8) we have

$$U \ll \sum_{a=1}^{z} L(a) \frac{\exp\big(-\frac{\pi a}{\sqrt{6n}}\big)\big(1-\exp\big(-\frac{\pi}{\sqrt{6n}}\big)\big)}{\big(1-\exp\big(-\frac{\pi a}{\sqrt{6n}}\big)\big)\big(1-\exp\big(-\frac{\pi(a+1)}{\sqrt{6n}}\big)\big)} + \frac{z}{(\log z)^2}$$

$$\ll \sum_{a\leqslant\sqrt{n}} L(a)\frac{n^{-1/2}}{a^2 n^{-1}} + \sum_{\sqrt{n}<a\leqslant z} \frac{a}{(\log n)^2}\exp\big(-\frac{\pi a}{\sqrt{6n}}\big)n^{-1/2} + \frac{\sqrt{n}}{\log n} \tag{5.9}$$

$$\ll \sqrt{n}\big(1+\sum_{2\leqslant a\leqslant n}\frac{1}{a(\log a)^2}\big)$$

$$+ n^{-1/2}(\log n)^{-2}\sum_{\sqrt{n}<a} a\exp\big(-\frac{\pi a}{\sqrt{6n}}\big) + \frac{\sqrt{n}}{\log n}$$

$$\ll \sqrt{n} + \sqrt{n}(\log n)^{-2} + n^{1/2}(\log n)^{-1}$$

$$\ll \sqrt{n}.$$

Finally we have

$$V \ll \sum_{a\leqslant z}\exp\big(-\frac{\pi z}{\sqrt{6n}}\big)\big\{1-\exp\big(-\frac{\pi a}{\sqrt{6n}}\big)\big\}^{-1}$$

$$\ll \exp(-6\log n)\Big(\sum_{a\leqslant\sqrt{n}}\frac{1}{1-\exp\big(-\frac{\pi a}{\sqrt{6n}}\big)} \tag{5.10}$$

$$+\sum_{\sqrt{n}<a\leqslant z}\frac{1}{1-\exp\big(-\frac{\pi a}{\sqrt{6n}}\big)}\Big)$$

$$\ll n^{-6}\Big(\sum_{a\leqslant\sqrt{n}}\frac{\sqrt{n}}{a} + \sum_{\sqrt{n}<a\leqslant z} 1\Big) \ll n^{-4}.$$

(5.5) follows from (5.6), (5.8), (5.9), and (5.10). This completes the proof of Lemma 5.3.

5.3. Estimation of the second moment.

Lemma 5.4. *For $n\to\infty$ we have*

$$M(\xi_n^2) = O(n). \tag{5.11}$$

Proof. Define $m(a,\gamma)$ as in the proof of Lemma 5.2. Then as in [DSS] and in the proof of Lemma 5.2 above, we have

$$M(\xi_n^2) = \frac{1}{p(n)}\sum_{\gamma\in\mathcal{P}(n)} S^2(n,\gamma)$$

$$= \frac{1}{p(n)}\sum_{\gamma\in\mathcal{P}(n)}\Big(\sum_{i=1}^{s(\gamma)}\lambda(\gamma_i)\Big)^2$$

$$= \frac{1}{p(n)} \sum_{\gamma\in\mathcal{P}(n)} \Big(\sum_{a_1\leqslant n} m(a_1,\gamma)\lambda(a_1)\Big)\Big(\sum_{a_2\leqslant n} m(a_2,\gamma)\lambda(a_2)\Big)$$

$$= \frac{1}{p(n)} \sum_{a_1=1}^{n}\sum_{a_2=1}^{n} \lambda(a_1)\lambda(a_2) \sum_{\gamma\in\mathcal{P}(n)} m(a_1,\gamma)m(a_2,\gamma)$$

$$= \frac{1}{p(n)} \sum_{a_1=1}^{n}\sum_{a_2=1}^{n} \lambda(a_1)\lambda(a_2) \sum_{t_1=1}^{n}\sum_{t_2=1}^{n} \sum_{\substack{\gamma\in\mathcal{P}(n)\\ m(a_1,\gamma)\geqslant t_1\\ m(a_2,\gamma)\geqslant t_2}} 1.$$

By (5.4) we have

$$M(\xi_n^2) = \frac{1}{p(n)} \sum_{\substack{1\leqslant a_1,a_2\leqslant n\\ a_1\neq a_2}} \lambda(a_1)\lambda(a_2) \sum_{\substack{t_1,t_2\geqslant 1\\ t_1a_1+t_2a_2\leqslant n}} p(n-t_1a_1-t_2a_2)$$
$$+ \frac{1}{p(n)} \sum_{a=1}^{n} \lambda^2(a) \sum_{1\leqslant t\leqslant n/a} (2t-1)p(n-ta).$$

We remove in the first term the condition $a_1 \neq a_2$.

$$M(\xi_n^2) = \frac{1}{p(n)} \sum_{a_1=1}^{n}\sum_{a_2=1}^{n} \lambda(a_1)\lambda(a_2) \sum_{\substack{t_1,t_2\geqslant 1\\ t_1a_1+t_2a_2\leqslant n}} p(n-t_1a_1-t_2a_2)$$

$$(5.12) \qquad + \frac{1}{p(n)} \sum_{a=1}^{n} \sum_{1\leqslant t\leqslant n/a} tp(n-ta)$$

$$= \sum_{a_1=1}^{n}\sum_{a_2=1}^{n} \lambda(a_1)\lambda(a_2) \sum_{\substack{1\leqslant t_1\leqslant z/a_1\\ 1\leqslant t_2\leqslant z/a_2}} \frac{p(n-t_1a_1-t_2a_2)}{p(n)}$$
$$+ \sum_{a=1}^{n} \sum_{1\leqslant t\leqslant z/a} t\frac{p(n-ta)}{p(n)} + O(n^{-2}).$$

Whence, by Lemma 5.1,

$$M(\xi_n^2) = \sum_{a_1=1}^{n}\sum_{a_2=1}^{n} \lambda(a_1)\lambda(a_2)(1+O(n^{-3/4}))$$

$$(5.13) \qquad \times \sum_{\substack{1\leqslant t_1\leqslant z/a_1\\ 1\leqslant t_2\leqslant z/a_2}} \Big(1 + \frac{t_1a_1+t_2a_2}{n} - \frac{\pi(t_1a_1+t_2a_2)^2}{4n\sqrt{6n}}\Big)$$

$$\times \exp\Big(-\frac{\pi(t_1a_1+t_2a_2)}{\sqrt{6n}}\Big)$$

$$+\sum_{a=1}^{n}\sum_{t\leqslant z/a} t(1+O(n^{-3/4}))(1+\frac{ta}{n}-\frac{\pi(ta)^2}{4n\sqrt{6n}})\exp\big(-\frac{\pi ta}{\sqrt{6n}}\big)$$
$$+O(n^{-2}).$$

In the first term we have

$$1+\frac{t_1a_1+t_2a_2}{n}-\frac{\pi(t_1a_1+t_2a_2)^2}{4n\sqrt{6n}}=(1+\frac{t_1a_1}{n}-\frac{\pi t_1^2a_1^2}{4n\sqrt{6n}})$$
$$\times(1+\frac{t_2a_2}{n}-\frac{\pi t_2^2a_2^2}{4n\sqrt{6n}})$$
$$-\frac{\pi t_1t_2a_1a_2}{2n\sqrt{6n}}+O(\frac{(\log n)^4}{n}).$$

Thus it follows from (5.13) that

$$M(\xi_n^2)=\sum_{a_1=1}^{n}\sum_{a_2=1}^{n}\lambda(a_1)\lambda(a_2)$$
$$\times\sum_{\substack{1\leqslant t_1\leqslant z/a_1\\1\leqslant t_2\leqslant z/a_2}}(1+\frac{t_1a_1}{n}-\frac{\pi t_1^2a_1^2}{4n\sqrt{6n}})(1+\frac{t_2a_2}{n}-\frac{\pi t_2^2a_2^2}{4n\sqrt{6n}})$$
$$\times\exp\big(-\frac{\pi(t_1a_1+t_2a_2)}{\sqrt{6n}}\big)$$
$$-\sum_{a_1=1}^{n}\sum_{a_2=1}^{n}\lambda(a_1)\lambda(a_2)\sum_{\substack{1\leqslant t_1\leqslant z/a_1\\1\leqslant t_2\leqslant z/a_2}}\frac{\pi t_1t_2a_1a_2}{2n\sqrt{6n}}\exp\big(-\frac{\pi(t_1a_1+t_2a_2)}{\sqrt{6n}}\big)$$
$$+\sum_{a=1}^{n}\sum_{t\leqslant z/a} t(1+\frac{ta}{n}-\frac{\pi(ta)^2}{4n\sqrt{6n}})\exp\big(-\frac{\pi ta}{\sqrt{6n}}\big)$$
$$+\sum_{a_1=1}^{n}\sum_{a_2=1}^{n}\sum_{\substack{1\leqslant t_1\leqslant z/a_1\\1\leqslant t_2\leqslant z/a_2}}O(n^{-3/4}+\frac{(\log n)^4}{n})+\sum_{a=1}^{n}\sum_{t\leqslant z/a}tO(n^{-3/4})$$
$$+O(n^{-2}).$$

Whence we have

$$M(\xi_n^2)=\Big(\sum_{a=1}^{n}\lambda(a)\sum_{1\leqslant t\leqslant z/a}(1+\frac{ta}{n}-\frac{\pi t^2a^2}{4n\sqrt{6n}})\exp\big(-\frac{\pi ta}{\sqrt{6n}}\big)\Big)^2$$
$$-\Big(\sum_{a=1}^{n}\lambda(a)\sum_{t\leqslant z/a}\frac{\sqrt{\pi}ta}{\sqrt{2}6^{1/4}n^{3/4}}\exp\big(-\frac{\pi ta}{\sqrt{6n}}\big)\Big)^2 \tag{5.14}$$

$$
\begin{aligned}
&+\sum_{a=1}^{n}\sum_{t\leqslant z/a} t\Big(1+\frac{ta}{n}-\frac{\pi(ta)^2}{4n\sqrt{6n}}\Big)\exp\Big(-\frac{\pi ta}{\sqrt{6n}}\Big)\\
&+O(z^2(\log z)^2 n^{-3/4}+z^2n^{-3/4}+n^{-2})\\
=\;& X+Y+Z+O(n^{1/4}(\log n)^4),
\end{aligned}
$$

say. By Lemma 5.2 and Lemma 5.3 we have

$$X=(M(\xi_n)+O(n^{-1/4}(\log n)^2))^2=O(n). \tag{5.15}$$

Clearly we have

$$Y\leqslant 0. \tag{5.16}$$

The estimate of Z:

$$
\begin{aligned}
Z&\ll\sum_{a=1}^{z}\sum_{t\leqslant z/a} t\exp\big(-\pi ta/\sqrt{6n}\big)\\
&\ll\sum_{a=1}^{z}\sum_{t=1}^{\infty} t\exp\big(-\pi ta/\sqrt{6n}\big)\\
&\ll\sum_{a=1}^{z}\big(1-\exp\big(-\pi a/\sqrt{6n}\big)\big)^{-2}\\
&\ll\sum_{a\leqslant\sqrt{n}}\frac{n}{a^2}+\sum_{\sqrt{n}<a\leqslant z}1\ll n.
\end{aligned}
\tag{5.17}
$$

(5.11) follows from (5.14), (5.15), (5.16) and (5.17) which completes the proof of Lemma 5.4.

5.4. Completion of the proof of Theorem 1.3.

Lemma 5.5. *For $n\to\infty$ we have*

$$D(\xi_n)=O(\sqrt{n}).$$

Proof. This follows from

$$D^2(\xi_n)=M(\xi_n^2)-M^2(\xi_n)$$

and Lemma 5.3 and Lemma 5.4.

Finally, by Chebyshev's inequality we have for any $\sigma>0$

$$P(|\xi_n-M(\xi_n)|>\sigma D(\xi_n))<\frac{1}{\sigma^2}.$$

(5.1) follows from this inequality by Lemma 5.3 and Lemma 5.5, and this completes the proof of Theorem 1.3.

References

[DSS] C. Dartyge, A. Sárközy, M. Szalay, *On the distribution of the summands of partitions in residue classes*. Acta Math. Hungar. **109** (2005), 215–237.

[EL] P. Erdős, J. Lehner, *The distribution of the number of summands in the partitions of a positive integer*. Duke Math. Journal **8** (1941), 335–345.

[ST2] M. Szalay, P. Turán, *On some problems of the statistical theory of partitions with application to characters of the symmetric group II*. Acta Math. Acad. Sci. Hungar. **29** (1977), 381–392.

[ST3] M. Szalay, P. Turán, *On some problems of the statistical theory of partitions with application to characters of the symmetric group III*. Acta Math. Acad. Sci. Hungar. **32** (1978), 129–155.

[T] G. Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres, 2e édition*. Cours spécialisés **n° 1**, Société mathématique de France (1995).

Cécile Dartyge
Institut Élie Cartan
Université Henri Poincaré–Nancy 1
BP 239
54506 Vandœuvre Cedex, France
E-mail: dartyge@iecn.u-nancy.fr

András Sárközy, Mihály Szalay
Department of Algebra and Number Theory
Eötvös Loránd University
H-1117 Budapest
Pázmány Péter sétány 1/C, Hungary
E-mail: sarkozy@cs.elte.hu
E-mail: mszalay@cs.elte.hu