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On sum-sets and product-sets of complex numbers

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RÉSUMÉ. On donne une preuve simple que pour tout ensemble fini de nombres complexes A, la taille de l'ensemble de sommes $A+A$ ou celle de l'ensemble de produits $A \cdot A$ est toujours grande.

ABSTRACT. We give a simple argument that for any finite set of complex numbers A, the size of the the sum-set, $A+A$, or the product-set, $A \cdot A$, is always large.

1. Introduction

Let A be a finite subset of complex numbers. The *sum-set* of A is $A+A = \{a+b : a, b \in A\}$, and the *product-set* is given by $A \cdot A = \{a \cdot b : a, b \in A\}$. Erdős conjectured that for any n-element set the sum-set or the product-set should be close to n^2. For integers, Erdős and Szemerédi [7] proved the lower bound $n^{1+\varepsilon}$.

$$\max(|A+A, |A \cdot A|) \geq |A|^{1+\varepsilon}.$$

Nathanson [9] proved the bound with $\varepsilon = 1/31$, Ford [8] improved it to $\varepsilon = 1/15$, and the best bound is obtained by Elekes [6] who showed $\varepsilon = 1/4$ if A is a set of real numbers. Very recently Chang [3] proved $\varepsilon = 1/54$ to finite sets of complex numbers. For further results and related problems we refer to [4, 5] and [1, 2].

In this note we prove Elekes' bound for complex numbers.

Theorem 1.1. *There is a positive absolute constant c, such that, for any finite sets of complex numbers A, B, and Q,*

$$c|A|^{3/2}|B|^{1/2}|Q|^{1/2} \leq |A+B| \cdot |A \cdot Q|,$$

whence $c|A|^{5/4} \leq \max\{|A+A|, |A \cdot A|\}$.

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2. Proof

For the proof we need some simple observations and definitions. For each $a \in A$ let us find "the closest" element, an $a' \in A$ so that $a' \neq a$ and for any $a'' \in A$ if $|a-a'| > |a-a''|$ then $a = a''$. If there are more then one closest elements, then let us select any of them. This way we have $|A|$ ordered pairs, let us call them *neighboring pairs.*

Definition. We say that a quadruple (a, a', b, q) is *good* if (a, a') is a neighboring pair, $b \in B$ and $q \in Q$, moreover

$$\left|\{u \in A+B : |a+b-u| \leq |a-a'|\}\right| \leq \frac{28|A+B|}{|A|}$$

and

$$\left|\{v \in A \cdot Q : |aq-v| \leq |aq-a'q|\}\right| \leq \frac{28|A \cdot Q|}{|A|}.$$

When a quadruple (a, a', b, q) is good, then it means that the neighborhoods of $a+b$ and aq are not very dense in $A+B$ and in $A \cdot Q$.

Lemma 2.1. *For any $b \in B$ and $q \in Q$ the number of good quadruples (a, a', b, q) is at least $|A|/2$.*

Proof. Let us consider the set of disks around the elements of A with radius $|a-a'|$ (i.e. for every $a \in A$ we take the largest disk with center a, which contains no other elements of A in it's interior). A simple geometric observation shows that no complex number is covered by more then 7 disks. Therefore

$$\sum_{a \in A} \left|\{u \in A+B : |a+b-u| \leq |a-a'|\}\right| \leq 7|A+B|$$

and

$$\sum_{a \in A} \left|\{v \in A \cdot Q : |aq-v| \leq |aq-a'q|\}\right| \leq 7|A \cdot Q|$$

providing that at least half of the neighboring pairs form good quadruples with b and q. Indeed, if we had more then a quarter of the neighboring pairs so that, say,

$$\left|\{v \in A \cdot Q : |aq-v| \leq |aq-a'q|\}\right| > \frac{28|A \cdot Q|}{|A|}$$

then it would imply

$$7|A \cdot Q| \geq \frac{|A|}{4} \left|\{v \in A \cdot Q : |aq-v| \leq |aq-a'q|\}\right| > 7|A \cdot Q|.$$

□

Proof of Theorem 1 To prove the theorem, we count the good quadruples (a, a', b, q) twice. For the sake of simplicity let us suppose that $0 \notin Q$. Such a quadruple is uniquely determined by the quadruple $(a+b, a'+b, aq, a'q)$. Now observe that there are $|A+B|$ possibilities for the first element, and given the value of $a+b$, the second element $a'+b$ must be one of the $28|A+B|/|A|$ nearest element of the sum-set $A+B$. We make the same argument for the third and fourth component to find that the number of such quadruples is at most

$$|A+B|\frac{28|A+B|}{|A|}|A \cdot Q|\frac{28|A \cdot Q|}{|A|}.$$

On the other hand, by Lemma 1 the number of such quadruples is at least

$$\frac{|A|}{2}|B||Q|$$

that proves the theorem.

A similar argument works for quaternions and for other hypercomplex numbers. In general, if T and Q are sets of similarity transformations and A is a set of points in space such that from any quadruple $(t(p_1), t(p_2), q(p_1), q(p_2))$ the elements $t \in T$, $q \in Q$, and $p_1 \neq p_2 \in A$ are uniquely determined, then

$$c|A|^{3/2}|T|^{1/2}|Q|^{1/2} \leq |T(A)| \cdot |Q(A)|,$$

where c depends on the dimension of the space only.

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